United Nations University

WIDER

World Institute for Development Economics Research

Research Paper No. 2005/02

Measuring Inequality Without the Pigou–Dalton Condition

Alain Chateauneuf[1] and Patrick Moyes[2]

January 2005

Abstract

Typical welfare and inequality measures are required to be Lorenz consistent which guarantees that inequality decreases and welfare increases as a result of a progressive transfer. We explore the implications for welfare and inequality measurement of substituting the weaker absolute differentials, deprivation and satisfaction quasi-orderings for the Lorenz quasi-ordering. Restricting attention to distributions of equal means, we show that the utilitarian model – the so-called expected utility model in the theory of risk – does not permit one to make a distinction between the views embedded in the differentials, deprivation, satisfaction and Lorenz quasi-orderings. In contrast it is possible within the dual model of M. Yaari (*Econometrica* 55 (1987), 99–115) to derive the restrictions to be placed on the weighting function which guarantee that the corresponding welfare orderings are consistent with the differentials, deprivation and satisfaction quasi-orderings, respectively. Finally we drop the equal mean condition and indicate the implications of our approach for the absolute ethical inequality indices.

Keywords: income differentials, deprivation, satifaction, Lorenz dominance, progressive transfers, expected utility, generalized Gini social welfare functions

JEL classification: D31, D63

[1] CERMSEM, Université Paris 1 Panthéon-Sorbonne, France, Email. chateaun@univ-paris1.fr. [2] CNRS, IDEP and GRAPE, Université Montesquieu Bordeaux IV, France, Email. moyes@u-bordeaux4.fr.

This is a revised version of the paper originally prepared for the UNU-WIDER Conference on Inequality, Poverty and Human Well-Being, 30-31 May 2003, Helsinki.

UNU-WIDER acknowledges the financial contributions to the research programme by the governments of Denmark (Royal Ministry of Foreign Affairs), Finland (Ministry for Foreign Affairs), Norway (Royal Ministry of Foreign Affairs), Sweden (Swedish International Development Cooperation Agency—Sida) and the United Kingdom (Department for International Development).

ISSN 1810-2611 ISBN 92-9190-665-4 (internet version)

Acknowledgements

A preliminary version of this paper was presented at the WIDER Conference on Inequality, Poverty and Human Well-Being, Helsinki, Finland, 30-31 May 2003. We are in particular indebted to Stephen Bazen, Guillaume Carlier, Udo Ebert, Alain Trannoy and Claudio Zoli for useful conversations and suggestions. Needless to say, the authors bear the entire responsibility for remaining errors and deficiencies.

UNU World Institute for Development Economics Research (UNU-WIDER)
Katajanokanlaituri 6 B, 00160 Helsinki, Finland

Measuring Inequality Without the Pigou-Dalton Condition♯

This version, October 2004.

Alain Chateauneuf† and Patrick Moyes‡

Abstract

Typical welfare and inequality measures are required to be Lorenz consistent which guarantees that inequality decreases and welfare increases as a result of a progressive transfer. We explore the implications for welfare and inequality measurement of substituting the weaker absolute differentials, deprivation and satisfaction quasi-orderings for the Lorenz quasi-ordering. Restricting attention to distributions of equal means, we show that the utilitarian model – the so-called expected utility model in the theory of risk – does not permit one to make a distinction between the views embedded in the differentials, deprivation, satisfaction and Lorenz quasi-orderings. In contrast it is possible within the dual model of M. Yaari (*Econometrica* 55 (1987), 99–115) to derive the restrictions to be placed on the weighting function which guarantee that the corresponding welfare orderings are consistent with the differentials, deprivation and satisfaction quasi-orderings, respectively. Finally we drop the equal mean condition and indicate the implications of our approach for the absolute ethical inequality indices.
Journal of Economic Literature Classification Number: D31, D63. *Keywords*: Income Differentials, Deprivation, Satisfaction, Lorenz Dominance, Progressive Transfers, Expected Utility, Generalized Gini social welfare functions.

1. Introduction

1.1. Motivation and Relationship to the Literature

Following Atkinson (1970) and Kolm (1969) there is a wide agreement in the literature to appeal to the Lorenz curve for measuring inequality. A distribution of income is typically considered as being no more unequal than another distribution if its Lorenz curve lies nowhere below that of the latter distribution. Besides its simple graphical representation, much of the popularity of the so-called Lorenz criterion originates in its relationship with the notion of progressive transfers. It is traditionally assumed that inequality is reduced by a progressive transfer i.e., when income is transferred from a richer to a poorer individual whithout affecting their relative positions on the ordinal income scale. The principle of tranfers, which captures this judgement, is closely associated with the Lorenz quasi-ordering of distributions of equal means. Indeed half a century ago, Hardy, Littlewood and Polya (1952) have demonstrated that, if a distribution Lorenz dominates another distribution, then the former can be obtained

♯ A preliminary version of this paper was presented at the WIDER Conference on *Inequality, Poverty and Human Well-Being*, Helsinki, Finland, 30-31 May 2003. We are indebted to Stephen Bazen, Udo Ebert, Alain Trannoy and Claudio Zoli for useful conversations and suggestions. Needless to say, the authors bear the entire responsibility for remaining errors and deficiencies.

† CERMSEM, Université Paris 1 Panthéon-Sorbonne, 106-112 Boulevard de l'Hôpital, F-75647 Paris Cédex 13, France, Tel. (+33) [0]1.55.43.42.97, Fax. (+33) [0]1.55.43.43.01, Email. chateaun@univ-paris1.fr.

‡ CNRS, IDEP and GRAPE, Université Montesquieu Bordeaux IV, Avenue Léon Duguit, F-33608 Pessac, France, Tel. (+33) [0]5.56.84.29.05, Fax. (+33) [0]5.56.84.29.64, Email. moyes@u-bordeaux4.fr, HomePage: http://moyes.u-bordeaux4.fr.

from the latter by means of a finite sequence of progressive transfers, and conversely[1]. This relationship between progressive transfers and the Lorenz quasi-ordering constitutes the cornerstone of the modern theory of welfare and inequality measurement. As a consequence, the literature has concentrated on Lorenz consistent inequality measures i.e., indices such that a progressive transfer is always recorded as reducing inequality or increasing welfare.

Notwithstanding its wide application in theoretical and empirical work the approach based on the Lorenz curve is not immune to criticism. Whereas most of the literature on inequality and welfare measurement imposes the principle of transfers, one may however raise doubts about the ability of such a condition to capture the very idea of inequality in general. Though a progressive transfer unambiguously reduces inequality between the individuals involved in the transfer, it is far from being obvious that everyone would agree that inequality on the whole has declined as a result. This is due to the fact that in general making two incomes closer increases the gap between each of these two incomes and the incomes of the rest of the society, so that it is difficult to admit that inequality on the whole has declined. It is to some extent surprising that the profession has been assimilating overall inequality reduction with local pairwise inequality reduction for such a long time. The fact that progressive transfers are not universally approved has been confirmed by recent experimental studies [see e.g. Amiel and Cowell (1992), Ballano and Ruiz-Castillo (1993), Harrison and Seidl (1994), Gaertner and Namezie (2003) among others].

However the experimental studies fail to provide information about the subjects' preferences towards equality with the exception that these preferences are at variance with the views captured by the principle of tranfers used by the theory of inequality measurement. Different ideas come to mind in order to reconcile the theory with the conclusions of these experimental studies. A first possibility would be to declare that inequality unambiguously decreases if and only if the income differentials between *any* two individuals in the population are reduced, assuming that all individuals occupy the same positions on the income scale in the situations under comparison. This is a kind of unanimity point of view: overall inequality decreases if and only if the inequalities between any two individuals in the society decrease. This rules out the limitation of the principle of transfers we pointed out above since now, not only should the gap between the donor and the recipient of a transfer be reduced, but also the gaps between these two individuals and the individuals not taking part in the transfer. This still leaves open the question to know which kind of income differentials are thought of relevance when making inequality judgements. The relative and absolute differentials quasi-orderings introduced by Marshall, Olkin and Proschan (1967) constitute two possible candidates. But there are other possible views – e.g. along the lines suggested by Bossert and Pfingsten (1990) – that might constitute alternative grounds for constructing a theory of inequality measurement more in line with common sense.

[1] Although parts of this general result appeared in different places in Hardy, Littlewood and Polya (1952), one had to wait until Berge (1963) who collected these scattered statements and provided a self-contained proof of what is know now as the Hardy-Littlewood-Polya theorem. Related results in the field of inequality measurement have been provided by Kolm (1969), Atkinson (1970), and Fields and Fei (1978) among others [see also Dasgupta, Sen and Starrett (1973), Sen (1973), and Foster (1985)].

On the other hand, there is evidence that the social status of an individual – approximated by her position in the social hierarchy – plays an important role in the determination of her well-being [see e.g. Weiss and Fershtman (1998)]. Attitudes such as envy, deprivation, resentment and satisfaction have been argued to be important components of individual judgements and they might be taken into account as far as distributive justice is concerned. In particular the notion of individual deprivation originating in the work of Runciman (1966) accommodates such views making the individual's assessment of a given social state depend on her situation compared with the situations of all the individuals who are treated more favourably than her. The deprivation profile, which indicates the level of deprivation felt by each individual, might therefore constitute the basis of social judgement. Drawing upon previous work by Yitzhaki (1979, 1982), Hey and Lambert (1980), Kakwani (1984), Chakravarty (1997), and Chakravarty and Moyes (2003), one can propose two deprivation quasi-orderings depending on the way individual deprivation is defined. Individual deprivation in a given state formally ressembles the aggregate poverty gap where the poverty line is set equal to other individuals' incomes[2]. So stated, one may conceive of absolute individual deprivation, which is simply the sum of the gaps between the individual's income and the incomes of all individuals richer than her, and relative deprivation, where the income gaps are deflated by the individual's income. Then the deprivation quasi-ordering is based on the comparisons of the individual deprivation curves and social deprivation unambiguously decreases as the individual deprivation curve is moving downwards.

Rather than comparing herself with individuals who are richer than her – equivalently who occupy a higher position on the social status scale – an individual can consider those who are poorer. The larger the aggregate gap between her income and the incomes of poorer individuals, the higher her satisfaction will be. More precisely one may conceive of absolute individual satisfaction, which is simply the sum of the gaps between the individual's income and the incomes of all individuals poorer than her, and relative individual satisfaction, where the income gaps are deflated by the individual's income [see Chakravarty (1997)]. The notion of satisfaction may be considered the dual of the notion of deprivation. Then the satisfaction quasi-ordering is based on the comparisons of the individual satisfaction curves and social satisfaction unambiguously decreases as the individual satisfaction curve moves downwards. A natural objective of the society will be to make individual satisfaction and deprivation as small as possible, the minimum being attained when all incomes are equal.

1.2. The Theoretical Approach Developed in the Paper

Assuming that we subscribe to these more primitive notions of inequality, the next step is to identify the welfare and inequality indices that are consistent with the differentials, deprivation and/or satisfaction quasi-orderings. In this paper we restrict attention to ethical

[2] Most scholars take for granted that individual deprivation is simply the sum – possibly normalized in a suitable way – of the income gaps between the individual's income and the incomes of all individuals richer than her. An axiomatic characterization of the absolute deprivation profile is provided by Ebert and Moyes (2000).

inequality indices, which means that we start with a given welfare ordering of income distributions – more precisely a given social welfare function – and derive an inequality index in an appropriate way[3]. We assume in addition that this ordering can be represented by a member of the class of rank-dependent expected utility social welfare functions introduced by Quiggin (1993), which admits as particular cases the utilitarian and the generalized Gini social welfare functions known as the expected utility and the Yaari models respectively in the theory of choice under risk[4]. Then we look for the restrictions that have to be imposed on the social welfare function – equivalently on the social welfare ordering – that guarantee that the implied ethical inequality index is consistent with the primitive views captured by the differentials, deprivation and satisfaction quasi-orderings. It is argued that the standard expected utility model does not permit to distinguish the different concepts of inequality discussed above. In other words, the utilitarian social welfare function is not sufficiently flexible to accommodate such distinct attitudes as those encompassed by the differentials, deprivation and satisfaction quasi-orderings. On the contrary the dual model of choice introduced by Yaari (1987, 1988) permits to derive measures which are consistent with the differentials, deprivation and satisfaction quasi-orderings[5]. More precisely, the paper identifies the restrictions to be imposed on the weighting function that guarantee that inequality will not increase when incomes are more equally distributed according to the three former quasi-orderings.

1.3. Organization of the Paper

Section 2 introduces our conceptual framework consisting of distributions for finite populations of possibly different sizes where every individual is associated with a given income. In addition to the Lorenz criterion, we distinguish different inequality views which we identify with quasi-orderings defined on the set of income distributions. The quasi-orderings we consider are all weaker than the Lorenz quasi-ordering as they all imply it. Section 3 defines the inequality quasi-orderings used in the paper, explores their relationships and hints at some connections with progressive transfers. We examine in Section 4 different ways of weakening the notion of equalizing transfer – equivalently of strenghening the principle of transfers – which are related to our inequality quasi-orderings. Section 5 contains our main results and investigates the implications for the social welfare functions of the inequality views captured by the differentials, deprivation and satisfaction quasi-orderings in the particular case where distributions have equal means. It is shown that the utilitarian model, which frames most of the theory of welfare and inequality measurement, does not allow to distinguish between these views and the traditional one captured by the Lorenz quasi-ordering. On the contrary the model proposed by Yaari (1987, 1988) allows the ethical planner to make a distinction

[3]See Blackorby, Bossert and Donaldson (1999) for a recent survey of the literature on the ethical approach to inequality measurement.

