TURKISH ECONOMIC ASSOCIATION

DISCUSSION PAPER 2004/18
http://www.tek.org.tr

LECTURE NOTES ON MACROECONOMICS

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October, 2004

Lecture Notes on Macroeconomics

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July 21, 2004

[1]These notes were written while I was an Assistant Professor at the Department of Economics at the University of Minnesota. I developed the material for these notes as part of my teaching of Graduate Macroeconomics in the Economics Department.

Some Preliminaries

In this course, we have the ambitious task of explaining how the aggregate economy works. We would also like to evaluate the impact of government policies designed to correct potential inefficiencies that lead to welfare losses among individuals. For this purpose, we will integrate individual behavior into our macroeconomic models and try to see the consequences of market equilibrium.

Our approach will be different from the approach followed in undergraduate courses. There the approach is to study individual behavior (the firm and the consumer) in micro classes, and then employ a somewhat different set of models in the macro courses. For example, we have the consumption function, the demand for money, an investment schedule, the IS *curve and the* LM *curve.*

These functions are typically only loosely connected to an underlying choice problem. Furthermore, these models do not take into account that individuals live in an environment under uncertainty, i.e., that individuals do not for sure what the future will bring and have to predict future values of prices, government policies, etc. when making current decisions. Also, they do not take into account the fact that individuals live in a fundamentally dynamic environment in that the action taken today has implications for the payoff that one receives tomorrow. Traditional macroeconomics has not dealt adequately with either of these issues.

*Prior to Keynes, there was little dynamic theory, except for Ramsey who examined problems of optimal taxation. Keynes did not have an adequate framework dealing with problems under uncertainty, so he basically abandoned the theory of choice. (***General Theory***, 1936.) Hicks tried to address the problem of economies moving over time, how people form expectations, etc. (***Value and Capital***, 1937.) This book influenced scholars such as Lucas and Prescott who were instrumental in bringing dynamic, stochastic analysis into macroeconomics.*

Some mathematical developments also furthered the developments in economics. Bellman came up with the method of dynamic programming in the 1950s. This method can be used to transform a complicated intertemporal optimization problem into a much simpler and seemingly static looking optimization problem. This is done by summarizing features of the environment which change over time into a small set of state variables. Also, linear prediction theory was developed during this time by Wiener and Kolmogorov. We will use this technique when modelling the way individuals make forecasts over time and the way they use a signal or a noisy piece of information to infer the values of variables they are interested in. Finally, we will try to integrate economic theory with the econometric analysis of date which it generates.

1 Introduction

What are some data which macroeconomists look at?

Typically we have *time series* observations on a set of economic variables. Clearly, if we wish to describe how investment behavior changes over time, we have to postulate some model or equation which moves over time. Linear difference equations provide a way to model the movement of variables over time. Hence, we will study them.

$$i_t = \lambda i_{t-1} + \epsilon_t, \;\; 0 < \lambda < 1,$$

i_0 given. Here ϵ_t may be a constant or it may vary over time. While this difference equation provides a way of generating a time path of observations for investment, the time path will not necessarily look what we observe. In particular, it will be too smooth relative to the behavior of the actual series. Now, in the 1930's, Slutsky and Frisch showed that if you took several white noise or serially uncorrelated stochastic processes, i.e., $\{\epsilon_t\}_{t=0}^{\infty}$, $E(\epsilon_t) = 0$, $Var(\epsilon_t) = \sigma^2$, and $E(\epsilon_t \epsilon_{t-k}) = 0$ for all $k > 0$, and summed them in various ways, you could replicate the behavior of many observed time series. Hence, one way to improve our difference equation is to make it stochastic, i.e., assume that ϵ_t is a random variable and hence, the sequence $\{\epsilon_t\}_{t=0}^{\infty}$ is a stochastic process (or a collection of random variables). Then i_t will bounce around with shocks to ϵ_t. Furthermore, even though ϵ_t is itself serially uncorrelated, the effect of past ϵ shocks will persist over time through the dependence of i_t on i_{t-1}.

But what is still missing? Suppose we can in fact match up the observed investment series with the time path of our stochastic difference equation. Suppose we would like to predict the effect of a new tax scheme on the behavior of investment expenditures. Can we do it with our stochastic difference equation? No, because this equation does not incorporate the implications of individual behavior. For example, if firms know that the tax on new investment will be lower relative to buying old machines, they will change their investment plans. So we have to have a *model* based on *individual choice* which takes into account the *uncertainty in the environment* (for example, firms may be uncertain about real future tax rates due to the existence of inflation, they may be unsure about the price that they can get for their product in the future). Furthermore, this model must be *dynamic* because current investment decisions by firms will have ramifications for their future decsions, i.e., firms make decisions on new machines or new technology which last for many periods. This is the basic notion behind capital theory – the existence of durable goods, whether they are machines, a reputation, human capital (augmented by education), organizational capital (the little procedures and augmented wisdom which make organizations function smoothly), etc. Finally, we want our solution to this model to look like a stochastic difference equation. If it is linear as well, we are in great shape because then we can use *econometric methods* (OLS) to estimate parameters and to test the implications of one model against another.

2 A Mathematical Framework

Modern macroeconomics is concerned with the problem of introducing dynamics and uncertainty into models which try to describe the aggregate economy. Now, it turns out that the solution to such models take the form of sequences over time. We may formulate dynamic problems with a finite time horizon. On the other hand, we may not wish to deal with the problem of the last period T. Hence, we may let the economy proceed out to infinity and approximate individual optimization problems which occur in a finite horizon as infinite horizon problems. For example, a consumer may choose how much to consume and how many securities to hold in the following problem:

$$\begin{aligned} &\max_{\{c_t\},\{s_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\qquad\quad \text{subject to} \\ &c_t + p_t s_t \leq (p_t + d_t) s_{t-1} + w_t, \\ &\{w_t\}, \{d_t\} \ \text{ given.} \end{aligned}$$

The *solution* to the above problem consists of a set of sequences $c = \{c_t\}_{t=0}^{\infty}$, $s = \{s_t\}_{t=0}^{\infty}$. The solution is not in terms of some scalars c and s as in a static problem. Hence, we have to have a way of dealing with the properties of the entire sequence if we want to characterize the behavior of the optimal consumption and security holdings choices.

An easy way to deal with models which have sequences or stochastic processes as their solution is to learn some functional analysis. Functional analysis studies spaces whose elements are functions. How is a function defined on Euclidean space? A function f is a rule which associates a unique real number $y \in Y \subset R$ denoted $y = f(x)$ with each element of a set of real numbers $X \subset R$.

$$X \subset R \ \ f : X \to Y, \ \ Y \subset R.$$

Now consider the price sequence $\{p_t\}_{t=1}^{\infty} = \{P_t(T)\}_{t=1}^{\infty}$. Then P is a rule which associates a unique real sequence $p \in \mathcal{P}$ such that

$$p_1 = P_1(T), \ p_2 = P_2(T), \ldots, p_t = P_t(T)$$

with each element of the set of integers $T = \{1, 2, 3, \ldots\}$.

$$P : T \to \mathcal{P} \ \text{ s.t. } \ p_t = P_t(T) \ \ \forall \in T.$$

If $P_t(\cdot) = P(\cdot)$, then P is a stationary or time-invariant rule.

We will also be dealing with objects called *difference equations*. Let $\epsilon = \{\epsilon_t\}_{t=1}^{\infty}$ be an element from the space of real sequences S, i.e., $S = (\{\epsilon\}_{t=1}^{\infty} : e_t \in R \ \forall t)$. Thus, a first-order linear difference equation for the sequence $i = \{i_t\}_{t=1}^{\infty} \in S$ is a function g which maps S into itself, i.e.,

$$g : S \to S \text{ s.t } [(g(i)]_t = \lambda i_{t-1} + \epsilon_t \text{ for } |\lambda| < 1 \text{ and } \forall t \geq 1,$$

or

$$i_t = \lambda i_{t-1} + \epsilon_t \ \ \forall t \geq 1.$$

Now, a difference equation is a function of functions because, as noted before, $\{\epsilon_t\}_{t=1}^{\infty}$ and $\{i_t\}_{t=1}^{\infty}$ are themselves functions of time. These examples show that we have to deal with more abstract spaces than the ones we are used to. In particular, we cannot just deal with the set of real number $\Re$ or $\Re^+$. This brings us to a discussion of metric spaces.

2.1 Metric spaces

A *metric space* is a pair of objects (X, d) if X is a set and $d(x, y)$ is a real-valued function, called the *metric*, defined for all $x, y \in X$ and satisfying the following conditions or axioms:

1. *Positivity*: $d(x, y) \geq 0$ and $d(x, x) = 0$ for all $x, y \in X$.
2. *Strictly Positive*: If $d(x, y) = 0$, then $x = y$ for all $x, y \in X$.
3. *Symmetry*: $d(y, x) = d(x, y)$ for all $x, y \in X$.
4. *Triangle Inequality*: $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

What is the concept of distance in Euclidian space?

$$d(x, y) = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right]^{1/2}.$$

For future reference, we have the following:

Definition 2.1 *Let E be an ordered set, i.e., a set on which order is defined. If there exists an α s.t. $x \leq \alpha$ for all $x \in E$, then E is bounded above and α is an upper bound on E.*

Definition 2.2 *Let E be an ordered set and let E be bounded above. Suppose there exists an α s.t.*

(i) α is an upper bound on E.

(ii) If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is the least upper bound on E, or $\alpha = \sup(E)$.

Other examples are given by:

1. $\left(\Re^2, d(x,y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2}\right)$.
2. $(\Re^2, d(x,y) = |(x_1 - y_1)| + |(x_2 - y_2)|)$.
3. $(\Re^+, d_\infty(x,y) = \max(|x_1 - y_1)|, |x_2 - y_2|))$.
4. $X = C(0,T)$: space of all bounded, continuous functions on the interval $(0,T)$. Metrics:

   $$d_p(x,y) = \left[\int_0^T |x(t) - y(t)|^p \, dt\right]^{1/p}, \quad p \in [1, \infty),$$

   or

   $$d_\infty(x,y) = \sup_{0 \le t \le T} |x(t) - y(t)|.$$

5. Define $l_p(0,\infty)$ as the set of all sequences $\{x_t\}_{t=0}^\infty$ s.t. $\sum_{t=0}^\infty |x_t|^p < \infty$, and the metric as

   $$d_p(x,y) = \left\{\sum_{t=0}^\infty |x_t - y_t|^p\right\}^{1/p}.$$

   Then (l_p, d_p) is a metric space.

6. Define l_∞ as the set of all bounded sequences of real or complex numbers, i.e., $\{x_t\}_{t=}^\infty \in l_\infty$ iff $\exists$ a real number M s.t. $|x_t| \le M$ for all t. Define

   $$d_\infty(x,y) = \sup |x_t - y_t|.$$

   Then (l_∞, d_∞) is a metric space.

How do you show that (l_∞, d_∞) is a metric space? We have to show that d_∞ satisfies the properties of a metric, i.e., P, SP, S, TE. Note that $l_p(0,\infty) \supset l_\infty(0,\infty)$. We will use $(l_2(0,\infty), d_2)$ where

$$d_2(x,y) = \left\{\sum_{t=0}^\infty |x_t - y_t|^2\right\}^{1/2},$$

where $l_2(0,\infty)$ is the set of sequences $\{x_t\}_{t=0}^\infty$ s.t. $\sum_{t=0}^\infty |x_t|^2 < \infty$.

Definition 2.3 *A sequence $\{x_n\}$ in a metric space (X,d) is said to be a* **Cauchy sequence** *if for each $\epsilon > 0$, there exists an $N(\epsilon)$ s.t. $d(x_n, x_m) < \epsilon$ for any $n, m \ge n$.*

Thus, a sequence $\{x_n\}$ is said to be Cauchy if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

Definition 2.4 *A sequence $\{x_n\}$ in a metric space (X, d) is said to converge to a limit $x_0 \in X$ if for every $\epsilon > 0$ there exists an $N(\epsilon)$ s. t. $d(x_n, x_m) < \epsilon$ for $n \geq N(\epsilon)$.*

Lemma 2.1 *Let $\{x_n\}$ be a convergent sequence in a metric space (X, d). Then $\{x_n\}$ is a Cauchy sequence.*

Proof. Let x_0 be the limit of the sequence $\{x_n\}$ in (X, d). Then for any n and m, one has $d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m)$ by the Triangle Inequality. Since x_0 is the limit of $\{x_n\}$, given $\epsilon > 0$, there is an N s.t. $n, m \geq N$ implies $d(x_n, x_0) \leq \epsilon/2$ and $d(x_0, x_m) \leq \epsilon/2$. Then $d(x_n, x_m) \leq \epsilon$ whenever $n, m \geq N$. Hence, $\{x_n\}$ is a Cauchy sequence. ∎

But is every Cauchy sequence in a metric space (X, d) convergent in the same space? No!

Consider the metric space $(C[0, 2], d_2(x, y))$. Let $\{x_n\}$ be the sequence of continuous functions

$$x_n(t) = \begin{cases} t^n & 0 \leq t \leq 1 \\ 1 & 1 < t \leq 2. \end{cases}$$

Clearly, $\{x_n(t)\}$ converges to

$$x_0(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t \leq 2. \end{cases}$$

Now in $(C[0, 2], d_2(x, y))$, $\{x_n(t)\}$ is a Cauchy sequence. Note that

$$\begin{aligned} d_2(t^m, t^n)^2 &= \int_0^1 (t^n - t^m)^2 dt \\ &= \int_0^1 (t^{2n} + t^{2m} - 2t^{n+m}) dt \\ &= \left(\frac{1}{2n+1} t^{2n+1} + \frac{1}{2m+1} t^{2m+1} - \frac{2}{n+m+1} t^{n+m+1} \right) \Big|_0^1 \\ &= \frac{1}{2n+1} + \frac{1}{2m+1} - \frac{2}{n+m+1}. \end{aligned}$$

Hence, for any $\epsilon > 0$, it is possible to choose $N(\epsilon)$ such that the RHS is less than ϵ whenever m and n both exceed N, i.e., choose $\epsilon = 2/(2N+3)(2N+4)(2N+5)$.

