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When Winning is the Only Thing:

Pure Strategy Nash Equilibria in a Three-Candidate Spatial Voting Model

Richard A. Chisik[*]

and

Robert J. Lemke[**]

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Abstract: It is well-known that there are no pure strategy Nash equilibria (PSNE) in the standard three-candidate spatial voting model when candidates maximize their share of the vote. When all that matters to the candidates is winning the election, however, we show that PSNE do exist. We provide a complete characterization of such equilibria and then extend our results to elections with an arbitrary number of candidates. Finally, when two candidates face the potential entrant of a third, we show that PSNE no longer exist, however, they do exist when the number of existing candidates is at least three.

Key words: Voting, spatial equilibrium, location models, entry.

JEL Classification: C7, D0, H8, R1

[*]Department of Economics DM-309C, Florida International University, Miami, FL, 33199; Phone: (305) 348-3286; Fax: (305) 348-1524; E-mail: chisikr@fiu.edu.
[**]Department of Economics and Business, Lake Forest College, Box M3, 555 N. Sheridan Road, Lake Forest, IL, 60045; Phone: (847) 735-5143; Fax: (847) 735-6193; E-mail: lemke@lakeforest.edu.

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1. Introduction

Among the many extensions of Hotelling's (1929) classic introduction to the spatial location of firms is contained the well-known result that pure strategy Nash equilibria (PSNE) exist only when the number of firms differs from three (Eaton and Lipsey, 1975). An alternative direction of research, first suggested by Hotelling himself, is the application of the spatial model to the location of candidates for political office. It is now recognized, however, that candidates may have other objectives than vote maximization, which is the natural counterpart in the political realm to the assumption that firms maximize their market share.[1] Combining these two directions of research, we show that if winning is the candidates' only objective, then PSNE do exist in three-candidate (or three-party) elections. Although this result is straightforward, it does not appear to have been previously recognized in the literature.

We characterize the entire set of PSNE when there are three candidates, each maximizing her expected probability of winning the election. We show that in all of the PSNE, the winner is the sole candidate on her half of the political spectrum. In this way, the additional candidate on the losing half of the political spectrum may serve as an election spoiler for the other candidate on her side of the political spectrum (a result that may reflect some recent election experiences).

When there are more than three candidates, there are many possible PSNE; however, when the winner is the most extreme candidate on one side of the spectrum, the conditions for PSNE are closely related to those for the three-candidate case. There are other equilibria, however, and a PSNE whereby the winner adopts the platform most preferred by the median voter is obtainable. An interesting implication of the PSNE we construct for the general case, when the winner adopts the left most or right most platform, is that the number of candidates can be allowed to approach infinity without altering the general structure of the equilibrium set.

[1] Cox (1985, 1987), Denzau, Kats, and Slutsky (1985), Greenberg and Shepsle (1987), Myerson and Weber (1993), Osborne (1993, 2000) and Palfrey (1984) all analyze multi-candidate elections under various voting procedures and entry rules. For general surveys, see Cox (1990), Enelow and Hinich (1990), Myerson (1999), Osborne (1995), Shepsle (1991), and Shepsle and Cohen (1990).

We also note how an increase in the number of candidates allows the winner to adopt a more extreme position. First, we show that, in comparison with the two-candidate case, none of the candidates in the three-candidate case locates at the platform most preferred by the median voter. Hence, as the number of candidates increases from two to three the winner must adopt a more extreme position and the distance between the chosen platforms of the left most and the right most candidate must expand as well. With three candidates there is an upper bound on this distance, however, as the number of candidates becomes large the allowable distance grows to encompass the entire political spectrum and the winner's position can approach the farthest, or most extreme, reaches of the spectrum.

Finally, we analyze the potential entry of additional candidates. A PSNE does not exist when a third candidate can enter the election after observing the simultaneously chosen platforms of two existing candidates. This non-existence result is closely related to the reason that a PSNE fails to exist in a simultaneous move three-candidate election when candidates maximize their share of the vote (see Osborne, 1993).[2] On the other hand, when there are at least three existing candidates, we show that any platform configuration that is a PSNE in our general case also prevents the entry of additional candidates.

In the next section we develop our main result for the three-candidate case. In the third section we extend these results to the general *N*-candidate case. We consider entry in the fourth section and make our conclusions in the fifth section.

2. Pure Strategy Nash Equilibria in Three-Candidate Elections

The voting environment we have in mind is easily summarized. There exists a continuum of voters whose measure is normalized to one. The symmetric, single-peaked preferences of these voters are defined along a single dimension and are distributed uniformly

[2] It stands in contrast to the results of other models with entry, such as in Palfrey (1984) where candidates maximize their share of the vote or in Osborne (2000) where candidates are uncertain about the voters' preferences.

along the unit interval. Voters have no strategic incentives and, therefore, each casts her vote for the candidate whose platform is closest to her most preferred platform. If two or more candidates declare the same platform, then the candidates split equally all votes garnered by their platform.

