Revisting the Rate of Change

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Abstract:

The Mean Value Theorem is a great theory and guide for any body who deals with the rate of change. This paper aims to add something to the theory. Even a small addition is better than none, only time will tell the significance of the addition.

1.1

It is understood from the Mean Value Theorem that for any polynomial with a degree 2 or higher the derivative is not accurate. Table 1 shows Y = X and the difference, *diff,* between each x accumulation as well as the derivative. The last column shows the difference in percentage terms of $f'(X)$/ difference, as can be seen from table 1 $f'(X)$/ difference in percentage terms is 100%, they are equal, therefore the derivative is equal.

Difference Between 'real difference' and differentiation of Y = X				
X	X = Y	diff	$f'(X) = 1$	%
0	0	n/a	0	N/A
1	1	1	1	100.00
2	2	1	1	100.00
3	3	1	1	100.00
4	4	1	1	100.00
5	5	1	1	100.00
6	6	1	1	100.00
7	7	1	1	100.00
8	8	1	1	100.00
9	9	1	1	100.00
10	10	1	1	100.00
11	11	1	1	100.00
12	12	1	1	100.00
13	13	1	1	100.00
14	14	1	1	100.00
15	15	1	1	100.00

Table 1

Table 1 is as expected from theory, however let us look at table 2 that shows the difference between the derivative and the 'real difference' for $Y = X^2$.

Difference Between 'real difference' and differentiation of $Y = X^2$				
X	$X^2 = Y$	diff	$f'(X) = 2X$	%
0	0	n/a	0	N/A
1	1	1	2	200.00
2	4	3	4	133.33
3	9	5	6	120.00
4	16	7	8	114.29
5	25	9	10	111.11
6	36	11	12	109.09
7	49	13	14	107.69
8	64	15	16	106.67
9	81	17	18	105.88
10	100	19	20	105.26
11	121	21	22	104.76
12	144	23	24	104.35
13	169	25	26	104.00
14	196	27	28	103.70
15	225	29	30	103.45
20	400	39	40	102.56
30	900	59	60	101.69
50	2,500	99	100	101.01
Table 2				

Table 2 confirms the derivative will always be higher than the ‘real difference’. As well as the fact that between X = 0, and X = 1, the derivative can greatly exaggerate what the rate of change is and must continuously recover. Therefore as the independent variable gets larger, as expected in ratio terms the derivative gets closer to the ‘real difference’, though in the case of $Y = X^2$, the derivative is always 1 greater than the ‘real difference’.

Tables 3, 4, and 5 illustrate the real difference and the derivative for $Y = X^3$, $Y = X^4$, and $Y = X^5$ respectively. As can be seen from tables, 3, 4, and 5, the derivative always higher than the real difference, however the ratio of the difference falls as X, or the independent variable increases. This would be a real problem for example if one is using the derivative to illustrate the marginal revenue.

Difference Between 'real difference' and differentiation of $Y = X^3$				
X	$X^3 = Y$	diff	$f'(X) = 3X^2$	%
0	0	n/a	0	N/A
1	1	1	3	300.00
2	8	7	12	171.43
3	27	19	27	142.11
4	64	37	48	129.73
5	125	61	75	122.95
6	216	91	108	118.68
7	343	127	147	115.75
8	512	169	192	113.61
9	729	217	243	111.98
10	1,000	271	300	110.70
11	1,331	331	363	109.67
12	1,728	397	432	108.82
13	2,197	469	507	108.10
14	2,744	547	588	107.50
15	3,375	631	675	106.97
20	8,000	1,141	1,200	105.17
30	27,000	2,611	2,700	103.41
50	125,000	7,351	7,500	102.03
Table 3				

Difference Between 'real difference' and differentiation of $Y = X^4$				
X	$X^4 = Y$	diff	$f'(X) = 4X^3$	%
0	0	n/a	0	N/A
1	1	1	4	400.00
2	16	15	32	213.33
3	81	65	108	166.15
4	256	175	256	146.29
5	625	369	500	135.50
6	1,296	671	864	128.76
7	2,401	1,105	1,372	124.16
8	4,096	1,695	2,048	120.83
9	6,561	2,465	2,916	118.30
10	10,000	3,439	4,000	116.31
11	14,641	4,641	5,324	114.72
12	20,736	6,095	6,912	113.40
13	28,561	7,825	8,788	112.31
14	38,416	9,855	10,976	111.37
15	50,625	12,209	13,500	110.57
20	160,000	29,679	32,000	107.82
30	810,000	102,719	108,000	105.14
50	6,250,000	485,199	500,000	103.05
Table 4				