Another question.

Is every sequence which is Cauchy in one metric space also Cauchy in another metric space? The answer again is no!

Example 1 Consider the space $(C[0,2], d_\infty(x,y))$ and put the sequence of functions

$$x_n(t) = \begin{cases} t^n & 0 \le t \le 1 \\ 1 & 1 < t \le 2 \end{cases}$$

each of which are elements of $(C[0,2], d_\infty(x,y))$. In $(C[0,2], d_\infty)$, $x_n(t)$ is not Cauchy, i.e., for any fixed $m > 0$, there is a $\delta > 0$ s.t.

$$\sup_{n>0} \sup_{0 \le t \le 1} |t^n - t^m| > \delta.$$

More precisely,

$$\begin{aligned} \sup_{n>0} \sup_{0 \le t \le 1} |t^n - t^m| &= \sup_{0 \le t \le 1} \sup_{n>0} |t^n - t^m| \\ &= \sup_{0 \le t \le 1} \lim_{n \to \infty} |t^n - t^m| \\ &= \sup_{0 \le t \le 1} \lim_{n \to \infty} t^m |t^{n-m} - 1| \\ &= \sup_{0 \le t \le 1} t^m \lim_{n \to \infty} |t^{n-m} - 1|. \end{aligned}$$

But

$$\lim_{n \to \infty} |t^{n-m} - 1| = \begin{cases} 1 & \text{for } 0 \le t \le 1 \\ 0 & \text{for } t = 1 \end{cases}$$

Therefore,

$$\sup_{0 \le t \le 1} \lim_{n \to \infty} t^m |t^{n-m} - 1| = \sup_{0 \le t \le 1} t^m = 1.$$

Hence, $x_n(t) = t^n$, $0 \le t \le 2$ is not Cauchy in $(C[0,2], d_\infty(x,y))$.

We can also examples in which a sequence converges to a limit in one metric space but it does not converge in another metric space.

Example 2 Consider the sequence of functions $(C[0,1], d_2)$ where

$$x_n(t) = \begin{cases} 1 - nt & 0 \le t \le \frac{1}{n} \\ 0 & \frac{1}{n} < t \le 1. \end{cases}$$

Claim: $x_n(t) \to x_0$ where $x_0 = 0$. We have that

$$\begin{aligned} d_2(x_n, x_m) &= \left\{ \int_0^1 |x_n(t) - 0|^2 dt \right\}^{1/2} \\ &= \left\{ \int_0^{1/n} (1 - nt)^2 dt \right\}^{1/2} \\ &= \left\{ \int_0^{1/n} (1 - 2nt + n^2 t^2) dt \right\}^{1/2} \end{aligned}$$

$$
\begin{aligned}
&= \left\{ \left(t - nt^2 + \frac{n^2 t^3}{3} \right) \Big|_0^{1/n} \right\}^{1/2} \\
&= \left\{ \frac{1}{n} - n\frac{1}{n^2} + \frac{n^2}{3}\frac{1}{n^3} \right\}^{1/2} = (3n)^{-1/2}.
\end{aligned}
$$

Given $\epsilon > 0$, if $n \geq N$ where $N > \epsilon^{-2}$ then $d(x_n, x_m) < \epsilon$. Hence, $\lim_{n\to\infty} x_n(t) = x_0(t) = 0$.

Does $\{x_n\}_{n=0}^{\infty}$ converge to x_0 is $(C[0,1], d_\infty(x,y))$?

Now

$$
\begin{aligned}
d_\infty(x_n, x_m) &= \sup_{0\leq t\leq 1} |x_n(t) - x_m(t)| \\
&= \sup_{0\leq t\leq 1} |1 - nt - 0| \\
&= 1 \ \text{ for each } n.
\end{aligned}
$$

More precisely, $\{x_n\}$ does not converge to x_0 because for any arbitrary m we can find an $N > n$ s.t.

$$
d_\infty(x_n, x_m) \geq \frac{1}{2} \ \text{ for all } m > N.
$$

Hence, Cauchy sequences need not converge in a given metric space nor do sequences which are Cauchy in one metric space need to be Cauchy in another metric space. In many cases, we want Cauchy sequences to converge to points in the same space.

Definition 2.5 *A metric space* (X, d) *is said to* **complete** *if each Cauchy sequence in* (X, d) *is a convergent sequence in* (X, d).

The spaces $(l_p(0,\infty), d_p)$ and (l_∞, d_∞) are complete.

Example 3 The space of rational numbers with the absolute value is not a complete metric space, i.e. the sequence $\{3, 3.14, 3.141, 3.1415, \ldots\}$ is a Cauchy sequence but it is not convergent in this space because π is not a rational number.

Example 4 Consider $(C[0,T], d_\infty)$. How we show that this space is complete?

Let $\{x_n\}$ be an arbitrary Cauchy sequence in $(C[0,T], d_\infty)$. Hence, there is an $N(\epsilon)$ s.t. for $n \geq N(\epsilon)$ implies

$$
|x_n(t) - x_m(t)| \leq d_\infty(x_n, x_m) \equiv \sup |x_n(t) - x_m(t)| \leq \epsilon \ \text{ for all } t.
$$

Then for fixed t $\{x_n(t)\}$ converges to $x_0(t)$. Since t is arbitrary, the sequence of functions $\{x_n(\cdot)\}$ converges point-wise to $x(\cdot)$. But $N(\epsilon)$ beng independent of t implies that $\{x_n(\cdot)\}$ converges uniformly to $x_0(\cdot)$. But from real analysis, if a sequence of continuous functions $\{x_n(\cdot)\}$ converges uniformly to a function $x_0(\cdot)$, then $x_0(\cdot)$ is continuous. Then every Cauchy sequence in $(C[0,T], d_\infty)$ is convergent, and $(C[0,T], d_\infty)$ is complete.

Definition 2.6 *A function f mapping a metric space (X, d) into itself is called an* **operator**.

Definition 2.7 *Let $f : X \to X$ be an operator on a metric space (X, d). The operator is said to be* **continuous at a point** *$x_0 \in X$ if for $\epsilon > 0$, there exists a $\delta > 0$ s.t. $d(f(x), f(x_0)) < \epsilon$ whenever $d(x, x_0) < \delta$. The operator f is said to a* **continuous** *if it is continuous at each point $x \in X$.*

Now we wish to study contraction mappings which are a particular type of operator useful for economics.

Definition 2.8 *Let (X, d) be a metric space and let $f : X \to X$. Then f is a* **contraction** *or contraction mapping if there exists a real number k, $0 < k < 1$, s.t.*

$$d(f(x), f(y)) \leq kd(x, y) \ \text{ for } x, y \in X.$$

Can you show that a contraction is a continuous operator? Easy. Consider any $x, y \in X$ s.t. $d(x, y) < \delta$. Then $d(f(x), f(y) \leq kd(x, y) < k\delta$. Therefore, for any $\epsilon > 0$, there exists a $\delta > 0$ s.t. $d(f(x), f(y)) < \delta$ whenever $d(x, y) < \delta$, i.e. pick $\delta = \epsilon/k$.

Theorem 1 *(Contraction Mapping): Let (X, d) be a compete metric space and $f : X \to X$ be a contraction. Then there exists one and only one point $x_0 \in X$ s.t.*

$$f(x_0) = x_0.$$

Moreover, if x is any point in X and x_n is defined inductively by $x_1 = f(x_0)$, $x_2 = f(x_1), \ldots, x_n = f(x_{n-1})$, then $x_n \to x_0$ as $n \to \infty$. That is, f has a unique fixed point and every sequence of iterations of f converge to this fixed point.

Proof. We show that every sequence of iterations of f converges to a fixed point. Then we show that f can have only one fixed point.

Let x be any point in X and define $x_1 = f(x_0)$, $x_2 = f(x_1)$, and, in general, $x_n = f(x_{n-1})$. Then $x_n = f^n(x)$. We will show that $\{x_n\}$ is a Cauchy sequence. Assume $n > m$; then

$$\begin{aligned} d(x_n, x_m) &= d(f^n(x), f^m(x)) \\ &= d(f^m(x_{n-m}), f^m(x)) \\ &\leq d(f^{m-1}(x_{n-m}), f^{m-1}(x)) \ \text{[Contraction property]}. \end{aligned}$$

By induction,

$$d(x_n, x_m) \leq k^m d(x_{n-m}, x).$$

Using the triangle inequality,

$$\begin{aligned} d(x_n, x_m) &\leq k^m \left[d(x_{n-m}, x_{n-m-1}) + \ldots + d(x_2, x_1) + d(x_1, x)\right] \\ &\leq k^m \left[k^{n-m-1} + \ldots + k + 1\right] d(x_1, x). \end{aligned}$$

Since $0 \leq k < 1$, we have

$$d(x_n, x_m) \leq k^m \sum_{i=0}^{\infty} k^i d(x_1, x) = \frac{k^m}{1-k} d(x_1, x).$$

Since $0 \leq k < 1$, it is possible to find an N s.t. for $n, m > N$, $d(x_n, x_m) < \epsilon$. Since the space (X, d) is complete, the sequence $\{x_n\}$ converges to an element of (X, d). Let $x_0 = \lim_{n\to\infty} x_n$. Since f is continuous, we know that

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

But

$$f(\lim_{n\to\infty} x_n) = f(x_0)$$

and

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x_0.$$

Therefore, $F(x_0) = x_0$. Hence, x_0 is a fixed point of f.

To show that the fixed point is unique, we argue by contradiction. Assume x_0 and y_0 are two distinct fixed points of f. Then we obtain the contradiction:

$$0 < d(x_0, y_0) = d(f(x_0), f(y_0)) \leq kd(x_0, y_0) < d(x_0, y_0).$$

Therefore, f has only one fixed point. ■

Corollary 1 *Let (X, d) be a complete metric space, and let f be a (not necessarily continuous) function, $f : X \to X$. If for some integer p, f^p is a contraction, then f has a unique fixed point.*

We will now look at sets X whose elements are functions. The functions $C[0, T]$ and $l_p[0, \infty)$ for $1 \leq p \leq \infty$ are some examples.

Definition 2.9 *Let X be the space of functions, and let $x, y \in X$. Then $x \geq y$ iff $x(t) \geq y(t)$ for all t in the domain of definition of the function.*

Consider the metric:

$$d_\infty(x, y) = \sup_t |x(t) - y(t)|.$$

Theorem 2 *(Blackwell's sufficient conditions for T to be a contraction) Let T be an operator on a metric space (X, d) where X is a space of functions. Assume T has the following two properties:*

(i) Monotonicity: For any $x, y \in X$, $x \geq y$ implies $T(x) \geq T(y)$.

(ii) Discounting: Let c denote a function that is constant at the real value c for all points in the domain of definition of functions in X. Therefore, for any real c, and every $x \in X$,

$$T(x+c) \leq T(x) + \beta c \text{ for some } 0 \leq \beta \leq 1.$$

Then T is a contraction mapping with modulus β.

Proof. For any $x, y \in X$, notice that

$$\begin{aligned} x(t) &\leq y(t) + d_\infty(x, y) \text{ for all } t \\ &= y(t) + \sup_t |x(t) - y(t)|, \end{aligned}$$

which implies that $x \leq y + d_\infty(x, y)$. Now

$$\begin{aligned} T(x) &\leq T(y) + d_\infty(x, y) \text{ by monotonicity} \\ &\leq T(y) + \beta d_\infty(x, y) \text{ by discounting} \end{aligned}$$

which implies that

$$T(x) - T(y) \leq \beta d_\infty(x, y).$$

Interchange x and y, which implies that

$$T(y) - T(x) \leq \beta d_\infty(x, y),$$

or

$$-\beta d_\infty(x, y) \leq T(x) - T(y) \leq \beta d_\infty(x, y).$$

But (2.1) holds for all t elements in the domain of definition of x and y. Thus, we have that

$$|T(x) - T(y)| \leq \beta d_\infty(x, y),$$

which implies that

$$\sup_t |T(x) - T(y)| \leq \beta d_\infty(x, y),$$

. Thus,

$$d_\infty(T(x) - T(y)) \leq \beta d_\infty(x, y).$$

Thus, T is a contraction. ■

2.2 Uncertainty

We will be dealing with so-called random phenomena or with variables whose values or conditions we cannot predict perfectly. To study such phenomena, we need to develop some mathematical tools.

Probability spaces are the basic mathematical models for random phenomena. Consider a collection of random phenomena. We assume that there exists a set Ω which is called the *sample space*. In a given experiment, a particular element $\omega \in \Omega$ is chosen. This point in turn determines the values associated with a random phenomena in that experiment.

We are also interested in events which are subsets of Ω. $\omega \in A$ where $A \subset \Omega$ means that the event A occurred during the experiment. The probability that the event A will occur is then a real number $P(A)$ satisfying $0 < P(A) < 1$.

Definition 2.10 *Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a field (or algebra) if and only if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and finite union:*

(i) $\Omega \in \mathcal{F}$;

(ii) If $A \in \mathcal{F}$, *then* $A^c \in \mathcal{F}$.

(iii) If $A_1, A_2, \ldots, A_n \in \mathcal{F}$, *then* $\cup_{i=1}^n A_i \in \mathcal{F}$

Notice that $\mathcal{F}$ is closed under finite intersection, i.e.

$$\bigcap_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n} A_i^c\right)^c \in \mathcal{F}.$$

If (iii) is replaced by closure under countable union, i.e.,

(iv) If $A_1, A_2, \ldots, \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$,

then $\mathcal{F}$ is called a *σ-field* or *σ-algebra*.