[4]The rank-dependent expected utility model is flexible enough to accommodate most of the inequality views one encounters in the literature.

[5]Yaari's (1987) model was introduced in the fields of choice under risk and then applied to the measurement of inequality in Yaari (1988). Related approaches have been proposed in the inequality literature by Ebert (1988) and Weymark (1981).

between these competing views. Section 6 indicates how the analysis could be extended in order to cover the general case where the distributions under comparison do not necessarily have the same mean. We summarize our results in Section 7 which also hints at some directions for future work.

2. Preliminary Notation and Definitions

We assume throughout that incomes are drawn from an interval D which is a compact subset of $\mathbb{R}$. An *income distribution* or *situation* for a population consisting of n identical individuals ($n \geq 2$) is a list $\mathbf{x} := (x_1, x_2, \ldots, x_n)$ where $x_i \in D$ is the income of individual i. We indicate by $\mathbf{1_n} := (1, \ldots, 1)$ the unit vector in $\mathbb{R}^n$. Letting $\mathcal{Y}_n(D)$ represent the set of income distributions for a population of size n, the set of all income distributions of finite size will be denoted as $\mathcal{Y}(D) := \bigcup_{n=2}^{\infty} \mathcal{Y}_n(D)$. The *dimension* of distribution $\mathbf{x} \in \mathcal{Y}(D)$ is indicated by $n(\mathbf{x})$ and its arithmetic *mean* by $\mu(\mathbf{x}) := \sum_{i=1}^{n(\mathbf{x})} x_i / n(\mathbf{x})$. Given $\mathbf{x} := \big(x_1, \ldots, x_{n(\mathbf{x})}\big) \in \mathcal{Y}(D)$, we use $\mathbf{x}_{(\,)} := \big(x_{(1)}, x_{(2)}, \ldots, x_{(n(x))}\big)$ to indicate its non-decreasing rearrangement defined by $\mathbf{x}_{(\,)} = P\mathbf{x}$ for some $n(\mathbf{x}) \times n(\mathbf{x})$ permutation matrix P such that $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n(\mathbf{x}))}$[6]. We denote as $F(\,\cdot\,; \mathbf{x})$ the *cumulative distribution function* of $\mathbf{x} \in \mathcal{Y}(D)$ defined by $F(z; \mathbf{x}) := q(z; \mathbf{x})/n(\mathbf{x})$, for all $z \in (-\infty, +\infty)$, where $q(z; \mathbf{x}) := \#\big\{i \in \{1, 2, \ldots, n(\mathbf{x})\} \,\big|\, x_{(i)} \leq z\big\}$. We let $F^{-1}(\,\cdot\,; \mathbf{x})$ represent the *inverse cumulative distribution function* – equivalently the *quantile function* – of $\mathbf{x}$ obtained by letting $F^{-1}(0; \mathbf{x}) := x_{(1)}$ and

$$F^{-1}(p; \mathbf{x}) := \operatorname{Inf}\big\{z \in (-\infty, +\infty) \,\big|\, F(z; \mathbf{x}) \geq p\big\}, \ \forall\, p \in (0, 1] \tag{2.1}$$

[see Gastwirth (1971)].

We are interested in the comparisons of income distributions from the point of view of social welfare and inequality. A *social welfare function* $W : \mathcal{Y}(D) \longrightarrow \mathbb{R}$ associates to every distribution a real number $W(\mathbf{x})$ that represents the social welfare attained in situation $\mathbf{x} \in \mathcal{Y}(D)$. When $W(\mathbf{x}) \geq W(\mathbf{y})$, then we will say that situation $\mathbf{x}$ is at least as good as $\mathbf{y}$ from the point of view of W. Similarly an *inequality index* $I : \mathcal{Y}(D) \longrightarrow \mathbb{R}$ indicates for every distribution the degree of inequality attained with the convention that $I(\mathbf{x}) \leq I(\mathbf{y})$ means that situation $\mathbf{x}$ is no more unequal than situation $\mathbf{y}$. Here social welfare functions and inequality indices are considered as particular cardinal representations of orderings – complete, reflexive and transitive binary relations – on the set of income distributions, and no cardinal significance should be attributed to the values taken by these indices. We denote as $\mathbb{W}(D)$ and $\mathbb{I}(D)$ the set of social welfare functions and the set of inequality indices respectively.

Although our primary concern is to make comparisons of arbitrary distributions whose dimensions may differ, it is worth emphasizing that there is no loss of generality restricting attention to distributions with the same dimension. This is a consequence of the *principle of population* according to which a replication does not affect inequality and welfare [see

[6] A permutation matrix $P := [p_{ij}]$ is an $n \times n$ (some $n \geq 2$) matrix such that (i) for all i, j, either $p_{ij} = 0$, or $p_{ij} = 1$, (ii) for all i, $p_{ij} = 1$ implies $p_{ik} = 0$, for all $k \neq j$, and (iii) for all j, $p_{ij} = 1$ implies $p_{hj} = 0$, for all $h \neq i$.

Dalton (1920)][7]. Similarly because all individuals are identical in all respects other than their incomes, one usually imposes the condition of *symmetry* which requires that exchanging incomes between two individuals would not affect the levels of welfare and inequality[8].

In a number of cases it is impossible to reach a unanimous agreement regarding the appropriate ordering of situations and the only consensus that one might reasonably expect only yields a partial ranking. A *quasi-ordering* is a reflexive and transitive binary relation defined on the set of distributions which may result in a partial ranking of the situations under consideration[9]. Given the quasi-ordering $\geq_J$ over $\mathcal{Y}(D)$, we denote as $>_J$ and $\sim_J$ its asymmetrical and symmetrical components defined in the usual way. We are mostly concerned with welfare and inequality indices that are compatible with certain given inequality views that will be expressed by means of quasi-orderings. Precisely, given a quasi-ordering $\geq_J$ over $\mathcal{Y}(D)$ and a social welfare function $W \in \mathbb{W}(D)$ [resp. an inequality index $I \in \mathbb{I}(D)$], we will say that W *[resp. I] is consistent with* $\geq_J$, if

$$\forall\, \mathbf{x}, \mathbf{y} \in \mathcal{Y}(D) \,:\, \mathbf{x} \geq_J \mathbf{y} \Longrightarrow W(\mathbf{x}) \geq W(\mathbf{y}) \text{ [resp. } I(\mathbf{x}) \leq I(\mathbf{y})]. \tag{2.2}$$

All the quasi-orderings $\geq_J$ we consider in the paper have the property that (i) $\mathbf{x}^r \sim_J \mathbf{x}$, for all $\mathbf{x} \in \mathcal{Y}(D)$ and all $r \in \mathbb{N}$ $(r \geq 2)$, and (ii) $P\mathbf{x} \sim_J \mathbf{x}$, for all $\mathbf{x} \in \mathcal{Y}(D)$ and all $n(\mathbf{x}) \times n(\mathbf{x})$ permutation matrices P. Therefore there is no loss of generality in restricting attention to distributions of the same dimension n $(n \geq 2)$ that are non-decreasingly arranged.

3. From Lorenz to More Primitive Inequality Views

3.1. Introductory Example and Definitions

It is typically assumed in normative economics that inequality is reduced and welfare increased by a transfer of income from a richer individual to a poorer individual. More precisely, we have:

Definition 3.1. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we will say that $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a progressive transfer*, if there exists $\Delta > 0$ and two individuals i, j such that

$$x_k = y_k, \ \forall\, k \neq i, j; \tag{3.1.a}$$

$$x_i = y_i + \Delta; \quad x_j = y_j - \Delta; \text{ and} \tag{3.1.b}$$

$$\Delta \leq (y_j - y_i)/\,2. \tag{3.1.c}$$

[7]Given two distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}(D)$ we will say that $\mathbf{x}$ is a *replication* of $\mathbf{y}$ if there exists $r \in \mathbb{N}$ $(r \geq 2)$ such that $\mathbf{x} = \mathbf{y}^r := (\mathbf{y}; \ldots; \mathbf{y}) \in \left[\mathcal{Y}_{n(\mathbf{y})}(D)\right]^r$. Then the index $M \in \mathbb{W}(D) \cup \mathbb{I}(D)$ satisfies the principle of population if, for all $\mathbf{x} \in \mathcal{Y}(D)$ and all $r \in \mathbb{N}$ $(r \geq 2)$: $M(\mathbf{x}^r) = M(\mathbf{x})$.

[8]Precisely the index $M \in \mathbb{W}(D) \cup \mathbb{I}(D)$ satisfies the condition of symmetry if, for all $\mathbf{x} \in \mathcal{Y}(D)$ and all $n(\mathbf{x}) \times n(\mathbf{x})$ permutation matrices P, $M(P\mathbf{x}) = M(\mathbf{x})$.

[9]We adopt throughout the paper the terminology proposed by Sen (1970, Chap. 1*), but we recognize that there are other possibilities.

By definition, a progressive transfer does not reverse the relative positions of the individuals involved. However, although the donor cannot be made poorer than the recipient, it may be the case that their positions relative to the positions of the other individuals are modified. It is convenient to assume that the progressive transfer is *rank-preserving* in the sense that the relative positions of *all* the individuals are unaffected, which amounts to impose the additional condition:

$$(x_k - x_h)\,(y_k - y_h) \geq 0,\ \forall\ h \neq k. \tag{3.2}$$

PRINCIPLE OF TRANSFERS. For all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we have $W(\mathbf{x}) \geq W(\mathbf{y})$ and $I(\mathbf{x}) \leq I(\mathbf{y})$, whenever $\mathbf{x}$ is obtained from $\mathbf{y}$ by a [rank-preserving] progressive transfer.

The notion of a progressive transfer is closely associated with that of the Lorenz quasi-ordering. The *Lorenz curve* of distribution $\mathbf{x} \in \mathcal{Y}_n(D)$, which we denote as $L(p;\mathbf{x})$, is defined by

$$L(p;\mathbf{x}) := \int_0^p F^{-1}(s;\mathbf{x})\,ds,\ \forall\, p \in [0,1]. \tag{3.3}$$

By definition $L(p;\mathbf{x})$ represents the total income possessed by the fraction p of poorest individuals deflated by the population size in situation $\mathbf{x}$[10].

DEFINITION 3.2. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we will say that $\mathbf{x}$ *Lorenz dominates* $\mathbf{y}$, which we write $\mathbf{x} \geq_L \mathbf{y}$, if and only if

$$L(p;\mathbf{x}) \geq L(p;\mathbf{y}),\ \forall\ p \in (0,1) \text{ and } L(1;\mathbf{x}) = L(1;\mathbf{y}). \tag{3.4}$$

The higher its associated Lorenz curve, the less unequal a distribution is according to the Lorenz criterion. Condition (3.4) can be equivalently rewritten as

$$\sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j,\ \forall\, k = 1, 2, \ldots, n-1, \text{ and } \mu(\mathbf{x}) = \mu(\mathbf{y}). \tag{3.5}$$

As we already insisted, much of the popularity of the Lorenz criterion originates in the fact that it is closely associated with progressive transfers. Hardy, Littlewood and Polya (1952) were the first to show that a distribution Lorenz dominates another one if and only if it can be obtained from the latter by means of successive applications of progressive transfers [see also Berge (1963), Kolm (1969), Fields and Fei (1978) and Marshall and Olkin (1979) among others)]. Precisely, they proved the following:

[10] Our definition of the Lorenz curve is different from the standard one which requires that the total income possessed by the fraction p of poorest individuals is deflated by the total income. The difference is immaterial as long as we focus on comparisons of distributions with equal means.

Proposition 3.1. *Let* $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ *such that* $\mu(\mathbf{x}) = \mu(\mathbf{y})$. *Then, the following two statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a finite sequence of progressive transfers.*

(b) $\mathbf{x} \geq_L \mathbf{y}$.

It is important to note that there is no particular restriction is imposed on the way the progressive transfers are combined in the result above: any sequence of progressive transfers results in an improvement in terms of Lorenz dominance. A direct implication of Proposition 3.1 is that any measure that verifies the principle of transfers is Lorenz-consistent [see Foster (1985)].

However despite its wide use in theoretical and empirical work on inequality the Lorenz quasi-ordering is not the only criterion that can be used in order to rank distributions in an unambiguous way. Other criteria have been proposed in the literature that pay attention to different features of the distributions under comparisons and that may be considered potential candidates for measuring inequality. Particular instances are the deprivation and satisfaction indices – due to Kakwani (1984) and Chakravarty (1997) respectively – that are concerned with the feelings of the individuals with respect to their personal situations relative to the situations of the other individuals. It must also be recognized that in some circumstances the application of the Lorenz quasi-ordering has implications that go far beyond the judgements embedded in this criterion. A typical example is given in public finance, where it has been shown that an a non-decreasing average tax rate is a necessary and sufficient condition for the after tax distribution to Lorenz dominate the before tax distribution whatever the circumstances [see Jakobsson (1976)]. Actually the application of the Lorenz criterion implies that all pairwise relative income differences be not larger in the after tax distribution than in the before tax one [see Moyes (1994)]. A test involving the comparisons of all pairwise differences – leaving aside for the moment the way we measure these differerences – appears to be an uncontroversial criterion for passing inequality judgements.

