No. 36 august 2008

**Erwin Amann**
**Heng Qiao**

# Parallel Contests

Research Paper Series
Thurgau Institute of Economics and Department of Economics
at the University of Konstanz

THURGAU INSTITUTE
OF ECONOMICS
at the University of Konstanz

# Parallel Contests *

Heng Qiao
Department of Economics, University of Konstanz, Germany

Erwin Amann
Department of Economics, University of Duisburg-Essen, Germany

July 2008

**Abstract**

A problem of *Parallel Contests* is raised and modeled. The equilibria in final situations of parallel contests are analyzed and characterized and the behaviours of contestants with different abilities' parameters are explained. Given that the values of the prizes in the two contests are different, in equilibrium a group of strong players prefer entering into the contest with a higher prize. However, except the group of stronger ones, in equilibrium others will enter into both contests because they obtain equal expected revenue from the two contests, though these weak ones do not have equal probabilities to enter into the two parallel contests. Under the condition of rationalizability, this paper characterizes the respective distributions of contestants' abilities in the two parallel contests, proves the existence of the equilibrium in parallel contests and completes the analysis of the parallel contests from the perspective of contestants.

*Keywords*: Parallel contests, Contest, Strategic behaviours

*JEL Classification*: C72; D44; D82

## 1 Introduction

In real world it is easy to observe numerous cases in which there exists competition among contest designers. For instance, in sporting contests two or more tournaments are hold closely in time or even synchronously. Facing this situation, since one player's energy and preparation time are limited, it may not be a good strategy to participate two different tournaments in a very short period. Therefore, one prefers to choose one tournament to enter even if time schedule

*We wish to thank Prof. Elmar Wolfstetter and discassants for helpful comments in the seminar of Department of Economics, Humboldt-University at Berlin. Email contact: heng.qiao@uni-konstanz.de (Heng Qiao), erwin.amann@uni-due.de (Erwin Amann)

permits participation of both. Similar cases happen to lots of economic environments, such as the competition of two or more employers in a same industry. One can also find examples in research contests.

Theoretically, Gradstein and Konrad (1999)[2] focus on the optimal structure of multiple-round contests with identical individuals. Moldovanu and Sela (2006)[5] introduce more complicated model to deal with the optimal architecture of contests, involving one-stage and two-stage contests. Fullerton and McAfee (1999)[1] investigate the optimal number of contestants in a research tournament model with heterogeneous contestants and also consider the optimal design of tournment. Moldovanu and Sela (2006)[5] point to the significance of analyzing competition among contest designers. However, it is rare to find a paper dealing with the above problem of parallel contests.

Here we introduce a whole picture of the *parallel contests*. Assume that there are two parallel contests denoted by contest 1 and contest 2, which are both open for all potential contestants. The designers of the two contests have their own objective function, represented by their utility functions respectively, which are usually the expected total efforts or the expected highest efforts in their contests. A standard process of the parallel contest is that firstly the two designers decide the prizes' values of the two contest and declare them to all contestants. After knowing the values of the prizes, players make an entry decision between the two contests. A Player competes with others within one of the contests. Finally according to the normal winning rule of contests, each of the two designers claims that the player with a maximal effort in the specific contest wins the prize. And the designers also obtain the realized revenue (or efforts) from the players.

In this paper we focus only on the entry choice of players into the two parallel contests, i.e. the problem that given the value of the two prizes, how contestants will choose one of the two contests to enter. We analyze and characterize the equilibrium in equilibrium of the parallel contests.

Now we compare the information settings of one- and multiple-stage contests with that of parallel contests. Firstly, in a one-stage contest the two typical assumptions are (1) that the types of contestants are drawn from an interval according to a distribution (*iid*), which is common knowledge and (2) that the total number of players is also common knowledge. These common knowledges allow the rationalizability in one-stage contest. Secondly, in a contest with more than one stage, or called elimination tournaments, without lose of generality, in a typical two stage contest $n$, contestants are split into $t$ parallel and symmetric sub-contests. The number of players in each subcontest is common knowledge. Using *oder statistics*, the distribution function at the second stage can be derived and is also common knowledge. Therefore, from the perspective of information setting, the structure of one-stage contests and tournaments are essentially similar, because the number of players and distribution functions are all common knowledge.

In a parallel contest, i.e., a competition between two contests with different settings, we also assume that the number and the distribution of total population are common knowledge. However, the types' distribution before all players' entry into parallel contests transforms into two different distributions in the parallel contests after all players' entry into the two contests. And generally the above transformation of distributions can not be directly derived without an analysis of the equilibrium bidding strategies in the two contests. This is the prominent feature in the information structure of parallel contests, which is distinguished from typical one-stage contests and tournaments.

This paper assume that the numbers of players and the distributions in the parallel contests are also common knowledge, because without the above assumption a player can not make a rational decision on her entry choices between the two contests except certain special case when the prizes in the parallel contests are equal. In other words, these common knowledges allow the rationalizability in parallel contests.

The rest of this paper is organized as follows. In Section 2 we introduce a grand contest as the basis of all following contest models. The eqilibrium bid function and payoff function are derived and some basic results are proved. In Section 3 we introduce some assumptions for parallel contests. An important concept, Final Situation, is defined. In Section 4 the parallel contests are analyzed qualitatively and then quantitatively. An equilibrium situaiton of parallel contests in final situation is characterized. Section 5 concludes.

# 2 Basic Model and Analysis

## 2.1 Description of the basic model

First we introduce a general one-stage contest with the participation of all contestants, which is often called a grand contest in contests literature. The set of contestants is $K = \{1, 2, ..., k\}$. Each player $i$ makes an effort $x_i$. All players' efforts are submitted simultaneously. An effort $x_i$ causes a cost, denoted by $c_i\gamma(x_i)$ [4], where $\gamma : R_+ \to R_+$ is a strictly increasing function with $\gamma(0) = 0$, and where $c_i > 0$ is an ability parameter and also seen as the type of this contestant. Note that a low $c_i$ means that $i$ has a high ability and vice versa.