Definition 2.11 *A measure on a σ-field $\mathcal{F}$ is a non-negative, extended real-valued function μ on $\mathcal{F}$ such that whenever $A_1, A_2, \ldots,$ form a finite or countably infinite collection of disjoint sets in $\mathcal{F}$, we have*

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n). \quad \textit{[Countably additive]}$$

If $\mu(\Omega) = 1$, then μ is a probability measure.

Definition 2.12 *A* **probability space** *is a triple $(\Omega, \mathcal{F}, P)$ where Ω is a non-empty set, $\mathcal{F}$ is a σ-field of subsets of Ω and P is a positive measure s.t. $P(\Omega) = 1$. In this case, P is sometimes called a probability measure.*

Example 1 Let Ω denote the collection of all possible outcomes of an experiment involving 50 flips of a coin. A typical sample point is then an ordered 50-tuple, i.e., $(H, H, T, \ldots, T)$. Let p denote the probability of getting H on any toss and $q = 1 - p$ be the probability of getting T. Let A be the event consisting of all outcomes with n heads and $50 - n$ tails. Then

$$P(A) = \frac{50!}{(50-n)!n!} p^n q^{50-n}.$$

Instead of dealing with the abstract notions of probability spaces, we will deal with random variables and their distributions. We begin with some definitions.

Definition 2.13 *A Borel field $\mathcal{B}$ in $\Re$ is the smallest σ field of sets from $\Re$ that contains all open intervals (a, b).*

The sets in the Borel field are called *Borel sets*. It can be shown that

(i) every interval is a Borel set,

(ii) every open set is a Borel set,

(iii) every closed set is a Borel set.

Definition 2.14 *A real random variable X is a real-valued function defined on the sample space Ω such that all sets*

$$A = \{\omega \in \Omega : X(\omega) \in B \ \ for\ all\ Borel\ sets\ B \subset \Re\} \in \mathcal{F},$$

or,

$$X : \Omega \to \Re \ \ s.t.\ \ X^{-1}(B) \in \mathcal{F}.$$

Definition 2.15 *The probability distribution function of X, denoted by $F(x)$, is*

$$F(x) = P[X(\omega) \leq x].$$

Then $F(x)$ is the probability of the event $A = \{\omega \in \Omega : X(\omega) \leq x\}$.

$$\begin{aligned} 0 \leq F(x) &\leq 1 \ \ \forall x \\ F(-\infty) &= 0 \\ F(\infty) &= 1. \end{aligned}$$

If $x \leq y$, then

$$\{\omega \in \Omega : X(\omega) \leq x\} \subseteq \{\omega \in \Omega : X(\omega) \leq y\} \Rightarrow F(x) \leq F(y).$$

Thus, F is monotonic increasing. Also, if $x_1 < x_2$,

$$P[x_1 \leq X(\omega) \leq x_2] = F(x_2) - F(x_1).$$

If $F(x)$ is absolutely continuous, then there exists a function $f \in L_1(-\infty, \infty)$ such that

$$F(x) = \int_{-\infty}^{x} f(t)dt,$$

where $L_1(-\infty, \infty)$ is the set of functions $f : \Re \to \Re$ s.t.

$$\int_{\Re} |f(t)|dt < \infty.$$

Let X and Y be two real random variables on Ω. The joint probability distribution is

$$F(x, y) = P(X(\omega) \leq x \text{ and } Y(\omega) \leq y).$$

Expectation

If X is a random variable, its expectation, if it exists, is

$$E(X) = \int_{\Omega} X(\omega)dP(\omega) < \infty.$$

If $E(|X|) = \int_{\Omega} |X(\omega)|dP(\omega) < \infty$, then $E(X) < \infty$ and is well defined. Thus, if $X \in L_1(\Omega, \mathcal{F}, P)$, then X has a well-defined expected value.

A real-valued random variable X is said to be an element of $L_2(\Omega, \mathcal{F}, P)$ if and only if

$$E(|X|^2) = \int_{\Omega} |X(\omega)|^2 dP(\omega) < \infty.$$

If $X \in L_2(\Omega, \mathcal{F}, P)$, then X has a finite second moment. Since $L_2(\Omega, \mathcal{F}, P) \subset L_1(\Omega, \mathcal{F}, P)$, random variables which have finite second moments also have finite expected values. In this case, the variance of X, denoted $\sigma^2(X)$, is

$$\sigma^2(X) = E[(X - E(X)]^2.$$

If X and Y are two real-valued random variables in $L_2(\Omega, \mathcal{F}, P)$, then the covariance of X and Y is defined as

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)].$$

Stochastic Independence

Let X and Y be two real-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$. X and Y are stochastically independent if and only if

$$P[X(\omega) \leq x \text{ and } Y(\omega) \leq y] = P[X(\omega) \leq x]P[Y(\omega) \leq y].$$

Theorem 3 *Let X and Y be two stochastically independent random variables in $L_2(\Omega, \mathcal{F}, P)$. Then $E(XY) = E(X)E(Y)$.*

Also,

$$P[X(\omega) \geq x \text{ and } Y(\omega) \geq y] = P[X(\omega) \geq x]P[Y(\omega) \geq y].$$

Example 2 Let A and B be two events in Ω and let χ_A and χ_B denote their characteristic functions, i.e.

$$\begin{aligned} \chi_A &= 1 \text{ if } \omega \in A \\ &= 0 \text{ if otherwise.} \end{aligned}$$

Claim χ_A and χ_B are independent $\Leftrightarrow P(A \bigcap B) = P(A)P(B)$,

$\Rightarrow$ Since the stochastic independence condition is valid for all x and y, then for $x < 1$ and $y < 1$,

$$\begin{aligned} P(A \bigcap B) &= P[\chi_A \geq x \text{ and } \chi_B \geq y] \\ &= P[\chi_A > x]P[\chi_B > y] \\ &= P(A)P(B). \end{aligned}$$

$\Leftarrow$ Now $P(A \bigcap B) = P[\chi_A \geq x \text{ and } \chi_B \geq y]$ for all $x < 1$ and $y < 1$, and

$$P(A)P(B) = P[\chi_A > x]P[\chi_B > y],$$

which yields the required condition.

Also check cases with $x \geq 1$ and $y \geq 1$.

Conditional Expectation Operator

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{B}$ be a sub-collection of $\mathcal{F}$. Then $\mathcal{B}$ is a sub-σ-field if $\mathcal{B}$ is itself a σ-field of $\mathcal{F}$.

Example 1 $\mathcal{B} = \{\emptyset, \Omega\}$.

Example 2 $\mathcal{B} = \{\mathcal{F}\}$.

Example 3 Let $A \in \mathcal{F}$. Then

$$\mathcal{B} = \{\emptyset, A, A^c, \Omega\}$$

is the σ-field generated by A.

Example 4 Let $A, B \in \mathcal{F}$, where $A \bigcap B = \emptyset$ and let $C = (A \bigcup B)^c = A^c \bigcap B^c$. Then

$$\mathcal{B} = \{\emptyset, A, B, C, A^c, B^c, C^c, \Omega\}$$

is the sub-σ-field generated by A and B, that is, $\mathcal{B}$ is the smallest σ-field containing A and B.

Example 5 Let Y be a random variable on $(\Omega, \mathcal{F}, P)$, i.e., for every Borel set $A \subset \Re$, the set $Y^{-1}(A) \in \mathcal{F}$. Now let $\mathcal{B}$ be the smallest σ-field in $\mathcal{F}$ that contains all events of the form $Y^{-1}(A)$ where A is a Borel set in $\Re$. Then $\mathcal{B}$ is said to be the sub-σ-field generated by Y.

To see how we can generate a sub-σ-field from the inverse images of random variables, notice that if $Y = \chi_A$ is the characteristic function of an event A, then $\mathcal{B}$ is $\{\emptyset, A, A^c, \Omega\}$. If $Y = \alpha\chi_A + \beta\chi_B$, $0 < \alpha < \beta$ and $A \bigcap B = \emptyset$, then

$$\mathcal{B} = \{\emptyset, A, B, C, A^c, B^c, C^c, \Omega\}.$$

To define conditional expectation, let $\mathcal{B}$ be the sub-σ-field of $\mathcal{F}$ and let $X \in L_1(\Omega, \mathcal{F}, P)$. Then define the real-valued function $\nu(B)$ by

$$\nu(B) = \int_B X(\omega)dP(\omega), \quad B \in \mathcal{B}.$$

We need to show that ν is a measure on $\mathcal{B}$ such that it is absolutely continuous with respect to the restriction of the underlying probability measure P to $\mathcal{B}$, $P^{\mathcal{B}}$.[1] Let $P^{\mathcal{B}}$ be the restriction of P to $\mathcal{B}$ or $P^{\mathcal{B}}$ is the probability measure which assigns probability to events $B \in \mathcal{B}$. Then it can be shown by the Radon-Nykodym Theorem that there exists a unique random variable $E^{\mathcal{B}}(X)$ that is measurable with respect to $\mathcal{B}$ and which satisfies

$$\nu(B) = \int_B E^{\mathcal{B}}[X](\omega)P^{\mathcal{B}}(\omega) = \int_B X(\omega)P(d\omega).$$

The random variable $E^{\mathcal{B}}(X)$ is called the *conditional expectation* of X with respect to $\mathcal{B}$. If $\mathcal{B}$ is the σ-field generated by the random variable Y, then we shall denote $E^{\mathcal{B}}[X]$ by $E^Y[X]$ and call it the conditional expectation of X with respect to Y.

Example 6 If $\mathcal{B} = \{\emptyset, \Omega\}$, then $E^{\mathcal{B}}(X) = E(X)$, i.e., $E^{\mathcal{B}}$ maps X onto the constant function $E(X)$.

Example 7 If $\mathcal{B} = \mathcal{F}$, then $E^{\mathcal{B}}[X] = X$.

Example 8 If $\mathcal{B} = \{\emptyset, A, A^c, \Omega\}$ and if $0 < P(A) < 1$, then

$$\begin{aligned} E^{\mathcal{B}}[X](\omega_0) &= \frac{1}{P(A)} \int_A X(\omega P(d\omega) \text{ if } \omega_0 \in A \\ &= \frac{1}{P(A)} \int_{A^c} X(\omega)P(d\omega) \text{ if } \omega_0 \in A^c. \end{aligned}$$

Example 9 Let $X, Y \in L_2(\Omega, \mathcal{F}, P)$ and assume that X is measurable with respect to $\mathcal{B}$. Then

$$E^{\mathcal{B}}[XY] = XE^{\mathcal{B}}[Y].$$

[1]See, for example, Section 14, Appendix D, Naylor and Sell, *Linear Operator Theory in Engineering and Science*, New York: Springer Verlag.

Example 10 Let $X, Y \in L_2(\Omega, \mathcal{F}, P)$. Then

$$E^{\mathcal{B}}\left[E^{\mathcal{B}}[X]Y\right] = E^{\mathcal{B}}[X]E^{\mathcal{B}}[Y].$$

Example 11 $E^{\mathcal{B}}$ is a linear operator on $L_1(\Omega, \mathcal{F}, P)$.

<u>Stochastic Processes</u>
A stochastic process is a family or collection of random variables $X_t(\omega)$ where t is an element of some index set T. If $t = 0, 1, 2, \ldots$ we have a discrete stochastic process. The distribution for stochastic processes refers to distributions over a finite number of the elements of the stochastic process.

Example 1 Let $\{\epsilon_t(\omega)\}_{t=0}^{\infty}$ be a sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. Then $\{\epsilon_t\}_{t=0}^{\infty}$ is defined to be a sequence of identically and independently distributed random variables if and only if

$$Pr(\epsilon_t < a_0, \ldots, \epsilon_{t+k} < a_k) = Pr(e_t < a_0) \cdots Pr(\epsilon_{t+k} < a_k).$$

Example 2 $\{\epsilon_t\}_{t=0}^{\infty}$ is said to be a stationary stochastic process if and only if

$$Pr(\epsilon_{t+h} < a_0, \ldots, \epsilon_{t+h+k} < a_k) = Pr(\epsilon_t < a_0, \ldots, \epsilon_{t+k} < a_k)$$

for finite h and k.

For t in the interval I, let $X(t, \omega)$ be a random variable with finite second moment defined on $(\Omega, \mathcal{F}, P)$ for all t, i.e.,

$$E\left[|X(t,\cdot)|^2\right] = \int_{\Omega} |X(t,\omega)|^2 P(d\omega) < \infty.$$

Since $X(t, \omega)$ has a finite second moment, it also has a finite first moment or expected value, i.e.,

$$E\left(|X(t,\cdot)|\right) = \int_{\Omega} |X(t,\omega)| P(d\omega) < \infty,$$

which implies that

$$E\left(X(t,\cdot)\right) = \int_{\Omega} X(t,\omega) P(d\omega)$$

is finite and well-defined. Hence, we can define the variance of $X(t, \omega)$ as

$$E\left(|X(t,\cdot) - E\left[X(t,\cdot)\right]|^2\right) = \int_{\Omega} |X(t,\cdot) - E\left[X(t,\cdot)\right]|^2 P(d\omega).$$

We can also define the covariance between $X(t, \omega$ and $X(t+s, \omega)$ as

$$Cov\left[X(t,\cdot), X(t+s,\cdot)\right] = E\left[(X(t,\cdot) - E[X(t,\cdot)])\left(X(t+s,\cdot) - E[X(t+s,\cdot)]\right)\right].$$

A realization from some stochastic process $\{\epsilon\}_{t\in T}$ is a time series if the index set T varies over time.

We may have stochastic processes which are defined over individuals. This is termed a *cross section*. We can also have stochastic processes which are defined over individuals and time. In this case, the underlying probability space is defined in terms of a sample space which is the product space of two underlying probability spaces, i.e., $\Omega = \Omega_1 \times \Omega_2$. and

$$\{\epsilon_{tn}(\omega_1, \omega_2)\}_{t \in T, n \in N}.$$

The realization from this stochastic process is termed a *panel data set* for a given number of individuals over a given time span.