Therefore the Lorenz criterion does not exhaust all the possibilities for measuring inequality and one may think of other potential candidates. Given the relationship between the Lorenz criterion and progressive transfers, this calls into question the very principle of transfers that is at the heart of the theory of inequality measurement and that sustain the Lorenz criterion. This suggests that not everyone would agree on the fact that a progressive transfer decreases inequality in all circumstances. This has been exemplified in a number of experimental studies by means of questionnaires where it has been demonstrated that the principle of transfers is largely rejected by the respondents [see Amiel and Cowell (1992), Ballano and Ruiz-Castillo (1993), Harrison and Seidl (1994), Gaertner and Namezie (2003) among others]. The following example, which captures the main features of the situations presented to the interviewed in these experiments, might help to convince the reader that the principle of transfers is debatable.

Example 3.1. Let $n = 4$ and consider the distributions $\mathbf{x}^1 = (1, 3, 5, 7)$, $\mathbf{x}^2 = (1, 3, 6, 6)$, $\mathbf{x}^3 =$

$(1, 4, 4, 7)$, $\mathbf{x}^4 = (2, 2, 5, 7)$, and $\mathbf{x}^5 = (2, 3, 5, 6)$. It is immediate that each of the distributions $\mathbf{x}^2$, $\mathbf{x}^3$, $\mathbf{x}^4$ and $\mathbf{x}^5$ obtains from $\mathbf{x}^1$ by means of a single [rank-preserving] progressive transfer of one income unit. It follows from Proposition 3.1 that $\mathbf{x}^g \geq_L \mathbf{x}^1$, for all $g = 2, 3, 4, 5$, so that everyone who subscribes to the principle of transfers – equivalently to the Lorenz criterion – will consider that distributions $\mathbf{x}^2$, $\mathbf{x}^3$, $\mathbf{x}^4$ and $\mathbf{x}^5$ are less unequal than distribution $\mathbf{x}^1$.

Inspection of the above distributions reveals that $\mathbf{x}^2$ is obtained from $\mathbf{x}^1$ by transferring one unit of income from the richest individual to the second richest, which actually amounts to equalise the incomes of the two richest individuals. Inequality between individuals 3 and 4 has therefore been eliminated but at the same time the income gap – or income differentials – between individuals 1 and 3 on the one hand, and individuals 2 and 3 on the other hand, has been widened. Although the Lorenz criterion would say that $\mathbf{x}^2$ is unambiguously more equal than $\mathbf{x}^1$, there might be – and there are actually – people who disagree with this conclusion invoking the fact that the pairwise income differentials are not all made smaller as a result of the progressive transfer.

Suppose we agree with the above view according to which inequality unambiguously decreases if and only if the absolute difference between any two incomes is reduced. Precisely, given the income distribution $\mathbf{x} \in \mathcal{Y}_n(D)$, we define:

$$AD(p, s; \mathbf{x}) := F^{-1}(s; \mathbf{x}) - F^{-1}(p; \mathbf{x}), \ \forall\ 0 \leq p < s \leq 1. \tag{3.6}$$

Thus $AD(p, s; \mathbf{x})$ measures the absolute income gap between the richer individual occupying rank s and the poorer individual ranked p in situation $\mathbf{x}$. It is our contention that nobody would object to the judgement that inequality does not increase when the absolute income gaps between any richer and any poorer individuals are made smaller. Actually it is a simple matter to verify that the following definition captures precisely this idea.

Definition 3.3. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we will say that $\mathbf{x}$ *dominates* $\mathbf{y}$ *in absolute differentials*, which we write $\mathbf{x} \geq_{AD} \mathbf{y}$, if and only if

$$AD(p, s; \mathbf{x}) \leq AD(p, s; \mathbf{x}), \ \forall\ 0 \leq p < s \leq 1. \tag{3.7}$$

According to condition (3.6) the differences between any two adjacent individuals' incomes are no larger in situation $\mathbf{x}$ than in situation $\mathbf{y}$. Actually condition (3.7) is equivalent to

$$x_{k+1} - x_k \leq y_{k+1} - y_k, \ \forall\ k = 1, 2, \ldots, n-1. \tag{3.8}$$

This quasi-ordering, first introduced by Marshall, Olkin and Proschan (1967) in the fields of majorization [see also Bickel and Lehmann (1976)], has been considered a suitable inequality criterion [see e.g. Thon (1987), Preston (1990), and Moyes (1994, 1999)].

Although it is difficult to object against its consensual nature, the absolute differentials quasi-ordering might be viewed far too strong a criterion and one might want consider criteria that lay half way between it and the Lorenz quasi-ordering. When comparing two distributions by means of the absolute differentials quasi-ordering, every individual compares her situation with that of *all* the individuals richer than her. It might be that what is important is not really by how much every poorer individual falls below every richer individual, but rather by how much *on average* she is away from the richer individuals. This is reminiscent of the notion of deprivation introduced by Runciman (1966) according to whom the individual's assessment of a given social state depends on her situation compared with the situations of individuals who are treated more favourably than her. The *absolute deprivation curve* of distribution $\mathbf{x} \in \mathcal{Y}_n(D)$ – denoted as $ADP(p;\mathbf{x})$ – is defined by

$$\text{(3.9)} \qquad ADP(p;\mathbf{x}) := \int_p^1 \left[F^{-1}(s;\mathbf{x}) - F^{-1}(p;\mathbf{x})\right]\, ds \equiv \int_p^1 AD(p,s;\mathbf{x})\, ds,\ \forall\, p \in [0,1]$$

[see Kakwani (1984)]. We can interpret $ADP(p;\mathbf{x})$ as a measure of the absolute deprivation felt by individual with rank p in situation $\mathbf{x}$. By definition, the best-off individual is never deprived and thus $ADP(1;\mathbf{x}) = 0$, for all $\mathbf{x} \in \mathcal{Y}(D)$. Following Chakravarty (1997), we introduce:

Definition 3.4. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we will say that *there is no more absolute deprivation in* $\mathbf{x}$ *than in* $\mathbf{y}$, which we write $\mathbf{x} \geq_{ADP} \mathbf{y}$, if and only if

$$\text{(3.10)} \qquad ADP(p;\mathbf{x}) \leq ADP(p;\mathbf{y}),\ \forall\, p \in [0,1).$$

Actually condition (3.10) simply states that overall deprivation decreases if the individual deprivation felt by any member of the society decreases. In the particular case where $n(\mathbf{x}) = n(\mathbf{y}) = n$, condition (3.10) reduces to

$$\text{(3.11)} \qquad \sum_{j=k+1}^{n} [x_j - x_k] \leq \sum_{j=k+1}^{n} [y_j - y_k]\,,\ \forall\ k = 1,2,\ldots,n-1.$$

Rather than comparing herself with the individuals richer than her, an individual might find some comfort in comparing her situation with the situations of the individuals who are in a position worse than her. The *absolute satisfaction curve* of distribution $\mathbf{x} \in \mathcal{Y}_n(D)$ – denoted as $ASF(p;\mathbf{x})$ – is defined by

$$\text{(3.12)} \qquad ASF(p;\mathbf{x}) := \int_0^p \left[F^{-1}(p;\mathbf{x}) - F^{-1}(s;\mathbf{x})\right]\, ds \equiv \int_0^p AD(p,s;\mathbf{x})\, ds,\ \forall\, p \in [0,1].$$

We can interpret $ASF(p;\mathbf{x})$ as a measure of the absolute satisfaction felt by individual ranked p in situation $\mathbf{x}$. By definition, the worst-off individual is never satisfied and $ASF(0;\mathbf{x}) = 0$, for all $\mathbf{x} \in \mathcal{Y}(D)$. Following Chakravarty (1997), we introduce:

Definition 3.5. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we will say that *there is no more absolute satisfaction in* $\mathbf{x}$ *than in* $\mathbf{y}$, which we write $\mathbf{x} \geq_{ASF} \mathbf{y}$, if and only if

$$\text{(3.13)} \qquad ASF(p;\mathbf{x}) \leq ASF(p;\mathbf{y}),\ \forall\, p \in (0,1].$$

Actually condition (3.13) simply states that overall satisfaction decreases if the individual satisfaction felt by any member of the society decreases. In the case of discrete distributions of fixed dimension n, condition (3.13) reduces to

$$\sum_{j=1}^{k-1}[x_k - x_j] \leq \sum_{j=1}^{k-1}[y_k - y_j]\,,\ \forall\ k = 2, 3, \ldots, n. \tag{3.14}$$

Applying the preceding quasi-orderings to the comparisons of the distributions introduced in Example 3.1 gives the following rankings:

Table 3.1. Inequality Rankings of Distributions of Example 3.1.

$\geq_{AD}$

	$\mathbf{x}^1$	$\mathbf{x}^2$	$\mathbf{x}^3$	$\mathbf{x}^4$
$\mathbf{x}^2$	#			
$\mathbf{x}^3$	#	#		
$\mathbf{x}^4$	#	#	#	
$\mathbf{x}^5$	1	#	#	#

$\geq_{ADP}$

	$\mathbf{x}^1$	$\mathbf{x}^2$	$\mathbf{x}^3$	$\mathbf{x}^4$
$\mathbf{x}^2$	1			
$\mathbf{x}^3$	#	#		
$\mathbf{x}^4$	#	#	#	
$\mathbf{x}^5$	1	#	#	1

$\geq_{ASF}$

	$\mathbf{x}^1$	$\mathbf{x}^2$	$\mathbf{x}^3$	$\mathbf{x}^4$
$\mathbf{x}^2$	#			
$\mathbf{x}^3$	#	#		
$\mathbf{x}^4$	1	#	#	
$\mathbf{x}^5$	1	1	#	#

$\geq_{L}$

	$\mathbf{x}^1$	$\mathbf{x}^2$	$\mathbf{x}^3$	$\mathbf{x}^4$
$\mathbf{x}^2$	1			
$\mathbf{x}^3$	1	#		
$\mathbf{x}^4$	1	#	#	
$\mathbf{x}^5$	1	1	1	1

The symbol "1" at the intersection of row i and column j means that "$\mathbf{x}^i >_J \mathbf{x}^j$", while the occurence of the symbol "#" indicates that the distributions $\mathbf{x}^i$ and $\mathbf{x}^j$ are not comparable.

Table 3.1 makes clear that, depending on the way we measure it, the change in inequality caused by a progressive transfer may be ambiguous. In particular anyone who subscribes to the views captured by the differentials, the deprivation or the satisfaction quasi-ordering may feel unable to accept the common view that inequality decreases as a result of a progressive transfer. This is in accordance with the findings of the experimental studies we referred to above which aimed at confronting the axioms laid down by the theorist with the society's values. A recurrent conclusion of all these studies is that the public rejects to a large extent the *principle of transfers*.

3.2. Properties and Relationships Between the Inequality Quasi-Orderings

Table 3.1 suggests that our inequality quasi-orderings might be nested as the rankings obtained are more or less finer depending on the chosen quasi-ordering. Leaving aside the case

$n = 2$, where the preceeding quasi-orderings provide the same ranking of distributions, we have:

Remark 3.1. *Let $n > 2$ and suppose all the distributions under comparison have equal means. Then, we have: (i) $\geq_{AD} \subset \geq_{ADP}$; (ii) $\geq_{AD} \subset \geq_{ASF}$; (iii) $\geq_{ADP} \subset \geq_L$; (iv) $\geq_{ASF} \subset \geq_L$; and (v) $\geq_{ADP} \neq \geq_{ASF}$.*

Proof: Suppose that $\mathbf{x} \geq_{AD} \mathbf{y}$, so that

$$x_{k+1} - x_k \leq y_{k+1} - y_k, \ \forall\, k = 1, 2, \ldots, n-1. \tag{3.15}$$

Summing the inequalities above over h for $h = 1, 2, \ldots, k-1$ and $k = 1, 2, \ldots, n-1$, we obtain:

$$x_k - x_h \leq y_k - y_h, \ \forall\, h = 1, 2, \ldots, k-1, \ \forall\, k = 2, 3, \ldots, n. \tag{3.16}$$

(i) $\geq_{AD} \subseteq \geq_{ADP}$. Suppose that $\mathbf{x} \geq_{AD} \mathbf{y}$, so that (3.16) holds. Summing the inequalities in (3.16) over j for $j = h+1, h+2, \ldots, n-1$ and $h = 1, 2, \ldots, n-1$, and upon simplifying, we obtain (3.11) so that $\mathbf{x} \geq_{ADP} \mathbf{y}$.

(ii) $\geq_{AD} \subseteq \geq_{ASF}$. Suppose that $\mathbf{x} \geq_{AD} \mathbf{y}$, so that (3.16) holds. Summing the inequalities in (3.16) over i for $i = 1, 2, \ldots, k-1$ and $k = 2, 3, \ldots, n$, and upon simplifying, we obtain (3.14) so that $\mathbf{x} \geq_{ASF} \mathbf{y}$.

(iii) $\geq_{ADP} \subseteq \geq_L$. Developing (3.11), we obtain

$$(n-k+1)\, x_k - [x_k + x_{k+1} + \cdots + x_n] \geq (n-k+1)\, y_k - [y_k + y_{k+1} + \cdots + y_n], \tag{3.17}$$

for all $k = 1, 2, \ldots, n-1$. The proof then proceeds in $(n-1)$ successive steps.