The ability of contestant $i$ is her private information (also her type). Abilities are drawn independently of each other from an interval $[m, 1]$ according to the distribution functions $F_j$, which is common knowledge. (in parallel contests, $j = 1, 2$ and in a grand contest we neglect the subscript $j$). We assume that any distribution function in this paper has continuous densities, when they are strictly larger than 0. In order to avoid infinite bids caused by zero costs, we assume that $m$, the type with the highest possible ability, is strickly positive.

In this paper we only consider the single-prize case. In a contest with a single prize $V_j$, the payoff of contestant $i$, who has ability $c_i$ and exerts an effort $x_i$ is either $V_j - c_i\gamma(x_i)$ if $i$ wins prize $v_j$, or $-c_i\gamma(x_i)$ if $i$ does not win a prize, i.e.,

$$R_i = \begin{cases} V_j - c_i\gamma(x_i), & \text{if } i \text{ wins the prize;} \\ -c_i\gamma(x_i), & \text{if } i \text{ wins nothing.} \end{cases}$$

Note $R_i$ as the payoff of contestant $i$. Here we consider the case of linear cost functions, i.e. the cost functions are linear in effort, $\gamma(x_i) = x_i$. Each contestant $i$ chooses her effort in order to maximize expected utility, given the other competitors' actions and the values of the prizes. In a grand contest with only one prize $V$, it is known that there exists a symmetric equilibrium bid function, denoted by $x_i = b(c_i)$. Then contestant $c_i$'s payoff can be written as

$$R_i = \max_{x_i} V(1 - F(b^{-1}(x_i)))^{k-1} - c_i x_i$$

## 2.2 Analysis

Now we first analyze the basic static model. Assume that the set of all the contestants is $K = \{1, 2, ..., k\}$. Abilities are drawn independently from an interval $[m, 1]$ according to distribution functions $F$, which means that if all $k$ contestants enter into a grand contest, any of them knows her own ability but looks any other one's ability as a variable randomly drawn from the distribution $F$.

Let $y$ denote the inverse of $b$, i.e. $y(x) = b^{-1}(x_i)$. Using strict monotonicity and symmetry, the first-order condition (FOC) is:

$$1 = -(k-1)V(1 - F(y))^{k-2} F'(y) y' \frac{1}{y}$$

Since a contestant with the lowest possible ability $c = 1$ never wins a prize (note: he can win a second prize, if more than one prizes are assumed. However here single prize is assumed for each contests in the beginning of analysis), the optimal effort of this type is always zero, which yields the boundary condition $y(0) = 1$. Denote

$$G(y) = (k-1)V \int_y^1 \frac{1}{t}(1 - F(t))^{k-2} F'(t) dt$$

The solution to the above differential equation with the boundary condition is given by

$$\int_x^0 dt = -G(y)$$

Then we have $x = G(y) = G(b^{-1}(x_i))$ and therefore $b \equiv G$, which defines the equilibrium bid function.

$$\begin{aligned} b(c_i) &= (k-1)V \textstyle\int_{c_i}^1 \frac{1}{t}(1 - F(t))^{k-2} dF(t) \\ &= -V \textstyle\int_{c_i}^1 \frac{1}{t} d(1 - F(t))^{k-1} \\ &= -V \left\{ \left. \frac{(1-F(t))^{k-1}}{t} \right|_{c_i}^1 - \textstyle\int_{c_i}^1 (1 - F(t))^{k-1} d\frac{1}{t} \right\} \\ &= V \frac{(1-F(c_i))^{k-1}}{c_i} + V \textstyle\int_{c_i}^1 (1 - F(t))^{k-1} d\frac{1}{t} \end{aligned}$$

Furthermore, we obtain the expression of the expected payoff

$$
\begin{aligned}
R(c_i) &= V(1-F(c_i))^{k-1} - c_i b(c_i) \\
&= -c_i V \int_{c_i}^{1} (1-F(t))^{k-1} d\frac{1}{t} \\
&= c_i V \int_{c_i}^{1} \frac{(1-F(t))^{k-1}}{t^2} dt
\end{aligned}
$$

**Lemma 1 (Type)** *According to the equilibrium bidding function, a contestant with a higher type (or lower ability) bids less, i.e. the equilibrium bid decreases in the type of the contestant. And the payoff of a contestant also decreases in her type.*

*Proof of the Lemma*
From the expression of the bidding function, we have

$$b'(c_i) = -V(k-1)\frac{1}{c_i}(1-F(c_i))^{k-2} dF(c_i) < 0$$

Then we have the result in this *Lemma* that the equilibrium bid decreases in the type of the contestant. In the following we prove that the payoff of a contestant decreases in her type. Firstly, we get the first derivative of the revenue,

$$R'(c_i) = V\int_{c_i}^{1} \frac{(1-F(t))^{k-1}}{t^2} dt - c_i V \frac{(1-F(c_i))^{k-1}}{c_i^2}$$

Then we obtian the second derivative of the revenue function, i.e.

$$
\begin{aligned}
&R''(c_i) = -V\tfrac{(1-F(c_i))^{k-1}}{c_i^2} - V\tfrac{-(k-1)c_i(1-F(c_i))^{k-2}F'(c_i)-(1-F(c_i))^{k-1}}{c_i^2} \\
&= V\tfrac{(k-1)(1-F(c_i))^{k-2}F'(c_i)}{c_i} \\
&> 0
\end{aligned}
$$

Therefore, the revenue function is convex. We assume that there exists a continuous interval in the definitional domain of function $\frac{(1-F(t))^{k-1}}{t^2}$. Using the Mean-Value Theorem for integration, there exists a number in this specific domain such that $c_i < c_s < 1$ and from the first derivative of the function, we then have

$$
\begin{aligned}
R'(c_i) &= V\tfrac{(1-F(c_s))^{k-1}}{c_s^2}(1-c_i) - V\tfrac{(1-F(c_i))^{k-1}}{c_i} \\
&= V\tfrac{(1-F(c_s))^{k-1}}{c_s}\tfrac{(1-c_i)}{c_s} - V\tfrac{(1-F(c_i))^{k-1}}{c_i}
\end{aligned}
$$

