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Three Sequential Cases: from Symmetry to Asymmetry *

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Abstract

Three critical cases, involving asymmetric and symmetric cases, in the sequential stages of the n-player repeated auctions are analyzed and compared. These cases might arise in a process of sequential, identical or equivalent auctions, where the auction result may reveal information about the strength or competitiveness of the participants. The behaviours of different players are characterized. Generally a player bids more aggressively when facing a strong player rather than a weak player. However a player favours competing with a weak one rather than a strong one. By applying the concept of *Conditional Stochastic Dominance*, revenues of players and the seller between the three stages are compared. It is proved that in this sequential process the information structure of the auctions changes and the seller's revenue increases. Finally, this n-player asymmetric auction model can also be used to compare the revenues between high-bid and open auctions and especially the results first derived by Maskin and Riley (2000) in two-player case are proved to be valid in the n-player case.

Keywords: Asymmetric auction; Revenue comparison

JEL Classification: C72; D44; D82

1 INTRODUCTION

In this paper we discuss the transformation of information structure during sequential stages of repeated auctions. The private information of the winners

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in previous stages releases gradually, which changes the information structure and the behaviours of players in later stages of auctions. By the analysis of the three typical and critical cases during all stages of the repeated auctions, this paper shows the changes of information structure and behaviours of different players. Firstly we assume that each stage of the repeated auctions is a sealed-bid first-price auction for a single and equivalent item, and assume that the ability (or the bidding cost) of any participant remains the same throughout all stages of the sequential auctions. In real world, this indicates that the process of the repeated auctions is restricted in a limited period so that the ability (or cost structure) of any player changes not much. Technically it permits the ability parameter of any player to be a constant through the analysis of all the stages.

Now we introduce the three cases of the repeated auctions. The first stage of the repeated auctions, as analyzed in *Case 1*, shows a symmetric case, where no private information about the valuation of the item is known except that any one's valuation of the item can be seen as randomly derived from a common distribution. After the first round or when the same auction repeats for some time, the information of the winner's valuation of the item is released and the situation described by *Case 2* emerges. In this stage, valuation of the winners in previous stages can be seen as drawn from a stochastically dominant distribution, comparing to the common distribution. After some times of the repeat, more information is released about the group of winners in the previous auctions. In the final round, weak players, knowing the strong players' private informtion, withdraw automatically or are ruled out of the final list by the seller. Then the *Case 3* emerges. Specifically we analyze the 2-player case in *Case 3*. In the following we describle the three cases in details.

Case 1: Symmetric n-player Case There are n players, participating in the sealed-bid auction for a single item. Any of the n players, $i = 1, 2, ..., n$, has private information on the valuation of the item, v_i. From other players' perspective, it is a random variable $\tilde{v}_i$ with c.d.f. $F(\cdot)$ on the support $[\underline{v}_i, \bar{v}_i]$, $0 \le \underline{v}_i < \bar{v}_i$. In other words, all the n players are symmetric since anyone 's valuation of the item is distributed according to the same function $F(\cdot)$ on the same support. Assume that $F(\cdot)$ is twice continuously differentiable on $(\underline{v}_i, \bar{v}_i]$ and the density function $F'(\cdot)$ is strictly positive on $[\underline{v}_i, \bar{v}_i]$. Therefore, it is easy to know that there exists a symmetric equilibrium, characterized by player i's bidding strategy $b_i(v_i)$, which can be easily derived explicitly.

We ssume that the sequential auctions are efficient, i.e. in any of the auctions the seller selects the strongest player of all participants as the winner. The process of the sequential auctions will reach the end of *Case 1*, if only the winner's information is revealed publicly, i.e. the distribution of her valuation is known by all other players. Therefore, the auctions in *Case 1* may repeat for more than one round until the above critical information of the winner is released. Additionally, we assume that in *Case 1* the distribution function is labelled as $F_i(\cdot)$, which indicates that in this case the seller see all players as 'weak player'. We have $F(\cdot) = F_i(\cdot)$. As a counterpart of the this situation, we conceive a symmetric auction with all strong players, in which the distribution function is labelled as $F_1(\cdot)$. This situation with purely strong players is only used for equilibrium and revenue comparison in later sections.

Case 2: Asymmetric n-player Case In this case there exist one strong player and $n-1$ normal players. Without loss of generality, the strong player is labelled as *Player 1*, whose valuation stochastically dominates those of other players. In other words, the strong player and other players are distributed according to the *c.d.f.* $F_1(\cdot)$ and $F_i(\cdot)$, respectively.

Comparing with the denotation in Case 1, we have $F(\cdot) = F_i(\cdot)$, which indicates that for $n-1$ normal players the distribution function does not change while the information of the strong player's distribution is revealed in this case for all participants.

Case 3 emerges, when the information of the second strong player releases in the previous auctions. The following *Case 3* is a two-player final round, where the final winner is selected. Generally there are several reasons, which may expain why other players do not participate any longer. One possible reason is that when knowing the information of the two strong players, other weak players are deferred from participating the later round any longer because participating in the following stages of auctions is much less or no more profitable for them. However, this is not a sufficient condition for all other players to draw off. In the real world the seller often specifies certian rules to select the final list of players in order to maximize seller's revenue. Theoretically, many literatures on auction and contests indicate that for a large of contests and tournaments the optimal number of contestants is two (see Fullerton and McAfee (1999)[2], Gradstein and Konrad(1999)[3]). Therefore, this paper specifies that in the final *Case 3*, only two strong players are final-listed.

Case 3: Final Round — 2-player Case In this case there are only two players active while other $(n-2)$ players have already drawn off or been ruled out from the game. We label the two players as *Player 1* and *Player 2*, whose valuations of the item are drawn from *c.d.f.* $F_1(\cdot)$ and $F_2(\cdot)$, respectively. In this final round, there are two possible situations, i.e. one symmetric situation and one asymmetric situation. In the asymmetric situation, the valuation of *Player 1* stochastically dominates that of *Player 2*. In the symmetric situation, two strong players are in the final round, whose valuations are both drawn from the same distribution function $F_1(\cdot)$.

Additionally, we conceive a symmetric situation involving two weak players, which can be seen as a counterpart of the two-strong-player situation and exists just for the convenience and completeness of analysis and comparison. This is not a real situation in the sequential auctions because if all the players are all weak ones, *Case 2* and *case 3* will not emerge.

In the real world, online auctions attract much attention of researchers. The results in this paper is helpful to explain phenomena in repeated online transactions and especially the repeated procurement auctions. On the other hand, from certain perspective some of the sequential stages in repeated auctions share similar information structure with multiple rounds of contests or tournaments. The change of information structure discussed here is useful to explain players' behaviour in different stages of tournament. Generally this paper prove that in a n-player case, a player will bid more aggressively when facing strong players rather than weak ones. For the designer or the seller, they obtain higher revenue, when players with higher abilities compete with each other.

Theoretically, this paper pay much attention on the anylysis of the asymmetric situations of auctions. Auction literature considering asymmetry is not so rich compared to other auction fields (see review [4] at pp. 236–237). In 1980s, some important studies assuming asymmetry developed the approach of marginal revenues and obtained important comparison results (see [9], [1] and [5]). As written By Klemperer[4], the large variety of different possible kinds of asymmetries makes it difficult to develop general results, but Maskin and Riley [6] make large strides, in which the concept of *Conditional stochastic Dominance* is applied in the analysis of asymmetric 2-player case, which solves the problem of revenue comparison between high-bid and open auction effectively.

In this paper, a n-player asymmetric auction is established and analyzed. Since the advantage of *CSD* is on revenue comparison, we apply the approach by Maskin and Riley (2000)[6] and extend the 2-player results of Maskin and Riley (2000) to n-player case.This paper gives the general results for n-player asymmetric auctions and makes revenue comparisons from the perspectives of both the players and the seller in the background of three cases. Finally, the results of revenue comparison between high-bid auction and open auction are generalized from 2-player case to the n-player case.

The rest of this paper is organized as follows. Section 2 analyzes the equilibrium bidding functions in the three cases and under the condition of *CSD* Section 3 makes further analysis and comparison of the equilibrium distributions and characterize the equilibrium inverse bid functions in various symmetric and asymmetric cases. Section 4 gives the players' equilibrium revenues and the seller's revenues in various symmetric and asymmetric cases. The seller's revenues in these cases are compared and ranked. In this section, the players' and seller's bahaviours in the three cases analyzed in previous sections are explained from the perspective of revenue comparison. Section 5 makes comparison between the revenue in open auction and that in first-price sealed-bid auction in various asymmetric n-player situations of auctions. This part theoretically complete the analysis of asymmetric auctions. Section 6 concludes.