Step 1: $h = 1$. Then (3.17) reduces to

$$n x_1 - [x_1 + x_2 + \cdots + x_n] \geq n y_1 - [y_1 + y_2 + \cdots + y_n]. \tag{3.18}$$

Since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we deduce from (3.18) that $x_1 \geq y_1$.

Step 2: $h = 2$. Then (3.17) reduces to

$$(n-1)\, x_2 - [x_2 + x_3 + \cdots + x_n] \geq (n-1)\, y_2 - [y_2 + y_3 + \cdots + y_n]. \tag{3.19}$$

Adding $x_1 - x_1$ and $y_1 - y_1$ to the *lhs* and the *rhs* respectively of (3.19) and since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we obtain

$$x_1 + (n-1)\, x_2 \geq y_1 + (n-1)\, y_2. \tag{3.20}$$

Adding $(n-2)x_1 \geq (n-2)y_1$, which follows from Step 1, to inequality (3.20), we get finally

$$(n-1)\,[x_1 + x_2] \geq (n-1)\,[y_1 + y_2], \tag{3.21}$$

hence $x_1 + x_2 \geq y_1 + y_2$.

$$\vdots$$

STEP h: $h = k$. Then (3.17) reduces to

$$(n-k+1)\, x_k - [x_k + x_{k+1} + \cdots + x_n] \geq (n-k+1)\, y_k - [y_k + y_{k+1} + \cdots + y_n]\,. \tag{3.22}$$

Adding $[x_1 + \cdots + x_{h-1}] - [x_1 + \cdots + x_{h-1}]$ and $[y_1 + \cdots + y_{h-1}] - [y_1 + \cdots + y_{h-1}]$ to the *lhs* and the *rhs* respectively of (3.22) and since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we obtain

$$x_1 + \cdots + x_{k-1} + (n-k+1)\, x_k \geq y_1 + \cdots + y_{k-1} + (n-k+1)\, y_k. \tag{3.23}$$

Adding $(n-k)\,[x_1 + \cdots + x_{k-1}] \geq (n-k)\,[y_1 + \cdots + y_{k-1}]$, which follows from Step $k-1$, to inequality (3.23), we get finally

$$(n-k)\,[x_1 + x_2 + \cdots + x_{n-k+1}] \geq (n-k)\,[y_1 + y_2 + \cdots + y_{n-k+1}]\,, \tag{3.24}$$

hence $x_1 + x_2 + \cdots + x_{n-k+1} \geq y_1 + y_2 + \cdots + y_{n-k+1}$.

$$\vdots$$

STEP $n-1$: $h = n-1$. Then (3.17) reduces to

$$2x_{n-1} - [x_{n-1} + x_n] \geq 2y_{n-1} - [y_{n-1} + y_n]\,. \tag{3.25}$$

Adding $[x_1 + \cdots + x_{n-2}] - [x_1 + \cdots + x_{n-2}]$ and $[y_1 + \cdots + y_{n-2}] - [y_1 + \cdots + y_{n-2}]$ to the *lhs* and the *rhs* respectively of (3.25) and since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we obtain

$$x_1 + \cdots + x_{n-2} + 2x_{n-1} \geq y_1 + \cdots + y_{n-2} + 2y_{n-1}. \tag{3.26}$$

Adding $[x_1 + \cdots + x_{n-2}] \geq [y_1 + \cdots + y_{n-2}]$, which follows from Step $n-2$, to inequality (3.26), we get finally

$$2\,[x_1 + x_2 + \cdots + x_{n-1}] \geq 2\,[y_1 + y_2 + \cdots + y_{n-1}]\,, \tag{3.27}$$

hence $x_1 + x_2 + \cdots + x_{n-1} \geq y_1 + y_2 + \cdots + y_{n-1}$.

We have shown that $x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k$, for all $k = 1, 2, \ldots, n-1$. Since by assumption $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$, we conclude that $\mathbf{x} \geq_L \mathbf{y}$.

(iv) $\geq_{ASF} \subseteq \geq_L$. Developing (3.14), we obtain

$$k\, x_k - [x_1 + x_2 + \cdots + x_k] \leq k\, y_k - [y_1 + y_2 + \cdots + y_k]\,, \tag{3.28}$$

for all $k = 2, 3, \ldots, n$. The proof then proceeds in $(n-1)$ successive steps.

Step 1: $h = n$. Then (3.28) reduces to

$$n\,x_n - [x_1 + x_2 + \cdots + x_n] \leq n\,y_n - [y_1 + y_2 + \cdots + y_n]\,. \tag{3.29}$$

Since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we deduce from (3.29) that $x_n \leq y_n$.

Step 2: $h = n - 1$. Then (3.28) reduces to

$$(n-1)\,x_{n-1} - [x_1 + x_2 + \cdots + x_{n-1}] \leq (n-1)\,y_{n-1} - [y_1 + y_2 + \cdots + y_{n-1}]\,. \tag{3.30}$$

Adding $x_n - x_n$ and $y_n - y_n$ to the *lhs* and the *rhs* respectively of (3.30) and since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we obtain

$$x_n + (n-1)\,x_{n-1} \leq y_n + (n-1)\,y_{n-1}. \tag{3.31}$$

Adding $(n-2)\,x_n \leq (n-2)\,y_n$, which follows from Step 1, to inequality (3.31) we get finally

$$(n-1)\,[x_{n-1} + x_n] \leq (n-1)\,[y_{n-1} + y_n]\,, \tag{3.32}$$

hence $x_{n-1} + x_n \leq y_{n-1} + y_n$.

$\vdots$

Step $n - k + 1$: $h = k$. Then (3.28) reduces to

$$k\,x_k - [x_1 + x_2 + \cdots + x_k] \leq k\,y_k - [y_1 + y_2 + \cdots + y_k]\,. \tag{3.33}$$

Adding $[x_{k+1} + \cdots + x_n] - [x_{k+1} + \cdots + x_n]$ and $[y_{k+1} + \cdots + y_n] - [y_{k+1} + \cdots + y_n]$ to the *lhs* and the *rhs* respectively of (3.33) and since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we obtain

$$k\,x_k + [x_{k+1} + \cdots + x_n] \leq k\,y_k + [y_{k+1} + \cdots + y_n]\,. \tag{3.34}$$

Adding $(k-1)\,[x_{k+1} + \cdots + x_n] \leq (k-1)\,[y_{k+1} + \cdots + y_n]$, which follows from Step $n-k+2$, to inequality (3.34), we get finally

$$k\,[x_k + x_{k+1} + \cdots + x_n] \leq k\,[y_k + y_{k+1} + \cdots + y_n]\,, \tag{3.35}$$

hence $x_k + x_{k+1} + \cdots + x_n \leq y_k + y_{k+1} + \cdots + y_n$.

$\vdots$

Step $n - 2$: $k = 2$. Then (3.28) reduces to

$$2x_2 - [x_1 + x_2] \leq 2y_2 - [y_1 + y_2]\,. \tag{3.36}$$

Adding $[x_3 + \cdots + x_n] - [x_3 + \cdots + x_n]$ and $[y_3 + \cdots + y_n] - [y_3 + \cdots + y_n]$ to the *lhs* and the *rhs* respectively of (3.36) and since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we obtain

$$2x_2 + [x_3 + \cdots + x_n] \leq 2y_2 + [y_3 + \cdots + y_n]\,. \tag{3.37}$$

Adding $[x_3 + \cdots + x_n] \leq [y_3 + \cdots + y_n]$, which follows from Step $n-2$, to inequality (3.37) and since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we obtain finally

$$2\,[x_2 + x_3 + \cdots + x_n] \leq 2\,[y_2 + y_3 + \cdots + y_n]\,, \tag{3.38}$$

hence $x_2 + x_3 + \cdots + x_n \leq y_2 + y_3 + \cdots + y_n$.

We have shown that $x_k + x_{k+1} + \cdots + x_n \leq y_k + y_{k+1} + \cdots + y_n$, for all $k = 2, 3, \ldots, n$. Since by assumption $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$, we conclude that $\mathbf{x} \geq_L \mathbf{y}$.

(v) $\geq_{ADP} \neq \geq_{ASF}$. Consider the following table where we have made use of the distributions defined in Example 3.1. By convention the symbol "1" at intersection of line "$\{\mathbf{x}^i, \mathbf{x}^j\}$" and row "$\geq_J$" indicates that "$\mathbf{x}^i >_J \mathbf{x}^j$", while a "#" means that $\mathbf{x}^i$ and $\mathbf{x}^j$ are not comparable, where $J \in \{AD, ADP, ASF, L\}$. Inspection of Table 3.2 reveals that $\geq_{ADP}$ and $\geq_{ASF}$ are logically independent.

TABLE 3.2.

Pairs of Distributions	$\geq_{AD}$	$\geq_{ADP}$	$\geq_{ASF}$	$\geq_L$
$\{\mathbf{x}^5, \mathbf{x}^1\}$		1	1	
$\{\mathbf{x}^2, \mathbf{x}^1\}$	#	1	#	
$\{\mathbf{x}^5, \mathbf{x}^2\}$	#	#	1	
$\{\mathbf{x}^3, \mathbf{x}^1\}$		#	#	1
$\{\mathbf{x}^4, \mathbf{x}^1\}$			#	1

Finally to make the proof complete, it remains to prove that the inclusions in statements (i) to (iv) are strict, which follows from Table 3.2. □

The preceding discussion demonstrates that the three quasi-orderings we have considered so far are at variance with the Lorenz criterion and thus capture dimensions of inequality that are not embedded in the latter criterion. A second issue is to determine the structure of the individuals' preferences towards more equality which support the views expressed by the differentials, satisfaction and deprivation quasi-orderings and inconsistent with those captured by the Lorenz quasi-ordering. Both issues are of particular importance for understanding the normative content of these three quasi-orderings and they will be the subject of the two following sections.

4. INEQUALITY, SOLIDARITY AND EQUALIZING TRANSFORMATIONS

4.1. A Preliminary Result

Although it is typically assumed that inequality is reduced and welfare increased by a progressive transfer, Example 3.1 points at good reasons for challenging this common view. On the one hand, depending on the way we measure inequality, the effect of a transfer of income from a richer to a poorer individual may be ambiguous. In particular anyone who subscribes to the views captured by the differentials quasi-ordering may feel unable to accept the common view that inequality unambiguously decreases as a result of a progressive transfer. On the other hand, one might even consider that inequality has increased as the result of an elementary progressive transfer. It must be stressed that $\mathbf{x} \geq_L \mathbf{y}$ and $\mathbf{y} >_{AD} \mathbf{x}$ cannot hold simultaneously. This follows from Remark 3.1 according to which, for all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}(D)$, one has that $\mathbf{x} >_{AD} \mathbf{y}$ implies $\mathbf{x} >_L \mathbf{y}$. However this does not preclude the possibility that $I(\mathbf{x}) > I(\mathbf{y})$ for some inequality index $I \in \mathbb{I}$ consistent with the absolute differentials quasi-ordering. This is because for $\mathbf{x} >_{AD} \mathbf{y}$ it is necessary that all inequality indices in a given class – to be determined – declare distribution $\mathbf{x}$ as being less unequal than distribution $\mathbf{y}$ [see Section 5]. Since these three quasi-orderings are all subrelations of the Lorenz quasi-ordering, it is clear that, if a distribution is ranked above another one according to either of the former quasi-orderings, then a sequence of progressive transfers will be needed in order to construct the dominating distribution starting from the dominated one. However the precise way these progressive transfers have to be combined for such domination to hold has to be determined.

It might be helpful to begin with a benchmark result that constitutes a first step towards a more general solution. By construction distributions $\mathbf{x}^2$, $\mathbf{x}^3$, $\mathbf{x}^4$ and $\mathbf{x}^5$ in Example 3.1 obtain from $\mathbf{x}^1$ by means of a *single* rank-preserving progressive transfer of one income unit. Close inspection reveals that the only case where the resulting distribution dominates the original distribution according to the differentials quasi-ordering is when the progressive transfer involves the richest and the poorest individual. The three cases where the progressive transfers generate an improvement according to the deprivation quasi-ordering is when income is taken from the richest individual and given to someone poorer. Finally a transfer from any richer individual to the poorest one results in a reduction of inequality as measured by the satisfaction quasi-ordering. The positions on the income scale of the individuals taking part in the transfer seem to play a crucial role in the redistributive impact of the progressive transfer. The result below – which we state without proof – indicates the restrictions one has to introduce for inequality to decrease as a result of a single progressive transfer in a very particular case.

Remark 4.1. *Let $n > 2$ and $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that, such that $y_1 < y_2 \leq \ldots \leq y_{n-1} < y_n$. Suppose we are only permitted to use a single rank-preserving progressive transfer in order to obtain $\mathbf{x}$ from $\mathbf{y}$. Then, we have:*

(a) $\mathbf{x} \geq_{AD} \mathbf{y}$ *if and only if* $1 = i < j = n$.

(b) $\mathbf{x} \geq_{ADP} \mathbf{y}$ *if and only if* $i < j = n$.

(c) $\mathbf{x} \geq_{ASF} \mathbf{y}$ *if and only if* $1 = i < j$.

(d) $\mathbf{x} \geq_L \mathbf{y}$ *if and only if* $i < j$.