Since the function $\frac{(1-F(t))^{k-1}}{t^2}$ is decreasing in its variable $t$ and $c_i < c_s$, we have

$$\frac{(1-F(c_s))^{k-1}}{c_s} < \frac{(1-F(c_i))^{k-1}}{c_i}$$

When $\frac{(1-c_i)}{c_s} \leq 1$ or $c_i + c_s \geq 1$, we obtain $R'(c_i) \leq 0$. Additionally, when $c_i + c_s < 1$, from the convexity of the revenue function, we can also conclude that in the definitional domain of the distribution function, the revenue function is decreasing in the type of contestant. ■

This *Lemma* implies that the grand contest is a typical and usually-used mechanism, in which the strong contestants exert more effort and also obtain more payoff. In other words, this mechanism provides incentives for stronger participants to exert more and get more.

**Lemma 2 (Number of players)** *In equilibrium, a contestant's payoff decreases in the total number of the contestants. But the equilibrium bid increases in the total number of the contest.*

*Proof of Lemma 2*

From the expression of $b(c_i)$, we have

$$\begin{aligned}
\frac{\partial b_j(c_i)}{\partial k} &= \tfrac{V_j}{c_i}(1-F(c_i))^{k-1}\ln(1-F(c_i) + V_j\int_{c_i}^1 (1-F(t))^{k-1}\ln(1-F(t))d\tfrac{1}{t} \\
&= \tfrac{V_j}{c_i}(1-F(c_i))^{k-1}\ln(1-F(c_i) + V_j\ (1-F(t))^{k-1}\ln(1-F(t))\tfrac{1}{t}\Big|_{c_i}^1 \\
&\quad -V_j\int_{c_i}^1 \tfrac{1}{t}d\left[(1-F(t))^{k-1}\ln(1-F(t))\right] \\
&= \tfrac{V_j}{c_i}(1-F(c_i))^{k-1}\ln(1-F(c_i) + V_j\lim_{t\to 1}(1-F(t))^{k-1}\ln(1-F(t))\tfrac{1}{t} \\
&\quad -\tfrac{V_j}{c_i}(1-F(c_i))^{k-1}\ln(1-F(c_i) - V_j\int_{c_i}^1 \tfrac{1}{t}d\left[(1-F(t))^{k-1}\ln(1-F(t))\right] \\
&= V_j\lim_{t\to 1}\tfrac{\ln(1-F(t))}{t(1-F(t))^{1-k}} - V_j\int_{c_i}^1 \tfrac{1}{t}d\left[(1-F(t))^{k-1}\ln(1-F(t))\right] \\
&= \frac{\tfrac{1}{1-F(t)}\left[-F'(t)\right]}{(1-F(t))^{1-k} + t(1-k)(1-F(t))^{-k}\left[-F'(t)\right]} \\
&\quad -V_j\int_{c_i}^1 \tfrac{1}{t}d\left[(1-F(t))^{k-1}\ln(1-F(t))\right] \\
&= \frac{\left[-F'(t)\right](1-F(t))^{k-1}}{(1-F(t)) + t(1-k)\left[-F'(t)\right]} - V_j\int_{c_i}^1 \tfrac{1}{t}d\left[(1-F(t))^{k-1}\ln(1-F(t))\right] \\
&= -V_j\int_{c_i}^1 \tfrac{1}{t}d\left[(1-F(t))^{k-1}\ln(1-F(t))\right] \\
&\geq 0
\end{aligned}$$

In the above deduction, l'Hopital' rule are applied. Notice that $\lim_{t\to 1} F'(t) \geq 0$. Then differentiating the expression of the contestant's revenue under the integral sign by the variable "$k$", we have

$$\frac{dR(c_i)}{dk} = c_i V\int_{c_i}^1 \frac{(1-F(t))^{k-1}\ln(1-F(t)}{t^2}dt \leq 0$$

Since $\ln(1-F(t) \leq 0$ when $t \in [m,1)$, we know that the above expression is less than or equal to zero, and then we obtain the result in Lemma. ■

From the the above *Lemma*, the holder of the contest hope more contestants to participate when her aim is to maximize the expected total efforts or the expected maximal effort. Therefore, the grand contest is a standard contest, which is highly accordant to the intuition of a normal contest. Based on this model, we will analyse the extended model of *Parallel Contests*.

# 3 Parallel Contests

Following the introduction of the parallel contests in the first section, the total number of players is $n$ and the prizes in the two parallel contests are $V_1$ and $V_2$, respectively. We use vector $(k_j, V_j, F_j)$ to describe a situation in one of the parallel contests $j$, $j = 1, 2$, in which $V_j$ denotes the prize, $k_j$ denotes the number of contestants and all players hold the belief that their abilities are drawn from the distribution $F_j$. Each of the contestants will make a decision to enter into one of the two parallel contests. The basic rules of the contestants are listed below.

1. A contestant can choose one and only one of the two contests to enter;
2. A contestant knows the two abilities' distributions of the parallel contests in final situation, i.e. $F_1$ and $F_2$;
3. A contestant knows the final number of participants in the parallel contests.

Then the above three assumptions indicate that for any contestant the vectors $(k_j, V_j, F_j)$, $j = 1, 2$, of the parallel contests are common knowledge, which suffice the rationalizability of the game. According to the analysis of the basic contest model in the previous section, given the vector $(k_j, V_j, F_j)$ of a contest $j$, there exists equilibrium in contest $j$, which can be explicitly derived. So the equilibrium bidding function and equilibrium revenue of contestants in the two parallel contests can be explicitly derived. More specifically, all contestants hold the belief that it is available to observe and compute the vector $(k_j, V_j, F_j)$ in any situation of both parallel contests for all players. This belief allows all contestants making a rational decision in this game. In other words, the vector $(k_j, V_j, F_j)$ implies the indispensable information in the information structure of parallel contests, based on which contestants can make a rational decision on her entry problem. Since contestants' entry decision can change the vectors $(k_j, V_j, F_j)$, $j = 1, 2$, of the parallel contests, every contestant faces a strategic entry choice, which is interacting with all other contestants' entry choices.