2 EQUILIBRIUM BIDDING

2.1 Asymmetry, imperfect, 1, $N-1$

Without loss of generality, buyer 1 is assumed to be the buyer whose valuation is publicly known to stochastically dominate that of the others by the concept of *Conditional Stochastic Dominance*. Specifically suppose that for all $x < y$ in $[\underline{v}_1, \bar{v}_1]$.

$$Pr\{\tilde{v}_1 < x | \tilde{v}_1 < y\} = \frac{F_1(x)}{F_1(y)} < \frac{F_i(x)}{F_i(y)} = Pr\{\tilde{v}_i < x | \tilde{v}_i < y\}, \tag{2.1}$$

from which it is easy to obtain,

Corollary 1 CSD *implies*

$$F_i(v) > F_1(v), \text{ for all } v \in (\underline{v}_i, \bar{v}_1). \tag{2.2}$$

$$\underline{v}_i \leq \underline{v}_1 \text{ and } \bar{v}_i \leq \bar{v}_1 \tag{2.3}$$

$$\frac{F_1(x)}{F_i(x)} < \frac{F_1(y)}{F_i(y)} \text{ for all } x < y \text{ in } (\underline{v}_i, \bar{v}_1) \tag{2.4}$$

Notice (2.2) means that buyer 1's valuation first order stochastically dominates that of any others. And (2.2) implies (2.3). (2.4) indicates the monotony of increasing for $F_1(v)/F_i(v)$.

More specifically in this model, combined with the results from *CSD* we have

Corollary 2 *1. If* $\underline{v}_i < \underline{v}_1$

(a) $F_i(v) > 0 = F_1(v)$ *for all* $v \in (\underline{v}_i, \underline{v}_1)$

(b) $\frac{F_1(v)}{F_i(v)} < \frac{F_1(\bar{v}_1)}{F_i(\bar{v}_1)} = 1$ *for all* $v \in [\underline{v}_1, \bar{v}_1)$

(c) $\frac{d}{dv}\frac{F_1(v)}{F_i(v)} > 0$ *for all* $v \in [\underline{v}_1, \bar{v}_1]$

2. If $\underline{v}_i = \underline{v}_1 = \underline{v}$

(a) $F_1 = F_i = 0$

(b) $\exists \lambda \in (0,1)$ *and* $\gamma \in [\underline{v}, \bar{v}_i]$

$$\begin{cases} F_1 = \lambda F_i, & \text{for all } v \in [\underline{v}, \gamma]; \\ \frac{d}{dv}\frac{F_1(v)}{F_i(v)} > 0, & \text{for all } v \in [\gamma, \bar{v}_1]. \end{cases} \tag{2.5}$$

We assume that there exists an equilibrium in which buyer 1 uses the bidding strategy $b_1(v)$ and others use a symmetric bidding strategy $b_i(v)$, $i = 2, 3...n$.

Then buyer 1's and i's expected revenue, when bidding, b is

$$\begin{cases} U_1(b|v_1) \equiv (v_1 - b)Prob(b > b_i(v_i), i = 2, 3...n) \\ \quad = (v_1 - b)F_i^{n-1}(b_i^{-1}(b)) \\ U_i(b|v_i) \equiv (v_i - b)Prob(b > b_j(v_j), i \neq j)Prob(b > b_1) \\ \quad = (v_i - b)F_i^{n-2}(b_i^{-1}(b))F_1(b_1^{-1}(b)) \end{cases} \tag{2.6}$$

Rather than solving directly for the equilibrium bid functions, it is convenient to deal with inverse bid functions, which identify for some equilibrium bid the bidders' true valuation. Note that $\phi_i(b) = b_i^{-1}(b), i = 1, 2, ...n$. We then have

$$\begin{cases} U_1(b|v_1) = (v_1 - b)F_i^{n-1}(\phi_i(b)) \\ U_i(b|v_i) = (v_i - b)F_i^{n-2}(\phi_i(b))F_1(\phi_1(b)) \end{cases} \tag{2.7}$$

Taking logarithms and differentiating by b, we obtain the first-order condition *FOC*. Similar to the case of *Maskin and Riley (2000 and 2003)*, under certain conditions, that $\underline{v}_1$ must not be much larger relative to $\bar{v}_i$, there are unique minimum and maximum winning bids b_* and b^* for which there exists a solution to the following pair of differential equations.

$$\begin{cases} (n-1)\frac{F_i'(\phi_i)}{F_i(\phi_i)}\phi_i' = \frac{1}{\phi_1 - b} \\ (n-2)\frac{F_i'(\phi_i)}{F_i(\phi_i)}\phi_i' + \frac{F_1'(\phi_1)}{F_1(\phi_1)}\phi_1' = \frac{1}{\phi_i - b} \end{cases} \tag{2.8}$$

(All functions ϕ_i and ϕ_i' are evaluated at b for all $b \in [b_*, b^*]$.)Satisfying the boundary conditions

$$\begin{gathered} F_i(\phi_i(b^*)) = 1, i = 1, 2..., n. \\ \underline{v}_1 = \underline{v}_i \Rightarrow b_* = \phi_i(\underline{v}_1) = \phi_1(\underline{v}_1) = \underline{v}_1 \\ \underline{v}_1 > \underline{v}_i \Rightarrow b_* = \max \arg \max_b \{(\underline{v}_1 - b)F_i(b)\}, \phi_i(b_*) = b_*. \end{gathered} \tag{2.9}$$

Moreover, this solution constitutes the unique equilibrium of inverse bid functions, i.e.

$$\phi_i(b) = b_i^{-1}(b) \tag{2.10}$$

where $b_i(\cdot)$ is buyer i' equilibrium bid as a function of his valuation.

2.2 Symmetry, imperfect, N

Now we go back to consider the initial situation of the repeated auction, when all buyers' values are assumed to be symmetrically distributed. The random variable $\tilde{v}$ possibly has a higher or lower *c.d.f*, i.e. $F_1(\cdot)$ or $F_i(\cdot)$, which indicates that there are possibly two participation pools. Actually, before a buyer enter any of the two pools, he knows which one he chooses to enter. Let $y_i(b)$ be the symmetric equilibrium inverse bid function. Similar to (2.8), we have

$$(n-1)\frac{F_i'(y_i)}{F_i(y_i)}y_i' = \frac{1}{y_i - b}, \tag{2.11}$$

with boundary condition $y_i(\underline{v}_i) = \underline{v}_i$.

The subscript $i = 1$ implies that it is a pool of higher ability and $i > 1$ implies a pool of lower ability.

Rearranging (2.11), we obtain

$$(n-1)bF_i'(y_i)\frac{dy_i}{db} + F_i(y_i) = (n-1)y_iF_i'(y_i)\frac{dy_i}{db}$$

Multiply both sides of the equation by $F_i^{n-2}(y_i)$, it is easy to obtain

$$y_i dF_i^{n-1}(y_i) = bdF_i^{n-1}(y_i) + F_i^{n-1}(y_i)db$$

Let $\bar{b}_i(v)$ be the corresponding equilibrium bid function, i.e. the inverse of $y_i(b)$. Integrating the last equation, we get

$$\bar{b}_i(v)F_i^{n-1}(v) = \int_{\underline{\mathrm{v}}_i}^{v} y_i dF_i^{n-1}(y_i) \tag{2.12}$$

It follows immediately that in symmetric equilibrium a buyer's maximum possible bid is equal to the mean valuation

$$\bar{b}_i(\bar{v}_i) = \int_{\underline{\mathrm{v}}_i}^{\bar{v}_i} y_i dF_i^{n-1}(y_i) = E\{\tilde{v}_i\} \equiv \mu_i$$

From (2.2), we then have

$$\mu_1 > \mu_i \quad i = 2, 3, ..., n. \tag{2.13}$$

We rewrite (2.12) in a usually used form in literature

$$\bar{b}_i(v) = v - \frac{\int_{\underline{\mathrm{v}}_i}^{v} F_i^{n-1}(x)dx}{F_i^{n-1}(v)} \quad i = 1, 2, ..., n. \tag{2.14}$$

From *CSD*, we know

$$\bar{b}_i(v) \leq \bar{b}_1(v),\ \ for\ all\ v \in (\underline{\mathrm{v}}_1, \bar{v}_i), \quad i = 2, 3, ..., n. \tag{2.15}$$

2.3 Asymmetry, perfect, 2, $N-2$

Finally, there are only two active buyers left in this auction, because other buyers all leave the auction when information of the two strongest buyers is released from former auction, which indicates that other weaker buyers never profit by participation. Therefore, the repeated auction shrinks to a two-player asymmetric case.

Actually, the final situation may be either an imperfect Information case or a perfect Information case. Such cases are analyzed respectively here.