This result uncovers the rationale behind the construction of Example 3.1 and also hints at potential explanations why the public might reject the principle of transfers in some given situations. But above all Remark 4.1 confirms that there is little room for reducing inequality as measured by either of the differentials, deprivation and satisfaction quasi-orderings if one is only allowed to make use of elementary progressive transfers.

4.2. General Results

Given Remark 4.1 we know that, for inequality as measured by any of the differentials, deprivation and satisfaction quasi-orderings to decrease, a single progressive transfer will generally not be sufficient and progressive transfers will have to be combined in one way or another. Therefore our first task is to identify possible transformations of distributions which combine progressive transfers in such a way that domination in terms of the differentials, deprivation and satisfaction quasi-orderings obtains. A related task is to derive the appropriate sequence of such transformations that permits to obtain the dominating distribution from the dominated one. We impose to ourselves two requirements: (i) the transformations must admit as a particular case the progressive transfers exhibited in Remark 4.1, and (ii) the transformations must be elementary in the sense that they are as simple as possible. The latter requirement has the effect that in general a single transformation would not suffice to convert the dominated distribution into the dominating one: successive applications of such transformations will be needed.

Considering first those transformations, which successive applications of result in a distributional improvement according to the differentials quasi-ordering, we propose:

Definition 4.1. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we will say that $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a* T1*-transformation*, if there exists $\delta, \epsilon > 0$ and two individuals h, k $(1 \leq h < k \leq n)$ such that condition (3.2) holds and:

(4.1.a) $$x_g = y_g,\ \forall\ g \in \{h+1, \dots, k-1\};$$

(4.1.b) $$x_i = y_i + \delta,\ \forall\ i \in \{1, \dots, h\}; \quad x_j = y_j - \epsilon,\ \forall\ j \in \{k, \dots, n\};$$

(4.1.c) $$h\delta = (n-k+1)\epsilon.$$

Therefore T1-transformations comprise as a particular case the progressive transfers identified in statement (a) of Remark 4.1. Such elementary transformation impose a lot of solidarity in the society. There is solidarity among the rich: if some income is taken from a rich individual, then the same amount has to be taken from every richer individual. Symmetrically, there is solidarity among the poor: if some income is given to a poor individual, then the same amount has to be given to every poorer individual. This solidarity among the donors and the benificiaries is typically broken down in the progressive transfer. The following result identifies the relationship between the absolute differentials quasi-ordering and T1-transformations.

Proposition 4.1. *Let $n > 2$ and $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$. Then, the following two statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a finite sequence of* T1*-transformations.*

(b) $\mathbf{x} \geq_{AD} \mathbf{y}$.

The proofs of this result and of the next two propositions are particularly tedious and they are not given here[11].

Turning now attention to those transformations, which successive applications of would result in a distributional improvement according to the deprivation quasi-ordering, we propose:

Definition 4.2. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we will say that $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a* T2*-transformation*, if there exists $\delta, \epsilon > 0$ and two individuals h, k ($1 \leq h < k \leq n$) such that condition (3.2) holds and:

$$x_g = y_g, \ \forall\ g \in \{1, \dots, h-1\} \cup \{h+1, \dots, k-1\}; \tag{4.2.a}$$

$$x_h = y_h + \delta; \quad x_j = y_j - \epsilon, \ \forall\ j \in \{k, \dots, n\}; \tag{4.2.b}$$

$$\delta = (n - k + 1)\epsilon. \tag{4.2.c}$$

Again a progressive transfer constitutes a particular case of a T2-transformation, which in turn is a particular T1-transformation. Although solidarity is still present in a T2-transformation, it is now limited to the donors. If some income is taken from a rich individual, then the same amount is to be taken from every richer individual. However it is no longer necessary that individuals poorer than the transfer recipient do benefit also from some [equal] additional income. The result below establishes the connection between dominance in terms of the absolute deprivation quasi-ordering and T2-transformations.

Proposition 4.2. *Let $n > 2$ and $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$. Then, the following two statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a finite sequence of* T2*-transformations.*

(b) $\mathbf{x} \geq_{ADP} \mathbf{y}$.

Finally we consider the transformations, which successive applications of would result in a distributional improvement according to the satisfaction quasi-ordering. We propose:

Definition 4.3. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we will say that $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a* T3*-transformation*, if there exists $\delta, \epsilon > 0$ and two individuals h, k

[11] We invite the interested reader to consult the longer version of the paper which contains the details of the proofs [see Chateauneuf and Moyes (2004)].

$(1 \leq h < k \leq n)$ such that condition (3.2) holds and:

$$x_g = y_g,\ \forall\ g \in \{h+1, \ldots, k-1\} \cup \{k+1, \ldots, n\}; \tag{4.3.a}$$

$$x_i = y_i + \delta,\ \forall\ i \in \{1, \ldots, h\}; \quad x_k = y_k - \epsilon;\ \text{and} \tag{4.3.b}$$

$$h\delta = \epsilon. \tag{4.3.c}$$

The progressive transfers identified in statement (c) of Remark 4.1 are particular instances of a T3-transformation. A T3-transformation is to some extent the dual of a T2-transformation: solidarity concerns the beneficiaries of the transfer. If some additional income is given to a poor individual, then the same amount has to be given to every poorer individual. But there is no need that individuals richer than the donor give away some [equal] amount of income. Dominance in terms of the absolute satisfaction quasi-ordering and T3-transformations are related as it shown below.

Proposition 4.3. *Let $n > 2$ and $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$. Then, the following two statements are equivalent:*

(a) $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a finite and non-empty sequence of T3*-transformations.*

(b) $\mathbf{x} \geq_{ASF} \mathbf{y}$.

Propositions 4.1 to 4.3 show that it is possible to associate to the differentials, deprivation and satisfaction quasi-orderings relatively simple transformations, which by successive applications allow to derive the dominating distribution starting from the dominated one. Although these transformations comprise as particular cases the usual progressive transfers, they are in general more complicated. Invoking Remark 3.1 it is clear that any such transformation can be decomposed into a finite sequence of progressive transfers.

5. Welfare Comparisons for Equal Mean Distributions

5.1. Two Classes of Social Welfare Functions

Basically, two general families of social welfare functions have been studied in the inequality literature up to now. The first approach – the expected utility model or the utilitarian social welfare function – assumes linearity in the weights so that social welfare in situation $\mathbf{x} \in \mathcal{Y}_n(D)$ is given by

$$W_U(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} U(x_i), \tag{5.1}$$

where the *utility function* U is increasing and defined up to an increasing and affine transformation. It is convenient to define

$$P_i := \frac{i}{n}, \tag{5.2.a}$$

$$Q_i := 1 - P_{i-1} = \frac{n-i+1}{n}, \tag{5.2.b}$$

for all $i = 1, 2, \ldots, n$, with $P_0 = 0$ and $Q_{n+1} := 0$. Letting $x_0 := 0$, the second approach – the Yaari model – assumes linearity in incomes so that

$$W_f(\mathbf{x}) = \sum_{i=1}^{n} \left[f\left(Q_i\right) - f\left(Q_{i+1}\right)\right] x_i \equiv \sum_{i=1}^{n} f\left(Q_i\right) \left[x_i - x_{i-1}\right], \tag{5.3}$$

where $f \in \mathcal{F} := \{f : [0,1] \longrightarrow [0,1] \mid f \text{ continuous, non-decreasing}, f(0) = 0 \text{ and } f(1) = 1\}$ is the *weighting function*. Inequality aversion is fully captured by the utility function U in the utilitarian framework and by the weighting function f in the Yaari model. The two former models are actually particular cases of the rank-dependent expected utility model [see e.g. Quiggin (1993)].

5.2. An Almost Impossibility Result

The egalitarian nature of the utilitarian rule has long been recognized when all individuals have the same concave utility function. Precisely a progressive transfer improves social welfare as measured by the utilitarian rule if and only if the common utility function is concave. In combination with Proposition 3.1 this implies that Lorenz domination holds if and only if, whatever the concave utility function one chooses, the sum of utilities generated by the dominating distribution is always greater than the sum of utilities generated by the dominated distribution. However the equalizing implications of the utilitarian social welfare function are based on a particular notion of inequality reduction and one may wonder what happens when alternative inequality views – such as the ones considered in this paper – are substituted for the ones captured by the Lorenz criterion. Because the quasi-orderings we are interested in are weaker than the Lorenz quasi-ordering – they imply but are not implied by it – one expects that the class of utility functions that guarantee that the utilitarian rule is equalizing is larger than the class of the concave utility functions. Actually the utilitarian model is not flexible enough to distinguish between the views captured by the differentials, deprivation and satisfaction quasi-orderings. Precisely we have the following result:

Proposition 5.1. *Let $n > 2$ and $J \in \{AD, ADP, ASF, L\}$. Then, the following two statements are equivalent:*

(a) For all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$: $\mathbf{x} \geq_J \mathbf{y} \Longrightarrow W_U(\mathbf{x}) \geq W_U(\mathbf{y})$.

(b) U is concave.

Proof: The technique of proof builds on standard arguments used for instance in Ebert and Moyes (2002, Prop. 4.8). Consider the four following statements:

(a-1) For all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$: $\mathbf{x} \geq_{AD} \mathbf{y} \Longrightarrow W_U(\mathbf{x}) \geq W_U(\mathbf{y})$.

(a-2) For all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$: $\mathbf{x} \geq_{ADP} \mathbf{y} \Longrightarrow W_U(\mathbf{x}) \geq W_U(\mathbf{y})$.

(a-3) For all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$: $\mathbf{x} \geq_{ASF} \mathbf{y} \Longrightarrow W_U(\mathbf{x}) \geq W_U(\mathbf{y})$.

(a-4) For all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$: $\mathbf{x} \geq_L \mathbf{y} \Longrightarrow W_U(\mathbf{x}) \geq W_U(\mathbf{y})$.

The proof consists in establishing the four chains of implications: (b) $\Longrightarrow$ (a-4), (a-4) $\Longrightarrow$ (a-3) $\Longrightarrow$ (a-1), (a-4) $\Longrightarrow$ (a-2) $\Longrightarrow$ (a-1), and (a-1) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (a-4). This is a well-known result in the theory of inequality measurement [see e.g. Marshall and Olkin (1979, B.1)].

(a-4) $\Longrightarrow$ (a-3). We argue a contrario and show that $\neg$(a-3) $\Longrightarrow$ $\neg$(a-4). Suppose that:

$$\exists \mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D) \text{ with } \mu(\mathbf{x}) = \mu(\mathbf{y}) \mid \mathbf{x} \geq_{ASF} \mathbf{y} \wedge \neg \left[W_U(\mathbf{x}) \geq W_U(\mathbf{y})\right] . \tag{5.9}$$

Since $\geq_{ASF} \subset \geq_L$, this implies that

$$\exists \mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D) \text{ with } \mu(\mathbf{x}) = \mu(\mathbf{y}) \mid \mathbf{x} \geq_{L} \mathbf{y} \wedge \neg \left[W_U(\mathbf{x}) \geq W_U(\mathbf{y})\right] . \tag{5.10}$$

(a-4) $\Longrightarrow$ (a-2). Similar to the proof that (a-4) $\Longrightarrow$ (a-3).

(a-3) $\Longrightarrow$ (a-1). Similar to the proof that (a-4) $\Longrightarrow$ (a-3).

(a-2) $\Longrightarrow$ (a-1). Similar to the proof that (a-4) $\Longrightarrow$ (a-3).

(a-1) $\Longrightarrow$ (b). We argue a contrario and show that $\neg$(b) $\Longrightarrow$ $\neg$(a-1). Suppose that there exists $u, v \in D$ $(u < v)$ such that

$$2U\left(\frac{u+v}{2}\right) < U(u) + U(v). \tag{5.11}$$

Consider next distributions $\mathbf{x} = ((u+v)/2, (u+v)/2, \ldots, (u+v)/2, (u+v)/2)$ and $\mathbf{y} = (u, (u+v)/2, \ldots, (u+v)/2, v)$. By construction $\mu(\mathbf{x}) = \mu(\mathbf{y})$ and

$$x_1 - y_1 = \frac{v-u}{2} > x_2 - y_2 = 0 = \cdots = 0 = x_{n-1} - y_{n-1} > -\frac{v-u}{2} = x_n - y_n, \tag{5.12}$$

so that $\mathbf{x} >_{AD} \mathbf{y}$. Using (5.11), we obtain $W_U(\mathbf{x}) - W_U(\mathbf{y}) < 0$ and we conclude that condition (a-1) is violated, which makes the proof complete. $\square$

The consistency of the utilitarian social welfare function with the different inequality views captured by our three quasi-orderings leads to the same restriction as the one implied by Lorenz-consistency: the utility function has to be concave. Thus the utilitarian model does not permit to distinguish between the views embedded in the Lorenz quasi-ordering and its three competitors: the differentials, the deprivation and the satisfaction quasi-orderings. On the contrary, as we will demonstrate in a while, an ethical observer endowed with the Yaari social welfare function will be able to make a difference between these alternative views.