Given the three assumptions above, we discuss the equilibrium concept in this paper. In game theory, an equilibrium is defined as a solution, in which no player can benefit from changing her own strategy unilaterally. Similarly, in an *Equilibrium* of parallel contests, no player can obtain more payoff by unilaterally changing her entry choice between the two contests. Then a *Final Situation* in parallel contests is defined by a situation vector $(k_j, V_j, F_j)$ in equilibrium. In other words, a final situation is represented by a vector when any participant in one contest do not have incentive to enter the other contest.

Now we discuss the characteristics of the equilibrium concept in parallel contests. Firstly we compare it with the equilibrium concepts in a single-stage contest and a multiple-stage contest. Generally in a single-stage contest model, a contestant's payoff can be represented by certain von Neumann-Morgenstern utility function, which is depending on the contestant's own strategy and others' strategies. In a multiple-stage contest, such as the two-stage contest in Moldlvanu and Sela (2006)[5], the symmetric equilibrium bidding function in the first stage and the sub-game-perfect equilibrium bidding function in the second stage can be derived explicitly. Given the number of contestants ( the total number and the numbers in each contests ), values of prizes and the distribution of contestants' abilities (for all contestants) as common knowledge, the payoff function in each stage can be expressed by a formular by the use of certain techniques, such as *Order Statistics*.

In parallal contests, a contestant can not rationally make a optimal decision on his entry choice until she can specify the vector $(k_j, V_j, F_j)$ in the equilibrium of final situation. In most cases the equilibrium in final situation is not unique. It is easy to conclude that when the vector $(k_j, V_j, F_j)$ is given as common knowledge, the equilibrium concept in this paper is Nash Equilibrium, i.e. the equilibrium concept here is self-enforcing and the result of a dynamic process. However, when any parameter in the vector $(k_j, V_j, F_j)$ is not given as common knowledge, contestants can not make an entry decision rationally, the game of parallel contests will not satisfy the condition of rationalizability and, therefore, the concept of Nash equilibrium can not apply to such a game. However, it will be much interesting to apply some methodology of irrationality to that case. Additionally, in the former case, i.e. when the equilibrium in parallel contests is accordant to Nash Equilibrium, it is still more complicated than the usually used concept of equilibrium, because it indicates an equilibrium between the two equilibria of the parallel contests.

Later on we will prove the existence of equilibrium in parallel contests. Here we discuss the characteristics of the final situation, given the existence of equilibrium. Note that the equilibrium in final situation means that no one will change her entry decision even when she is permitted to enter the other contest. One possible way of the existence of equilibrium is that the two prizes of the parallel contests are equal and then the two vectors $(k_j, V_j, F_j)$ of final situation in the parallel contests will be the same, i.e. $(\frac{n}{2}, V, F)$. Though this is an equilibrium, we are more interested in the case when the two prizes of the parallel contests are different, i.e. when some or all parameters $(k_j, V_j, F_j)$ of the parallel contests are different.

Since the equilibrium in final situation is based on the rationality of the entry decision of all players into the parallel contests, there are two possible ways for the existence of equilibrium. In the equilibrium of a final situation a player either finds that the payoff of one contest is definitely higher than that of the other one, or she finds that the payoffs of the parallel contests are equal. More specifically, in the former case, when a contestant with type $c$ finds that the payoff in contest 1 is definitely higher than that in contest 2, she will choose to enter into contest 1 rather than contest 2. Then she has no incentive to change her decision in the final situation. In the later case, if in equilibrium the payoff from the two contests are equal for a player with type $c$, she also has no incentive to change her previous entry decision. Though in this case the eqilibrium in final situation is a Nash equilibrium, this equilibrium is much sensitive to the change of the vector $(k_j, V_j, F_j)$, because a small change of any parameter may break the equivalence of the two payoff from the contests, which will give some contestants an incentive to change their entry decisions. For example, a change of entry decision by player with type $c$ will change the number of contests in the two contests and then make others having incentives to change. Therefore, it can be concluded that the equilibrium in final situation is unique only when the former case applies to every possible ability along the interval $[m, 1]$. In other words, if all possible contestant only strictly prefers one of the contests, the equilibrium must be a unique one. However, if there exists any contestant, whose payoff of the two contests are equal, the equilibrium in final situation will not be unique.

The following *Lemma* shows a basic result.

**Lemma 3** *When $V_1 \neq V_2$, the abilities' distribution function in the grand contest $F$ is different from the abilities' distribution functions in the equilibrium of the parallel contests, i.e. $F_1$ and $F_2$, which are also different between each other.*

*Proof of this Lemma*

We prove this Lemma by contradiction. Firstly, we assume that the distributions in equilibrium remain the same, i.e. still $F$, in the parallel contests. For any of the contests $j$ ($j = 1, 2$ ), any contestant with a type $c_i$ chooses her effort $x_i$ to maxmize her payoff, i.e.

$$\max_{x_i}\{V_j(1 - F(c_i))^{k_j - 1} - c_i x_i\}$$

From previous analysis, we have the equilibrium bid function,

$$b_j(c_i) = (k_1 - 1)V_j \int_{c_i}^{1} \frac{1}{t}(1 - F(t))^{k_j - 2} F'(t)dt$$

and the contestant's payoff expression

$$R_j(c_i) = c_i V_j \int_{c_i}^{1} \frac{(1 - F(t))^{k_j - 1}}{t^2} dt$$

Based on the above analysis, firstly we discuss the numbers of the participants in the two contests. In the final situation of the parallel contests, if there are equal numbers of contestants existing in both contests, from the above equilibrium bid function and payoff expression, any contestant has an incentive to enter the contest with a higher prize, because it provides higher payoff. Therefore, as mentioned at the beginning of the proof, if we assume that the distribution remains the same in equilibrium, the number of participants in the contest with a higher prize must be larger than that in the contest with a lower one, i.e. without loss of generality, if we assume $V_1 > V_2$, then we obtain $k_1 \geq k_2$.