Difference Between 'real difference' and differentiation of $Y = X^5$				
X	$X^5 = Y$	diff	$f'(X) = 5X^4$	%
0	0	n/a	0	N/A
1	1	1	5	500.00
2	32	31	80	258.06
3	243	211	405	191.94
4	1,024	781	1,280	163.89
5	3,125	2,101	3,125	148.74
6	7,776	4,651	6,480	139.32
7	16,807	9,031	12,005	132.93
8	32,768	15,961	20,480	128.31
9	59,049	26,281	32,805	124.82
10	100,000	40,951	50,000	122.10
11	161,051	61,051	73,205	119.91
12	248,832	87,781	103,680	118.11
13	371,293	122,461	142,805	116.61
14	537,824	166,531	192,080	115.34
15	759,375	221,551	253,125	114.25
20	3,200,000	723,901	800,000	110.51
30	24,300,000	3,788,851	4,050,000	106.89
50	312,500,000	30,024,751	31,250,000	104.08
Table 5				

Now a theory has been established as to why the derivative is always higher, (The Mean Value Theorem), a theory and real prove, the prove being in tables 1 – 5, it is time to correct the mistake and look for a new derivative function that will suite our needs for precision and accuracy.

1.2 Finding the Accurate Function for Rate of Change

To find the accurate differential we must find the function that will define $f'(X)$ – real difference (*diff*). Remember that in a simple linear function and the basic general rule of thumb:

Given a function $f(X) = aX^n$ (1)
Then the derivative $f'(X)$ is:
$f'(X) = naX^{n-1}$ (2)

We as model builders and social scientist like to be accurate as possible when we can, we want to know the real derivative, $f'_k(X)$.

$f'_k(X) = f'(X) - \text{diff}$ (3)

We can build a general formulae step by step, but first we need to understand why there is a difference between the derivative and the real change, a more compelling reason than merely the tangent, because that will not allow us to extract a real derivative. Take a polynomial say of degree 3, say the function $Y = X^3$. $X^3 = X \square X \square X$. Therefore $Y = X^3$ has properties of $Y = X^2$ and $Y = X^2$ in turn has properties that influence it from X. These properties need to be taken out because they are included in the derivative. The larger the polynomial function the more amplified are these properties that need to be taken out. We need to sort of distill the derivative in order to arrive at the real difference, to distill implies purify. In the case of $Y = X^3$, we need to get rid of the properties from the derivative that defines $Y = X^2$ and $Y =X$.

One can see this effect if one looks at tables 1 – 5 and by how much percentage points, by how high the ratio is that defines derivative over the real difference. When X = 1, Y = 1, and the derivative is 100% in table 1, 200% in table 2, 300% in table 3, 400% in table 4, and 500% in table 5. Table 1, is when X^1, table 2, X^2, table 3, X^3, table 4, X^4 and table 5, X^5. One can see the influence gets larger and larger, it is this influence that must be removed, this residual influence of $Y = X^n$, and the residual influence is equal to $Y = X^{n-1}$ as well as the influence of $\square(X^{n-2} \square X^1)$. Obviously the greatest influence would be the preceding polynomial. $\square(X^{n-2} \square X^1)$ is the residual effect of all the other preceding polynomials without including the immediate preceding polynomial, $X^1 = X$.

Knowing what causes the difference between the derivative and the real derivative we can then attempt to build a general formula for the rate of change, in polynomials, but the conclusions will be universal, for all derivatives. Let us look at $Y = X^2$.