2.3.1 Imperfect Information case

As assumed in the *Introduction*, we still assume that this specific information is in the sense of first-order stochastic dominance and, more specifically, the valuations of the two strong buyers first-order stochastically dominate that of the other buyers and the valuations of the strongest buyer first-order stochastically dominates that of the second strongest one.

$$\begin{cases} U_1(b|v_1) = (v_1 - b)F_2(\phi_2(b)) \\ U_2(b|v_2) = (v_2 - b)F_1(\phi_1(b)) \end{cases} \tag{2.16}$$

3 ANALYSIS

Define

$$p_i(b) \equiv F_i(\phi_i(b)),\ i = 1, 2, ..., n.$$

$$H_i(\cdot) \equiv F_i^{-1}(\cdot),\ i = 1, 2, ..., n.$$

Therefore, $\phi_i(b) = H_i(p_i(b))$. We have stochastic dominance, $H_1(p) > H_i(p)$ for all $p \in (0, 1)$. Then we can rewrite (2.8) as

$$\begin{cases} (n-2)\frac{p_i'}{p_i} + \frac{p_1'}{p_1} = \frac{1}{H_i(p_i)-b} \\ (n-1)\frac{p_i'}{p_i} = \frac{1}{H_1(p_1)-b} \end{cases} \tag{3.1}$$

Similarly, for the symmetric equilibrium, we define

$$\pi_i(b) \equiv F_i(y_i(b)),\ i \text{ can be 1 or larger than 1.}$$

Then, we have

$$(n-1)\frac{\pi_i'}{\pi_i} = \frac{1}{H_i(\pi_i) - b} \tag{3.2}$$

The following results, i.e. *Lemma 1*, *Proposition 1*, *Corollary 3* and *Proposition 2* are the n-player version of the 2-player results in Maskin and Riley (2000)[6], (specifically the extension of *Lemma 3.2*, *Proposition 3.3*, *Corollary 3.4* and *Proposition 3.5*, respectively.) See proofs of the results in appendix, which follow similar approach of Maskin and Riley (2000). Therefore, these results apply to both 2-player case and n-player case well. In other words, the following two *Propositions* in this section apply both to *Case 2* and to *Case 3* and explain the behaviours in various situations.

Lemma 1 *If*

$$\begin{array}{l} F_i(v) > f_1(v),\ for\ all\ v \in (\underline{v}_i, \bar{v}_1). \\ \underline{v}_i = \underline{v}_1 = \underline{v}\ and\ F_1 = F_i = 0 \\ \frac{d}{dv}\frac{F_1(v)}{F_i(v)}\Big|_{v=\underline{v}} > 0 \end{array}$$

then there exists $\delta > \underline{v}$ *such that for all* $b \in [\underline{v}, \delta]$

1. $\pi_i(b) > \pi_1(b)$;

2. $p_i(b) > p_1(b)$;

3. $\pi_i(b) > p_i(b)$;

4. $p_1(b) > \pi_1(b)$.

in which $i = 2, 3, ..., n$.

Proposition 1 *Comparison of equilibrium bid distributions. Given* CSD, *then*

1. $\pi_i(b) > \pi_1(b)$, *For all* $b \in (\underline{v}_1, \mu_1)$;

2. $p_i(b) > p_1(b)$, *For all* $b \in (b_*, b^*)$;

3. $\pi_i(b) > p_1(b)$, *For all* $b \in (b_*, b^*)$;

4. $p_i(b) > \pi_1(b)$, *For all* $b \in (b_*, b^*)$.

in which $i = 2, 3, ..., n$.

Proposition 1 explains the behaviours of different players facing different situations in the n-player case, i.e. *Case 3*. The first result of *Proposition 1* is obvious. Comparing two symmetric auctions invloving purely strong players and weak players, respectively, the equilibrium bid distribution of the strong player stochastically donimates that of the weak player. The second result of *Proposition 1* compares the behaviours of different players in asymmetric case and indicates that the equilibrium bid distribution of the strong player stochastically donimates that of the weak player.

The third result of *Proposition 1* compares a asymmetric situation with a symmetric situation involving only weak players and characterizes the behaviour of weak players, i.e. if a weak player faces a strong buyer rather than another weak player, she responds with a more aggressive bid distribution in the sense of stochastic dominance.

The fourth result of *Proposition 1* compares an asymmetric situation with a symmetric situation involving only weak players and characterizes the behaviour of the strong player, i.e. if a strong buyer faces a weak player rather than another strong player, she will respond with a less aggressive bid distribution.

Since this proposition can applies to *Case 3*, it is easy to conclude that for the designer of the auction it is more profitable that the final round of the sequential auctions will turn out to be two strong players, because this result implies that the two strong players will bid more aggressively than other situations.

Corollary 3 *Given* CSD, *the following inequality holds*

$$\mu_i \leq b^* \leq \mu_1,$$

with at least one strict inequality and $i = 2, 3, ..., n$.

Proof. Suppose $b^* < \mu_i$. From (2.10), $1 = p_1(b^*) = \pi_i(\mu_i) > \pi_i(b^*)$ and so $p_1(b) > \pi_i(b)$ for b near b^*, a contradiction of *Proposition* 1 *part* 3. A similar contradiction follows from $b^* \geq \mu_1$. $\|$

Proposition 2 *Characterization of equilibrium inverse bid functions. Given* CSD*,*

1. $y_i(b) \geq y_1(b)$*, For all* $b \in (\underline{v}_1, \mu_1)$*;*
2. $\phi_1(b) > \phi_i(b)$*, For all* $b \in (b_*, b^*)$ *;*
3. $y_i(b) > \phi_i(b)$*, For all* $b \in (b_*, b^*)$ *;*
4. $\phi_1(b) > y_1(b)$*, For all* $b \in (\underline{v}_1, b^*)$ *.*

in which $i = 2, 3, ..., n$*.*

By the comparison of equilibrium inverse bid functions, this *Proposition* characterizes the bidding behaviours of different players in various situations of the n-player case, i.e. *Case 2*. The first result of *Proposition 2* compares two symmetric auctions invloving purely strong players and weak players, respectively, and indicates that the weak player shades her bid further below her valuation than the strong player.

The second result of *Proposition 2* characterizes the behaviours of different players in asymmetric equilibrium, i.e. in the asymmetric equilibrium, the strong player shades her bid further below her valuation than the weak player. The third result of *Proposition 2* compares a asymmetric situation with a symmetric situation involving only weak players and characterizes the behaviour of weak players, i.e. if a weak player faces a strong player rather than a weak player he will bid more aggressively (closer to her valuation). The fourth result of *Proposition 2* indicates that if a strong player faces a weak player rather than a strong player she will bid less aggressively. As a conclusion of the above two *Propositions*, in general a player bids more aggressively, when she faces strong players rather than weak players.

From *Proposition* 1 and *Part* 3, *Part* 4 of *Proposition* 2, we obtain the following *Corollary*.

Corollary 4 *For* $b \in (\underline{v}_1, b^*)$*, we have*

$$\pi_i(b) > p_i(b) > p_1(b) > \pi_1(b).$$

in which $i = 2, 3, ..., n$*.*

This *Corollary* can be seen as a conclusion of the above two *Propositions* and gives the basic principle of different players' behaviours in *Case 2* and *Case 3* in the sense of stochastic dominance.

4 REVENUE COMPARISON

4.1 Players' Equilibrium Revenue

For any player i, $i = 1, 2, ..., n$, denote $U_i^{As}(v, F_1, F_i)$ as buyer i's expected equilibrium revenue from an auction with asymmetric players when his reservation price is v. The strong player's reservation price is distributed according to F_1 and other symmetric buyers' reservation prices are distributed according to F_i), $i = 2, ..., n$.

(1) Player's revenue in asymmetric n-player case

Note $i = 2, 3, ..., n$, and we have,

$$R_1^{As}(v, F_1, F_i) = \max_b p_i^{n-1}(b)(v - b);$$

and

$$R_i^{As}(v, F_1, F_i) = \max_b p_i^{n-2}(b)p_1(b)(v - b).$$

(2) Player's revenue in symmetric n-player case

When all buyers are strong, we have

$$R_1^S(v, F_1, F_1) = \max_b \pi_1^{n-1}(b)(v - b) = \pi_1^{n-1}(\bar{b}_1)(v - \bar{b}_1).$$

In a symmetric auction with all weak players, we have

$$R_i^S(v, F_i, F_i) = \max_b \pi_i^{n-1}(b)(v - b) = \pi_i^{n-1}(\bar{b}_i)(v - \bar{b}_i).$$

where $i = 2, 3, ..., n$.

From *Corollary* 4, it is easy to obtain (see proof in Appendix)

Proposition 3 *We rank the expected players' revenues of the above asymmetric and symmetric auctions as*

$$R_1^{As}(v, F_1, F_i) > R_1^S(v, F_1, F_1);$$

and

$$R_i^{As}(v, F_1, F_i) < R_i^S(v, F_i, F_i).$$

This proposition implies that for a strong player it is more profitable to compete with weak players in an asymmetric auction than to compete with strong players in a symmetric auction. And it is the same for a weak player to prefer competing with less strong opponents. Since *Proposition 1* and *Proposition 2* indicate that a player bids more aggressively when she faces strong players rather than weak ones, it is easy to know that it is more profitable to compete with weak players than with strong ones.

4.2 Expected seller revenue

(1) Expected seller revenue in asymmetric n-player case

Any of the strong players' expected payment, if he bids $b \geq b_*$, is $bF_i^{n-1}(\phi_i(b))$. Since her equilibrium bid distribution has $c.d.f.F_1(\phi_1(b))$, the expectation over all bids is

$$R_1^{As} = \int_{b_*}^{b^*} bF_i^{n-1}(\phi_i(b))dF_1(\phi_1(b))$$

in which R_1^{As} note the expected seller revenue from the strong player in the asymmetric model and $i = 2, 3..., n$.