Each candidate's objective is to maximize her expected chance of winning the election. Hence, candidates prefer winning the election to losing the election, but do not otherwise care about their share of the vote or the implications of their chosen platform. The winner of the election is the candidate who receives a plurality of the vote. In the case of a tie, the winner is selected randomly among all the candidates receiving the plurality share.

To see that a PSNE can exist when each of three candidate's sole objective is to win the election, consider platforms of {.3, .4, .7}. The shares of the vote are then {.35, .20, .45}. When candidates act so as to maximize their share of the vote, this configuration of platforms is not a PSNE, in part, because the right-most candidate can increase her share of the vote by choosing any platform between .4 and .7. When candidates only care about winning the election, however, this configuration is a PSNE. The left-most candidate cannot unilaterally deviate and capture a plurality of the vote (although she could make the middle candidate the winner). Likewise, the middle candidate cannot win or tie the election by deviating, and the right-most candidate has no incentive to deviate as she currently wins the election.

With Proposition 1 below, we provide a complete characterization of the PSNE in a three-candidate election in which winning the election is all that matters to the candidates. In our characterization, the winner's platform is greater than ½. A symmetric set exists when the winner's location is less than ½. Let the three candidates be labeled *A*, *B*, and *C*. Candidates simultaneously choose a platform, denoted by $P(x)$ where $P(x) \in [0, 1]$ for all $x \in \{A, B, C\}$. Without loss of generality, order the candidates so that $P(A) \leq P(B) \leq P(C)$. Given a vector of pure strategies, $\boldsymbol{P} = \{P(A), P(B), P(C)\}$, let $S(x|\boldsymbol{P})$ represent candidate *x*'s share of the vote.

Proposition 1:* *a. *There exists a set of pure strategy Nash equilibrium whereby C is the unique winner of the election and* $0 < P(A) \pounds P(B) < ½ < P(C) < 1$ *if and only if:*

(i) $P(C) – P(A) < 2/3$,

(ii) $3P(B) < 2 – P(C)$,

(iii) $3P(C) > 2 – P(A)$.

b. *There exists a symmetric set of PSNE whereby A is the unique winner of the election and* $0 < P(A) < ½ < P(B) \pounds P(C) < 1$.

c. *There are no other PSNE.*

Proof: a. In the first two steps we show that if a vector of strategies, $\boldsymbol{P}$, satisfies conditions *(i)* and *(ii)*, then *C* is the unique winner of the election.

Step 1. *B* cannot win. Suppose $P(A) < P(B)$. From condition *(i)*, candidate *B*'s share of the vote, $S(B|\boldsymbol{P})$, is less than 1/3. As the winner's share when there are three candidates must be at least 1/3, *B* cannot win or tie the election. If $P(A) = P(B)$, then $S(A|\boldsymbol{P}) = S(B|\boldsymbol{P}) = ½[P(B) + ½[P(C) – P(B)]]$ while $S(C|\boldsymbol{P}) = 1 – P(C) + ½[P(C) – P(B)]$. Thus, $S(C|\boldsymbol{P}) > S(A|\boldsymbol{P}) = S(B|\boldsymbol{P})$ if $3B < 4 – 3C$, a condition implied by condition *(ii)*.

Step 2. *A* cannot win. If $A = B$, then *A* cannot win for the same reason as above. For *A* to not win when $P(A) < P(B)$ requires $S(A|\boldsymbol{P}) = P(A) + ½ [P(B) – P(A)] = ½[P(A) + P(B)] < 1 – P(C) + ½[P(C) – P(B)] = S(C|\boldsymbol{P})$. Clearly *A*'s share is maximized at $P(B) - \boldsymbol{e}$, for $\boldsymbol{e}$ vanishingly small. Hence, we require that $S(A|\boldsymbol{P}) < ½[P(B) + P(B)] = P(B) < 1 - ½[P(C) + P(B)]$ or $3P(B) < 2 - P(C)$ which is condition *(ii)*. Thus, *C* is the unique winner.

In the next three steps we show that $\boldsymbol{P}$ is a PSNE under conditions *(i) – (iii)*.

Step 3. First, we show that no one gains by deviating to a position greater than $P(C)$. A deviation by *B* to $P(C) + \boldsymbol{e}$, for $\boldsymbol{e}$ vanishingly small yields a share for *B* of $1 – P(C) - \boldsymbol{e}/2$, however, *A*'s share becomes $P(A) + ½[P(C) – P(A)] > 1 – P(C)$ by condition *(iii)* allowing *A* to win. Furthermore, a deviation by *A* to $P(C) + \boldsymbol{e}$ would leave a share for *B* of $½[P(C) + P(B)] >$

$½[P(C) + P(A)]$ so that B would win. Hence, neither A nor B can win or tie the election by undertaking such a deviation.