In a final situation any contestant $c_i$ in one contest has no incentive to enter the other, i.e.

$$R_1(c_i) = R_2(c_i)$$

Then we have

$$c_i V_1 \int_{c_i}^{1} \frac{(1-F(t))^{k_1-1}}{t^2} dt = c_i V_2 \int_{c_i}^{1} \frac{(1-F(t))^{k_2-1}}{t^2} dt$$

Then for any $c_i$, we have

$$\int_{c_i}^{1} \frac{(1-F(t))^{k_1-1} - \frac{V_2}{V_1}(1-F(t))^{k_2-1}}{t^2} = 0$$

which means that for any $c_i$ the numerator under the integral sign also equals zero or $(1-F(t))^{k_1-k_2}$ equals a constant. This is not possible except that $k_1 = k_2$, which contradicts the assumption that in equilibrium $F$ remains the same in the parallel contests and also proves that the abilities' distributions in both contests are different between each other except the case when $k_1 = k_2$. ■

# 4 Analysis of Parallel Contests

In this section, we will characterize the equilibrium of parallel contests and firstly, we consider the entry problem from the perspective of contestants. Since the equilibrium in final situation comes from the rational entry decision of all contestants, in a final situation if any contestant with type $c_i$ has chosen contest 1 to enter, the following condition needs to be fulfilled.

$$U_{1,c_i}(V_1, k_1, F_1) \geq U_{2,c_i}(V_2, k_2 + 1, F_2)$$

which means that the payoff of the contestant's entering contest 1 is larger or at least equal to that of his entering contest 2, when there will be the number of $k_2 + 1$ contestants, competing with each other in contest 2. Then we have

$$c_i V_1 \int_{c_i}^{1} \frac{(1-F_1(t))^{k_1-1}}{t^2} dt \geq c_i V_2 \int_{c_i}^{1} \frac{(1-F_2(t))^{k_2}}{t^2} dt$$

Similarly, if any contestant with type $c_i$ has chosen contest 2 to enter, the following condition needs to be fulfilled

$$U_{2,c_i}(V_2, k_2, F_2) \geq U_{1,c_i}(V_1, k_1 + 1, F_1).$$

## 4.1 The relations among distribution functions

We assume that the total number of the contestants is $n$. Before their entry into the parallel contests, all the $n$ contestants' abilities are drawn from the interval $[m, 1]$ according to the distribution (*i.i.d.*) $F$. After the $n$ contestants choose contests to enter, we assume that in the final situation, there are $k_1$ contestants in contest 1 and $k_2$ contestants in contest 2. In the parallel contests, the contestants' abilities are drawn from the interval $[m, 1]$ according to the distributions $F_1$ and $F_2$, respectively.

Now we discuss the general relations among the three distributions, $F$, $F_1$ and $F_2$. We know that $F(c_i) = P(c \leq c_i)$ represents the probability when the random variable $c$ is less than $c_i$. Then $n \cdot F(c_i)$ implies the expected number of contestants whose types are less than $c_i$. We assume that all the contestants $n$ enter the two contests, i.e. $n = k_1 + k_2$. We have

$$n \cdot F(c_i) = k_1 \cdot F_1(c_i) + k_2 \cdot F_2(c_i)$$

This result can also be extended to a specific interval $[c_1, c_2] \in [m, 1]$, i.e.

$$n \cdot [F(c_2) - F(c_1)] = k_1 \cdot [F_1(c_2) - F_1(c_1)] + k_2 \cdot [F_2(c_2) - F_2(c_1)]$$

Intuitively, the above result impies that the expected number of contestants in a certain interval before contestants' entry into the parallel contests is equivalent to the expected total number of contestants in the same interval in the two parallel contests. In the special case when $k_1 = k_2 = \frac{n}{2}$, we obtain the following expression:

$$2F(c_i) = F_1(c_i) + F_2(c_i)$$

## 4.2 Qualitative Analysis of the entry process

Now we analyze a conceived process in which all players enter into the parallel contests. This process is not a real one, because we assume that at first all players enter into contest 1. From the previous section we know that it is easy to identify if an equilibrium is unique in parallel contests. Because all equilibriua share the same characteristics as they are defined, i.e. a situation when all players do not have incentive to change their entry decision, what is more important is to find a equilibrium no matter by a conceived way or in the spontaneous way.

**First Step:** *The Original Situation*

We consider here a conceived situation, in which all $n$ contestants enter the contest 1 and no one enters contest 2. In this situation, we have the revenue curve in contest 1:

$$R_1(c_i) = c_i V_1 \int_{c_i}^{1} \frac{(1 - F(t))^{n-1}}{t^2}$$

However, the revenue curve (where the payoff-axis is vertical and the ability-axis is horisontal) in contest 2 is a horizontal line, whose intersection with the payoff-axis is $V_2$, which means that any contestant can get a payoff of $V_2$ by exerting a very little effort, if only she is the only participant in contest 2. The comparison of the payoffs in the two contests shows an incentive for those, whose payoff in contest 1 is less than $V_2$, to enter contest 2. Superficially, this situation is not a real one, because it is easy to be broken when any player enters contest 2 and at the same time the distribution in contest 1 changes from $F$ to be $F_1$.

Notice that by the assumption of our parallel contests model, this change of distribution is observable for all contestants.

**Second Step:** *Effects when more players enter* contest 2.

Here we discuss the behaviour of contestants with different abilities, in the process when more players come from *contest 1* into *contest 2*, i.e. when $k_1$, the number of players in contest 1 decreases while $k_2$, the number of players in contest 2 increases.

We assume that $c_{2max}$ is the lowest type of contestant, who will enter contest 2 (or $c_{2max}$ is the strongest contestant who will enter contest 2).