3 Some Additional Methods

In this section, we will consider the solution of linear difference equations and the solution of dynamic linear-quadratic (LQ) optimization problems under uncertainty. LQ problems have the property that the optimizing sequence can be described as a linear stochastic difference equation. Furthermore, such problems satisfy the property of *certainty equivalence* in that the solution to the problem under uncertainty is equivalent to finding the solution to the dynamic optimization problem in which all exogenous random variables are replaced by their expectations. An original treatment of some of these issues may be found in Sargent (1979).

3.1 Solution to Linear Stochastic Difference Equations

Consider a general stochastic process $\{x_t\}_{t=0}^{\infty}$. If $\{x_t\}_{t=0}^{\infty}$ is a covariance stationary stochastic process with $E(x_t) = 0$, then the Wold representation theorem says that it can be represented as

$$x_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j} + \eta_t, \tag{3.1}$$

where $c_0 = 1$, $\sum_{j=0}^{\infty} c_j^2 < \infty$, $E(\epsilon_t^2) = \sigma^2 > 0$, $E(\epsilon_t \epsilon_s) = 0$ for $t \neq s$, $E(\epsilon_t) = 0$, and $E(\eta_t \epsilon_s) = 0$ for all t and s. Here η_t is a process which can be predicted arbitrarily well by a linear function of only past values of x_t so that η_t is linearly deterministic. Also, $\epsilon_t = x_t - \hat{E}[x_t | x_{t-1}, x_{t-2}, \ldots]$, where $\hat{E}(\cdot)$ denotes the best linear predictor of x_t on its past.

We wish to solve the expectational difference equation

$$y_t = \lambda \hat{E}(y_{t+1} | x_t, x_{t-1}, \ldots,) + x_t. \tag{3.2}$$

We will seek a solution of the form

$$y_t = \sum_{j=0}^{\infty} b_j \epsilon_{t-j} \quad \text{with} \quad \sum_{j=0}^{\infty} b_j^2 < \infty. \tag{3.3}$$

Notice that this is equivalent to seeking solutions to the difference equation which are stochastic processes measurable with respect to the information set generated by current and past values of x_t and which have finite second moments, i.e.,

$$E(y_t^2) = E\sum_{j=0}^{\infty} b_j\epsilon_{t-j} = \sigma^2\sum_{j=0}^{\infty} b_j^2,$$

and

$$\begin{aligned} E(y_t y_{t+h}) &= E\left\{\left[\sum_{j=0}^{\infty} b_j\epsilon_{t-j}\right]\left[\sum_{j=0}^{\infty} b_j\epsilon_{t-j+h}\right]\right\} \\ &= E\left\{[b_0\epsilon_t + b_1\epsilon_{t-1} + \ldots][b_0\epsilon_{t+h} + b_1\epsilon_{t+h-1} + \ldots]\right\} \\ &= b_0 b_h\sigma^2 + b_1 b_{h+1}\sigma^2 + \ldots \\ &= \sigma^2\sum_{j=0}^{\infty} b_j b_{j+h} \le \left(\sum_{j=0}^{\infty} b_j\right)\left(\sum_{j=0}^{\infty} b_{j+h}^2\right) < \infty, \end{aligned}$$

where the last result follows by the Schwartz Inequality.

Hence, our solution will come from the space of square summable sequences (l_2, d_2) with metric

$$d_2(f,g) = \left(\sum_{j=0}^{\infty}(f_j - g_j)^2\right)^{1/2} \quad \text{for } f, g \in l_2.$$

To define an operator from the right-hand side of (3.2), notice that

$$y_{t+1} = b_0\epsilon_{t+1} + b_1\epsilon_t + b_2\epsilon_{t-1} + \ldots$$

and

$$\hat{E}(y_{t+1}|x_t, x_{t-1}, \ldots,) = b_1\epsilon_t + b_2\epsilon_{t-1} + \ldots.$$

Substituting this result into the right-hand side of (3.2) yields

$$\lambda(b_1\epsilon_t + b_2\epsilon_{t-1} + b_3\epsilon_{t-2} + \ldots) + (c_0\epsilon_t + c_1\epsilon_{t-1} + \ldots).$$

But

$$y_t = b_0\epsilon_t + b_1\epsilon_{t-1} + b_2\epsilon_{t-2} + \ldots.$$

Therefore, we obtain an operator $T : (l_2, d_2) \to (l_2, d_2)$ such that the j'th element of T is given by

$$\left[T(b^1)\right]_j = \lambda b_{j+1}^0 + c_j, \quad c \ge 0. \tag{3.4}$$

We can show that there exists a unique sequence b^* such that $T(b^*) = b^*$ because this equation defines a contraction on a complete metric space.

To obtain the $b^* = \{b_j^*\}_{j=0}^{\infty}$ sequence, start with the zero sequence. i.e., $\{b_j^0\}_{j=0}^{\infty} = \{0\}_{j=0}^{\infty}$. Then

$$\begin{aligned}
b_j^1 &= c_j \\
b_j^2 &= \lambda b_{j+1}^1 + c_j \\
&= \lambda c_{j+1} + c_j \\
b_j^2 &= \lambda b_{j+1}^2 + c_j \\
&= \lambda^2 c_{j+2} + \lambda c_{j+1} + c_j \\
\vdots &= \vdots \\
b_j^n &= \lambda^{n-1} c_{j+n-1} + \lambda^{n-2} c_{j+n-2} + \ldots + c_j,
\end{aligned}$$

or

$$\lim_{n\to\infty} b_j^* = c_j + \lambda c_{j+1} + \lambda^2 c_{j+2} + \ldots$$

More generally,

$$\begin{aligned}
b(L) &= \sum_{j=0}^{\infty} (c_j + c_{j+1}\lambda + c_{j+2}\lambda^2 + \ldots) L^j \\
&= \sum_{j=0}^{\infty} c_j L^j + \lambda \sum_{j=0}^{\infty} c_{j+1} L^j + \lambda^2 \sum_{j=0}^{\infty} c_{j+2} L^j + \ldots \\
&= c(L) + \lambda \left[\frac{c(L)}{L} - \frac{c_0}{L}\right] + \lambda^2 \left[\frac{c(L)}{L^2} - \frac{c_0 + c_1 L}{L^2}\right] \\
&\quad + \lambda^3 \left[\frac{c(L)}{L^3} - \frac{c_0 + c_1 L + c_2 L^2}{L^3}\right] + \ldots \\
&= \frac{c(L)}{1 - \lambda L^{-1}} - \\
&\quad \left(1 + \lambda L^{-1} + \lambda^2 L^{-2} + \ldots\right) \lambda L^{-1} \left(c_0 + c_1 \lambda + c_2 \lambda^2 + \ldots\right),
\end{aligned}$$

which implies

$$b(L) = \frac{c(L) - \lambda L^{-1} c(\lambda)}{1 - \lambda L^{-1}}. \tag{3.5}$$

Suppose x_t has an underlying autoregressive representation given by

$$a(L) x_t = \epsilon_t,$$

with $a(L)c(L) = I$ and $\sum_{j=0}^{\infty} a_j^2 < \infty$. Then we can express the solution as

$$y_t = \frac{1 - \lambda L^{-1} a(\lambda)^{-1} a(L)}{1 - \lambda L^{-1}} x_t. \tag{3.6}$$

3.2 Linear-Quadratic Optimization Problems under Certainty

Before we introduce uncertainty, we will study some infinite horizon dynamic optimization problems under certainty. This will help in the analysis of examples which we will consider later.

Consider the problem of selecting a sequence $\{y_t\}_{t=0}^{\infty}$ to maximize

$$J\left(\{y_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \left\{-(f/2)y_t^2 - (d/2)(y_t - y_{t-1})^2 + y_t x_t\right\}, \quad f > 0, d > 0$$

subject to $y_{-1} = \bar{y}_1$ given. Assume that $\{x_t\}_{t=0}^{\infty}$ is a given sequence in $(l_2[0,\infty), d_2)$. We will maximize this objective function with respect to sequences which are elements of $\Re^{\infty}$, i.e., $y_t \in \Re$ for all t.

First, we note that if there is a sequence $\{\tilde{y}_t\}_{t=0}^{\infty}$ which solves the above problem, then it must lie in the space $(l_2[0,\infty), d_2)$. Suppose the contrary, that $\tilde{y}$ is not an element of (l_2, d_2). Then

$$\sum_{t=0}^{n} \tilde{y}_t^2 \to \infty \ \text{ as } \ n \to \infty,$$

which implies that

$$-(f/2)\sum_{t=0}^{n} \tilde{y}_t^2 \to -\infty \ \text{ as } \ n \to \infty.$$

But if $\sum_{t=0}^{\infty} y_t^2$ diverges with n, then it can be established that for some $\epsilon > 0$,

$$\sum_{t=0}^{\infty} y_t x_y \leq \left(\sum_{t=0}^{\infty} x_t^2\right)\left(\sum_{t=0}^{\infty} y_t^2\right)$$

does not diverge to $+\infty$ fast enough to prevent the value of the objective function from going to $-\infty$, because $\{x_t\}_{t=0}^{\infty} \in (l_2[0,\infty), d_2)$. Therefore, we seek solutions to the problem which are elements of $(l_2[0,\infty), d_2)$.

Differentiating with respect to y_t for $r = 0, 1, \ldots,$ and equating to zero to obtain the Euler equations:

$$-dy_{t+1} + (f_1 + d_1)y_t - dy_{t-1} = x_t, \quad t \geq 0. \tag{3.1}$$

Collecting terms, we have

$$\left(-d + (f + 2d)L - dL^2\right) y_{t+1} = x_t,$$

or

$$\left(1 - \frac{f + 2d}{d}L + L^2\right) y_{t+1} = -\frac{1}{d}x_t, \tag{3.2}$$

We can factor the characteristic polynomial as

$$\begin{aligned}\left(1-\frac{f+2d}{d}L+L^2\right) &= (1-\lambda_1 L)(1-\lambda_2 L)\\ &= 1-(\lambda_1+\lambda_2)L+\lambda_1\lambda_2 L^2,\end{aligned}$$

where

$$\lambda_1+\lambda_2 = \frac{f+2d}{d} \tag{3.3}$$

$$\lambda_1\lambda_2 = 1, \tag{3.4}$$

which implies that

$$\lambda_1+\frac{1}{\lambda_1}=\frac{f+2d}{d}. \tag{3.5}$$

Solving

$$\min \lambda_1+\frac{1}{\lambda_1} \quad\Rightarrow\quad 1-\frac{1}{\lambda^2}=0,$$

which yields $\lambda_1 = \pm 1$. For $\lambda = 1$, we have that

$$\lambda_1+\frac{1}{\lambda_1}=2.$$

From (3.5), we see that when $f = 0$, $(f + 2d)/d = 2$, implying that the solution is $\lambda_1 = \lambda_2 = 1$. When $f > 0$, we have that $(f + 2d)/d > 2$, and $\lambda_1 < 1$ and $\lambda_2 > 1$. But $\lambda_2 = 1/\lambda_1$. Therefore, the solution is

$$(1-\lambda_1 L)(1-\lambda_2 L)=(1-\lambda L)(1-\lambda^{-1}L). \tag{3.6}$$

Therefore, we can write

$$\begin{aligned}\left(1-\frac{f+2d}{d}L+L^2\right)y_{t+1} &= -\frac{1}{d}x_t\\ (1-\lambda L)(1-\lambda^{-1}L)y_{t+1} &= -\frac{1}{d}x_t\\ (1-\lambda L)(1-\lambda L^{-1})(\lambda^{-1}L)y_{t+1} &= -\frac{1}{d}x_t\\ (1-\lambda L)(1-\lambda L^{-1})y_t &= \frac{\lambda}{d}x_t\\ (1-\lambda L^{-1})y_t^* &= \frac{\lambda}{d}x_t, \end{aligned} \tag{3.7}$$

where $y_t^* = (1-\lambda L)y_t$. We note that $y^* = \{y_t^*\}_{t=0}^{\infty} \in (l_2, d_2)$ whenever $y = \{y_t\}_{t=0}^{\infty} \in (l_2, d_2)$. In other words,

$$\sum_{t=0}^{\infty}(y_t^*)^2=\sum_{t=0}^{\infty}(y_t-\lambda y_{t-1})^2 \equiv d_2(y,y'),$$

where $y = \{y_t\}_{t=0}^{\infty}$ and $y' = \{\lambda y_{t-1}\}_{t=0}^{\infty}$. But

$$\begin{aligned} d_2(y, y') &\leq d_2(0, y) + d_2(0, y') \\ &= \sum_{t=0}^{\infty} y_t^2 + \lambda^2 \sum_{t=0}^{\infty} y_{t-1}^2. \end{aligned}$$

But if $\sum_{t=0}^{\infty} y_t^2 < \infty$, then $\sum_{t=0}^{\infty} y_{t-1}^2 = \sum_{t=1}^{\infty} y_t^2 - y_{-1}^2 < \infty$. Then the Euler equation can be represented as

$$y_t^* = \lambda y_{t+1}^* + \frac{\lambda}{d} x_t, \quad |\lambda| < 1. \tag{3.8}$$

But we know that difference equation defines a contraction mapping on the complete metric space (l_2, d_2). Therefore, a unique fixed point of the form

$$y_t^* = \frac{\lambda}{d} \sum_{j=0}^{\infty} \lambda^j x_{t+j} \tag{3.9}$$

exists. Hence, the optimizing sequence is defined as

$$y_t = \lambda y_{t-1} + y_t^*. \tag{3.10}$$

Thus, we obtain the feedback-feedforward form of the solution

$$y_t = \lambda y_{t-1} + \frac{\lambda}{d} \sum_{i=0}^{\infty} \lambda^i x_{t+i}, \quad t \geq 0. \tag{3.11}$$