Step 4. We now show that no one deviates to $P(C)$. *S*uppose B deviates to $P(C)$. In this case, condition *(iii)* implies that A's share of the vote, $P(A) + ½[P(C) – P(A)]$ is greater than B's share of the vote, namely, $½[1 – P(C) + ½[P(C) – P(A)]]$. Moreover, as $P(A)$ £ $P(B)$, A also cannot win or tie by deviating to $P(C)$.

Step 5. We now show that no one deviates to a position less than $P(C)$. From step 1 we know that A (B) will not deviate to a position between $P(B)$ and $P(C)$ (between $P(A)$ and $P(C)$) and that A (B) will not deviate to $P(B)$ $(P(A))$. From step 2 we know that A cannot win at their best position of $P(B) - e$ and clearly they have no incentive to move further to the left. Furthermore, because $P(A)$ £ $P(B)$ a deviation by B to the left of $P(A)$ will yield a vote share for B that is smaller than A's previous losing vote share and, therefore, they will not make this deviation. Hence, we have shown that any $\boldsymbol{P}$ that satisfies conditions *(i) – (iii)* is a PSNE.

Step 6. We now show that for any PSNE satisfying conditions *(i) – (iii)*, it must be that $0 < P(A), P(B) < ½$, and $½ < P(C) < 1$. If $P(A) = 0$, then $P(C) < 2/3$ by condition *(i)*, but then $3P(C) < 2 – P(A)$, which contradicts condition *(iii)*. Similarly, if $P(C) = 1$, then $P(A) > 1/3$ by condition *(i)*, but condition *(ii)* implies that $3P(B) < 1$ or, more specifically, that $P(B) < 1/3$, which contradicts $P(A)$ £ $P(B)$. Finally, if $P(C)$ £ $½$, then condition *(iii)* implies that $P(A) > ½$ which contradicts $P(A)$ £ $P(C)$. Therefore, $P(C) > ½$ which, from condition *(ii)*, yields that $P(B) < ½$. Hence, $0 < P(A)$ £ $P(B) < ½ < P(C) < 1$.

Step 7. It remains to show that if $\boldsymbol{P}$ is a PSNE where C is the unique winner and $0 < P(A)$ £ $P(B) < ½ < P(C) < 1$, then conditions *(i) – (iii)* must hold. By the construction of the proof it is evident that conditions *(ii)* and *(iii)* are necessary. To see that condition *(i)* is necessary as well, suppose instead that $P(C) – P(A) > 2/3$. It is then the case that either B wins or ties the election, which contradicts C being the unique winner, or, if not, B can deviate to $4/3 – C$ so that B would win the election as *B*'s share would be greater than 1/3 following the deviation while C's share would equal 1/3. Thus, such a configuration is not a PSNE. Hence, condition *(i)* is necessary.

b. The proof of the case where *A* wins is straightforward and proceeds identical to the proof of part a.

c. To see that there are no other PSNE, suppose instead that *B* wins the election. If *P(B)* £ ½, then a deviation by *C* to *P(B) + e*, for *e* small enough, would allow *C* to win the election. Similarly, if *P(B)* ³ ½, then there is a deviation by *A* towards ½ that would generate a victory for *A*. Hence, there is no PSNE whereby *B* wins the election. □

An immediate implication of Proposition 1 is that, in a three-candidate PSNE, the middle candidate can never be the winner of an election and that the winner does not locate at the median of the distribution of voters. We state these results in Corollary 1.

Corollary 1: *In a three-candidate plurality rule election, the candidate taking the middle position does not win or tie the election in any PSNE. Furthermore, the winning candidate does not adopt the platform that is most preferred by the median voter.*

Corollary 1 provides an important contrast to a two-candidate election whereby both candidates (and the eventual randomly chosen winner) locate at the median position. The addition of one more candidate moves all three candidates from the center and provides that the winner must adopt an extreme position. In addition, the allowable distance between the equilibrium platforms increases from 0 (in the two-candidate case) to almost 2/3 (in the three-candidate case). As will be seen, this important effect of the increase in the number of candidates on the viability of extreme positions is not limited to the $N = 3$ case.

An additional implication of Proposition 1 is that the position of the winning candidate places upper and lower bounds on the possible equilibrium platforms of the two losing candidates. Moreover, the amount by which the losing candidate's platforms can differ is not monotonic with respect to the winning candidate's position. The closer the winning candidate locates to ½ or 1, the closer the losing candidates must locate to each other. When the winner's platform is 2/3, the possible difference in the losing positions is maximized as both losers can locate anywhere above 0 and below 4/9.