It is easy to prove the following results of the change of distributions (More specifically in the proof of these results, the continuous distribution functions can be seen as discrete distribution functions and the results are obvious).

*In Contest 1:*

- For a strong contestant, i.e. one with a lower type $c_{low} \leq c_{2max}$, $F_1(c_{low})$ increases. It impies that more strongers stay in contest 1.
- But for a contestant with a higher type $c_{high} \geq c_{2max}$, $F_1(c_{high})$ decreases. It implies that more weakers enter into contest 2. We can say that the payoff of weakers in contest 1 increases .

We can not predict the payoff of strongers, because their winning probability decreases owing to the increase of the average ability in contest 1, while $k_1$ decreases. Intuitively, because no stronger $c_i < c_{2max}$ will leave contest 1, we can not predict that the strongers will bid less aggressive for the prize in contest 1.

It is easy to prove that $R_1''$, the second derivative in contest 1, decreases. This indicates the revenue curve in contest 1 changes to be less convex when $c_i$ changes to be larger and larger.

*In Contest 2:* When $k_2$, the number of players in contest 2 increases, we have the following results about the change of distribution $F_2(c_i)$.

- For player with type $c_i$, if one with a type less than $c_i$ enters into contest 2, $F_2(c_i)$ increases. The payoff of $c_i$ decreases.
- For player with type $c_i$, if one with a type larger than $c_i$ enters into contest 2, $F_2(c_i)$ decreases. It is hard to predict the change of payoff of $c_i$.

Therefore, the entry of players with middle abilities can decrease the payoff of weak ones. However, weak players can play much less influence on those stronger, because weak ones can play influence on strongers' revenue only through the change of the total number of players in a contest.

**The Third Step:** *The Equilibrium in the Final Situation*

As assumed, the distribution functions of the two contests are common knowledge. Here, we also assume that there exist enough contestants and discuss the equilibrium in final situation. The contestants in contest 1 have incentive to enter contest 2, until for any type $c_i$, her payoff in contest 2 approaches to be equal with that in contest 1, because if the payoffs between the two contests are different, the movement of contestants between the two contests will continue until no one has an incentive to change their decision.

We denote a contestant with type $c_{2max}$ as the contestant with the lowest type, who has an incentive to enter contest 2. In other words, $c_{2max}$ is the lowest type in contest 2, which represents the strongest player in contest 2. It is easy to know that in equilibrium the players will obtain equal payoffs from the two contests, except some strong players, who only exist in contest 1. More specifically, we have the following results.

- When $c_i < c_{2max}$, we have that $R_2(c_i) < R_1(c_i)$, which means that the contestant with type $c_i$ will only enter contest 1 and has no incentive to enter contest 2.
- When $c_i \geq c_{2max}$, we have that $R_1(c_i) = R_2(c_i)$. (We will discuss this critical type in later subsection.)

And the relation between the distributions of the two contests are

1. When $c_i < c_{2max}$, we have $F_2(c_i) = 0$;
2. When $c_i \geq c_{2max}$, we have $F_2(c_i) \geq 0$

## 4.3 An approximate result

In this subsection we prove a basic result, which will be used in later analysis.

**Lemma 4** *When the total number of the contestants is large enough and the difference of the two prizes is not so large, we have the following approximate result: In equilibrium, the numbers of the players in the parallel contests are equal, i.e.* $k_1 = k_2$*, or approximately equal.*

*Proof of the Lemma*

When the number of all the contestants is large enough, we can approximately ignore the difference between one's payoff in a contest with $k$ contestants and the payoff in a contest with $k+1$ contestants. This approximation simplifies the proof process.

For the parallel contests with prizes $V_1$ and $V_2$, i.e. contest 1 and contest 2, we denote the abilities' distributions are $F_1$ and $F_2$, respectively. Without loss of generality, we assume that $V_1 > V_2$. In equilibrium of final situation, for any $c_i \in (c_{2max}, 1]$ we have $R_1(c_i) = R_2(c_i)$, i.e.

$$V_1 \int_{c_i}^{1} \frac{(1 - F_1(t))^{k_1 - 1}}{t^2} dt = V_2 \int_{c_i}^{1} \frac{(1 - F_2(t))^{k_2 - 1}}{t^2}$$

Then

$$\int_{c_i}^{1} \frac{(1 - F_1(t))^{k_1 - 1} - \frac{V_2}{V_1}(1 - F_2(t))^{k_2 - 1}}{t^2} = 0$$

Since for any $c_i \in (m, 1)$ the above expression holds, we can make sure that for any $t$

$$(1 - F_1(t))^{k_1 - 1} - \frac{V_2}{V_1}(1 - F_2(t))^{k_2 - 1} = 0 \tag{1}$$

Rearranging the above expression, we obtain that

$$\frac{(1 - F_1(t))^{k_1 - 1}}{(1 - F_2(t))^{k_2 - 1}} = \frac{V_2}{V_1} > 0$$

However, because $\lim_{t \to 1}(1 - F_i(t))^{k_i - 1} = 0$, for any $i = 1, 2$, using the l'Hopital' rule to the left part of the above expression, we then get the following two results which contridict the the above inequality.

When $k_1 > k_2$, we obtain

$$\begin{aligned}\lim_{t \to 1} \frac{(1 - F_1(t))^{k_1 - 1}}{(1 - F_2(t))^{k_2 - 1}} &= \lim_{t \to 1} \frac{(k_1 - 1) \cdots (k_1 - k_2 + 1)(1 - F_1(t))^{k_1 - k_2}}{(k_2 - 1)! F_2'(t)} \\ &= 0\end{aligned}$$

Similar to this result, when $k_1 < k_2$, this expression will approach infinite.