Correcting the derivative of Y = X2 should be a fairly simple affair as the only effect that has to be taken out will be the effect of Y = X. Looking at table 1, we look for the pattern that defines the difference between the derivative and the real difference, we see that it is 1, the derivative of Y = X, that also happens to be the real difference. Therefore the rate of change for $Y = X^2$ the real derivative, f'_k is:

$f'_k(X) = f'(X) - f'_k(X^{n-1}) - R$ (4)
where R is defined as the residual created by $\square(X^{n-2} \square X)$
in this case $X^{n-2} = 0$ and X is already included. Therefore equation (4) becomes

$f'_k(X) = 2X - 1$ (5)

Equation (4) can be the general formula, but it is better to make it look more simple. Not that for $Y = X^2$, the real difference is just the derivative minus 1. That 1 is the real change of $Y = X$.

It becomes more complex when we move up to the real change of $Y = X^3$. How do we get the real change, the real derivative. We have a guide in equation (4) Therefore we know that:

$f'k(X) = f'(X) - f'k(X^{n-1}) - R$ that leads to
$f'k(X) = 3X^2 - 2X - R$ where
$3X^2 = f'(X)$ and
$2X = f'k(X^{n-1})$ from equation (5)
Now we need to find R the function that defines the residual effects of all the other residual effects.

Table 6 is an extension of table 3, it gives us a step by step solution on finding R.

				1	2	3	4	
X	$X^3 = Y$	diff	$f'(X) = 3X^2$	2X	$3X^2$ - 2X	residual	add 1	$f'_k(X)$
0	0	0	0	0	0	0		
1	1	1	3	2	1	0	1	1
2	8	7	12	4	8	1	2	7
3	27	19	27	6	21	2	3	19
4	64	37	48	8	40	3	4	37
5	125	61	75	10	65	4	5	61
6	216	91	108	12	96	5	6	91
7	343	127	147	14	133	6	7	127
8	512	169	192	16	176	7	8	169
9	729	217	243	18	225	8	9	217
10	1,000	271	300	20	280	9	10	271
11	1,331	331	363	22	341	10	11	331
12	1,728	397	432	24	408	11	12	397
13	2,197	469	507	26	481	12	13	469
14	2,744	547	588	28	560	13	14	547
15	3,375	631	675	30	645	14	15	631

table 6

As mentioned above we first must subtract 2X from $3X^2$ and we arrive at what is the residual. This is under 3 in table 6. Trying to make sense of the residual we see from table 6 that the residual is 1 less than X. To get a function of the residual one will find that to create a function divisible by X one must always add 1 or subtract 1, in all polynomial functions by taking this action one will find the residual will then be divided by X. Adding 1 to the residual we get X. Therefore when $Y = X^3$, the residual is:
$R = (X - 1)$
Therefore taking our guide from equation (4) we get

$f'_k(X) = f'(X) - f'_k(X^{n-1}) - R$ (4)
$f'_k(X) = 3X^2 - 2X - (X - 1)$ (6)
we have to subtract 1 from X because we added 1 in table six in order to get the X in the first place. From equation 6 we get the real derivative for $Y = X^3$:
$f'_k(X) = 3X^2 - 2X - X + 1$ and by simplification we get
$f'_k(X) = 3X^2 - 3X + 1$ (7)

We can test it out to be sure, we can take X = 7 and we get
$3(7^2) - 3(7) + 1$
$= 147 - 21 + 1 = 127$ and it is dead on. At one it equals 1.

To clarify our understanding let us take one more example, let us take the polynomial $Y = X^4$.We take guidance from equation (4).

$f'_k(X) = f'(X) - f'_k(X^{n-1}) - R$ (4)
Therefore when $Y = X^4$ the real rate of change would be
$f'_k(X) = 4X^3 - 3X^2 - 3X - R$. where
$4X^3 = f'(X)$
$3X^2 - 3X = f'_k(X^{n-1})$
We now will need to solve for R.
Solving for R, that is to say to see the function that defines the residual we shall take the same route as above, the same route as was used to solve for R to find the real rate of change for $Y = X^3$, the $f'_k(X^3)$. The process is made easy to understand by including a table to show each step by step process. We include table 7.