3. Pure Strategy Nash Equilibria in *N*-Candidate Elections

We define $\boldsymbol{P}_N$ as the set of N platforms. To construct PSNE in N-candidate elections when $N > 3$, we start by considering a three-candidate PSNE that satisfies Proposition 1, such as $\boldsymbol{P}_3 = \{.25, .3, .7\}$ with vote shares of {.275, .225, .5}. Now add two additional candidates to yield $\boldsymbol{P}_5 = \{.25, .26, .28, .3, .7\}$ with shares of {.255, .015, .02, .21, .5}. Although the right most candidate still has a plurality share, the candidate at .26 (for example) could win by deviating to .71 yielding $\boldsymbol{P}_5 = \{.25, .28, .3, .7, .71\}$ with shares of {.265, .025, .21, .205, .295}. One way to preclude this type of deviation with N candidates is to require the candidate on the far right to be at least as extreme as the candidate on the far left. A weaker condition is that the share to the right of the winner is less than the share of the vote received by the candidate(s) on the far left. For example, $\boldsymbol{P}_5 = \{.25, .26, .28, .3, .8\}$ with shares of {.255, .015, .02, .26, .45} precludes profitable deviations to the right of the winning candidate at .8. The candidate at .3, however, could move to .76 which yields $\boldsymbol{P}_5 = \{.25, .26, .28, .76, .8\}$ with shares of {.255, .015, .25, .26, .22}. This deviation does not increase the mover's share but it does reduce the previous winner's share by enough to make this deviation profitable. To preclude this type of deviation, we require that one-half the distance between the winner (on the far right) and the candidate third from the right (second closest to the winning candidate) is less than the share of the candidate on the far left. For example, $\boldsymbol{P}_5 = \{.25, .26, .28, .3, .75\}$ with shares of {.255, .015, .02, .235, .475} satisfies this additional condition as (½)(0.75 – 0.28) < 0.255, and it is a PSNE whereby the candidate on the far right wins the election.

With the two conditions provided above as well as the condition that the candidate on the far left cannot win by moving closer to their closest neighbor and the condition that no other interior candidate has a winning share, we can provide a set of sufficient conditions for a PSNE in which the winner is on the far end of the political spectrum. In Proposition 2, these conditions are provided for a winner on the far right, and, of course, a symmetric set exists for a winner on the far left. To facilitate our analysis we label the candidates X_j, $j = \{1, 2,\ldots,N\}$, and we order the candidates so that $P(X_1)$ £ ·, ·, ·£ $P(X_N)$. Furthermore, to be able to consider the share of the leftmost candidate when more than one candidate adopts that position we define A as the number of candidates locating at $P(X_1)$. Hence, if three candidates locate at $P(X_1)$, then $A + 1 = 4$ and $X_{A+1} = X_4$ is the first candidate distinct from X_1. Finally, we denote B as the number of candidates locating at $P(X_{A+1})$, so that candidate X_{A+B+1} is the candidate in the third leftmost position.

Proposition 2: *$\boldsymbol{P}_N = \{P(X_j)\}$, $j = \{1, 2,\ldots,N\}$, is a PSNE in an N-candidate election, whereby the unique plurality share winner is X_N, if the following conditions hold:*

(i) $1 - ½[P(X_N) + P(X_{N-1})] > P(X_2)$;

(ii) $1 - ½[P(X_N) + P(X_{N-1})] > ½[P(X_{j+1}) - P(X_{j-1})]$ *for every* $j = \{2, 3,\ldots,N-1\}$;

(iii) $1 - P(X_N) < \frac{1}{2A}[P(X_{A+1}) + P(X_1)]$;

(iv) If $A = 1$, *then* $1 - P(X_N) < \frac{1}{2B}[P(X_{A+B+1}) + P(X_{A+B})]$;

(v) $½[P(X_N) - P(X_{N-2})] < \frac{1}{2A}[P(X_{A+1}) + P(X_1)]$.

Proof: In the first three steps, we show that if a vector of strategies, $\boldsymbol{P}_N$, satisfies conditions *(i) – (iii)*, then X_N is the unique winner of the election.