These results are all contradictory against the previous expression, which indicates that the above limit should be a finite constant $\frac{V_2}{V_1}$. ■

Though we easily obtain the approximate result, we must still pay much attention to the situation when the assumption of this *Lemma* does not hold. In an interesting situation, when the value of the prize $V_1$ is suffciently lager than that of $V_2$, or the ratio of them can be seen as approaching infinite, it is easy to obtain that in equilibrium much more contestants enter *contest 1* than *contest 2*. In other words, except that $V_2$ is infinitely small comparing to $V_1$, this approximate result, shown in the above *Lemma*, holds.

## 4.4 Quantitative Analysis of the entry process

From results in previous subsections, for $c_i \geq c_{2max}$ we have the following set of equalities.

$$\begin{cases} 2F(c_i) = F_1(c_i) + F_2(c_i) \\ \dfrac{(1 - F_1(c_i))^{k-1}}{(1 - F_2(c_i))^{k-1}} = \dfrac{V_2}{V_1} \end{cases}$$

The second equality can be written as

$$\frac{(1 - F_1(c_i))}{(1 - F_2(c_i))} = \sqrt[k-1]{\frac{V_2}{V_1}}$$

We label $\sqrt[k-1]{\frac{V_2}{V_1}}$ as $C$.

Therefore, we obtain the solution of the equations

$$\begin{cases} F_1(c_i) = \dfrac{2F(c_i)C - C + 1}{C+1} \\ F_2(c_i) = \dfrac{2F(c_i) + C - 1}{C+1} \end{cases}$$

Comparing the two distribution functions, we have

$$(1 - F_1(c_i)) - (1 - F_2(c_i)) = -2(1 - C)(1 - F(c_i)) < 0$$

This means that for any contestant with type $c_i > c_{2max}$, it is with a higher probability for her to win in contest 2 than in contest 1. However, she obtains equal payoffs from the two contests, because $V_1 > V_2$. In the special case when $C = 1$, i.e. $V_1 = V_2$, the possibilities for her to win in both contests are equal. Additionally, from the previous group of equalities, we obtain the derivative for any type $c_i \geq c_{2max}$:

$$\frac{dF_1(c_i)}{dF_2(c_i)} = \frac{V_2}{V_1} \frac{(1 - F_2(c_i))^{k-2}}{(1 - F_1(c_i))^{k-2}}$$

Then we have

$$\begin{aligned} \frac{\frac{dF_1(c_i)}{dc_i}}{\frac{dF_2(c_i)}{dc_i}} \frac{(1 - F_2(c_i))}{(1 - F_1(c_i))} &= \frac{dF_1(c_i)}{dF_2(c_i)} \frac{(1 - F_2(c_i))}{(1 - F_1(c_i))} \\ &= \frac{V_2}{V_1} \frac{(1 - F_2(c_i))^{k-1}}{(1 - F_1(c_i))^{k-1}} = 1 \end{aligned}$$

We rewrite it as

$$r_1 = \frac{F_1'}{1 - F_1} = \frac{F_2'}{1 - F_2} = r_2$$

in which $r_1$ and $r_2$ denote the *hazard rates* of the distributions $F_1$ and $F_2$, respectively.

Denote $C_1$ and $C_2$ as the variables with the distributions $F_1$ and $F_2$, respectively. Using the results in the appendix of the paper (Moldovanu and Sela, 2006 [5] ), esp. *Theorem* 5 , we have the following *Lemma*.

**Lemma 5** *In equilibirum, for any contestant with type $c_i \geq c_{2max}$, we have $r_1 = r_2$. Then the following results hold.*

1. *The variable $C_1$ is equal to $C_2$ in the hazerd rate order, i.e. $C_1 =_{hr} C_2$ .*

2. *The variable $C_1$ is equal to $C_2$ in the usual stochastic order, i.e. $C_1 =_{st} C_2$.*

3. *$E[C_1] = E[C_2]$.*

This Lemma indicates that in equilibrium a player with type $c_i \geq c_{2max}$ obtians equal expected payoff from any of the two contests. This is a critical result, which implies that the equilibrium in final situation is not unique. When the total number of contestants is large enough, for a certain individual with type $c_i \geq c_{2max}$, one can not say which contest is optimal for her. No matter which contest she finally enters, the two choices, entering contest 1 and entering contest 2, bring the contestants with type $c_i \geq c_{2max}$ the same payoff in equilibrium. However, this does not mean that any equilibrium permits contestants to make decision irrationally, since irrationality will break the basic condition of a Nash Equilibrium. Intuitively the equilibrium is a stable situation, i.e. from the perspective of the balance of the numbers of contests and abilities of contestants between the parallel contests.

As a short review and conclusion of this subsection, though in equilibrium those players with $c_i \geq c_{2max}$ obtain the same payoff in both contests, we can not say that for players with $c_i \geq c_{2max}$, contest 1 can be seen as a mechanism equal to contest 2. The reason is that in contest 1 those strong players $c_i < c_{2max}$ play a deterrent influence on entry into contest 1, and specifically the existence of strong participants changes the distribution in contest 1.

## 4.5 A Critical Type

As defined in the previous subsection, the type $c_{2max}$ is the lowest type (or with the highest ability), having an incentive to enter into contest 2 in equilibrium. Now we analyze the contestant with this critical type. As shown in the above subsection, she can get equal payoff from both contests, i.e. $R_2(c_{2max}) = R_1(c_{2max})$ and has the highest possibility of winning the prize $V_2$, i.e. $F_2(c_{2max}) = 0$.

$$c_{2max} V_2 \int_{c_{2max}}^{1} \frac{(1 - F_2(t))^{k_2 - 1}}{t^2} dt = c_{2max} V_1 \int_{c_{2max}}^{1} \frac{(1 - F_1(t))^{k_1 - 1}}{t^2} dt$$

Then we rewrite it as

$$V_2 \int_{c_{2max}}^{1} \frac{(1 - F_2(t))^{k_2 - 1}}{t^2} dt = V_1 \int_{c_{2max}}^{1} \frac{(1 - F_1(t))^{k_1 - 1}}{t^2} dt$$

Since $F_2(c_{2max})) = 0$, we differentiate the above equality by $c_{2max}$ and obtain,

$$V_2 \frac{1}{c_{2max}^2} = V_1 \frac{(1 - F_1(c_{2max}))^{k-1}}{c_{2max}^2}$$

Rearranging it, then we get

$$(1 - F_1(c_{2max})) = \sqrt[k-1]{\frac{V_2}{V_1}} = C$$

Therefore, we have that $F_1(c_{2max}) = 1 - C$ or $c_{2max} = F_1^{-1}(1 - C)$. When $C$ approaches 1, $c_{2max}$approaches the smallest type $m$, which implies that when the values of the two prizes in contests are close enough, the behaviour of contestants in the two contests will approach to be the same.