Any of the weak players' expected payment, if she bids $b \geq b_*$, is $bF_i^{n-2}(\phi_i(b))F_1(\phi_1($ Since her equilibrium bid distribution has $c.d.f.F_i(\phi_i(b))$, the expectation over all bids is

$$R_i^{As} = \int_{b_*}^{b^*} bF_i^{n-2}(\phi_i(b))F_1(\phi_1(b))dF_i(\phi_i(b))$$

Therefore, the total expected revenue that the seller obtain from this auction is

$$R^{As} = R_1^{As} + (n-1)R_i^{As} = \int_{b_*}^{b^*} bF_i^{n-1}(\phi_i(b))dF_1(\phi_1(b)) + \int_{b_*}^{b^*} bF_1(\phi_1(b))dF_i^{n-1}(\phi_i(b))$$

Integrating by parts to the last part and rearranging, we have

$$R^{As} = bF_1(\phi_1(b))F_i^{n-1}(\phi_i(b))\Big|_{b_*}^{b^*} - \int_{b_*}^{b^*} F_1(\phi_1(b))F_i^{n-1}(\phi_i(b))db;$$

(2) Expected seller revenue in symmetric n-player case

When all players are strong, the total expected seller revenue is

$$R^{S_1} = n\int_{\underline{\mathrm{v}}_1}^{\mu_1} bF_1^{n-1}(y_1(b))dF_1(y_1(b)) = \int_{\underline{\mathrm{v}}_1}^{\mu_1} bdF_1^n(y_1(b));$$

Rearranging, we have

$$R^{S_1} = bF_1^n(y_1(b))|_{\underline{\mathrm{v}}_1}^{\mu_1} - \int_{\underline{\mathrm{v}}_1}^{\mu_1} F_1^n(y_1(b))db;$$

And when all players are weak, the total expected seller revenue is

$$R^{S_i} = n\int_{\underline{\mathrm{v}}_i}^{\mu_i} bF_i^{n-1}(y_i(b))dF_i(y_i(b)) = \int_{\underline{\mathrm{v}}_i}^{\mu_i} bdF_i^n(y_i(b));$$

Rearranging, we have

$$R^{S_i} = bF_i^n(y_i(b))|_{\underline{\mathrm{v}}_i}^{\mu_i} - \int_{\underline{\mathrm{v}}_i}^{\mu_i} F_i^n(y_i(b))db;$$

(3) Expected seller revenue in asymmetric imperfect 2-player case

$$R_1^{As2} = \int_{b_*}^{b^*} bF_2(\phi_2(b))dF_1(\phi_1(b))$$

$$R_2^{As2} = \int_{b_*}^{b^*} bF_1(\phi_1(b))dF_2(\phi_2(b))$$

Therefore, the total expected revenue that the seller obtain from this auction is

$$R^{As2} = R_1^{As2} + R_2^{As2} = bF_1(\phi_1(b))F_2(\phi_2(b))|_{b_*}^{b^*} - \int_{b_*}^{b^*} bF_1(\phi_1(b))F_2(\phi_2(b))db$$

In the context described in the *introduction*, player 2's information is newly emerged and following the reasoning of *Corollary* 4, we denote $p_2(b) = F_2(\phi_2(b))$ and assume that $p_i(b) > p_2(b) > p_1(b)$. Concluding all the results above, we then have the following *Proposition* in the next subsection.

(4) Comparison and Explanations of different situations

From the results from the above subsection (1), (2), (3) and (4), we have the following *proposition*.

Proposition 4 *If the difference between players' valuations is small, i.e. the following differences*

1. *the difference between integral upper bounds b^*, μ_1 and μ_i*
2. *the difference between integral lower bounds b_*, $\underline{v}_1$ and $\underline{v}_i$*

are not big enough to change the signs of the following inequalities, then we have

$$R^{S_1} > R^{As2} > R^{As} > R^{S_i}$$

where $i = 2, 3, ..., n$.

We can rewrite the above result as

$$R^{S_1} > R^{Case3} > R^{Case2} > R^{Case1}$$

Specifically, from the conceived case, where there are n strong players, the seller obtains the highest revenue. However, it is not a real case. It is obvious that following the sequential stages of the repeated auctions, i.e. from *Case 1*, *Case 2* to *Case 3*, the seller increases her revenue. Intuitively the seller hopes the participants to be stronger, and meanwhile hopes that the strong participants to be more. In the following we discuss the influence of the information structure to the seller's revenue in this sequential process. The following analysis is from the perspective of the seller, though the seller does not have special information more than a normal player, i.e. the seller has the information as much as it is released from the process of the repeated auctions.

In *Case 1*, all the players are symmetric and the seller does not know private information of the participants. *Proposition* 4 tells that the seller's revenue in *Case 1* is the lowest among all the cases. When some private information, i.e. the type and distribution of the winner, releases, the symmetric situation of auction in *Case 1* will change to be a asymmetric case as in *Case 2*. The private information which can ascertain the winner's distribution can increase the seller's revenue. That explains why sellers or auctioneers always have incentive to know more information about critical participants.

The seller or the public assumes that *Case 1* is a symmetric case and all players' valuations are assumed to be distributed according to the function $F_i(\cdot)$. Since the distribution function $F_i(\cdot)$ describes the weak players' valuations and their behaviours, the seller in *Case 1* holds a pessimistic assumption that all the unkown players are supposed to be weak players. This perspective implie that the seller is risk averse, because she does not take account of the risk what kind of bid distribution the winner will hold, and how high the winner will bid.

The *Case 2* emerges when the winer's distribution information is released from previous auctions. At this time the seller revised his knowledge of the participants. In this asymmetric auction, for the seller, there is a upper bound of distribution $F_1(\cdot)$ and a lower bound of distribution $F_i(\cdot)$. The seller can apply any distribution function between the two distributions to describe any of the unkown players. However, the seller applies her pessimistic perspective for all unkown players again, i.e. except the winner others' valuation are assumed to be distributed with $F_i(\cdot)$. From *Proposition* 4, it is easy to know that the seller's revenue falls in the interval $[R_i^S, R_1^S]$. As a partly conclusion, under the assumption that the seller has no special information available in *Case 1* and believes that all players are weak ones, from *Proposition* 4, the revenue of *Case 2* increases, comparing with that of *Case 1*.

The repeated auctions move on to *Case 3*, when the information of the second winner is publicly released. At this case, the revenue of the seller reaches to the peak of all the cases in the repeated auctions, though the seller obtains an ideally high revenue in a conceived case, where purely strong players participate. In between, we emphasize that how much the second winner values the item is more important than the first winner does, because this specific value benefits the seller more, so that the seller waits until its emergence for the longest time, i,e, duration of all the sequential process.

As a conclusion, when the seller have no special information in the original *Case 1*, following the process of the repeated auction, the revenue of seller always increases. Therefore, the seller has an incentive to wait until the releasing of the information of the second winner. Otherwise, if the seller obtians important information in the middle of the process, i.e. when she expects that no stronger player emerges any longer, she will try to interrupt the process of the repeated auctions and does not wait any longer. For instance, at *Case 2*, if the seller knows that there exists no second strongest buyer but weak ones, he will stop holding another auction in the same pool because beginning another first-stage auction in another pool is more beneficial than holding one in the same pool hopelessly. Similarly, at the second stage, if the seller knows that there exists no second strongest player but weak ones, he will just quit after the strongest one emerges. Therefore, knowing more information of the players' is much critical for the seller.

5 RANKING OF OPEN AUCTION AND FIRST-PRIZE SEALED-BID AUCTION

In this section, we follow the *Section 4* in Maskin and Riley (2000)[6] and extend the 2-player results into n-player case. This additional work is just to theretically complete the revenue comparison for n-player asymmetric auctions.

Let $U_i^H(v, F_a, F_b)$ be the buyer's expected equilibrium surplus from the first-prize sealed-bid auction (here noted as high-bid auction), when $i = 1, 2, ..., n$, buyer's reserve price is v, and the reservation prices of the strongest and other buyers' are distributed according to F_a and F_b,respectively. And let $U_i^H(v, F_a, F_b)$ be the buyer's expected equilibrium surplus from the open auction.

$$U_1^H(v, F_1, F_i) \equiv \max_b p_i^{n-1}(b)(v-b) = p_i^{n-1}(b_1(v))(v - b_1(v));$$

$$U_i^H(v, F_1, F_i) \equiv \max_b p_i^{n-2}(b)p_1(b)(v-b) = p_i^{n-2}(b_i)p_1(b_i)(v - b_i);$$

where $b_1(\cdot)$ and $b_i(\cdot)$ are the stronger's and others' equilibrium bid functions in the high-bid auction when the distributions are (F_1, F_i), respectively.