Step 1. To see that $P(X_{N-1}) < P(X_N)$, suppose instead that $P(X_N) = P(X_{N-1})$. Condition *(i)* can then be written as $1 - P(X_N) > P(X_2)$. If $A = 1$, then condition *(iii)* implies that $1 - P(X_N) < ½[P(X_2) + P(X_1)] < P(X_2)$, which is a contradiction. If A > 1, then condition *(ii)* can be written as $1 - P(X_N) > ½[P(X_{A+1}) - P(X_1)]$. Adding condition *(ii)* to condition *(i)* and noting that $P(X_1) = P(X_2) = P(X_A)$ yields $1 - P(X_N) > ¼ [P(X_{A+1}) + P(X_1)]$ which again contradicts condition *(iii).*

Step 2. If $A = 1$, then $S(X_1|\boldsymbol{P}_N) = P(X_1) + ½[P(X_2) - P(X_1)] = ½[P(X_1) + P(X_2)] < P(X_2) < 1 - P(X_N) + ½[P(X_N) - P(X_{N-1})] = S(X_N|\boldsymbol{P}_N)$ by condition *(i).* Furthermore, note for later that a move by candidate X_1 to $P(X_2) - e$, will yield at most $P(X_2) - e/2$ which is still less than X_N's share. If, on the other hand, $A > 1$, then $S(X_1|\boldsymbol{P}_N) = \frac{1}{A}P(X_1) + \frac{1}{2A}[P(X_{A+1}) - P(X_A)] = \frac{1}{A}P(X_1) + \frac{1}{2A}[P(X_{A+1}) - P(X_{A-1})]$, which by conditions *(i)* and *(ii)* must be less than $S(X_N|\boldsymbol{P}_N)$.

Step 3. By condition *(ii)*, the share of each candidate X_j that is located by themselves at a position between $P(X_1)$ and $P(X_N)$ is less than the share of X_N. On the other hand, if there is a group of size k such that $P(X_j) = P(X_{j+1}) = \cdots = P(X_{j+k})$, then each candidate in the cluster will receive $\frac{1}{2k}[P(X_{j+k+1}) – P(X_{j-1})]$ which, by condition *(ii)* must be less than the share of X_N if $k > 1$. Thus, we have that X_N wins the election and that no candidate can win by deviating anywhere less than or equal to $P(X_{N-1})$.

Step 4. We now show that no candidate can gain by deviating to a platform to the right of $P(X_N)$. A candidate deviating to $P(X_N) + e$ obtains a share less than $1 - P(X_N)$. By condition *(iii)* this share is less than the share accruing to X_1 when the deviating candidate is X_{A+1} through X_{N-1}. If $A > 1$ and a candidate located at $P(X_1)$ makes this deviation, then the share of the remaining candidates at $P(X_1)$ will increase so that no candidate at $P(X_1)$ will make this deviation. If $A = 1$ and $B = 1$ and X_1 makes this deviation then X_2's share increases to $½[P(X_3) + P(X_2)] > 1 - P(X_N)$ by condition *(iii).* Finally, if $A = 1$ and $B > 1$, and X_1 makes this deviation, the share of the candidates at X_2 is $\frac{1}{2B}[P(X_{B+2}) + P(X_{B+1})] > 1 - P(X_N)$ by condition *(iv).*

Step 5. We now show that no candidate gains by deviating between $P(X_{N-1})$ and $P(X_N)$. A deviation by candidate X_{N-1} to $P(X_N) - e$ would reduce the share of X_N, without reducing the share of X_{N-1}. Even if the reduced share of X_N is less than that of X_{N-1}, by condition *(v)*, the share of X_{N-1} is less than the share of X_1. A deviation by any other candidate to $P(X_N) - e$ would yield a deviator's share of $½[P(X_N) - P(X_{N-1})] < ½[P(X_N) - P(X_{N-2})]$. Hence, no candidate would make this deviation.

Step 6. Finally, notice that X_{N-1} has the most to gain by deviating to X_N. Under this deviation, however, the share of X_1 is greater than that of X_{N-1} by conditions *(iii)* and *(v)*. □

It is illustrative to compare Proposition 2 with Proposition 1. In particular, with $N = 3$, condition *(i)* in Proposition 2 is equivalent to condition *(ii)* in Proposition 1 and condition *(iii)* in Proposition 2 implies condition *(iii)* in Proposition 1.[3] With $N = 3$, condition *(ii)* in Proposition 2 is irrelevant as it is implied by conditions *(i)* and *(v)*, while condition *(iv)* is irrelevant with $N = 3$ as $P(C) > ½$. Finally, with $N = 3$, conditions *(i)*, *(iii)*, and *(v)* in Proposition 2 imply that $P(X_3) - P(X_1) < 6/11 < 2/3$, which is more restrictive than condition *(i)* in Proposition 1. Hence, the conditions in Proposition 2 are only sufficient for a set of PSNE. They are not necessary.

One implication of Proposition 2 is that when $N > 3$ the losing candidates are not limited to positions on the same side of the spectrum. For example, $\boldsymbol{P_5}$ = {.25, .35, .52, .6, .7} with shares of {.3, .135, .125, .09, .35} is a PSNE and has more candidates on the winning side than there are on the losing side.