The smaller $C$ is, the larger $c_{2max}$ will be. This implies that when the difference of the two prizes is larger, the critical point will also be larger, i.e. in equilibrium there will be more efficient competitors only in contest 1 and less players are indifferent between the two contests.

## 4.6 The Existence of *Equilibrium* of Parallel Contests

In this subsection we prove that for any type of contestant in equilibrium, no one can gain by changing her entry choices between the two contests. Remember that in previous subsections the total number of contestants is assumed to be large enough and, therefore, it is approximate that the change of one player's payoffs in contest 1, which is incurred by the event that any other player moves from one contest to the other, is so small that the difference of payoffs can be ignored, i.e.

$$c_i V_1 \int_{c_i}^{1} \frac{(1 - F_1(t))^{k_1 - 1}}{t^2} dt \approx c_i V_1 \int_{c_i}^{1} \frac{(1 - F_1(t))^{k_1}}{t^2} dt$$

However, in this subsection we release this assumption and assume that in equilibrium the number of players in contest 1 is $k_1$ and that in contest 2 is $k_2$. In the following we prove that no one wants to change her entry choice in equilibrium. It is easy to prove that in equilibrium all player with type $c_i < c_{2max}$ will enter contest 1. Now we prove that for player with type $c_i \geq c_{2max}$, the results in previous subsections still holds.

**When the player is in contest 1:**

Her payoff in contest 1 is that:

$$c_i V_1 \int_{c_i}^{1} \frac{(1 - F_1(t))^{k_1 - 1}}{t^2} dt$$

If this player will enter contest 2 and others will not change, her payoff will be

$$c_i V_2 \int_{c_i}^{1} \frac{(1 - F_2(t))^{k_2}}{t^2} dt$$

From previous subsection we have the equality in equilibrium

$$c_i V_1 \int_{c_i}^{1} \frac{(1 - F_1(t))^{k_1 - 1}}{t^2} dt = c_i V_2 \int_{c_i}^{1} \frac{(1 - F_2(t))^{k_2 - 1}}{t^2} dt$$

Therefore, we obtain

$$c_i V_1 \int_{c_i}^{1} \frac{(1 - F_1(t))^{k_1 - 1}}{t^2} dt > c_i V_2 \int_{c_i}^{1} \frac{(1 - F_2(t))^{k_2}}{t^2} dt$$

This inequality implies that a player in contest 1 will prefer staying in contest 1 rather than entering into contest 2.

**When the player is in contest 2:** Similarly, we obtain

$$c_i V_1 \int_{c_i}^{1} \frac{(1 - F_1(t))^{k_1}}{t^2} dt < c_i V_2 \int_{c_i}^{1} \frac{(1 - F_2(t))^{k_2 - 1}}{t^2} dt$$

This implies that a player in contest 2 will also prefer staying in contest 2 rather than entering into contest 1.

We conclude with the following proposition.

**Proposition 1** *No matter how many players paticipate in parallel contests, there exist equilibria in final situations, which are normally not unique. These equilibria are based on each player's rational entry decision into the parallel contests.*

# 5 Conclusion

From the perspective of the contestants, this paper analyses the entry choice between the two parallel contests. In the entry process, contestants with different types (or abilities) have different behaviours. Specifically, a group of contestants with higher abilities prefers entering the contest with a higher prize because they have a higher expected payoff in the contest with higher prize, which can even be higher than the lower prize itself. Therefore, in equilibrium we can not find any of these strong players existing in the contest with a lower prize. However, other players face other incentives. Surprisingly, if we assume that the number of players and distributions of types in the two contests are common knowledge, in equilibrium these weaker players can obtian equal revenue from any of the parallel contests. This results from the rational decision of contestants.

As described in the *Introduction*, the whole picture of parallel contests as a competition among contest designers is not yet addressed here. The analysis of parallel contests in this paper only discusses one central problem, i.e. the entry choice of players between the two parallel contests and the characterization of the equilibrium in final situation. The results and research avenue in this paper will be helpful for a further analysis of the whole model of parallel contests and relative mechanism problems, which is interesting from its theoretical and practical aspects.

# References

[1] Fullerton, Richard L. , R. Preston McAfee, (1999), *Auctioning Entry into Tournaments*, Journal of Political Economy, **107**(3), 573–605.

[2] Gradstein, Mark and Kai A. Konrad, (1999), *Orchestrating Rent Seeking Contest*, The Economic Journal, **109**, 536–545.

[3] Klemperer, Paul, (1999), *Auction Theory: A Guide to the literrature*, Journal of Economic Surveys, **13**(3), 227–286.

[4] Moldovanu, Benny and Aner Sela, (2001), *The Optimal Allocation of Prizes in Contests*, American Economic Review, **91**(3), 542–558.

[5] Moldovanu, Benny and Aner Sela, (2006), *Contest Architecture*, Journal of Economic Theory, **126**(1), 70–96.

[6] Szymanski, Stefan, (2003), *The Economic Design of Sporting Contests*, Journal of Economic Literature, **XLI**, 1137–1187.

THURGAU INSTITUTE
OF ECONOMICS
at the University of Konstanz

Hauptstr. 90
CH-8280 Kreuzlingen 2

Telefon: +41 (0)71 677 05 10
Telefax: +41 (0)71 677 05 11

info@twi-kreuzlingen.ch
**www.twi-kreuzlingen.ch**