Lemma 2 *In the sealed high bid auction, the expected seller revenue from bidder i, $i = 2, 3, ..., n$ is*

$$R_1^{As} = b_*[1 - F_1(\underline{v}_1)]F_i^{n-1}(b_*) + \int_{b_*}^{\bar{v}_i} [1 - F_1(Q(v))]Q(v)\frac{d}{dv}[F_i^{n-1}(v)]dv;$$

and

$$R_i^{As} = -\int_{b_*}^{\bar{v}_i} F_i^{n-2}(v)\frac{d}{dv}[v(1-F_i(v))]dv + \int_{b_*}^{\bar{v}_i} [1-F_1(Q(v))]F_i^{n-2}(v)\frac{d}{dv}[v(1-F_i(v))]dv$$

Lemma 3 *In the open auction, the expected seller revenue from bidder i, $i = 2, 3, ..., n$ is*

$$R_1^O = \underline{v}_i[1 - F_1(\underline{v}_i)]F_i^{n-1}(\underline{v}_i) + \int_{\underline{v}_i}^{v_1} [1 - F_1(b)]bd[F_i^{n-1}(b)]\}$$

and

$$R_i^O = -\int_{\underline{v}_1}^{\bar{v}_i} F_i^{n-2}(v)d[v(1 - F_i(v))] + \int_{\underline{v}_1}^{\bar{v}_i} (1 - F_1(v))F_i^{n-2}(v)d[v(1 - F_i(v))]$$

Now we compare the revenues. When $\underline{\mathrm{v}}_1 = \underline{\mathrm{v}}_i = \underline{\mathrm{v}}$, the minimum bid, i.e. b_*, is equal to $\underline{\mathrm{v}}$. From the above *Lemma*s, the difference in expected revenue from the two auctions is

$$\begin{aligned} D &\equiv R_i^{As} + (n-1)R_i^{As} - R_1^O - (n-1)R_i^O \\ &= \textstyle\int_{b_*}^{\bar{v}_i} (n-1)[1-F_1(Q(v))]F_i^{n-2}(v)[(1-F_i(v)) + (Q(v)-v)F_i'(v)]dv \\ &\quad - \textstyle\int_{b_*}^{\bar{v}_i} (n-1)[1-F_1(v)]F_i^{n-2}(v)(1-F_i(v))dv \end{aligned}$$

Rearranging the above expression, we obtain

$$D = \textstyle\int_{b_*}^{\bar{v}_i} F_i^{n-2}(n-1)\{[1-F_1(Q(v))](Q(v)-v)F_i'(v) - (1-F_i(v))[F_1(Q(v)) - F_1(v)] \tag{5.1}$$

Since the above expression is similar to that obtained by *Maskin and Riley (2000)*[6] in two-player case, following the same rationale, we obtain similar conclusions on the comparison between high-bid and open auctions.

Proposition 5 *High-bid auction superior for distribution shifts. Assume that (1)Distribution shifts: given $a < \bar{v}_i - \underline{v}_i$, for all $v \in [\underline{v}_i, \bar{v}_i + a]$,*

$$F_1 = \begin{cases} 0, & v < \underline{v}_i + a; \\ F_i(v-a), & v \geq \underline{v}_i + a. \end{cases} \tag{5.2}$$

(2) CSD: $\frac{d}{dv}\frac{F_i'(v)}{F_i(v)} < 0$ on $[\underline{v}_i, \bar{v}_i]$.
(3) Convexity: $F_i''(v) \geq 0$ on $[\underline{v}_i, \bar{v}_i]$.
additionally,
(4) $-vF_i'(v) + 1 - F_i(v) \geq 0$ for all $v \in [\underline{v}_i, \underline{v}_i + a]$
Then the high-bid auction generates higher expected revenue than does the open auction.

Proposition 6 *High-bid auction superior for distribution stretches.*
(1)Distribution stretches: For $\lambda \in (0,1)$, let the strong buyer have distribution $F_1(v)$, where $v \in [\underline{v}_i, \bar{v}_1](\bar{v}_i < \bar{v}_1)$, such that

$$F_1(v) = \begin{cases} \lambda F_i(v), & v \in [\underline{v}_i, \bar{v}_i]; \\ G(v), & v \in [\underline{v}_i, \bar{v}_1]. \end{cases}$$

where $G(\underline{v}_i) = \lambda$, $G(\underline{v}_1) = 1$, and for all $v \in [\underline{v}_i, \bar{v}_i]$ and $w \in [\underline{v}_i, \bar{v}_1]$,

$$F_i'(v) \geq G'(w) > 0$$

(2)$F_i(\underline{v}_i) = 0$,
(3)$\frac{d}{dv}\frac{F_i'(v)}{F_i(v)} < 0$ on $[\underline{v}_i, \bar{v}_i]$.
Then the high-bid auction generates more expected revenue than the open auction.

Proposition 7 *Open auction superior for shifts of probability mass to the lower end point. Suppose the strong buyer's valuation $\tilde{v}_1$ is distributed on according to $F_1(v)$, $v \in [\underline{v}, \bar{v}_1]$ where $F_1(\underline{v}) = 0$, and $\frac{F_1'(v)}{1-F_1(v)}$ is increasing. Buyer i's valuation $\tilde{v}_i$ is distributed so that, for all $v \in [\underline{v}, \bar{v}_1]$. its density at v is a fraction $\theta(v) \in (0,1)$ with $\theta'(v) \geq 0$ of $F_1'(v)$ where the remaining density is reassigned to $\underline{v}$. That is,*

$$F_i(v) = \int_{\underline{v}}^{v} \theta(v) dF_1(t) + \gamma,$$

where

$$\gamma = \int_{\underline{v}}^{\bar{v}_1} (1 - \theta(v)) dF_1(t).$$

Then, the open auction generates higher revenue than the high-bid auction.

6 CONCLUSION

Different reasons can incur asymmetric behavior of players in competitive activities. Most literatures focus on two main reasons: (1) players differ in risk attitudes, or (2) players differ in their valuations respectively cost, or (3) valuations are chosen from different distributions, due to asymmetry between participants. There are however more aspects of asymmetry, for instance players might differ in entry costs, in availabilities of information or options or budgets they face. In this paper we only consider aspects (2) and (3) and still are far from completing all the relevant varieties. One of the gaps between theoretical results and its practical applications is superficially symmetry and asymmetry, especially in auction theory and its applications. By establishing a repeated contest, this paper tries to explain the evolutionary process from symmetry to asymmetry and compare the different stages of them. We see symmetry as a situation, where the seller lacks in information. When the process goes on and more information about players, especially the winners, is released, the strong player can be identified and, therefore, the auctions later on turn out to be asymmetric. In this sense auction, as an economic institute, functions as a cognitive or learning mechanism for seller to obtain information relative to his interests. One of the important results obtained in this paper shows the validity that further information about players, especially those with higher valuation, are worth holding an auction to obtain optimal revenues for the seller.

References

[1] Bulow, Jeremy and John Roberts, (1989), *Simple Economics of Optimal Auctions*, The Journal of Political Economy. **97**(5), 1060–1090.

[2] Fullerton, Richard L. , R. Preston McAfee, (1999), *Auctioning Entry into Tournaments*, Journal of Political Economy, **107**(3), 573–605.

[3] Gradstein, Mark and Kai A. Konrad, (1999), *Orchestrating Rent Seeking Contest*, The Economic Journal, **109**, 536–545.

[4] Klemperer, Paul, (1999), *Auction Theory: A Guide to the literrature*, Journal of Economic Surveys, **13**(3), 227–286.

[5] McAfee, R. Preston and John McMillan, *Government Procurement and International Trade*, Journal of International Economics, **26**, 291–308.

[6] Maskin, Eric and John Riley, (2000), *Asymmetric Auctions*, Review of Economic Studies, **67**(3), 413-438.

[7] Maskin, Eric and John Riley, (2003), *Uniqueness in Sealed High Bid Auctions*, Games and Economic Behavior, **45**, 395–409.

[8] Maskin, Eric and John Riley, 2000, *Equilibrium in Sealed High Bid Auctions*, Review of Economic Studies, **67**, 439–454.

[9] Myerson, Roger B., (1981), *Optimal Auction Design*, Mathematics of Operations Research, **6** (1), 58–73.

[10] Riley, John G., (1989), *Expected Revenue from Open and Sealed Bid Auctions*, The Journal of Economic Perspectives, **3** (3), 41–50.

[11] Riley, John G. and William F. Samuelson, (1981), *Optimal Auctions*, The American Economic Review, **71** (3), 381–392.

Appendix

Suppose that $\underline{v}_i = \underline{v}_i = \underline{v}$, Define

$$e_i(v) \equiv \frac{(v - \underline{v})F_i'(v)}{F_i(v)}$$

$i = 1, 2, ..., n$

Then from (2.8) we have

$$\begin{cases} (n-1)e_i(\phi_i(b))\phi_i'(b) = \frac{\phi_i(b)-\underline{v}}{\phi_1(b)-b}; \\ (n-2)e_i(\phi_i(b))\phi_i'(b)\frac{\phi_i-b}{\phi_i-\underline{v}} + e_1(\phi_1(b))\phi_1'(b)\frac{\phi_i-b}{\phi_1-\underline{v}} = 1. \end{cases} \tag{A.1}$$

in which $i = 2, 3, ..., n$.