Suppose we add the requirement that the $\{x_t\}_{t=0}^{\infty}$ sequence is described by the difference equation

$$A(L)x_t = 0,$$

or

$$x_t = \sum_{j=1}^{\infty} a_j x_{t-j} \text{ where } \sum_{j=0}^{\infty} a_j^2 < \infty, a_0 = -1. \tag{3.12}$$

Recall that the Euler equation can be expressed as in (3.8). We have already showed the existence of a fixed point in (3.9). Suppose we seek an alternative representation for y_t^* of the form

$$y_t^* = \sum_{j=0}^{\infty} b_j x_{t-j}, \text{ where } \sum_{j=0}^{\infty} b_j^2 < \infty. \tag{3.13}$$

To obtain this representation, notice that the Euler equation maps sequences $b^n = \{b_j^n\}_{j=0}^{\infty}$ into sequences $\{b_j^{n+1}\}_{j=0}^{\infty}$:

$$\sum_{j=0}^{\infty} b_j^{n+1} x_{t-j} = \lambda \left(\sum_{j=0}^{\infty} b_j^n x_{t-j+1} \right) + \frac{\lambda}{d} x_t.$$

Substituting for $x_{t+1} = \sum_{j=1}^{\infty} a_j x_{t-j}$ yields

$$\begin{aligned} & b_0^{n+1} x_t + b_1^{n+1} x_{t-1} + b_2^{n+1} x_{t-2} + \ldots = \\ & \quad \frac{\lambda}{d} x_t + \lambda b_0^n (a_1 x_t + a_2 x_{t-1} + \ldots) + \lambda b_1^n x_t + \lambda b_2^n x_{t-1} + \ldots \end{aligned} \tag{3.14}$$

Equating coefficients on x_{t-j} for $j \geq 0$ yields

$$\begin{aligned} b_0^{n+1} &= \lambda b_1^n + \lambda a_1 b_0^n + \frac{\lambda}{d} \\ b_j^{n+1} &= \lambda b_0 a_{j+1} + \lambda b_{j+1}^n, \quad j \geq 1 \end{aligned}$$

Notice that we can define an operator

$$[T(b)]_j = \begin{cases} \lambda b_1^n + \lambda a_1 b_0^n + \frac{\lambda}{d} & j = 1 \\ \lambda b_0 a_{j+1} + \lambda b_{j+1}^n & j \geq 1 \end{cases} \tag{3.15}$$

Once we have verified the existence of a fixed point to this operator, we can solve for $\{b_j\}_{j=0}^{\infty}$ from:

$$\begin{aligned} b_0(1 - a_1 \lambda) &= \frac{\lambda}{d} + \lambda b_1 \\ b_j &= \lambda b_{j+1} + \lambda b_0 a_{j+1}. \end{aligned}$$

The second equation implies

$$b_j = \lambda b_0 \sum_{k=0}^{\infty} \lambda^k a_{k+j+1} = b_0 \sum_{k=0}^{\infty} \lambda^{k+1} a_{j+k+1}, \tag{3.16}$$

and

$$b_1 = b_0 \sum_{k=0}^{\infty} \lambda^{k+1} a_{k+2}, \tag{3.17}$$

To find an alternative expression for b_1, notice that

$$a_1 \lambda + a_2 \lambda^2 + a_4 \lambda^3 + \ldots = \lambda^{-1} - a_1 - \lambda^{-1}[1 - a_1 \lambda - a_2 \lambda^2 - \ldots].$$

If $a(L) = 1 - a_1 L - a_2 L^2 - \ldots$, then

$$b_1 = b_0 \left(\lambda^{-1} - a_1 - \lambda^{-1} a(\lambda) \right).$$

But we also have that

$$b_1 = \frac{b_0(1 - \lambda a_1) - \lambda/d}{\lambda}.$$

Equating these expressions and solving for b_0 yields

$$b_0(1 - \lambda a_1) - \frac{\lambda}{d} = \lambda b_0 \left(\lambda^{-1} - a_1 - \lambda^{-1} a(\lambda) \right),$$

or

$$b_0 = \left(\frac{\lambda}{d}\right) a(\lambda)^{-1}. \tag{3.18}$$

Also, we have that

$$b_j = \left(\frac{\lambda}{d}\right) a(\lambda)^{-1} \sum_{k=0}^{\infty} \lambda^{k+1} a_{k+j+1} \quad j \geq 1. \tag{3.19}$$

In compact form, we have

$$b(L) = \left(\frac{\lambda}{d}\right) a(\lambda)^{-1} \left[1 + \sum_{j=1}^{\infty} \left(\sum_{k=j+1}^{\infty} \lambda^{k-j} a_k \right) L^j \right], \tag{3.20}$$

and

$$y_t = \lambda y_{t-1} + \left(\frac{\lambda}{d}\right) b(L) x_t, \tag{3.21}$$

where $a(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots$. This gives as the solution for y_t in the space of square summable linear combinations of current and lagged x's.

4 Dynamic Land Allocation Model

Let us apply some of these concepts to a substantive problem. We consider an Egyptian farmer who can allocate land between the production of wheat and cotton. The continuous cultivation of one of the crops tends to a deterioration of the land quality, and that yields a substantive dynamic stochastic optimization problem.

Let

$$\begin{aligned} X_{it} &= \text{production of crop } i \text{ at time } t \\ P_{it} &= \text{price the farmer receives for crop } i \text{ at time } t \\ A_{it} &= \text{land allocated at time } t-1 \text{ for the production} \\ & \qquad \text{of crop } i \text{ at time } t \\ \bar{A} &= \text{total cultivated land available at time } t \\ 0 < \beta < 1 &= \text{discount factor} \\ a_{it} &= \text{shock to production of crop } i \text{ at time } t \end{aligned}$$

The production functions for the two crops are described as follows. For the first crop,

$$X_{1t} = \left(f_{1t} + a_{1t} - \frac{g_1}{2} A_{1t}\right) A_{1t} + d_1 \left(1 - \frac{A_{1,t-1}}{\bar{A}} - \frac{A_{1t}}{\bar{A}}\right) A_{1t}. \tag{4.1}$$

Notice that the second term represents the deterioration in land quality. If $\frac{A_{1,t-1}}{\bar{A}} = \frac{A_{1t}}{\bar{A}} = \frac{1}{2}$, then the problem of the farmer becomes a static one. If the fractions of land from current and past period's cotton cultivation are greater than one, then average land productivity is reduced. Hence, the dynamic aspects of the problem come from the features of the production in cotton, the main crop.

The production function of wheat is

$$X_{2t} = (f_2 + a_{2t})A_{2t}. \tag{4.2}$$

At time -1, the farmer maximizes

$$E_{-1} \sum_{t=0}^{\infty} \beta^t \left(X_{1t} + \frac{P_{1t}}{P_{2t}} X_{2t} \right) \tag{4.3}$$

subject to (4.1), (4.2), and the land constraint $A_{1t} + A_{2t} = \bar{A}$, given $A_{1,-1}$, where E_{-1} denotes expectation conditional on information at -1. The farmer takes the joint stochastic process for $\{P_{2t}/P_{1t}, a_{1t}, a_{2t}\}_{t=0}^{\infty}$ as given. The information the farmer has at the decision period t is $\{A_{1,t-1}, A_{2,t-1}, a_{1,t-1}, a_{2,t-1}, P_{2,t-1}/P_{1,t-1}, P_{2,t-2}/P_{1,t-2}, \cdots, S_{t-1}, S_{t-2}\}$. Substituting for X_{1t} and X_{2t}

$$E_{-1} \sum_{t=0}^{\infty} \beta^t \left[(f_1 + a_{1t})A_{1t} - \frac{g_1}{2} A_{1t}^2 + \frac{d_1}{\bar{A}} \left(\bar{A} - A_{1,t-1} - A_{1t} \right) A_{1t} - R_t(A_{1t} - \bar{A}) \right]$$

where

$$R_t = \frac{1}{R_t} \left[P_{2t}(f_2 + a_{2t}) \right. \tag{4.4}$$

denotes the value of the marginal product of land in the production of wheat. Hence, R_t is the return to land in the production of wheat, evaluated in terms of the price of cotton.

Define the vector Z_t as

$$Z_t = (a_{1t}, R_t, S_t)', \tag{4.5}$$

where S_t is an $(n-2)$ vector of other variables which may be jointly distributed with a_{1t} and R_t and which may contain information about these variables. For example, S_t may include taxes, tariffs, prices of other agricultural inputs and exports. We assume that $\{Z_t\}_{t=0}^{\infty}$ is a vector stochastic process. The evolution of each Z_t is determined by the law of motion

$$\delta(L)Z_t = U_t,$$

or

$$(I - \delta_1 L - \delta_2 L^2 - \cdots - \delta_n L^n)Z_t = U_t, \tag{4.6}$$

where δ_i are $n \times n$ matrices for $i = 1, \ldots, n$. Let $I_{t-1} = \{A_{1,t-1}, A_{1,t-2}, \ldots, a_{1,t-1}, a_{1,t-2}, \ldots, R_{t-1}, R_{t-2}, \ldots, S_{t-1}, S_{t-2}\}$ be the information set possessed by the Egyptian farmer at time $t-1$. Since $\{Z_t\}_{t=0}^{\infty}$ is a stochastic process, we know that $Z_t(\omega)$ for $t = 0, 1, 2, \ldots$ are random variables defined on some underlying probability space $(\Omega, \mathcal{F}, P)$. We also know that the collection of random variables in the information set I_{t-1} generates a sub-σ-algebra of events denoted by $\mathcal{F}_{t-1} \subset \ldots \subset \mathcal{F}$. In other words, observing the values of the random variables in I_{t-1} provides us with information about only a subset of the events in $\mathcal{F}$. Then $\{U_t\}_{t=0}^{\infty}$ is defined to be a sequence of innovations in that

$$E(U_t | I_{t-1}) = 0. \tag{4.7}$$

Observing the realizations of the random variables in I_{t-1} provides us with no additional information about the set of underlying events generated by U_t. We further assume that $E(U_t U_t')$ exists and is equal to Σ_t for $t \geq 0$. We argue that the solution to the Egyptian farmer's dynamic land allocation problem will be a set of stochastic processes $\{A_{1t}\}_{t=0}^{\infty}$ and $\{A_{2t}\}_{t=0}^{\infty}$. We will restrict this solution in two ways.

- Each $A_{it}(\omega)$, $t \geq 0$, will be required to be a measurable function with respect to the underlying sub-σ algebra $\mathcal{F}_{t-1}$.
- Each $A_{it}(\omega)$, $t \geq 0$, will be required to be an element of $L_2^{\beta}(\Omega, \mathcal{F}, P)$, i.e., we require the discounted value of the solution to be square integrable:

$$\int_{\Omega} \beta |A_{1t}(\omega)|^2 P(d\omega). \tag{4.8}$$

By analogy with the deterministic case, this last condition will be a necessary and sufficient condition for $\{A_{1t}\}_{t=0}^{\infty}$ and $\{A_{2t}\}_{t=0}^{\infty}$ to be a solution. Provided such a solution exists, it must satisfy the Euler equations for this problem. Differentiating with respect to A_{1t}, we obtain

$$\begin{aligned} f_1 + E[a_{1t}|I_{t-1}] - g_1 E[A_{1t}|I_{t-1}] + \frac{d_1}{\bar{A}} \left[\bar{A} - A_{1,t-1} - \frac{1}{2} E(A_{1t}|I_{t-1}) \right] \\ - E[R_t|I_{t-1}] - \frac{\beta d_1}{\bar{A}} E[A_{1,t+1}|I_{t-1}] = 0, \quad t \geq 0. \end{aligned} \tag{4.9}$$

Simplifying

$$\begin{aligned} f_1 + d_1 + E\left[(a_{1t} - R_t)|I_{t-1}\right] - \frac{d_1}{\bar{A}} A_{1,t-1} \\ - \left(g_1 + \frac{d_1}{\bar{A}} \right) E[A_{1t}|I_{t-1}] - \frac{\beta d_1}{\bar{A}} E[A_{1,t+1}|I_{t-1}] = 0, \end{aligned}$$

or

$$\begin{aligned} E\left[\frac{\beta d_1}{\bar{A}} A_{1,t+1} + \left(g_1 + \frac{2d_1}{\bar{A}} \right) A_{1t} + \frac{d_1}{\bar{A}} A_{1,t-1} | I_{t-1} \right] \\ = f_1 + d_1 + E[(a_{1t} - R_t)|I_{t-1}]. \end{aligned} \tag{4.10}$$

We will solve this problem using the *certainty equivalence* property of LQ optimization problems. Notice that the characteristic polynomial associated with the deterministic version of this difference equation is

$$\begin{aligned}
&\frac{\beta d_1}{\bar{A}}L^{-1} + \left(g_1 + \frac{2d_1}{\bar{A}}\right) + \frac{d_1}{\bar{A}} = \frac{\beta d_1}{\bar{A}}\left[L^{-1} + \left(\frac{\bar{A}g_1}{\beta d_1} + \frac{2}{\beta}\right) + \frac{1}{\beta}L\right] \\
&= \frac{\beta d_1}{\bar{A}}(L^{-1} - \lambda_1)(1 - \lambda_2 L) \\
&= -\frac{\lambda_1 \beta d_1}{\bar{A}}(1 - \lambda_1^{-1}L^{-1})(1 - \lambda_2 L), \qquad (4.11)
\end{aligned}$$

which implies that

$$\lambda_1 + \lambda_2 = -\left(\frac{\bar{A}g_1}{\beta d_1} + \frac{2}{\beta}\right) \qquad (4.12)$$

$$\lambda_1 \lambda_2 = \frac{1}{\beta}. \qquad (4.13)$$

Thus, $\lambda_2 = 1/\beta\lambda_1$ and we can examine $f(\lambda) = \lambda + 1/\beta\lambda$. Differentiating $f(\lambda)$ with respect to λ yields

$$f'(\lambda) = 1 - \frac{1}{\beta\lambda^2}.$$

The solution is given by

$$\lambda^2 = \frac{1}{\beta} \quad \text{or} \quad \lambda = \pm\sqrt{\frac{1}{\beta}}.$$