5.3. Consistency of the Yaari Model with Different Inequality Views

Before we state our main results, we need first introduce some definitions concerning the weighting function. Given a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ and an interval $V \subseteq \mathbb{R}$, we will say that g is *convex over* V if

$$\forall\ u, v \in V, \forall\ \lambda \in [0,1]\ :\ g((1-\lambda)u + \lambda v) \leq (1-\lambda)g(u) + \lambda g(v). \tag{5.13}$$

Given $V := (\underline{v}, \overline{v}) \subseteq \mathbb{R}$ and $\xi \in V$, we will say that g is *star-shaped from above at* ξ if

$$\forall\, u, v \in (\underline{v}, \xi) \cup (\xi, \overline{v}) \,:\, u < v \Longrightarrow \frac{g(u) - g(\xi)}{u - \xi} \leq \frac{g(v) - g(\xi)}{v - \xi} \tag{5.14}$$

[see e.g. Landsberger and Meilijson (1990)]. Each of the four following classes of weighting functions will play a crucial role in subsequent developments:

(5.15.a) $\mathcal{F}_{AD} := \{f \in \mathcal{F} \mid f(Q) \leq Q, \forall\, Q \in (0, 1)\};$

(5.15.b) $\mathcal{F}_{ADP} := \{f \in \mathcal{F} \mid f \text{ is star-shaped from above at } 0\};$

(5.15.c) $\mathcal{F}_{ASF} := \{f \in \mathcal{F} \mid f \text{ is star-shaped from above at } 1\};$

(5.15.d) $\mathcal{F}_{L} := \{f \in \mathcal{F} \mid f \text{ is convex over } [0, 1]\}.$

It is a straightforward exercise to check that the classes of weighting functions defined above are nested in the way indicated below.

Remark 5.1. *(i)* $\mathcal{F}_L \subset \mathcal{F}_{ADP} \subset \mathcal{F}_{AD}$*; (ii)* $\mathcal{F}_L \subset \mathcal{F}_{ASF} \subset \mathcal{F}_{AD}$.

Typical members of the classes $\mathcal{F}_{AD}$, $\mathcal{F}_{ADP}$, $\mathcal{F}_{SFP}$ and $\mathcal{F}_L$ are represented in Figures 5.1, 5.2, 5.3 and 5.4, respectively. One can easily check that the weighting functions represented in Figure 5.4 are convex and thus star-shaped from above at 0 and 1, and that they also verify $f(Q) \leq Q$, for all $Q \in (0, 1)$. Figure 5.3 depicts weighting functions that are star-shaped from above at 1, verify $f(Q) \leq Q$, for all $Q \in (0, 1)$, but are neither star-shaped from above at 0 nor convex. On the other hand the weighting functions represented in Figure 5.2 depicts are star-shaped from above at 0 and thus verify $f(Q) \leq Q$, for all $Q \in (0, 1)$, but are neither star-shaped from above at 1 nor convex. Finally the weighting functions depicted in Figure 5.1 are not star-shaped but they lie everywhere below the main diagonal.

We are interested in identifying the restrictions to be placed on the weighting function $f \in \mathcal{F}$ for the Yaari social welfare function to be consistent with our absolute quasi-orderings. Actually we are able to provide more general results that establish the links between the different equalizing transformations we introduced in Section 4, subclasses of the Yaari social welfare functions and the inequality quasi-orderings.

Considering first the inequality view captured by the absolute differentials quasi-ordering, we obtain:

Proposition 5.2. *Let* $n > 2$ *and consider two distributions* $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ *such that* $\mu(\mathbf{x}) = \mu(\mathbf{y})$*. Then, the following three statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a finite sequence of* T1*-transformations.*

(b) $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$*, for all* $f \in \mathcal{F}_{AD}$.

(c) $\mathbf{x} \geq_{AD} \mathbf{y}$.

Proof: Since we know from Proposition 4.1 that statements (a) and (c) are equivalent, we only have to show that statements (b) and (c) are equivalent. Letting $\Delta W_f := W_f(\mathbf{x}) - W_f(\mathbf{y})$

and using the fact that by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we derive from (5.3) that

$$\Delta W_f = [f(Q_1) - Q_1](x_1 - y_1) + \sum_{h=2}^{n} [f(Q_h) - Q_h][(x_h - x_{h-1}) - (y_h - y_{h-1})]. \tag{5.16}$$

(c) $\Longrightarrow$ (b): Let that $f \in \mathcal{F}_{AD}$, which implies that $f(Q_i) - Q_i \leq 0$, for all $i = 2, 3, \ldots, n$. Upon substituting in (5.16) and using the fact that by definition $f(Q_1) - Q_1 = 0$, we deduce that $\mathbf{x} \geq_{AD} \mathbf{y}$ is sufficient for $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$.

(b) $\Longrightarrow$ (c): Choose $\phi^h(Q) := \alpha_1^h + \beta_1^h Q$, for all $Q \in [0, 1]$, where

$$\alpha_1^h := -\frac{Q_h - Q_{h+1}}{Q_{h-1} - Q_h} Q_{h-1} < 0 \quad \text{and} \quad \beta_1^h := \frac{Q_{h-1} - Q_{h+1}}{Q_{h-1} - Q_h} > 1, \tag{5.17}$$

for $h = 2, 3, \ldots, n$. Consider then the piecewise linear function $f^h : [0,1] \longrightarrow [0,1]$ defined by

$$f^h(Q) := \begin{cases} Q, & \text{for } 0 \leq Q < Q_{h+2}, \\ Q_{h-1}, & \text{for } Q_{h+2} \leq Q < Q_{h+1}, \\ \phi^h(Q), & \text{for } Q_{h+1} \leq Q < Q_h, \\ Q, & \text{for } Q_{h+1} \leq Q \leq Q_1, \end{cases} \tag{5.18}$$

for $h = 2, 3, \ldots, n$. Clearly $f^h \in \mathcal{F}_{AD}$, for $h = 1, 2, \ldots, n$ [see Figure 5.1]. Assuming that condition (b) holds and upon substitution into (5.16), we obtain

$$W_f(\mathbf{x}) - W_f(\mathbf{y}) = (Q_{h+1} - Q_h)[(x_h - x_{h-1}) - (y_h - y_{h-1})] \geq 0, \tag{5.19}$$

for all $h = 2, 3, \ldots, n$. Since by definition $Q_{h+1} - Q_h < 0$, for all $h = 2, 3, \ldots, n$, we conclude that $x_h - x_{h-1} \leq y_h - y_{h-1}$, for all $h = 2, 3, \ldots, n$, hence $\mathbf{x} \geq_{AD} \mathbf{y}$. □

The conditions that ensure inequality reduction in terms of the absolute income differentials are rather weak: the weighting function must lie below the main diagonal in the $[0,1] \times [0,1]$ space. Therefore the class of Yaari social welfare functions that are consistent with the views expressed by the differentials quasi-ordering is quite large.

We turn now to the absolute deprivation quasi-ordering and search for the restrictions to be placed on the weighting function f that guarantee that the Yaari social welfare function is consistent with this criterion.

Proposition 5.3. *Let $n > 2$ and consider two distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$. Then, the following three statements are equivalent:*

(a) $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a finite sequence of T2*-transformations.*

(b) $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$, for all $f \in \mathcal{F}_{ADP}$.

(c) $\mathbf{x} \geq_{ADP} \mathbf{y}$.

Proof: Since we know from Proposition 4.2 that statements (a) and (c) are equivalent, we only have to show that statements (b) and (c) are equivalent. Letting $\Delta W_f := W_f(\mathbf{x}) - W_f(\mathbf{y})$ and upon manipulating (5.3), we obtain

$$\Delta W_f = \frac{f(Q_1)}{Q_1}\left[\mu(\mathbf{x}) - \mu(\mathbf{y})\right] + \sum_{h=1}^{n-1}\left[\frac{f(Q_{h+1})}{Q_{h+1}} - \frac{f(Q_h)}{Q_h}\right]\left[\sum_{j=h+1}^{n}\left(\frac{x_j - x_h}{n}\right) - \sum_{j=h+1}^{n}\left(\frac{y_j - y_h}{n}\right)\right], \tag{5.20}$$

where by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$.

(c) $\Longrightarrow$ (b): Let $f \in \mathcal{F}_{ADP}$ which implies that $f(Q_{h+1})/Q_{h+1} \leq f(Q_h)/Q_h$, for all $h = 1, 2, \ldots, n-1$. Using the fact that by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, it follows from (5.20) that $\mathbf{x} \geq_{ADP} \mathbf{y}$ is a sufficient condition for $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$.

(b) $\Longrightarrow$ (c): Choose $\psi^h(Q) := \alpha_2^h + \beta_2^h Q$, for all $Q \in [0,1]$, where

$$\alpha_2^h := -\frac{Q_h Q_{h+1}}{Q_h - Q_{h+1}} < 0 \quad \text{and} \quad \beta_2^h := \frac{Q_h}{Q_h - Q_{h+1}} > 1, \tag{5.21}$$

for $h = 1, 2, \ldots, n-1$. Consider then the piecewise linear function $f^h : [0,1] \longrightarrow [0,1]$ defined by:

$$f^h(Q) := \begin{cases} 0, & \text{for } 0 \leq Q < Q_{h+1}, \\ \psi^h(Q), & \text{for } Q_{h+1} < Q < Q_h, \\ Q, & \text{for } Q_h \leq Q \leq 1, \end{cases} \tag{5.22}$$

for $h = 1, 2, \ldots, n-1$. Clearly $f^h \in \mathcal{F}_2$, for $h = 1, 2, \ldots, n-1$ [see Figure 5.2]. Assuming that condition (b) holds and upon substitution into (5.20), we obtain

$$W_f(\mathbf{x}) - W_f(\mathbf{y}) = -\left[\sum_{j=h+1}^{n}\left(\frac{x_j - x_h}{n}\right) - \sum_{j=h+1}^{n}\left(\frac{y_j - y_h}{n}\right)\right] \geq 0, \tag{5.23}$$

for $h = 1, 2, \ldots, n-1$, from which we conclude that $\mathbf{x} \geq_{ADP} \mathbf{y}$. □

The conditions that ensure inequality reduction in terms of the absolute deprivation quasi-ordering are weaker than convexity: the weighting function must be star-shaped at the origin on the interval $(0,1)$.

The next result identifies the restrictions to be placed on the weighting function f that guarantee that the Yaari social welfare function is consistent with the absolute satisfaction quasi-ordering.

Proposition 5.4. *Let $n > 2$ and consider two distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$. Then, the following three statements are equivalent:*

(a) $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a finite sequence of T3*-transformations.*

(b) $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$, for all $f \in \mathcal{F}_{ASF}$.

(c) $\mathbf{x} \geq_{ASF} \mathbf{y}$.

Proof: Since we know from Proposition 4.3 that statements (a) and (c) are equivalent, we only have to show that statements (b) and (c) are equivalent. Letting $\Delta W_f := W_f(\mathbf{x}) - W_f(\mathbf{y})$ and upon manipulation of (5.3), we obtain

$$\text{(5.24)}\quad \Delta W_f = \sum_{h=2}^{n} \left(\frac{1 - f(Q_{h+1})}{1 - Q_{h+1}} - \frac{1 - f(Q_h)}{1 - Q_h} \right) \left[\sum_{j=1}^{h-1} \left(\frac{x_h - x_j}{n} \right) - \sum_{j=1}^{h-1} \left(\frac{y_h - y_j}{n} \right) \right],$$

$$- \frac{1 - f(Q_{n+1})}{1 - Q_{n+1}} \left[\mu(\mathbf{x}) - \mu(\mathbf{y}) \right],$$

where by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$.

(c) $\Longrightarrow$ (b): Let $f \in \mathcal{F}_{ASF}$ so that $(1 - f(Q_{h+1}))/(1 - Q_{h+1}) \leq (1 - f(Q_h))/(1 - Q_h)$, for all $h = 2, 3, \ldots, n$. Then, since by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$, we conclude that $\mathbf{x} \geq_{ASF} \mathbf{y}$ guarantees that condition (5.24) holds, hence $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$.

(b) $\Longrightarrow$ (c): Choose $\varphi^h(Q) := \alpha_3^h + \beta_3^h Q$, for all $Q \in [0, 1]$, where

$$\text{(5.25)}\qquad \alpha_3^h := \frac{Q_{h+1} - Q_h}{1 - Q_h} < 0 \quad \text{and} \quad \beta_3^h := \frac{1 - Q_{h+1}}{1 - Q_h} > 1,$$

for $h = 2, 3, \ldots, n$. Consider then the piecewise linear function $f^h : [0, 1] \longrightarrow [0, 1]$ defined by:

$$\text{(5.26)}\qquad f^h(Q) := \begin{cases} Q, & \text{for } Q \leq Q < Q_{h+1}, \\ Q_{h+1}, & \text{for } Q_{h+1} \leq Q < Q_h, \\ \varphi^h(Q), & \text{for } Q_h \leq Q \leq 1, \end{cases}$$

for $h = 2, 3, \ldots, n$. Clearly $f^h \in \mathcal{F}_3$, for $h = 2, 3, \ldots, n$ [see Figure 5.3]. Assuming that condition (b) holds and upon substitution into (5.26), we obtain

$$\text{(5.27)}\qquad W_f(\mathbf{x}) - W_f(\mathbf{y}) = \left(1 - \frac{1 - Q_{h+1}}{1 - Q_h} \right) \left[\sum_{j=1}^{h-1} \left(\frac{x_h - x_j}{n} \right) - \sum_{j=1}^{h-1} \left(\frac{y_h - y_j}{n} \right) \right] \geq 0,$$

for $h = 2, 3, \ldots, n$. Since by definition $Q_{h+1} < Q_h$, for all $h = 1, 2, \ldots, n$, we deduce from (5.27) that $\mathbf{x} \geq_{ASF} \mathbf{y}$. □

The conditions that ensure inequality reduction in terms of the absolute satisfaction quasi-ordering are weaker than convexity: the weighting function must be star-shaped at 1 over $(0, 1)$.