From the definition of $e_i(v)$, by l'Hôpital's Rule we infer some properties of it.

$$F_i(\underline{v}) = 0 \Rightarrow \begin{cases} e_i(\underline{v}) = 1, \\ e_i'(\underline{v}) = \frac{F_i''(\underline{v})}{2F_i'(\underline{v})} \end{cases} \tag{A.2}$$

Now we infer some properties of the fraction $F_1(v)/F_i(v)$. By differentiating, we obtain

$$\frac{d}{dv}\frac{F_1(v)}{F_i(v)} = \frac{F_1'F_i - F_1F_i'}{F_i^2}, for\ F_1 > 0\ and\ F_i > 0.$$

Applying l'Hôpital's Rule, we obtain

$$F_1(\underline{v}) = F_i(\underline{v}) = 0 \Rightarrow \left.\frac{d}{dv}\frac{F_1(v)}{F_i(v)}\right|_{v=\underline{V}} = \frac{1}{2}\frac{F_1'(v)}{F_i'(v)}[\frac{F_1''(v)}{F_1'(v)} - \frac{F_i''(v)}{F_i'(v)}] \tag{A.3}$$

If

$$\left.\frac{d}{dv}\frac{F_1(v)}{F_i(v)}\right|_{v=\underline{V}} > 0,$$

we have

$$\frac{F_1''(\underline{v})}{F_1'(\underline{v})} > \frac{F_i''(\underline{v})}{F_i'(\underline{v})} \tag{A.4}$$

i.e. the bracketed expression in (A.3) is strictly positive.

Proof of Lemma 1

Since $F_1(\underline{v}) = F_i(\underline{v}) = 0$, $p_i(\underline{v}) = \pi_i(\underline{v}) = 0$, $i = 1, 2, ..., n$, applying l'Hôpital's Rule to (A.1), we obtain

$$\begin{cases} (n-1)e_i(\underline{v})\phi_i'(\underline{v}) = \frac{\phi_i'(\underline{V})}{\phi_1'(\underline{V})-1}, \\ (n-2)e_i(\underline{v})\phi_i'(\underline{v})\frac{\phi_i'(\underline{V})-1}{\phi_i'(\underline{V})} + e_1(\underline{v})\phi_1'(\underline{v})\frac{\phi_i'(\underline{V})-1}{\phi_1'(\underline{V})} = 1 \end{cases} \tag{A.5}$$

in which $i = 2, 3, ..., n$.

It follows from (A.2) that $\phi_1'(\underline{v}) = \frac{n}{n-1}$, $n = 1, 2, ..., n$. and in symmetric equilibrium $y_1'(\underline{v}) = \frac{n}{n-1}$, $i = 1, 2, ..., n$.

$$\phi_1'(\underline{v}) = y_1'(\underline{v}) = \frac{n}{n-1}, \quad i = 1, 2, ..., n. \tag{A.6}$$

Taking the logarithm of the first equation of (A.1), differentiating, and using the result(obtained by applying l'Hôpital's Rule)

$$\lim_{v\to\underline{V}} \frac{\frac{1}{(n-1)e_i(\phi_i)} - \phi_1' + 1}{\phi_1 - b} = -\frac{n}{n-1}\frac{F_i''(\underline{v})}{2F_i'(\underline{v})} - (n-1)\phi_1'',$$

we obtain the second equation in (A.7).

Meanwhile, differentiating the second equation of (A.1), taking use of the results (obtained by l'Hôpital's Rule)

$$\lim_{v\to\underline{V}} \frac{(\phi_i'-1)(\phi_i-\underline{v}) - (\phi_i-b)\phi_i'}{(\phi_i-\underline{v})^2} = \frac{\phi_i''(\underline{v})}{2[\phi_i'(\underline{v})]^2}$$

and

$$\lim_{v\to\underline{V}} \frac{(\phi_i'-1)(\phi_1-\underline{v}) - (\phi_i-b)\phi_1'}{(\phi_1-\underline{v})^2} = \frac{\phi_i''(\underline{v})\phi_1'(\underline{v}) - \phi_1''(\underline{v})(\phi_1'(\underline{v})-1)}{2[\phi_1'(\underline{v})]^2}$$

we obtain the first equation in (A.7).

$$\begin{cases} \frac{n(n-2)}{2(n-1)^2}\frac{F_i''}{F_i'} + \frac{n}{2(n-1)^2}\frac{F_1''}{F_1'} + \frac{n^2-2}{2n}\phi_i'' + \frac{1}{2n}\phi_1'' = 0, \\ \frac{n}{n-1}\frac{F_i''}{F_i'} + \frac{n-1}{n}\phi_i'' + (n-1)\phi_1'' = 0. \end{cases} \tag{A.7}$$

in which $i = 2, 3, ..., n$.

Solving these equations yields

$$\begin{cases} \phi_1'' = -\frac{n^2}{(n-1)(n^2-1)(n^2-n-1)}[n(n-1)\frac{F_i''}{F_i'} - \frac{F_1''}{F_1'}] \\ \phi_i'' = -\frac{n^2}{(n^2-1)(n^2-n-1)}[\frac{(n^2-2n-1)}{(n-1)}\frac{F_i''}{F_i'} + \frac{n}{n-1}\frac{F_1''}{F_1'}] \end{cases}$$

in which $i = 2, 3, ..., n$. And in symmetric equilibrium,

$$y_i'' = -\frac{n^2}{(n^2-1)(n-1)}\frac{F_i''}{F_i'}$$

in which $i = 1, 2, ..., n$.

For the convenience of comparison of ϕ_i'' and y_i'', we rearrange them as

$$\begin{cases} \phi_i'' = -\frac{n^2}{(n^2-1)(n-1)}[\frac{(n^2-2n-1)}{(n^2-n-1)}\frac{F_i''}{F_i'} + \frac{n}{(n^2-n-1)}\frac{F_1''}{F_1'}], \\ y_i'' = -\frac{n^2}{(n^2-1)(n-1)}[\frac{(n^2-2n-1)}{(n^2-n-1)}\frac{F_i''}{F_i'} + \frac{n}{(n^2-n-1)}\frac{F_i''}{F_i'}]. \end{cases} \quad \text{(A.8)}$$

in which $i = 2, 3, ..., n$.

From the hypotheses of this Lemma and (A.4), we then have

$$\phi_i''(\underline{v}) < y_i''(\underline{v}). \quad \text{(A.9)}$$

in which $i = 2, 3, ..., n$.

Similarly, for the convenience of comparison of ϕ_1'' and y_i'', we rearrange them as

$$\begin{cases} \phi_1'' = -\frac{n^2}{(n^2-1)(n-1)}[\frac{n(n-1)}{(n^2-n-1)}\frac{F_i''}{F_i'} - \frac{1}{(n^2-n-1)}\frac{F_1''}{F_1'}] \\ y_i'' = -\frac{n^2}{(n^2-1)(n-1)}[\frac{n(n-1)}{(n^2-n-1)}\frac{F_i''}{F_i'} - \frac{1}{(n^2-n-1)}\frac{F_i''}{F_i'}] \end{cases}$$

in which $i = 1$. Then we obtain

$$\phi_1''(\underline{v}) > y_i''(\underline{v}). \quad \text{(A.10)}$$

in which $i = 1$.

By definition of $p_i(b)$ and $\pi_i(b)$,

$$p_i'(b) = F_i'(\phi_i(b))\phi_i'(b) \textit{ and } \pi_i'(b) = F_i'(y_i(b))y_i'(b).$$

With (A.6), we have $p_i'(\underline{v}) = \pi_i'(\underline{v})$.

Further we have

$$\begin{cases} p_i''(b) = F_i''(\phi_i(b))(\phi_i'(b))^2 + F_i'(\phi_i(b))\phi_i''(b) \\ \pi_i''(b) = F_i''(y_i(b))\ (y_i'(b))^2 + F_i'(y_i(b))\ y_i''(b) \end{cases}$$

with (A.6) and (A.9),(A.10) , *part* 3 and *part* 4 of *Lemma* 1 holds.

From Corollary 2, we know that $F_i(v) > F_1(v)$ for all $v \in (\underline{v}_i, \underline{v}_1)$. Since $\underline{v}_i = \underline{v}_1 = \underline{v}$ *and* $F_1 = F_i = 0$, we have $F_i'(\underline{v}) \geq F_1'(\underline{v})$.

1. if $F_i'(\underline{v}) > F_1'(\underline{v})$, *part* 1 and *part* 2 of *Lemma* 1 holds;

2. if $F_i'(\underline{v}) = F_1'(\underline{v})$, from (A.4) we have $F_1'' > F_i''$, which contradicts (2.2). ‖

Proof of Proposition 1

To establish *part* 1, firstly from (2.13), $1 = \pi_i(b) > \pi_1(b)$ for all $b \in [\mu_i, \mu_1)$. Contrary to Part 1, we assume that there is some $\hat{b} \in [\underline{v}_1, \mu_i]$ such that $\pi_i(b)/\pi_1(b) = 1$. We will show that $\pi_i(b)/\pi_1(b)$ is increasing at $\hat{b}$. Since $H_1(p) > H_i(p)$ for all $p \in (0,1)$, $\pi_1 \geq \pi_i$ implies that $H_1(\pi_1) \geq H_1(\pi_i) > H_i(\pi_i)$. Then from (3.2) we have

$$(n-1)\frac{\pi_i'}{\pi_i} = \frac{1}{H_i(\pi_i) - b} > \frac{1}{H_1(\pi_1) - b} = (n-1)\frac{\pi_1'}{\pi_1} \quad at\ b = \hat{b}.$$

Hence

$$\frac{d}{db}\frac{\pi_i}{\pi_1} = [\frac{\pi_i'}{\pi_i} - \frac{\pi_1'}{\pi_1}]\frac{\pi_i}{\pi_1} > 0, \quad at \quad b = \hat{b}.$$