Define

$$\phi = -\left(\frac{\bar{A}g_1}{\beta d_1} + \frac{2}{\beta}\right).$$

Notice that

$$\phi < -\frac{2}{\sqrt{\beta}},$$

implying that $|\lambda_1| > 1/\sqrt{\beta}$. Since $0 < \beta < 1$, we also have that

$$-\left(\frac{\bar{A}g_1}{\beta d_1} + \frac{2}{\beta}\right) < -\left(1 + \frac{1}{\beta}\right),$$

which implies that

$$|\lambda_1| > \frac{1}{\sqrt{\beta}} > 1 > |\lambda_2|. \qquad (4.14)$$

Thus, we can write

$$\frac{-\beta d_1 \lambda_1}{\bar{A}}(1 - \lambda_1^{-1}L^{-1})(1 - \lambda_2 L)A_{1t} = f_1 + d_1 + a_{1t} - R_t,$$

which implies that

$$A_{1t} = \lambda_2 A_{1,t-1} - \frac{\bar{A}}{d_1 \beta \lambda_1} \sum_{i=0}^{\infty} (\lambda_1^{-1})^i (f_1 + d_1 + a_{1,t+i} - R_{t+i}).$$

Since $\lambda_2 = 1/(\beta\lambda_1)$, $\lambda_1^{-1} = \beta\lambda_2$. Therefore, we have

$$A_{1t} = \lambda_2 A_{1,t-1} - \frac{\bar{A}\lambda_2}{d_1} \sum_{i=0}^{\infty} (\beta\lambda_2)^i (f_1 + d_1 + a_{1,t+i} - R_{t+i}). \tag{4.15}$$

Thus, the solution to the deterministic version of the difference equation yields the solution for the optimal quantity of land devoted to cotton production as a linear function of last period's land allocated to cotton and the discounted future value of random shocks to cotton production and the return to wheat production. We note that the latter has a negative effect on the optimal quantity of land allocated to cotton production: if the current of future return to wheat production increases – due to changes in the prices of wheat or cotton or to random shocks to the production of wheat – we find that the quantity of land allocated to cotton production declines.

We now return to the stochastic version of the difference equation defined by (4.10). Multiplying this equation by $\beta^{t/2}$ yields

$$\frac{\beta d_1}{\bar{A}} \beta^{t/2} E[A_{1,t+1}|I_{t-1}] + \left(g_1 + \frac{2d_1}{\bar{A}}\right) \beta^{t/2} E[A_{1t}|I_{t-1}] + \frac{d_1}{\bar{A}} \beta^{t/2} E[A_{1,t-1}|I_{t-1}] = E\left[\beta^{t/2}(f_1 + d_1 + (a_{1t} - R_t)|I_{t-1}\right]. \tag{4.16}$$

Now define the new variables

$$\begin{aligned} A^*_{1,t+i} &= E\left[\beta^{(t+1)/2} A_{1,t+1}|I_{t-1}\right] \\ A^*_{1t} &= E\left[\beta^{t/2} A_{1t}|I_{t-1}\right] \\ A^*_{1,t-1} &= E\left[\beta^{(t-1)/2} A_{1,t-1}|I_{t-1}\right] \\ x^*_t &= E\left[\beta^{t/2}(f_1 + d_1 + (a_{1t} - R_t)|I_{t-1}\right]. \end{aligned}$$

Then the Euler equation can be written as

$$\frac{\beta^{1/2} d_1}{\bar{A}} A^*_{1,t+1} + \left(g_1 + \frac{2d_1}{\bar{A}}\right) A^*_{1t} + \frac{d_1 \beta^{-1/2}}{\bar{A}} A^*_{1,t-1} = x^*_t. \tag{4.17}$$

Notice that $A^*_{1,t+1}$, A^*_{1t}, $A^*_{1,t-1}$, and x^*_t are all random variables that are measurable with respect to the information set I_{t-1}. Define B to be the operator such that

$$\begin{aligned} BA^*_{1,t+1} &= BE\left[\beta^{(t+1)/2} A_{1,t+1}|I_{t-1}\right] = E\left[\beta^{t/2} A_{1t}|I+_{t-1}\right] \\ B^{-1}A^*_{1t} &= B^{-1}E\left[\beta^{t/2} A_{1t}|I_{t-1}\right] = E\left[\beta^{(t+1)/2} A_{1,t+1}|I_{t-1}\right]. \end{aligned}$$

Now consider the characteristic polynomial associated with the difference equation relating $A^*_{1,t+1}$, A^*_{1t}, and $A^*_{1,t-1}$ to x^*_t. This characteristic polynomial may be factored as

$$\frac{\beta^{1/2}d_1}{\bar{A}}B^{-1} + \left(g_1 + \frac{2d_1}{\bar{A}}\right) + \frac{d_1\beta^{-1/2}}{\bar{A}}B = \frac{-\beta^{1/2}d_1}{\bar{A}\tilde{\lambda}_1}(1-\tilde{\lambda}_1 B^{-1})(1-\tilde{\lambda}_2 B),$$

which implies

$$(1-\tilde{\lambda}_1 B^{-1})(1-\tilde{\lambda}_2 B)A^*_{1t} = \frac{-\bar{A}\tilde{\lambda}_1}{\beta^{1/2}d_1}x^*_t,$$

or

$$(1-\tilde{\lambda}_1 B^{-1})\tilde{A}_{1t} = \frac{-\bar{A}\tilde{\lambda}_1}{\beta^{1/2}d_1}x^*_t \text{ where } \tilde{A}_{1t} = (1-\tilde{\lambda}_2 B)A^*_{1t},$$

or

$$\tilde{A}_{1t} = \tilde{\lambda}_1 \tilde{A}_{1,t+1} - \frac{\bar{A}\tilde{\lambda}_1}{\beta^{1/2}d_1}x^*_t, \quad |\tilde{\lambda}_1| < 1. \tag{4.18}$$

But this is a functional equation which maps random variables that are elements of $L_2^\beta(\Omega, \mathcal{F}_{t-1}, P_{t-1})$ into itself. It is also a contraction mapping since $|\tilde{\lambda}_1| < 1$. Hence, there exists a unique solution such that

$$\tilde{A}_{1t} = \frac{-\bar{A}\tilde{\lambda}_1}{\beta^{1/2}d_1}\sum_{i=0}^{\infty}\tilde{\lambda}_1 x^*_{t+i}. \tag{4.19}$$

But

$$\begin{aligned}\tilde{A}_{1t} &= E\left[\beta^{t/2}A_{1t}|I_{t-1}\right] - \tilde{\lambda}_1 E\left[\beta^{(t-1)/2}A_{1,t-1}|I_{t-1}\right]\\ x^*_{t+i} &= E\left[\beta^{(t+1)/2}(f_1 + d_1 + a_{1,t+i} - R_{t+i})|I_{t-1}\right].\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&\beta^{t/2}E\left[A_{1t}|I_{t-1}\right] = \tilde{\lambda}_1\beta^{(t-1)/2}E\left[A_{1,t-1}|I_{t-1}\right]\\ &\quad -\frac{\bar{A}\tilde{\lambda}_1\beta^{t/2}}{\beta^{1/2}d_1}\sum_{i=0}^{\infty}(\tilde{\lambda}_1\beta^{1/2})^i\left\{(f_1+d_1) + E\left[(a_{1,t+i} - R_{t+i})|I_{t-1}\right]\right\}.\end{aligned}$$

But $\beta^{(t-1)/2}E\left[A_{1,t-1}|I_{t-1}\right] = \beta^{(t-1)/2}A_{1,t-1}$ since $A_{1,t-1}$ is an element of the information set I_{t-1}. Therefore,

$$\begin{aligned}&E\left[A_{1t}|I_{t-1}\right] = \tilde{\lambda}_1\beta^{-1/2}A_{1,t-1}\\ &\quad -\frac{\bar{A}\tilde{\lambda}_1}{\beta^{1/2}d_1}\sum_{i=0}^{\infty}(\tilde{\lambda}_1\beta^{1/2})^i\left\{(f_1+d_1) + E\left[(a_{1,t+i} - R_{t+i})|I_{t-1}\right]\right\}.\end{aligned}$$

Define λ such that $\lambda\beta^{1/2} = \tilde{\lambda}$. Then, we can write the solution to the stochastic difference equation as

$$\begin{aligned} E\left[A_{1t}|I_{t-1}\right] &= \lambda A_{1,t-1} \\ &\quad -\frac{A\lambda}{d_1}\sum_{i=0}^{\infty}(\lambda\beta)^i \left\{(f_1+d_1)+E\left[(a_{1,t+i}-R_{t+i})|I_{t-1}\right]\right\}. \end{aligned} \tag{4.20}$$

Recall that we chose the initial sequence $\{A_{1t}\}_{t=0}^{\infty}$ to be measurable with respect to the information set I_{t-1}. Therefore, we can replace $E(A_{1t}|I_{t-1}) = A_{1t}$. However, we still have to evaluate $E(a_{1,t+i}|I_{t-1})$ and $E(R_{t+i}|I_{t-1})$. In the model which Eckstein takes to data, he assumes that

$$\begin{bmatrix} a_{1t} \\ R_t \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix} + \begin{bmatrix} \rho & 0 \\ 0 & \alpha_1 \end{bmatrix}\begin{bmatrix} a_{1,t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}, \tag{4.21}$$

where $|\rho| < 1$, $|\alpha_1| < 1$ and $u_t = [u_{1t}, u_{2t}]' \sim N(0, \Sigma)$ for all $t \geq 0$. Given this representation, we can calculate

$$E(a_{1t}|I_{t-1}) = \rho a_{1,t-1} \;\text{ because } E(u_{1t}|I_{t-1}) = 0 \text{ by definition.}$$

Also,

$$E(a_{1,t+1}|I_{t-1}) = \rho E(a_{1t}|I_{t-1}) + E(u_{1,t+1}|I_{t-1}) = \rho^2 a_{1,t-1}.$$

Iterating in this way, we have that

$$E(a_{1,t+k}|I_{t-1}) = \rho^{k+1} a_{1,t-1}.$$

Similarly,

$$E(R_{t+1}|I_{t-1}) = \alpha_0 + \alpha_1 R_{t-1},$$

and

$$\begin{aligned} E(R_{t+1}|I_{t-1}) &= \alpha_0 + \alpha_1 E(R_t|I_{t-1}) \\ &= \alpha_0 + \alpha_0\alpha_1 + \alpha_1^2 R_{t-1}. \end{aligned}$$

Continuing in this way,

$$E(R_{t+i}) = \alpha_0 \sum_{j=0}^{i} \alpha_1^j + \alpha_1^{i+1} R_{t-1}.$$

Substituting into the formula for A_{1t} yields

$$A_{1t} = \lambda A_{1,t-1} - \frac{\bar{A}\lambda}{d_1}\sum_{i=0}^{\infty}(\lambda\beta)^i \left(\alpha_0 \sum_{j=0}^{i} \alpha_1^j - \alpha_1^{i+1} R_{t-1} + \rho^{i+1} a_{1,t-1}\right).$$

Collecting terms and simplifying yields

$$\begin{aligned}
A_{1t} &= \lambda A_{1,t-1} - \frac{\bar{A}\lambda\alpha_0}{d_1}\sum_{i=0}^{\infty}(\lambda\beta)^i\left(\sum_{j=0}^{i}\alpha_1^j\right) \\
&\quad + \frac{\bar{A}\lambda\alpha_1}{d_1}\sum_{i=0}^{\infty}(\lambda\beta\alpha_1)^i R_{t-1} - \frac{\bar{A}\lambda\rho}{d_1}\sum_{i=0}^{\infty}(\lambda\beta\rho)^i a_{1,t-1} \\
&= \lambda A_{1,t-1} - \frac{\bar{A}\lambda\alpha_0}{d_1}\sum_{i=0}^{\infty}(\lambda\beta)^i\left(\sum_{j=0}^{i}\alpha_1^j\right) \\
&\quad + \frac{\bar{A}\lambda\alpha-1}{d_1(1-\lambda\beta\alpha_1)}R_{t-1} - \frac{\bar{A}\lambda\rho}{d_1(1-\lambda\beta\rho)}a_{1,t-1}. \qquad (4.22)
\end{aligned}$$

Let

$$\begin{aligned}
\mu_0 &= -\frac{\bar{A}\lambda\alpha_0}{d_1}\sum_{i=0}^{\infty}(\lambda\beta)^i\left(\sum_{j=0}^{i}\alpha_1^j\right) \\
\mu_1 &= \frac{\bar{A}\lambda\alpha-1}{d_1(1-\lambda\beta\alpha_1)} \\
\mu_2 &= -\frac{\bar{A}\lambda\rho}{d_1(1-\lambda\beta\rho)}
\end{aligned}$$

Using this notation, we can the solution as

$$A_{1t} = \lambda A_{1,t-1} + \mu_0 + \mu_1 R_{t-1} + \mu_2 a_{1,t-1}.$$

But $A_{1,t-1} = \rho a_{1,t-2} + u_{1,t-1}$, which implies that

$$A_{1t} = \lambda A_{1,t-1} + \mu_0 + \mu_1 R_{t-1} + \mu_2\rho a_{1,t-2} + \mu_2 u_{1,t-1}.$$

From the solution, we also have that

$$\mu_1\rho a_{1,t-2} = \rho A_{1,t-1} - \rho\lambda A_{1,t-2} - \rho\mu_0 - \rho\mu_1 R_{t-2}.$$

Substituting back into the expression for A_{1t}, simplifying, and stacking yields

$$\begin{aligned}
\begin{bmatrix} A_{1t} \\ R_t \end{bmatrix} &= \begin{bmatrix} \mu_0(1-\rho) \\ \alpha_0 \end{bmatrix} + \begin{bmatrix} \lambda+\rho & \mu_1 \\ 0 & \alpha_1 \end{bmatrix}\begin{bmatrix} A_{1,t-1} \\ R_{t-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} -\rho\lambda & \rho\mu_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_{1,t-2} \\ R_{t-2} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}, \qquad (4.23)
\end{aligned}$$

where $\epsilon_{1t} = \mu_2 u_{1,t-1}$ and $\epsilon_{2t} = u_{2t}$.