Finally for the sake of completeness we recall the conditions to be met by the weighting function for the Yaari social welfare function to be consistent with the Lorenz quasi-ordering.

Proposition 5.5. *Let $n > 2$ and consider two distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ such that $\mu(\mathbf{x}) = \mu(\mathbf{y})$. Then, the following three statements are equivalent:*

(a) $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a finite sequence of single progressive transfers.

(b) $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$, for all $f \in \mathcal{F}_L$.

(c) $\mathbf{x} \geq_L \mathbf{y}$.

Proof: Since it is well-known that statements (a) and (c) are equivalent [see Proposition 3.1], it suffices to check that statements (b) and (c) are equivalent. Letting $\Delta W_f := W_f(\mathbf{x}) - W_f(\mathbf{y})$ and upon manipulation of (5.3), we obtain

$$\Delta W_f = \sum_{h=1}^{n} \left[\frac{f(Q_h)-f(Q_{h+1})}{Q_h - Q_{h+1}} - \frac{f(Q_{h+1})-f(Q_{h+2})}{Q_{h+1} - Q_{h+2}}\right] \left[\frac{\sum_{i=1}^{h} x_i}{n} - \frac{\sum_{i=1}^{h} y_i}{n}\right] + \left[\frac{f(Q_n) - f(Q_{n+1})}{Q_n - Q_{n+1}}\right] [\mu(\mathbf{x}) - \mu(\mathbf{y})], \tag{5.28}$$

where by assumption $\mu(\mathbf{x}) = \mu(\mathbf{y})$.

(c) $\Longrightarrow$ (b): Suppose that $f \in \mathcal{F}_L$ i.e., f is convex in Q over $[0,1]$, which implies that

$$\frac{f(Q_1) - f(Q_2)}{Q_1 - Q_2} \leq \frac{f(Q_3) - f(Q_4)}{Q_3 - Q_4}, \tag{5.29}$$

for all $Q_1, Q_2, Q_3, Q_4 \in [0,1]$ such that $Q_4 < Q_3 \leq Q_1$ and $Q_4 \leq Q_2 < Q_1$ [see e.g. Marshall and Olkin (1979, 16.B.3.a)]. Then $\mathbf{x} \geq_L \mathbf{y}$ guarantees that condition (5.29) holds, hence $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$.

(b) $\Longrightarrow$ (c): Let $\chi^h(Q) := \alpha_4^h + \beta_4^h Q$, for all $Q \in [0,1]$, where

$$\alpha_4^h := -\frac{Q_{h+1}}{1 - Q_{h+1}} \leq 0 \quad \text{and} \quad \beta_4^h := \frac{1}{1 - Q_{h+1}} \geq 1, \tag{5.30}$$

for $h = 2, 3, \ldots, n$. Consider then the piecewise linear function $f^h : [0,1] \longrightarrow [0,1]$ defined by:

$$f^h(Q) := \begin{cases} 0, & \text{for } 0 \leq Q < Q_{h+1}, \\ \chi^h(Q), & \text{for } Q_{h+1} \leq Q \leq 1, \end{cases} \tag{5.31}$$

for $h = 2, 3, \ldots, n$. Clearly $f^h \in \mathcal{F}_4$, for $h = 2, 3, \ldots, n$ [see Figure 5.4]. Assuming that condition (b) holds and upon substitution into (5.31), we obtain

$$W_f(\mathbf{x}) - W_f(\mathbf{y}) = \frac{1}{1 - Q_{h+1}} \left[\frac{\sum_{j=1}^{h} x_j}{n} - \frac{\sum_{j=1}^{h} y_j}{n}\right] \geq 0, \tag{5.32}$$

for $h = 1, 2, \ldots, n-1$, from which we conclude that $\mathbf{x} \geq_L \mathbf{y}$. □

The convexity of the weighting function is therefore necessary and sufficient for welfare as measured by the Yaari social welfare function to increase as the Lorenz curve moves upwards.

Contrary to the utilitarian model, which does not allow to make a distinction between the inequality views considered in this paper, the Yaari social welfare function permits to separate these different attitudes towards inequality. This is achieved by means of the weighting function which captures the planner's concern for inequality. Under the equal mean condition, Propositions 5.2 to 5.5 identify the restrictions to be placed on the weighting function that guarantee that welfare does not decrease when inequality as measured by our four quasi-orderings goes down. The propositions also identify the appropriate sequences of transformations that are needed in order to convert the dominated distribution into the dominating one for the differentials, deprivation and satisfaction quasi-orderings. These transformations are more complicated than – and are generally distinct from – the traditional progressive transfers.

6. Extensions When Distributions Have Different Means

So far we have restricted our attention to the comparison of distributions with equal means, in which case the notions of inequality and welfare are related in a one-to-one way. Here we turn to the more general case where the distributions under comparison do not necessarily have the same mean. Then the relationship between inequality and welfare is no longer unambiguous and we will focus here on inequality.

Following the ethical approach to inequality measurement, we restrict attention to those inequality indices that can be constructed from certain particular social welfare functions [see Blackorby, Bossert and Donaldson (1999)]. Given the social welfare function $W : \mathcal{Y}_n(D) \longrightarrow \mathbb{R}$, we let Ξ represent the *social evaluation function* implicitly defined by

$$W\big(\underbrace{x_1, \ldots, x_n}_{n}\big) = W\big(\underbrace{\Xi(\mathbf{x}), \ldots, \Xi(\mathbf{x})}_{n}\big),\ \forall\, \mathbf{x} \in \mathcal{Y}_n(D), \tag{6.1}$$

where $\Xi(\mathbf{x})$ is the *equally distributed equivalent income* corresponding to distribution $\mathbf{x}$. Solving (6.1) when W is the utilitarian social welfare function, we have

$$\Xi_U(\mathbf{x}) = U^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} U\left(x_i\right)\right),\ \forall\, \mathbf{x} \in \mathcal{Y}_n(D), \tag{6.2}$$

while, in the case of the Yaari social welfare function, we obtain

$$\Xi_f(\mathbf{x}) = \sum_{i=1}^{n} \left[f\left(Q_i\right) - f\left(Q_{i+1}\right)\right] x_i,\ \forall\, \mathbf{x} \in \mathcal{Y}_n(D). \tag{6.3}$$

The *absolute inequality index*, which measures the average income loss due to inequality, is defined by

$$I_U^A(\mathbf{x}) := \mu(\mathbf{x}) - \Xi_U(\mathbf{x}),\ \forall\, \mathbf{x} \in \mathcal{Y}_n(D), \tag{6.4}$$

in the case of the utilitarian social welfare function, and by

$$I_f^A(\mathbf{x}) := \mu(\mathbf{x}) - \Xi_f(\mathbf{x}),\ \forall\, \mathbf{x} \in \mathcal{Y}_n(D), \tag{6.5}$$

in the case of the Yaari social welfare function, respectively. We denote as $\tilde{\mathbf{x}} := (\tilde{x}_1, \ldots, \tilde{x}_n)$ the *mean-reduced* distribution of $\mathbf{x} \in \mathcal{Y}_n(D)$ obtained by letting $\tilde{x}_i := x_i - \mu(\mathbf{x})$, for all $i = 1, 2, \ldots, n$. The *absolute Lorenz curve* of distribution $\mathbf{x} \in \mathcal{Y}_n(D)$ – denoted as $AL(p; \mathbf{x})$ – is defined by

$$AL(p;\mathbf{x}) := L\left(p; \tilde{\mathbf{x}}\right) = \int_0^p \left[F^{-1}(s;\mathbf{x}) - \mu(\mathbf{x})\right] ds,\ \forall\, p \in [0,1]. \tag{6.6}$$

Actually $-AL(p; \mathbf{x})$ represents the average income shortfall of the fraction p of the poorest individuals in situation $\mathbf{x}$ [see Moyes (1987)]. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$, we will say that $\mathbf{x}$ *absolute Lorenz dominates* $\mathbf{y}$, which we write $\mathbf{x} \geq_{AL} \mathbf{y}$, if and only if $AL(p; \mathbf{x}) \geq AL(p; \mathbf{y})$, for all $p \in (0, 1)$, which in our framework is equivalent to:

$$\sum_{j=1}^{k} \left[x_j - \mu(\mathbf{x})\right] \geq \sum_{j=1}^{k} \left[y_j - \mu(\mathbf{y})\right],\ \forall\, k = 1, 2, \ldots, n-1. \tag{6.7}$$

The absolute differentials, deprivation and satisfaction quasi-orderings as well as the absolute Lorenz quasi-ordering have all the property that equal additions to incomes leave inequality unchanged. More precisely, they satisfy:

Translation Invariance. The binary relation $\geq_J$ on $\mathcal{Y}_n(D)$ is *translation-invariant* if, for all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$ and all $\gamma, \xi \in \mathbb{R}$:

$$\mathbf{x} \sim_J \mathbf{y} \Longrightarrow (\mathbf{x} + \gamma \mathbf{1_n}) \sim_J (\mathbf{y} + \xi \mathbf{1_n}). \tag{6.8}$$

We would like to know whether it is possible to find ethical inequality indices that are consistent with the views reflected by the absolute differentials, deprivation, satisfaction and Lorenz quasi-orderings.

As far as inequality measurement is concerned, we would like to know if definitions (6.4) and (6.5) introduce enough flexibility in order to distinguish between the views expressed by our different quasi-orderings. Actually substituting the [absolute] inequality index for the social welfare function in the utilitarian framework does not change anything. Before we present the formal results, we find it convenient to introduce some technicalities.

Remark 6.1. *Let $n \geq 2$. Then, the following two statements are equivalent:*

(a) For all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$: $\mathbf{x} \geq_L \mathbf{y} \Longrightarrow I_U(\mathbf{x}) \leq I_U(\mathbf{y})$.

(b) U is concave.

Remark 6.1 essentially says that the inequality index I_U is Lorenz consistent if and only if the utility function U is concave. The next result is the crucial step here and it goes back to Kolm's (1976) plea for the absolute inequality view.

REMARK 6.2. *Let $n \geq 2$. Then, the following two statements are equivalent:*

(a) For all $\mathbf{x} \in \mathcal{Y}_n(D)$ and all $\gamma \in \mathbb{R}$: $I_U(\mathbf{x} + \gamma \mathbf{1_n}) = I_U(\mathbf{x})$ [12].

(b) $U(y) = \frac{1}{\eta}\exp(\eta y)$ for some $\eta \in \mathbb{R}$ and all $y \in D$.

This result basically says that inequality is not affected by adding or substracting the same amount to all incomes if and only if the utility function U is of the *CARA* type and it is proved in its most general version in Ebert (1988, Theorem 4). The family of utility functions exhibited in statement (b) of Remark 6.2 is closely associated with what is known as the Kolm-Pollak family of inequality indices [see Kolm (1976)] We are now in a position to state the analogue to Proposition 4.1 for the measurement of inequality:

PROPOSITION 6.1. *Let $n > 2$ and $J \in \{AD, ADP, ASF, AL\}$. Then, the following two statements are equivalent:*

(a) For all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$: $\mathbf{x} \geq_J \mathbf{y} \Longrightarrow I_U(\mathbf{x}) \leq I_U(\mathbf{y})$.

(b) Either $U(y) = \frac{1}{\eta}\exp(\eta y)$ for some $\eta < 0$, or $U(y) = y$, for all $y \in D$.

We omit the proof of this result which is established by means of arguments similar to those used in the proof of Proposition 4.1. Proposition 6.1 confirms in the case of situations with possibly differing mean incomes what we learnt from Proposition 4.1: the utilitarian model does not allow one to distinguish between the views associated with the differentials, the deprivation, the satisfaction and the Lorenz [absolute] quasi-orderings.

In contrast the inequality indices derived from the Yaari social welfare function allow us to separate these different views, and one obtains the same restrictions on the weighting function as those pointed out in Propositions 4.2, 4.3, 4.4 and 4.5. Precisely, letting $\mathcal{F}_{AL} = \mathcal{F}_L$, we obtain:

PROPOSITION 6.2. *Let $n > 2$ and $J \in \{AD, ADP, ASF, AL\}$. Then, the following two statements are equivalent:*

(a) For all $\mathbf{x}, \mathbf{y} \in \mathcal{Y}_n(D)$: $\mathbf{x} \geq_J \mathbf{y} \Longrightarrow I_f(\mathbf{x}) \leq I_f(\mathbf{y})$.

(b) $f \in \mathcal{F}_J$.

7. CONCLUDING REMARKS

We have argued in the paper that the utilitarian model, that frames most of the theory of welfare and inequality measurement, is inappropriate when one is interested in inequality views that are more in accordance with the society's values than the one expressed by the Lorenz quasi-ordering. Considering three such concepts of inequality – captured by the differentials, the deprivation and the satisfaction quasi-orderings respectively – we have shown that the Yaari model allows to distinguish between these views. We have furthermore identified the classes of Yaari social welfare functions consistent with these three views. These appear to

[12] In other words this means that the inequality index I_U is *translatable of degre zero* over $\mathcal{Y}_n(D)$.

be subclasses of the general class of Lorenz consistent Yaari social welfare functions. It is expected that some of the views represented by these new criteria are more in line with the public's perception of inequality. The extent to which these criteria are closer to the public's view is a matter of empirical investigation which lies outside the scope of this paper. It is possible to consider other [absolute] inequality views – equivalently quasi-orderings – and investigate their implications for the properties of the social welfare function. We would rather point at three other directions which the present analysis could be extended in.