In table 7, the first step is to take out the known and be left with R, we therefore find $4X^3 - 3X^2 - 3X$. This value is clearly shown in table 7. When X = 13 from table 7 we see that $4X^3 - 3X^2 - 3X$ is 8242 and when X = 4, $4X^3 - 3X^2 - 3X$ is 196.

				take out of $3X^2$	take out of 3X	1st Residual $4X^3$-$3X^2$-3X	Ist Residual - diff	take out 1	take out X = R
X	$X^4 = Y$	diff	$f'(X) = 4X^3$	$3X^2$	3X	$4X^3$-$3X^2$-3X			
0	0	0	0	0	0	0	0	-1	#DIV/0!
1	1	1	4	3	3	-2	-3	-4	-4
2	16	15	32	12	6	14	-1	-2	-1
3	81	65	108	27	9	72	7	6	2
4	256	175	256	48	12	196	21	20	5
5	625	369	500	75	15	410	41	40	8
6	1,296	671	864	108	18	738	67	66	11
7	2,401	1,105	1,372	147	21	1,204	99	98	14
8	4,096	1,695	2,048	192	24	1,832	137	136	17
9	6,561	2,465	2,916	243	27	2,646	181	180	20
10	10,000	3,439	4,000	300	30	3,670	231	230	23
11	14,641	4,641	5,324	363	33	4,928	287	286	26
12	20,736	6,095	6,912	432	36	6,444	349	348	29
13	28,561	7,825	8,788	507	39	8,242	417	416	32
14	38,416	9,855	10,976	588	42	10,346	491	490	35
15	50,625	12,209	13,500	675	45	12,780	571	570	38

Table 7

Having arrived at the known solution of $4X^3 - 3X^2 - 3X$ we subtract the difference from the original table 4 and we arrive at what is called the 1st residual - diff. The diff being the real difference. Arriving at the solution we subtract 1. As mentioned above at this stage we must always add or subtract 1 in order to have a function that is related to X. At this stage after adding one we find that R at X = 15 is 570 and at say 3 is 6. After taking out 1 we find we can divide by X. The remaining solution follows the function $3X - 7$, therefore R for $Y = X^4$ is:
$(3X - 7)X + 1$. This can be simplified to:
$3X^2 - 7X + 1$

Taking guidance from equation (4)

$f'_k(X) = f'(X) - f'_k(X^{n-1}) - R$ (4) we get
$f'_k(X) = 4X^3 - 3X^2 - 3X - (3X^2 - 7X + 1)$, this can be simplified to:
$f'_k(X) = 4X^3 - 3X^2 - 3X - 3X^2 + 7X - 1$ further simplification:
$f'_k(X) = 4X^3 - 6X^2 + 4X - 1$ (8)

To get real change for $Y = X^5$ the $f'_k(X)$ taking guidance from above we get
$f'_k(X) = 5X^4 - 4X^3 - 6X^2 + 4X - R$
and we solve for R, not forgetting to add or subtract 1 before solving for R.

1.3 Why Is this Important in Economics

The simple exercise below will show the importance of $f'_k(X)$.
Take an economic phenomenon Y. This phenomenon is defined as
$Y = f(\Delta A, B, C)$ (a)
Where ΔA = change in A and
$A = X^3$
$B = -2X^2$
$C = X$
The function for Y is defined as
$Y = \Delta A + B + C$ (b)
Therefore:
$Y = \Delta(X^3) - 2X^2 + X$ (c)
Taking equation (1) from above, recalling equation (2) = $f'(X) = naX^{n-1}$. Y would be defined as
$Y = 3X^2 - 2X^2 + X$ (d) which simplified becomes
$Y = X^2 + X$ (e)

However we understand equation (e) to be wrong because we know that $\Delta(X^3)$ is not $3X^2$ but from equation (7) $\Delta(X^3)$ is = $3X^2 - 3X + 1$. Therefore the true function of Y is:
$Y = 3X^2 - 3X + 1 - 2X^2 + X$ (f) that when simplified becomes:
$Y = X^2 - 2X + 1$ (g)
(g) is correct and is fundamentally different from (e).

Conclusion

It is more prudent to use $f'_k(X)$, it is correct. Obviously if we had access to a supercomputer we would have solved for every polynomial, but the above is a good enough demonstration. In the symbol $f'_k(X)$ the k stands for Khumalo, therefore $f'_k(X)$ is known as the Khumalo derivative after the person who discovered it.

Reference:

Khumalo, B. (2009). The Variable Time: Crucial to Understanding Knowledge Economics. The Icfai University Journal of Knowledge Management, Volume 7, No 1, pp 34 -67.