It follows that, for some $\delta > 0$,

$$\pi_1(b) > \pi_i(b), \quad for \quad all \quad b \in (\hat{b} - \delta, \hat{b}). \tag{A.11}$$

Let δ be the biggest value for which (A.11) holds. If $\hat{b} - \delta > \underline{v}_1$, then

$$\pi_1(\hat{b} - \delta) = \pi_i(\hat{b} - \delta) \tag{A.12}$$

and from the above argument, we have $\pi_i(b) > \pi_1(b)$, for b near $\hat{b} - \delta$, such that $b > \hat{b} - \delta$, which is a contradiction of (A.11). Then we assume that $\hat{b} - \delta = \underline{v}_1$. In the symmetric auction of n strong buyers, any buyer bids above $\bar{v}_1$, if and only if they have valuations exceeding $\bar{v}_1$. Therefore, $pi_i(\bar{v}_1) \geq F_i(\bar{v}_1) \geq F_1(\bar{v}_1) = pi_i(\bar{v}_1)$ and so from(A.11), (A.12) holds.We conclude $F_i(\bar{v}_1) = F_1(\bar{v}_1)$ and so $\bar{v}_1 = \bar{v}_i = \bar{v}$. Thus from part 2 of $Corollary 2$, we have $F_i(\bar{v}) = F_1(\bar{v}) = 0$, and if $\gamma > \bar{v}$, then

$F_1(v) = \lambda F_i(v)$, for all $v \in [\underline{v}, \gamma]$.

Since from (2.11) $y_i(b) = y_1(b)$ for b in some neighborhood of $\bar{v}$, $\pi_i(b) > \pi_1(b)$ in that neighborhood, which is a contradiction of (A.11). Hence $\gamma = \bar{v}$. From part 1 of *Lemma* 1, $\pi_i(b) > \pi_1(b)$ for all b in a neighborhood of $\bar{v}$, a contradiction of (A.11). We conclude that $\hat{b} \in [\underline{v}_1, \mu_i]$ does not exist, and so *part* 1 is established.

To establish *part* 2, suppose that there exists $\hat{b} \in (b_*, b^*)$ such that $p_1(\hat{b})/p_i(\hat{b}) = 1$. Since $H_1(p) > H_i(p)$ for all $p \in (0,1)$, it follows from (3.1) that

$$(n-2)\frac{p_i'}{p_i} + \frac{p_1'}{p_1} = \frac{1}{H_i(p_i) - b} > \frac{1}{H_1(p_1) - b} = (n-1)\frac{p_i'}{p_i} \quad at\ b = \hat{b}.$$

Then we have $\frac{p_1'}{p_1} > \frac{p_i'}{p_i} \quad at\ b = \hat{b}$, or p_1/p_i is increasing at $\hat{b}$. Since the same argument applies to any $b^0 > \hat{b}$ for which $p_1(b^0)/p_i(b^0) = 1$, we have $p_1(b) > p_i(b)$ for all $b \in (\hat{b}, b^*)$. But from (2.9), $p_1(b^*) = p_i(b^*)$, and so $\hat{b}$ can not exist. Hence *part*2 holds unless $p_1(b) > p_i(b)$ for all $b \in (b_*, b^*)$, which would conflict with *part* 2 of *Lemma* 1.

To prove *part* 3, suppose that there exists $\hat{b} \in (b_*, b^*)$ such that $\pi_i(\hat{b})/p_1(\hat{b}) = 1$. If $\mu_i \leq \hat{b}$, then $\pi_i(\hat{b}) = p_1(\hat{b}) = 1$ and $p_1(\hat{b} < 1$, a contradiction. Hence, assume $\mu_i > \hat{b}$. From *part* 2, $\pi_i(\hat{b}) = p_1(\hat{b}) < p_i(\hat{b})$. Thus from

$$(n-2)\frac{p_i'}{p_i} + \frac{p_1'}{p_1} = \frac{1}{H_i(p_i) - b} < \frac{1}{H_1(p_1) - b} = (n-1)\frac{p_i'}{p_i} \quad at\ b = \hat{b},$$

we have $\frac{p_1'}{p_1} < \frac{p_i'}{p_i}$ and use it into

$$(n-1)\frac{\pi_i'}{\pi_i} = \frac{1}{H_i(\pi_i) - b} > \frac{1}{H_i(p_i) - b} = (n-2)\frac{p_i'}{p_i} + \frac{p_1'}{p_1} \quad at\ b = \hat{b}.$$

We then have $\frac{\pi_i'}{\pi_i} > \frac{p_1'}{p_1} \quad at\ b = \hat{b}.$

The rest of the proof parallels that of *part* 1 but uses *part* 3 instead of *part* 1 of *Lemma* 1.

To prove *part* 4, for $\hat{b} \in (b^*, \mu_1)$ we have $1 = p_i(\hat{b}) > \pi_1(\hat{b})$. Suppose that there exists $\hat{b} \in (b_*, b^*)$ such that $p_i(\hat{b})/\pi_1(\hat{b}) = 1$. From *part* 2, we have $\pi_1(\hat{b}) = p_i(\hat{b}) > p_1(\hat{b})$. Thus

$$(n-1)\frac{\pi_1'}{\pi_1} = \frac{1}{H_i(\pi_1) - b} = \frac{1}{H_i(p_i) - b} < \frac{1}{H_1(p_1) - b} = (n-1)\frac{p_i'}{p_i}$$

Following argument, which is similar to that of *part* 3, establishes *part* 4. ||

Proof of Proposition 2

For $b \in [\mu_i, \mu_1]$, $1 = y_i(b) > y_1(b)$. For $b \in [\underline{v}_1, \mu_i]$, *CSD* implies that *part* 1 immediately follows from (2.15).

To establish *part* 2, we first argue that *part* 2 holds in a punctured neighborhood of b^*. If $\bar{v}_i < \bar{v}_1$, this is immediate because $\bar{v}_i = \phi_i(b^*) < \phi_1(b^*) = \bar{v}_1$. If $\bar{v}_i = \bar{v}_1$, then $\phi_i(b^*) = \phi_1(b^*)$ and so, from(2.8),

$$(n-1)\frac{F_i'(\phi_i)}{F_i(\phi_i)}\phi_i' = \frac{1}{\phi_1 - b} = \frac{1}{\phi_i - b} = (n-2)\frac{F_i'(\phi_i)}{F_i(\phi_i)}\phi_i' + \frac{F_1'(\phi_1)}{F_1(\phi_1)}\phi_1' \ at\ b = b*. \tag{A.13}$$

Given CSD, it follows that $\phi_1' < \phi_i'$ and so *part* 2 holds in a punctured neighborhood of b^*, as claimed.

Suppose that there exists $\hat{b} \in (b_*, b^*)$ such that $\phi_i(\hat{b})/\phi_1(\hat{b}) = 1$. Then (A.13) holds at $b = \hat{b}$. Hence, the assumption of CSD implies that $\phi_i(\hat{b})/\phi_1(\hat{b}) \geq 1$ for all $b \in (\hat{b}, b^*)$, a contradiction of our former finding. Thus $\phi_i(\hat{b})/\phi_1(\hat{b}) < 1$ for all $b \in (b_*, b^*)$.

To prove *part* 3, for any $b \in (b_*, b^*)$ such that $y_i(b) < \phi_i(b)$, from (2.8), (2.11) and *part* 2 of this proposition ,we have

$$(n-1)\frac{F_i'(\phi_i(b))}{F_i(\phi_i(b))}\phi_i'(b) = \frac{1}{\phi_1(b)-b} < \frac{1}{\phi_i(b)-b} < \frac{1}{y_i(b)-b} = (n-1)\frac{F_i'(y_i(b))}{F_i(y_i(b))}y_i'(b)$$

Hence

$$y_i(b) < \phi_i(b) \ \Rightarrow \ \frac{d}{db}\frac{F_i(y_i)(b)}{F_i(\phi_i)(b)} \tag{A.14}$$

For some $\hat{\theta} \leq 1$, suppose that there exists $\hat{b} \in (b_*, b^*)$ satisfying

$$\hat{b} = \frac{F_i(y_i)(b)}{F_i(\phi_i)(b)}$$

By (A.14), $F_i(y_i)(b)/F_i(\phi_i)(b)$ is strictly increasing at $b = \hat{b}$, hence

$$y_i(b) < \phi_i(b) \quad and \frac{d}{db}\frac{F_i(y_i)(b)}{F_i(\phi_i)(b)} \quad for\ all\ b \in [b_*, b^*) \tag{A.15}$$

From (2.9), when $\underline{\mathrm{v}}_1 = \underline{\mathrm{v}}_i = b_*$, we have $\phi_i(b_*) = y_i(b_*) = b_*$, a contradiction of (A.15). We conclude that $\hat{b}$ can not exist, and so *part* 3 holds.

To prove *part* 4, note first that, by *Corollary* 3, $b^* \leq \mu_1$. Hence $\phi_1(b^*) \geq y_1(b^*)$.