These equations display the rational expectations cross-equation restrictions between the endogenously determined land allocation variable A_{1t} and the exogenous process for R_t. In particular, the coefficients μ_i for $i = 0, 1, 2$

display the effect of expectations of future values of R_t and a_{1t} on the optimal choice for A_{1t}.

Is $\{A_{1t}\}_{t=0}^{\infty}$ a covariance stationary process? Since $|\lambda| < 1$, we can write

$$A_{1t} = \frac{\mu_0}{1-\lambda} + \mu_1 \sum_{i=0}^{\infty} \lambda^i a_{1,t-1-i} + \mu_2 \sum_{i=0}^{\infty} \lambda^i R_{t-1-i}. \tag{4.24}$$

Using this representation and the fact that $E(a_{1t}) = 0$ for all t and $E(R_t) = \alpha_0/(1-\alpha_1)$, we find that

$$E(A_{1t}) = \frac{\mu_0}{1-\lambda} + \frac{\mu_2\alpha_0}{(1-\lambda)(1-\alpha_1)}.$$

Likewise,

$$\begin{aligned}
Var(A_{1t}) &= E\left[A_{1t} - E(A_{1t})\right]^2 \\
&= E\left[\mu_1 \sum_{i=0}^{\infty} \lambda^i a_{1,t-1-i} + \mu_2 \sum_{i=0}^{\infty} \lambda^i R_{t-1-i} - \frac{\mu_2\alpha_0}{(1-\alpha_1)(1-\lambda)}\right]^2 \\
&= \mu_1^2 \sum_{i=0}^{\infty} \lambda^{2i} E(a_{1,t-1-i}^2) + \mu_2^2 \sum_{i=0}^{\infty} \lambda^{2i} E\left(R_{t-1-i} - \frac{\alpha_0}{1-\alpha_1}\right)^2 \\
&\quad + 2\mu_1\mu_2 \sum_{i=0}^{\infty} \lambda^{2i} E\left[a_{1,t-1-i}\left(R_{t-1-i} - \frac{\alpha_0}{1-\alpha_1}\right)\right].
\end{aligned}$$

Now

$$\begin{aligned}
E(a_{1,t-1-i}^2) &= Var(a_{1,t-1-i}) = Var\left[\sum_{i=0}^{\infty} \rho^i u_{1,t-i}\right] = \frac{\sigma_1^2}{1-\rho^2}, \\
E\left[R_{t-1-i} - \frac{\alpha_0}{1-\alpha_1}\right]^2 &= Var(R_{t-1-i}) = E\left[\sum_{i=0}^{\infty} \alpha_1^i u_{2,t-i}\right] = \frac{\sigma_2^2}{1-\alpha_1^2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&E\left[a_{1,t-1-i}\left(R_{t-1-i} - \frac{\alpha_0}{1-\alpha_1}\right)\right] = Cov\left(a_{1,t-1-i}, R_{t-1-i}\right) \\
&= Cov\left[\left(\sum_{i=0}^{\infty} \rho^i u_{1,t-1-i}\right)\left(\frac{\alpha_0}{1-\alpha_1} + \sum_{j=0}^{\infty} \alpha_1^j u_{2,t-1-i}\right)\right] \\
&= \sum_{i=0}^{\infty} (\rho\alpha_1)^i Cov(u_{1,t-1-i}, u_{2,t-1-i}) \\
&= \frac{\sigma_{12}}{1-\rho\alpha_1}.
\end{aligned}$$

Substituting back into the expression for $Var(A_{1t})$ yields

$$Var(A_{1t}) = \frac{1}{1-\lambda^2}\left[\frac{\mu_1^2\sigma_1^2}{1-\rho^2} + \frac{\mu_2^2\sigma_2^2}{1-\alpha_1^2} + \frac{\mu_1\mu_2\sigma_{12}}{1-\rho\alpha_1}\right].$$

Thus, we find that the first and second (unconditional) moments of A_{1t} are constant and independent of t. We could also calculate $Cov\,(A_{1t}, A_{1,t-k})$ and show that it is independent of t and depends only on k. Thus, $\{A_{1t}\}_{t=0}^{\infty}$ is a covariance-stationary process.

Suppose that we had chosen the exogenous processes to be of mean exponential order less than $1/\sqrt{\beta}$, i.e.,

$$|E_t a_{1,t+j}| < k(x)^{t+j}, \quad |E_t R_{t+j}| < k(x)^{t+j},$$

where $k > 0$ and $1 \leq x < 1/\sqrt{\beta}$. Then the solution for $\{A_{1t}\}_{t=0}^{\infty}$ would also be of mean exponential order less than $1/\sqrt{\beta}$. What are some examples of such processes? Consider the trend process

$$a_{1t} = \alpha_0 g^t + u_{1t}, \quad \text{where } 1 < g < 1/\sqrt{\beta},\ \alpha_0 > 0, \text{ and } E(u_{1t}) = 0. \tag{4.25}$$

Then $|E_t a_{1,t+1}| = \alpha_0 g^{t+1}$.

5 Dynamic Industry Equilibrium

In the dynamic land allocation problem, we talked about the problem of a single decision-maker in a dynamic, stochastic environment. This decision-maker took prices as given and solved a maximization problem. Now we will look at simple models of equilibrium. We start out with models of industry equilibrium.

5.1 An LQ Model of Investment: Quadratic Adjustment Costs

We will use a simple investment model to discuss these issues. Suppose that there are N identical firms in the industry. Let output by a single firm be given by

$$y_t = f_0 k_t. \tag{5.1}$$

Let p_t be the price of output. Assume that firms buy capital goods in period t and sell off their capital carried oyer from the previous period at the price J_t. Let $(1/2)(K_t - K_{t-1})$ be the real cost of adjusting the capital stock. The firm is assumed to maximize the discounted value of net cash flows at time 0:

$$\max_{\{K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (p_t f_0 k_t - J_t(k_t - k_{t-1}) - \frac{d}{2}(k_t - k_{t-1})^2, \tag{5.2}$$

given k_{-1} and the price sequence $\{J_t\}_{t=0}^{\infty}$. The firm also takes the price of industry output as given, i.e., it treats $\{p_t\}_{t=0}^{\infty}$ as a given sequence. However, the price of industry output is now determined by the demand for output, i.e., by the inverse demand curve:

$$p_t = A_0 - A_1 Y_t + U_t, \tag{5.3}$$

where $\{U_t\}_{t=0}^{\infty}$ is a known sequence of demand shifters and $Y_t = Nf_0k_t = f_0K_t$. Assume that $\{J_t\}_{t=0}^{\infty}$ and $\{U_t\}_{t=0}^{\infty}$ are elements of l_2^{β}. Show for yourself that the optimal capital stock sequence for the firm will be given by

$$(1-L)k_{t+1} = \frac{-d^{-1}}{1-\beta L^{-1}}\left(J_{t+1} - bJ_{t+2} + p_{t+1}f_0\right), \ \ t \geq 0. \tag{5.4}$$

This capital stock sequence will itself be an element of l_2^{β} and it will satisfy the Euler equations:

$$p_t f_0 - J_t + \beta J_{t+1} - d(k_t - k_{t-1}) + d\beta(k_{t+1} - k_t), \ \ t \geq 0. \tag{5.5}$$

Notice that the firm takes the price of industry output as given but its decision influences the price of output. This simultaneity must be resolved in equilibrium.

An *industry equilibrium* is the pair of sequence $\{p_t\}_{t==}^{\infty}$ and $\{K_t\}_{t=0}^{\infty}$ such that

(i) given the representative firm's optimal capital stock sequence, prices $\{p_t\}_{t=0}^{\infty}$ clear the market:

$$p_t = A_0 - A_1 Y_t + U_t = A_0 - A_1 N f_0 k_t + U_t, \tag{5.6}$$

(ii) when the representative firm faces $\{p_t\}_{t=0}^{\infty}$ as a price-taker, the capital stock sequence $\{k_t\}_{t=0}^{\infty}$ defined by (5.4) maximizes the firm's present value.

Substitute for p_t in the Euler equation using the market-clearing condition and multiply by N:

$$\begin{aligned} &NA_0f_0 + Nf_0U_t - NJ_t + N\beta J_{t+1} - \{Nd(1+\beta) + A_1 f_0 N^2\}k_t \\ &\quad + Ndk_{t-1} + Nd\beta k_{t+1} = 0, \ \ t \geq 0. \end{aligned} \tag{5.7}$$

Re-write this equation as

$$\beta K_{t+1} + \phi K_t + K_{t-1} = Nd^{-1}\{J_t - \beta J_{t+1} - f_0 U_t - A_0 f_0\}, \tag{5.8}$$

where

$$\phi = -\left((1+\beta) + \frac{A_1 f_0 N}{d}\right), \tag{5.9}$$

or

$$\beta\left(1 + \frac{\phi}{\beta}L + \frac{1}{\beta}L^2\right)K_{t+1} = \frac{N}{d}\{J_t - \beta J_{t+1} - f_0 U_t - A_0 f_0\}. \tag{5.10}$$

We can factor the characteristic polynomial as

$$\left(1 + \frac{\phi}{\beta}L + \frac{1}{\beta}L^2\right) = (1 - \lambda_1 L)(1 - \lambda_2 L), \tag{5.11}$$

where $\lambda_2 > 1/\beta > 1 > \lambda_1$. We can write the solution as

$$(1 - \lambda_1 L)(1 - \lambda_2 L)K_{t+1} = \frac{N}{d\beta}\{J_t - \beta J_{t+1} - f_0 U_t - A_0 f_0\}. \tag{5.12}$$

To ensure that the solution is an element of l_2^β, i.e. $\sum_{t=0}^{\infty} \beta^t K_{t+1}^2 < \infty$, we solve the root λ_2 forward and the root λ_1 to obtain

$$K_{t+1} = \lambda_1 K_t - \frac{N}{d\beta\lambda_2}\sum_{i=0}^{\infty}(\lambda_2^{-i})\{J_{t+i} - \beta J_{t+i+1} - f_0 U_{t+i} - A_0 f_0\}. \tag{5.13}$$

Since $\lambda_1 = (\beta\lambda_2)^{-1}$, we can write the solution as

$$K_{t+1} = \lambda_1 K_t - \frac{N}{d\lambda_1}\sum_{i=0}^{\infty}(\lambda_2^{-i})\{J_{t+i} - \beta J_{t+i+1} - f_0 U_{t+i} - A_0 f_0\}. \tag{5.14}$$

One interesting that Lucas and Prescott did was to solve for the industry equilibrium indirectly. Instead of proceeding directly to derive (5.11), they used the fact that a competitive equilibrium sequence of prices and capital stocks will maximize the sum of consumer plus producer surplus with respect to K_t.

In the static version of the problem, we would have

$$\begin{aligned} Y &= N f_0 k && \text{output} \\ p &= A_0 - A_1 Y && \text{unit price} \\ c &= eY && \text{unit cost.} \end{aligned}$$

Thus, the problem is

$$\max_Y \int_0^Y (A_0 - A_1 x)dx - ex^2. \tag{5.15}$$

In our model, the area under the demand curve is given by

$$\begin{aligned} \int_0^{Y_t}(A_0 - A_1 X + U_t)dX &= A_0 Y_t - \frac{1}{2}A_1 Y_t^2 + U_t Y_t \\ &= A_0 N f_0 k_t - \frac{1}{2}A_1 N^2 f_0^2 k_t^2 + N f_0 k_t U_t. \end{aligned} \tag{5.16}$$

Total industry costs are

$$N J_t (k_t - k_{t-1})^2 - \frac{1}{2}N(k_t - k_{t-1})^2. \tag{5.17}$$

Therefore, the problem becomes

$$\max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left\{ A_0 N f_0 k_t - \frac{1}{2} A_1 N^2 f_0^2 k_t^2 + N f_0 U_t k_t \right.$$
$$\left. - N J_t (k_t - k_{t-1})^2 + \frac{1}{2} N (k_t - k_{t-1})^2 \right\}. \tag{5.18}$$

The Euler equations for this problem are given by

$$A_0 N f_0 - A_1 f_0^2 N^2 k_t + N f_0 U_t - N J_t + N \beta J_{t+1}$$
$$- dN(k_t - k_{t-1}) + \beta N d(k_{t+1} - k_t) = 0. \tag{5.19}$$

It is straightforward to show that the set of Euler equations can be simplified as in (5.7). Hence, the industry equilibrium capital sequence can be found as the solution of an optimal planning problem, as suggested by Lucas and Prescott.

What happens under *uncertainty*? In this case, $\{k_t\}_{t=0}^{\infty}$ and $\{p_t\}_{t=0}^{\infty}$ are *stochastic processes*. In the linear-quadratic dynamic optimization framework, certainty equivalence holds. Hence, the optimal decision rule under uncertainty is given by

$$(1 - L) k_{t+1} = -d \sum_{i=0}^{\infty} \beta^i E_{t+1} \left(J_{t+1+i} - \beta J_{t+2+i} - f_0 p_{t+1+i} \right). \tag{5.20}$$

Thus, firms decision rules depend on expectations of future price. But price depends on firms' decisions. There exists a *rational expectations equilibrium* if firms use the equilibrium price distribution or sequence to calculate these expectations.