Given the predominance of the relative approach in the literature it would be interesting to see if it were possible to find analogous results when one considers the relative versions of the criteria examined in the paper. The relative versions of the quasi-orderings examined in this paper can easily be derived. However, up to now we have not been able to identify the restrictions to be imposed on the weighting function which guarantee that social welfare increases as the result of more equally distributed incomes in this case [see however Chateauneuf (1996)]. The rank-dependent expected utility model introduced by Quiggin (1993), which comprises as particular cases the two approaches examined in this paper, might be helpful in this respect. Indeed it offers more flexibility as the chosen value judgements can be reflected by, either the utility function, or the weighting function. Finally the next step to go is surely to characterize by means of additional conditions particular elements of the different classes we have identified. Indeed one may raise doubts about the ability of the differentials quasi-ordering – and to a less extent the deprivation and satisfaction quasi-ordering – to generate conclusive verdicts in practice. Although it must be stressed that these criteria provide guidance in some cases such as taxation design [see e.g. Chakravarty and Moyes (2003)], it is equally true that their ability to rank arbitrary real-world distributions is limited. They must therefore be considered a first round approach, which should be supplemented by the use of particular indices in the general classes we have identified.

References

Amiel, Y., and F.A. Cowell, (1992): "Measurement of income inequality: Experimental test by questionnaire", *Journal of Public Economics*, Vol. 47, pp. 3–26.

Atkinson, A.B., (1970): "On the measurement of inequality", *Journal of Economic Theory*, Vol. 2, pp. 244–263.

Ballano, C., and J. Ruiz-Castillo, (1993): "Searching by questionnaire for the meaning of income inequality", *Revista Espanola de Economia*, Vol. 10, pp. 233–259.

Berge, C., (1963): *Topological Spaces, Including a Treatment of Multi-Valued Functions, Vector Spaces and Convexity*, Oliver and Boyd, Edinburgh.

Bickel, P.J., and E.L. Lehmann, (1976): "Descriptive statistics for non-parametric model III. Dispersion", *Annals of Statistics*, Vol. 4, pp. 1139–1158.

Blackorby, C., Bossert, W., and D. Donaldson, (1999): "Income inequality measurement: The normative approach", Chapter 3 in J. Silber (Ed.), *Handbook of Income Inequality Measurement*, Kluwer Academic Press, Dordrecht, pp. 133–157.

Bossert, W., and A. Pfingsten, (1990): "Intermediate Inequality: Concepts, Indices and Welfare Implications", *Mathematical Social Sciences*, 19, 117–134.

Chakravarty, S.R., (1997): "Relative deprivation and satisfaction orderings", *Keio Economic Studies*, Vol. 34, pp. 17–32.

Chakravarty, S.R., and P. Moyes, (2003): "Individual welfare, social deprivation and income taxation", *Economic Theory*, Vol. 21, pp. 843–869.

Chateauneuf, A., (1996): "Decreasing inequality: An approach through non-additive models", Cermsem Discussion Paper, Université Paris I.

Chateauneuf, A., and P. Moyes, (2004): "A non-welfarist approach to inequality measurement", forthcoming as Wider discussion paper.

Dalton, H., (1920): "The measurement of the inequality of incomes", *Economic Journal*, Vol. 30, pp. 348–361.

Dasgupta, P., Sen, A.K., and D. Starrett, (1973): "Notes on the measurement of inequality", *Journal of Economic Theory*, Vol. 6, pp. 180–187.

Ebert, U., (1988): "Measurement of inequality: An attempt at unification and generalization", *Social Choice and Welfare*, Vol. 5, pp. 59–81.

Ebert, U., and P. Moyes, (2000): "An axiomatic characterization of the Yitzhaki's index of individual deprivation", *Economics Letters*, Vol. 68, pp. 263–270.

Ebert, U., and P. Moyes, (2002): "Welfare, inequality and the transformation of incomes. The case of weighted income distributions", in P. Moyes, C. Seidl and A.F. Shorrocks (Eds.), *Inequalities: Theory, Experiments and Applications*, *Journal of Economics*, Supplement 9, pp. 9–50.

Foster, J.E., (1985): "Inequality measurement", in H.P. Young (ed.), *Fair Allocation*, American Mathematical Society Proceedings of Applied Mathematics, Volume 33.

Fields, G.S., and J.C.H. Fei, (1978): "On inequality comparisons", *Econometrica*, Vol. 46, pp. 305–316.

Gaertner, W., and C. Namezie, (2003): "Income inequality, risk, and the transfer principle: A questionnaire–experimental investigation", *Mathematical Social Science*, Vol. 45, pp. 229–245.

Gastwirth, J.L., (1971): "A General Definition of the Lorenz Curve", *Econometrica*, Vol. 39, pp. 1037–1039.

Hardy, G.H., Littlewood, J.E., and G. Polya, (1952): *Inequalities*, Cambridge University Press, Cambridge (2nd edition).

Harrison, E., and C. Seidl, (1994): "Perceptional inequality and preference judgements: An empirical examination of distributional axioms", *Public Choice*, Vol. 79, pp. 61–81.

Hey, J.D., and P.J. Lambert, (1980): "Relative deprivation and the Gini coefficiengt: Comment", *Quarterly Journal of Economics*, Vol. 95, pp. 567–573.

Jakobsson, U., (1976): "On the measurement of the degree of progression", *Journal of Public Economics*, Vol. 5, pp. 161–168.

Kakwani, N.C., (1984): "The relative deprivation curve and its applications", *Journal of Business and Economic Statistbics*, Vol. 2, pp. 384–405.

Kolm, S.-C., (1969): "The optimal production of social justice", in J. Margolis and H. Guitton (Eds.), *Public Economics*, Macmillan, London, pp. 145–200.

Kolm, S.-C., (1976): "Unequal inequalities I", *Journal of Economic Theory*, Vol. 12, pp. 416–442.

Landsberger, M., and I. Meilijson, (1990): "Lotteries, insurance, and star-shaped utility functions", *Journal of Economic Theory*, Vol. 52, pp. 1–17.

Marshall, A.W., and I. Olkin, (1979): *Inequalities: Theory of Majorization and its Applications*. Academic Press, New-York.

Marshall, A.W., Olkin, I., and F. Proschan, (1967): "Monotonicity of ratios of means and other applications of majorization", in O. Shisha (Ed.), *Inequalities*, Academic Press, New-York.

Moyes, P., (1987): "A new concept of Lorenz domination", *Economics Letters*, Vol. 23, pp. 203–207.

Moyes, P. (1994): "Inequality reducing and inequality preserving transformations of incomes: Symmetric and individualistic transformations", *Journal of Economic Theory*, Vol. 63, pp. 271–298.

Moyes, P., (1999): "Stochastic dominance and the Lorenz curve", Chapter 6 in J. Silber (Ed.), *Handbook of Income Inequality Measurement*, Kluwer Academic Press, Dordrecht, pp. 199–222.

Preston, I., (1990): "Ratios, differences and inequality indices", Institute for Fiscal Studies Working Paper No. W90–9.

Quiggin, J., (1993): *Generalized Expected Utility Theory. The Rank-Dependent Model*, Kluwer Academic Publishers, Boston/Dorrecht/London.

Runciman, W.G., (1966): *Relative Deprivation and Social Justice*, Routledge and Kegan Paul, London.

Sen, A.K., (1970): *Collective Choice and Social Welfare*, Holden-Day, San Francisco. Republished North-Holland, Amsterdam, 1979.

Sen, A.K., (1973): *On Economic Inequality*, Clarendon Press, Oxford.

Thon, D., (1987): "Redistributive properties of progressive taxation", *Mathematical Social Sciences*, Vol. 14, pp. 185–191.

Weiss, Y., and C. Fershtman, (1998): "Social status and economic performance: A survey", *European Economic Review*, Vol. 42, pp. 801–820.

Weymark, J., (1981): "Generalized Gini inequality indices", *Mathematical Social Sciences*, Vol. 1, pp. 409–430.

Yaari, M.E., (1987): "The dual theory of choice under risk", *Econometrica*, Vol. 55, pp. 99–115.

Yaari, M.E., (1988): "A controversial proposal concerning inequality measurement", *Journal of Economic Theory*, Vol. 44, pp. 381–397.

Yitzhaki, S., (1979): "Relative deprivation and the Gini coefficient", *Quarterly Journal of Economics*, Vol. 93, pp. 321–324.

Yitzhaki, S., (1982): "Relative deprivation and economic welfare", *European Economic Review*, Vol. 17, pp. 99–113.

Figure 5.1. Three Weighting Functions Compatible with the Absolute Differentials Quasi-Ordering

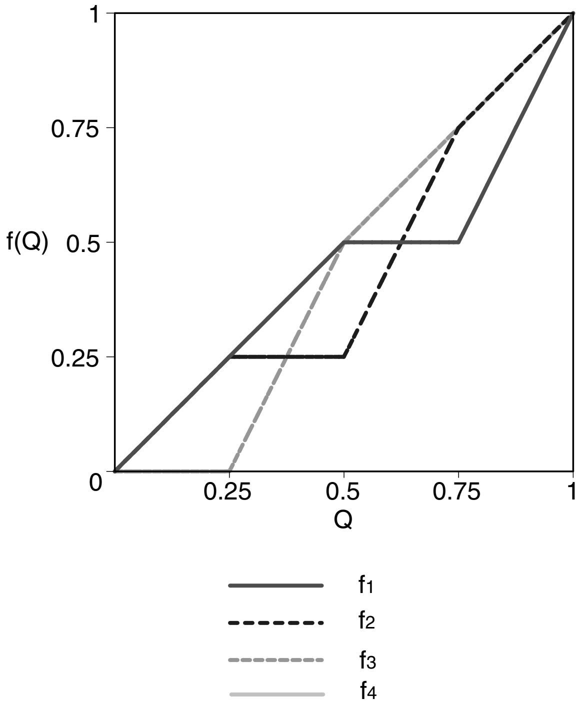

Figure 5.2. Three Weighting Functions Compatible with the Absolute Deprivation Quasi-Ordering

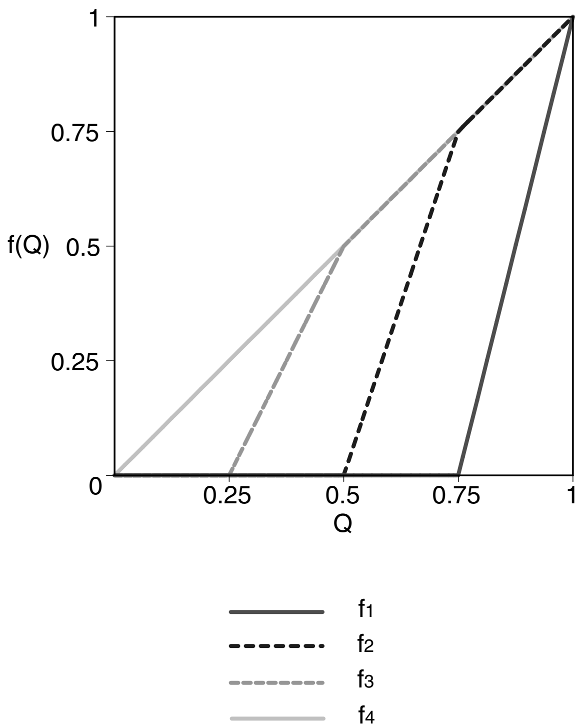

Figure 5.3. Three Weighting Functions Compatible with the Absolute Satisfaction Quasi-Ordering

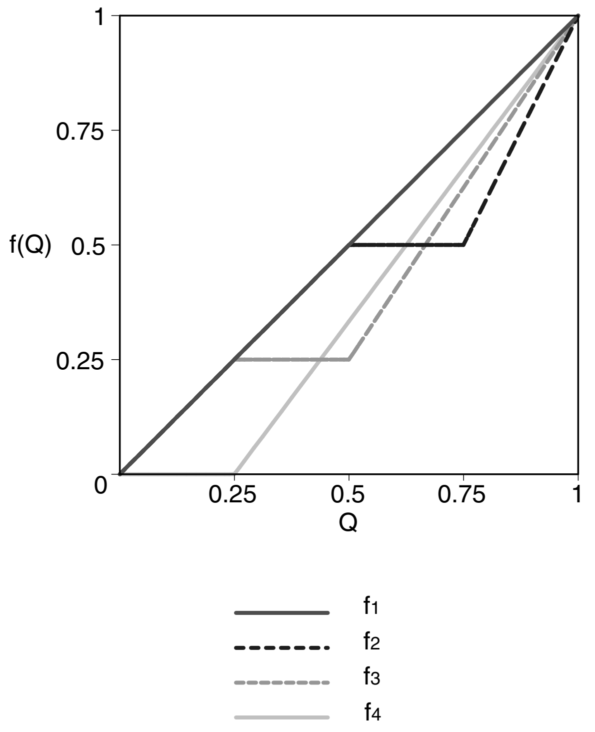

Figure 5.4. Three Weighting Functions Compatible with the Absolute Lorenz Quasi-Ordering