A more interesting implication of Proposition 2 is that additional candidates in the region between $P(X_{A+B+1})$ and $P(X_{N-1})$ have no effect on the set of equilibria (or, if $A = B = 1$, then between $P(X_2)$ and $P(X_{N-1})$). Consider, for example $\boldsymbol{P_5}$ = {.25, .26, .3, .4, .75} with shares of {.255, .025, .07, .225, .425}. This configuration continues to constitute a PSNE whereby the

[3] We more fully develop these comparisons in the Appendix and show as well that with $N = 3$ they imply that $P(X_1) > 1/11$ and $P(X_2) < 5/11$.

winner locates at .75 even if an infinite number of candidates are added at positions between .26 and .4. As Proposition 3 below makes clear, this result is not limited to the preceding example.

Proposition 3**:** *Given any PSNE that satisfies Proposition 2, it is possible to let N approach infinity without changing the general structure of equilibrium set.*

Proof: Let all additional candidates locate between $P(X_{A+B+1})$ and $P(X_{N-1})$. Then these additional candidates have no effect on conditions *(i)*, *(iii)*, and *(iv)* and they introduce slack into conditions *(ii)* and *(v)*. □

In contrast to the non-mutability property of Proposition 3, a policy relevant implication of the model is given by the alternative scenario whereby additional candidates do alter the equilibrium set. In particular, the PSNE configuration $\boldsymbol{P}_5$ = {.15, .25, .5, .55, .85} with vote shares of {.2, .175, .15, .175, .3} has a distance of .7 between the winner on the far right and the left most candidate. As a point of comparison, when N = 3, Proposition 1 requires that this space is less than 2/3 and the more restrictive Proposition 2 implies a space of less than 6/11. In contrast, we now show that, as N becomes large it is possible for the distance between the winner on the extreme right and the left most candidate to approach unity. For example, consider $\boldsymbol{P}_N$ = {.01, .02, .03,…,.96, .97, .986} with shares of {.015, .01, .01,…,.01, .012, .022}. This configuration satisfies Proposition 2 and the winner is located at .986. We develop this point more fully in Proposition 4.

Proposition 4: *As the number of candidates approaches infinity the space between the left most and right most candidates can approach unity in a PSNE whereby the winner adopts one of these extreme positions.*

Proof: Without loss of generality, consider the case where the winner is X_N. Let the N candidates locate at $\boldsymbol{P}_N = \{1/(N+2), 2/(N+2), \ldots, (N-1)/(N+2), (N+2/3)/(N+2)\}$. This configuration satisfies Proposition 2 and the winner locates at $P(X_N) = (N+2/3)/(N+2)$. As N approaches infinity, $P(X_N)$ approaches 1 and $P(X_1)$ approaches 0. □

Proposition 4 further develops an idea that was introduced in Corollary 1, which demonstrated that as the number of candidates increases from two to three the distance between the candidates must increase. Given Proposition 3, however, Proposition 4 must be seen as a somewhat weaker result. Furthermore, it is only applicable to configurations in which the winner is located at an extreme of the chosen platforms.

Thus far, we have considered N-candidate equilibria that resemble those when $N = 3$. We now briefly turn attention to other PSNE in which the winner is not the right most (or left most) candidate. As mentioned above, when $N = 3$, condition *(ii)* of Proposition 2 is irrelevant. This condition, however, is important when $N > 3$. For example, the PSNE, $\boldsymbol{P_4}$ = {.25, .26, .75, .8} with shares of {.255, .25, .27, .225} violates condition *(ii)* of Proposition 2 as 1 – ½ [.8 + .75] < ½ [.8 – .26], but it is a PSNE where the third candidate is the winner. This configuration is not an exception. For example, $\boldsymbol{P_5}$ = {.18, .2, .5, .8, .82} with shares of {.19, .16, .3, .16, .19} is a PSNE where the winner is at .5. Similarly, $\boldsymbol{P_9}$ = {.08, .12, .25, .4, .5, .6, .75, .88, .92} with shares of {.1, .085, .14, .125, .1, .125, .14, .085, .1} is a PSNE where the candidates at .25 and .75 are tied for the highest plurality share. With these examples we have established the following.