For any $b \in (\underline{\mathrm{v}}_1, b^*)$ such that $\phi_1(b^*) \leq y_1(b^*)$, from (2.8), (2.11) and *part* 2 of this proposition ,we have

$$(n-2)\frac{F_i'(\phi_i)}{F_i(\phi_i)}\phi_i' + \frac{F_1'(\phi_1)}{F_1(\phi_1)}\phi_1' = \frac{1}{\phi_i - b} > \frac{1}{\phi_1 - b} \geq \frac{1}{y_1 - b} = (n-1)\frac{F_1'(y_1)}{F_1(y_1)}y_1'$$

And from *part* 2 and (2.8), we have

$$(n-1)\frac{F_i'(\phi_i)}{F_i(\phi_i)}\phi_i' = \frac{1}{\phi_1 - b} < \frac{1}{\phi_i - b} = (n-2)\frac{F_i'(\phi_i)}{F_i(\phi_i)}\phi_i' + \frac{F_1'(\phi_1)}{F_1(\phi_1)}\phi_1',$$

and so

$$(n-1)\frac{F_1'(\phi_1)}{F_1(\phi_1)}\phi_1' > (n-2)\frac{F_i'(\phi_i)}{F_i(\phi_i)}\phi_i' + \frac{F_1'(\phi_1)}{F_1(\phi_1)}\phi_1' > (n-1)\frac{F_1'(y_1)}{F_1(y_1)}y_1'$$

Hence

$$\frac{d}{db}\frac{F_1(\phi_1)(b)}{F_1(y_1)(b)} > 0.$$

The rest of proof is similar to that of *part* 3. $\|$

Proof of Proposition 3

As defined in (2.12), $\bar{v}_1$ is the equilibrium bid of player 1 in symmetric case. From *Corollary* 4, we have

$$
\begin{aligned}
R_1^{As}(v,F_1,F_i) &= \max_b p_i^{n-1}(b)(v-b) \\
&\geq p_i^{n-1}(\bar{b}_1(v))(v-\bar{b}_1(v)) \\
&> \pi_1^{n-1}(\bar{b}_1)(v)(v-\bar{b}_1(v)) \\
&= R_1^S(v,F_1,F_1)
\end{aligned}
$$

We define $b_i(\cdot)$ as the equilibrium bid of the weak buyer in asymmetric case, i.e. $b_i(\cdot)$ maximizes $R_i^{As}(v,F_1,F_i)$. Then we have

$$
\begin{aligned}
R_i^S(v,F_i,F_i) &= \pi_i^{n-1}(\bar{b}_i)(v-\bar{b}_i) \\
&\geq \pi_i^{n-1}(b_i)(v-b_i) \\
&> p_i^{n-2}(b_i)p_1(b)(v-b_i) \\
&= R_i^{As}(v,F_1,F_i).
\end{aligned}
$$

$\|$

Proof of Proposition 4

Firstly we compare R^{As} and R_1^S.

From *subsection* $(1) and (2)$, We have

$$R^{As} = b^* - \int_{b_*}^{b^*} F_1(\phi_1(b))F_i^{n-1}(\phi_i(b))db;$$

and

$$R_1^S = \mu_1 - \int_{\underline{\mathrm{v}}_1}^{\mu_1} F_1^n(y_1(b))db;$$

Given the assumption in this *proposition*, i.e. neglecting the differences between integral upper and lower bounds, from *Corollary 4* $F_i^{n-1}(\phi_i(b)) > F_1(\phi_1(b)) > F_1(y_1(b))$, it is easy to obtain that $R_1^S > R^{As}$. Similarly, we obtain others results in this *proposition*. $\|$

Proof of Lemma 2

From the expression of R_1^{As} in *section 4.2*, integrating by parts, we have

$$R_1^{As} = bF_i^{n-1}(\phi_i(b))F_1(\phi_1(b))\Big|_{b_*}^{b^*} - \int_{b_*}^{b^*} F_1(\phi_1(b))\frac{d}{db}[bF_i^{n-1}(\phi_i(b))]db$$

Then we can obtain

$$R_1^{As} = [1-F_1(\underline{\mathrm{v}}_1)]b_*F_i^{n-1}(b_*) + \int_{b_*}^{b^*} [1-F_1(\phi_1(b)]\frac{d}{db}[F_i^{n-1}(\phi_i(b))]db$$

From (2.8), we have $\frac{d}{db}[1-F_1(\phi_1(b))] = \phi_1(b)\frac{d}{db}[F_i^{n-1}(\phi_i(b))]$.

Substituting this expression into the integral and using the definition $\phi_1(b) \equiv Q(\phi_i(b))$, we then obtain

$$R_1^{As} = [1 - F_1(\underline{v}_1)] b_* F_i^{n-1}(b_*) + \int_{b_*}^{\bar{v}_i} [1 - F_1(Q(v))] Q(v) \frac{d}{dv} [F_i^{n-1}(v)] dv$$

Symmetrically, for the weak buyer, we have

$$R_i^{As} = [1 - F_i(b_*)] b_* F_i^{n-2}(b_*) F_1(\underline{v}_1) + \int_{b_*}^{b^*} [1 - F_i(\phi_i(b))] \frac{d}{db} [b F_i^{n-2}(\phi_i(b)) F_1(\phi_1(b))] db$$

Since from (2.8) we have $\frac{d}{db}[bF_i^{n-2}(\phi_i(b))F_1(\phi_1(b))] = \phi_i(b)\frac{d}{db}[F_1(\phi_1(b))F_i^{n-2}(\phi_i(b))$ using the definition of $Q(\cdot)$ we obtain

$$R_i^{As} = b_* F_1(\underline{v}_1) F_i^{n-2}(b_*) [1 - F_i(b_*)] + \int_{b_*}^{\bar{v}_i} [1 - F_i(v)] v \frac{d}{dv} [F_1(Q(v)) F_i^{n-2}(v)] dv.$$

Then we have

$$R_i^{As} = - \int_{b_*}^{\bar{v}_i} F_1(Q(v)) F_i^{n-2}(v) \frac{d}{dv} [(1 - F_i(v)) v] dv$$

then,

$$R_i^{As} = b_* F_1(\underline{v}_1) F_i^{n-2}(b_*) [1 - F_i(b_*)] + \int_{b_*}^{\bar{v}_i} [1 - F_i(v)] v \frac{d}{dv} [F_1(Q(v)) F_i^{n-2}(v)] dv.$$

From this expression it is easy to obtain that one in *Lemma*. ‖

Proof of Lemma 3

If the weak buyer i ($i = 2, 3, ..., n$)'s valuation, v_i, is larger than $\underline{v}_1$, his expected payment is

$$\underline{v}_1 F_i^{n-2}(\underline{v}_1) F_1(\underline{v}_1) + \int_{\underline{v}_1}^{v_i} b \frac{d}{db} [F_i^{n-2}(b) F_1(b)] db.$$

Taking the expectation over v_i,

the expected revenue from the weak buyer is

$$\begin{aligned} R_i^O &= \textstyle\int_{\underline{v}_1}^{\bar{v}_i} \{\underline{v}_1 F_i^{n-1}(\underline{v}_1) F_1(\underline{v}_1) + \int_{\underline{v}_1}^{v_i} b d[F_i^{n-2}(b) F_1(b)]\} dF_i(v_i) \\ &= \underline{v}_1 F_i^{n-2}(\underline{v}_1) F_1(\underline{v}_1) [1 - F_i(\underline{v}_1)] + \textstyle\int_{\underline{v}_1}^{\bar{v}_i} [1 - F_i(b)] b d[F_i^{n-2}(b) F_1(b)] \\ &= - \textstyle\int_{\underline{v}_1}^{\bar{v}_i} F_1(v) F_i^{n-2}(v) d[v(1 - F_i(v))] \end{aligned}$$

where $i = 2, 3, ..., n$. Then

$$R_i^O = \underline{v}_1 F_1(\underline{v}_1) F_i^{n-2}(\underline{v}_1) [1 - F_i(\underline{v}_1)] + \int_{\underline{v}_1}^{\bar{v}_i} [1 - F_i(b)] b d[F_1(b) F_i^{n-2}(b)]$$

From this expression, it is easy to obtain the *Lemma*.
Symmetrically, the strong buyer's expected payment is

$$\underline{\mathrm{v}}_i F_i^{n-1}(\underline{\mathrm{v}}_i) + \int_{\underline{\mathrm{v}}_i}^{v_1} bd[F_i^{n-1}(b)]$$

Taking the expectation over v_1,
the expected revenue from the strong buyer is

$$\begin{aligned} R_1^O &= \int_{\underline{\mathrm{v}}_i}^{\bar{v}_1} \{\underline{\mathrm{v}}_i F_i^{n-1}(\underline{\mathrm{v}}_i) + \int_{\underline{\mathrm{v}}_i}^{v_1} bd[F_i^{n-1}(b)]\} dF_1(v_1) \\ &= \underline{\mathrm{v}}_i F_i^{n-1}(\underline{\mathrm{v}}_i)[1 - F_1(\underline{\mathrm{v}}_i)] + \int_{\underline{\mathrm{v}}_i}^{v_1} [1 - F_1(b)] bd[F_i^{n-1}(b)]\} \end{aligned}$$

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