Proposition 5: *In a PSNE, when N > 3, the plurality share winner does not need to be the candidate with the left most or right most of the chosen platforms. Furthermore, there exists PSNE where the winner chooses the platform most preferred by the median voter and where multiple candidates obtain the highest plurality share.*

Finally, with Proposition 1 we demonstrated that, for $N = 3$, there exists PSNE when the objective is winning the election that do not exist when the objective is vote maximization. When

$N > 3$, the opposite, it should be noted, can also occur. That is, a PSNE under vote maximization might not remain a PSNE when winning is the objective. For example, in a six-candidate election with vote maximization being the objective, $\boldsymbol{P_6}$ = {.15, .15, .45, .55, .85, .85} is a PSNE with shares of {.15, .15, .20, .20, .15, .15}. Thus, the third and fourth candidates tie the election. In this example, no candidate can unilaterally deviate and garner a greater share of the vote, however, either the third or the fourth candidates could unilaterally deviate and become the sole winner of the election. For example, suppose the third candidate deviates to a platform of .50. This deviation shifts 2.5 percent of the vote from the fourth candidate (giving 1.25 percent to each of the first two candidates) while leaving the third candidate's share unchanged. Thus, she now wins the election outright with 20 percent of the vote, while the fourth candidate receives 17.5 percent of the vote. This counter-example establishes Proposition 6.

Proposition 6: *If N is large enough, then there exists PSNE under vote maximization that are not PSNE when each candidate's objective is to win the election.*[4]

4. Entry

We now consider a modified game whereby candidates simultaneously choose their platforms and, after observing these choices, a sequence of additional candidates chooses whether or not to enter the election, and if entering, each chooses a platform that maximizes their probability of winning the election. The timing is such that each prospective entrant observes the entry and location choice of each candidate and previous potential entrant before making their own entry and location decision. We further assume that potential entrants face a positive cost of entry, and therefore will not enter unless they will win or tie the election. In this dynamic game

[4] The lowest N for which the proposition applies is 6. When $N = 4$ or 5, the only PSNE under vote maximization are $P_4 = (1/4, 1/4, 3/4, 3/4)$ and $P_5 = (1/6, 1/6, 1/2, 5/6, 5/6)$. Both are PSNE under vote maximization as well as maximizing one's probability of winning the election.

we refine our search to consider only subgame-perfect equilibria in pure strategies (PSPE). In particular, we are interested in identifying PSPE in which entry is precluded and we denote these as **entry-proof PSPE**.

First consider potential entry of a third candidate when $N = 2$. From Proposition 1, it is easy to see how two candidates can (and must) position themselves in order to prevent entry by a third candidate. Given that entry is deterred, however, the two positioned candidates both want to be the closer candidate to .5. Moreover, by Proposition 1, it can be seen that no matter how close one candidate locates to .5, the other can locate closer to .5 and continue to deter entry. This incentive to deviate towards .5 persists until both choose .5, at which time entry is no longer deterred. This is made formal in Proposition 7.

Proposition 7: *When two candidates face the potential entry of a third candidate, an entry proof PSPE does not exist.*

Proof: Let B be the potential entrant. First, if A or C chooses a platform of ½, then B can enter and win the election regardless of the other candidate's platform. Without loss of generality, therefore, suppose $P(A) < ½ < P(C)$. Using Proposition 1, it is easy to verify that if $P(A) = 1 - P(C) + \varepsilon$ where $\varepsilon < [\ 2 - P(C)\ ] / 3 - [\ 1 - P(C)\ ]$, then B cannot win or tie the election (and so does not enter) and A wins the election. By symmetry, however, C could win the election by deviating appropriately. □

When $N \geq 3$ it is possible to construct entry-proof PSPE. For example, when $N = 3$, the configuration $\boldsymbol{P} = \{.2, .4, .71\}$ is a PSPE in which the candidate at .71 wins and a potential entrant stays out of the contest. The configuration in this example satisfies Proposition 1 and, therefore, it is tempting to assume that any $\boldsymbol{P}$ that satisfies Proposition 1 is an entry-proof PSPE. The configuration $\boldsymbol{P} = \{.25, .4, .65\}$, however, satisfies Proposition 1 but does not preclude a winning entry of a fourth candidate at .66. On the other hand, in developing Proposition 2 we started with Proposition 1 and considered adding additional candidates. The first two conditions that we obtained became conditions *(iii)* and *(iv)* of Proposition 2. It is true that any $\boldsymbol{P}$ that satisfies Proposition 1 and these additional conditions is a PSPE in which a potential fourth candidate does not enter, however, we can generalize this idea even further. In particular, as Proposition 8 below makes clear, any $\boldsymbol{P_N}$ that satisfies the conditions of Proposition 2 constitutes an entry-proof PSPE.

Proposition 8: *When N candidates face the potential entry of a sequence of additional candidates, any configuration* $\boldsymbol{P_N}$ *that satisfies the following conditions is an entry-proof PSPE where the winner is* X_N*:*

(i) $1 - ½[P(X_N) + P(X_{N-1})] > P(X_2)$;

(ii) $1 - ½[P(X_N) + P(X_{N-1})] > ½[P(X_{j+1}) - P(X_{j-1})]$ *for every* $j = \{2, 3,\ldots,N-1\}$;

(iii) $1 - P(X_N) < \frac{1}{2A}[P(X_{A+1}) + P(X_1)]$;

(iv) *If* $A = 1$, *then* $1 - P(X_N) < \frac{1}{2B}[P(X_{A+B+1}) + P(X_{A+B})]$;

(v) $½[P(X_N) - P(X_{N-2})] < \frac{1}{2A}[P(X_{A+1}) + P(X_1)]$.

Proof: Conditions *(i) – (v)* imply that P_N is a PSNE by Proposition 2. It remains to show that a potential candidate cannot enter anywhere and win or tie the election. An entering candidate cannot win or tie by locating at, or to the left of, $P(X_{N-1})$ because locating there leaves X_N's share of the vote unchanged and yields to the entrant at most the same share as the second place finisher. Condition *(v)* implies that an entrant could not win by locating between $P(X_{N-1})$ and $P(X_N)$. Conditions *(iii)* and *(v)* implies that an entrant could not tie by locating at $P(X_N)$. And condition *(iii)* implies that an entrant could not win by locating to the right of $P(X_N)$. Given that an entrant could not choose a winning position, they will not enter. Given that a previous entrant does not enter, each new potential entrant will fact the same problem and also choose not to enter. Given that there is no entry, by Proposition 2 we know that the winner is candidate X_N. □

5. Conclusions

It is well-known that there are no pure strategy Nash equilibria in the standard three-candidate spatial voting model when candidates maximize their share of the vote. When all that matters to the candidates is winning the election, however, we show that PSNE do exist. Moreover, we provide a complete characterization of such equilibria. We next consider the implications of a plurality rule for $N > 3$ party elections. When the winner is on the end of the spectrum, the PSNE in the general case closely resembles the $N = 3$ case. In contrast to the $N = 3$ case, there are other PSNE in the $N > 3$ case where the winner does not take an extreme platform and in fact a winner locating at the median of the distribution is possible. An interesting implication of the PSNE we construct for the general case, when the winner is on the end of the spectrum, is that it is possible to unboundedly increase the number of candidates without altering the structure of the equilibrium set.

Finally, we consider the potential entry of additional candidates. When there are two existing candidates, we show that there are no PSNE in which entry of a third candidate is precluded. On the other hand we show that any configuration that is a PSNE in our general case also serves to avert the candidacy of a potential entrant.

APPENDIX: The relationship between Propositions 1 and 2 when N = 3.

When $N = 3$, condition *(i)* of Proposition 2 can be written as $1 - ½[P(X_3) + P(X_2)] > P(X_2)$, which is condition *(ii)* in Proposition 1. Similarly, condition *(iii)* of Proposition 2 implies that $1 - P(X_3) < ½[P(X_2) + P(X_1)] < ½[P(X_3) + P(X_1)]$, which is condition *(iii)* in Proposition 1. When $N = 3$, conditions *(i)* and *(v)* can be rewritten and combined as $1 - ½[P(X_3) + P(X_2)] > P(X_2) > ½[P(X_2) + P(X_1)] > ½[P(X_3) - P(X_1)]$, which yields condition *(ii)* for $j = 2$. When $N = 3$, however, condition *(ii)* is defined only for $j = 2$, therefore, conditions *(i)* and *(iv)* imply *(ii)*.

We now show that with $N = 3$, conditions *(i)*, *(iii)*, and *(v)* in Proposition 2 imply that $P(X_3) - P(X_1) < 6/11 < 2/3$. To this end define $Z = P(X_3) - P(X_1)$. Then using the Proposition 1 equivalents we have the following relationships.

Condition *(i)* of Proposition 2 implies that $Z = P(X_3) - P(X_1) < 2 - 3P(X_2) - P(X_1)$.

Condition *(iii)* of Proposition 2 implies that $3Z > 2 - 4P(X_1)$.

Condition *(v)* of Proposition 2 implies that $Z < P(X_2) + P(X_1)$.

These three inequalities describe a maximum level for Z in three dimensional space. Solving them simultaneously we find the following.

$Z = 2 - 3P(X_2) - P(X_1)$.

$3Z = 2 - 4P(X_1)$.

$Z = P(X_2) + P(X_1) \Rightarrow 3Z = 3P(X_2) + 3P(X_1)$.

Adding the first and third equations yields:

$3Z = 2 - 4P(X_1)$.

$4Z = 2 + 2P(X_1)$ Þ $8Z = 4 + 4P(X_1)$.

Adding these two equations yields $11Z = 6$. Substituting $Z = 6/11$ back into the original equations and observing the inequalities yields: $P(X_1) > 1/11$, $P(X_2) < 5/11$, and $Z < 6/11$.

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