In this case, we do not merely substitute for $p_t = A_0 - A_1 N f_0 k_t + U_t$ in the firm's first-order conditions. Instead, we use the market-clearing condition to calculate $E_{t+1}(p_{t+i})$ for $i \geq 0$, and substitute the expression for this expectation. Equivalently, we can substitute for p_t in the Euler equation for the stochastic problem. In the uncertainty case, the relevant set of Euler equations is

$$f_0 p_t - J_t + \beta E_t(J_{t+1}) - d(k_t - K_{t-1}) + d\beta E_t k_{t+1} - d\beta k_t = 0, \;\; t \geq 0.$$

Substituting for p_t using the market-clearing condition and multiplying by N yields

$$N A_0 f_0 - A_1 N^2 f_0^2 k_t + N f_0 U_t - N J_t + N \beta E_t(J_{t+1})$$
$$- N d(k_t - k_{t-1}) + d N \beta E_t(k_{t+1}) - d N \beta k_t = 0, \;\; t \geq 0. \tag{5.21}$$

Suppose $\{J_t\}_{t=0}^{\infty}$ and $\{U_t\}_{t=0}^{\infty}$ are chosen to be elements of $L_2^{\beta}(\Omega, \mathcal{F}, P)$, i.e.,

$$\int_{\Omega} \beta |J_t(\omega)|^2 P(d\omega) < \infty \;\text{ and }\; \int_{\Omega} \beta |U_t(\omega)|^2 P(d\omega) < \infty.$$

If the equilibrium sequence $\{k_{t+1}\}_{t=0}^{\infty}$ is also equired to be an element of this space and measurable with respect to the information set or sub-σ-algebra generated by $J_{t+1}, J_t, \ldots$ and $U_{t+1}, U_t, \ldots$, then the solution can be represented as

$$\begin{aligned} k_{t+1} &= \lambda_1 k_t - \lambda d^{-1} \sum_{i=0}^{\infty} E_{t+1}\{J_{t+i+1} \\ &\qquad -\beta J_{t+i+2} - f_0 U_{t+i+1} - A_0 f_0\}, \quad t \geq 0, \qquad (5.22) \\ K_t &= N k_t \qquad (5.23) \\ p_t &= A_0 - A_1 N f_0 k_t + U_t. \qquad (5.24) \end{aligned}$$

5.2 Externalities

Suppose that there are externalities in the production of industry output such that the average product of individual firms depends on the total available industry capital stock:

$$\frac{y_t}{k_t} = f_1 + f_2 \frac{K_t}{k_t}, \quad f_1, f_2 > 0. \qquad (5.25)$$

There are N firms in the industry, and each firm faces the inverse demand function

$$p_t = A_0 - A_1 Y_t + U_t, \qquad (5.26)$$

where $Y_t = N y_t$, and $\{U_t\}_{t=0}^{\infty}$ and $\{J_t\}_{t=0}^{\infty}$ are processes that are elements of l_2^{β}.

The firm solves the problem:

$$\max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \{p_t y_t - J_t(k_t - k_{t-1}) - \frac{d}{2}(k_t - k_{t-1})^2\} \qquad (5.27)$$

given k_{-1} and K_{-1}, $\{p_t\}_{t=0}^{\infty}$, $\{J_t\}_{t=0}^{\infty}$, and $\{K_t\}_{t=0}^{\infty}$.

Equilibrium in this industry is a set of sequences $\{\bar{p}\}_{t=0}^{\infty}$, $\{\bar{k}_t\}_{t=0}^{\infty}$, and $\{\bar{K}_t\}_{t=0}^{\infty}$ which satisfy the following three conditions:

(i) Given $\{\bar{p}\}_{t=0}^{\infty}$ and $\{\bar{K}_t\}_{t=0}^{\infty}$, $\{\bar{k}_t\}_{t=0}^{\infty}$ maximizes the present value of the firm;

(ii) the firm's behavior is consistent with the industry-wide capital stock:

$$N\bar{k}_t = \bar{K}_t;$$

(iii) given the firm's optimal sequence $\{\bar{k}_t\}_{t=0}^{\infty}$, the sequence $\{\bar{p}_t\}_{t=0}^{\infty}$ clears the markets, i.e.,

$$\bar{p}_t = A_0 - A_1[f_1 + N f_2]\bar{K}_t + U_t, \quad t \geq 0.$$

The Euler equations for the firm's problem are given by:

$$p_t f_1 - J_t + \beta J_{t+1} - d(k_t - k_{t-1}) + d\beta(k_{t+1} - k_t) = 0. \tag{5.28}$$

Substituting for p_t using the inverse demand function yields

$$\begin{aligned} &d\beta k_{t+1} - d(1+\beta)k_t + dk_{t-1} = \\ &\quad J_t - \beta J_{t+1} - A_0 f_1 + A_1 f_1(f_1 + N f_2)K_t + U_t. \end{aligned} \tag{5.29}$$

Multiply by N to solve for the industry capital stock:

$$\begin{aligned} &d\beta K_{t+1} - d(1+\beta)K_t + dK_{t-1} = \\ &\quad NJ_t - N\beta J_{t+1} - A_0 f_1 N + A_1 f_1 N(f_1 + N f_2)K_t + NU_t, \end{aligned} \tag{5.30}$$

which yields

$$\begin{aligned} &d\beta K_{t+1} - [d(1+\beta) + A_1 f_1 N(f_1 + N f_2)] K_t + dK_{t-1} = \\ &\quad N(J_t - \beta J_{t+1} - A_0 f_1 + U_t). \end{aligned} \tag{5.31}$$

What is the *social planner's problem* for this economy? This is given by

$$\max_{\{k_t\}_{t=0}^{\infty}} \int_0^{Y_t} (A_0 - A_1 x + U_t)dx - J_t(K_t - K_{t-1}) - \frac{d}{2n}(K_t - K_{t-1})^2 \tag{5.32}$$

s.t.

$$K = Nk_t \text{ and } Y_t = N(f_1 k_t + N f_2 k_t). \tag{5.33}$$

The Euler equations are given by:

$$\begin{aligned} &A_0 f_1 + A_0 N f_2 - A_1(f_1 + N f_2)^2 K_t + (f_1 + N f_2)U_t - J_t + \beta J_{t+1} \\ &\quad -\frac{d}{N}(K_t - K_{t-1})^2 + \frac{d\beta}{N}(K_{t+1} - K_t = 0, \end{aligned} \tag{5.34}$$

which can be simplified as

$$\begin{aligned} &d\beta K_{t+1} - \left[d(1+\beta) + A_1 N(f_1 + N f_2)^2\right] K_t + dK_{t-1} = \\ &\quad N(J_t - \beta J_{t+1} - (A_0 f_1 - U_t)(f_1 + N f_2)]. \end{aligned} \tag{5.35}$$

Comparing the Euler equations for the social planner's problem with the Euler equations for the industry-wide equilibrium stock, we note that they differ with respect to the coefficient on K_t. In particular we can factor the characteristic polynomial associated with the difference equation for each of these capital stocks as

$$\begin{aligned} &\beta L^{-1} - \phi + L = \\ &\quad d\beta\lambda_1 L^{-1}(1 - \lambda_1 L^{-1})(1 - \lambda_2 L), \end{aligned}$$

where

$$\begin{aligned} \phi &= -\left((1+\beta) + \frac{A_1 N(f_1 f_1 + N f_2)}{d}\right) \text{ industry equilibrium} \\ \tilde{\phi} &= -\left((1+\beta) + \frac{A_1 N(f_1 + N f_2)^2}{d}\right) \text{ social planner's problem.} \end{aligned}$$

Notice that $|\tilde{\phi}| > |\phi|$, which implies that $\tilde{\lambda}_2 > \lambda_2 > 1/\beta > 1 > \lambda_1 > \tilde{\lambda}_1$.

5.3 Dynamic Optimal Taxation

We now present a set of results with investment and capital income taxation. We consider the problem of a government which acts as a dominant player and sets tax rates to finance a given sequence of government expenditures subject to the behavior of the private sector. The government maximizes

the sum of consumer and producer surplus for a given industry by choosing a sequence of tax rates $\{\tau_t\}_{t=0}^{\infty}$ and the industry capital stock $\{K_t\}_{t=0}^{\infty}$ subject to the optimizing rues of firms in this industry and the sequence of government budget constraints. We initially assume that there is no tax on initial capital.

No Tax on Initial Capital.
Assume that the government receives the capital income tax

$$T_t = \tau_t N k_t, \tag{5.36}$$

and faces the sequence of budget constraints

$$bB_{t+1} - (g_t + B_t - \tau_t N k_t), \tag{5.37}$$

where $0 < b < 1$ is a discount factor, B_t is the existing stock of debt, B_{t+1} is the future stock of debt, and g_t is current government expenditures. Each firm produces output according to the production function

$$y_t = f_0 k_t, \quad f_0 > 0, \tag{5.38}$$

and faces the inverse demand function

$$p_t = A_0 - A_1 Y_t + U_t. \tag{5.39}$$

As before, assume that firms face quadratic costs of adjustment and a given sequence for the price of capital $\{J_t\}_{t=0}^{\infty}$.

The problem for the government can be expressed using the Lagrangian

$$\begin{aligned} L = \sum_{t=0}^{\infty} b^t \Bigg\{ \Bigg[& A_0 f_0 N k_t - \frac{1}{2} A_1 (f_0^2 N^2 k_t^2) + f_0 U_t k_t \\ & - N J_t (k_t - k_{t-1}) - \frac{1}{2} N d (k_t - k_{t-1})^2 \Bigg] \\ & + \theta_t \left[\lambda N k_{t-1} - \frac{\lambda}{d} \frac{N}{1 - \lambda b L^{-1}} (J_t - bJ_{t+1} - f_0 U_t - A_0 f_0 \tau_t) - k_t \right] \\ & + \mu_t \left[bB_{t+1} - (g_t + B_t - \tau_t N k_t) \right] \Bigg\} \end{aligned} \tag{5.40}$$

The FOC's with respect to k_t, τ_t, and $B_{t+1}$are:

$$\begin{aligned} N \Big\{ & -A_0 f_0^2 N^2 k_t - N d (k_t - k_{t-1}) - bNd(k_{t+1} - k_t) + \lambda b \theta_{t+1} \\ & -\theta_t + \mu_t \tau_t = (-A_0 f_0 N - f_0 N U_t + N J_t - N b J_{t+1}) \Big\}, \end{aligned} \tag{5.41}$$

$$-\frac{\lambda}{d} \frac{N}{1 - \lambda b L^{-1}} \theta_t + \mu_t N k_t = 0 \tag{5.42}$$

$$b\mu_t - b\mu_{t+1} = 0. \tag{5.43}$$

We seek a solution $\{\tau_t, k_t, \theta_t, \mu_t\}_{t=0}^{\infty}$ to (5.41) -(5.43), given the government budget constraint

$$\sum_{t=0}^{\infty} \tau_t N k_t = \sum_{t=0}^{\infty} b^t g_t + B_t \equiv G, \tag{5.44}$$

where G denotes the present value of government expenditures and where we have imposed the condition

$$\lim_{t+h\to\infty} b^{t+h} B_{t+h} \to 0,$$

and the initial conditions k_{-1} and $B_0 = 0$.

We can re-write (5.41) as

$$\mu_t \tau_t - \frac{Nd}{\lambda}(1-\lambda b L^{-1})(1-\lambda L)k_t - (1-\lambda b L^{-1})\theta_t = s_t, \tag{5.45}$$

where

$$s_t \equiv -A_0 f_0 N - f_0 N U_t + N J_t - N b J_{t+1}.$$

Using this definition of s_t, the equation describing the behavior of the industry capital stock can be expressed as

$$\frac{\lambda}{d}\frac{1}{1-\lambda b L^{-1}}\tau_t + (1-\lambda L)k_t = \frac{\lambda}{dN}\frac{1}{1-\lambda b L^{-1}} s_t. \tag{5.46}$$

Notice that condition (5.43) is satisfied with $\mu_t = \mu$ for all t. Hence we can write the remaining conditions as

$$\frac{\lambda}{d}\frac{1}{1-b\lambda L^{-1}}\tau_t + (1-\lambda L)k_t = -\frac{\lambda}{dN}\frac{1}{1-b\lambda L^{-1}} s_t, \quad t \geq 0 \tag{5.47}$$

$$\mu k_t - \frac{\lambda}{d}\frac{1}{1-\lambda L}\theta_t = 0, \quad t \geq 0. \tag{5.48}$$

For $t \geq 1$,

$$\tau_t = \left(-\frac{1}{N}\right) s_t - \frac{d}{\lambda}(1-b\lambda L^{-1})(1-\lambda L)k_t, \tag{5.49}$$

$$\theta_t = \frac{\mu d}{\lambda}(1-\lambda L)k_t. \tag{5.50}$$

Notice that we cannot operate on both sides of (5.47) for $t \geq 0$, only for $t \geq 1$. Now substitute these results into (5.48) to obtain

$$(1-\lambda L)k_t = -\frac{(1+\mu/N)\lambda}{(2\mu+N)d}\frac{1}{1-b\lambda L^{-1}} s_t, \quad t \geq 1. \tag{5.51}$$

Solving for τ_t and θ_t yields

$$\tau_t = -\left\{\frac{\mu/N}{(2\mu+N)}\right\} s_t, \quad t \geq 1 \tag{5.52}$$

$$\theta_t = -\frac{\mu(1+\mu/N)}{2\mu+N}\frac{1}{1-b\lambda L^{-1}} s_t, \quad t \geq 1. \tag{5.53}$$

In these expressions, μ is like an implicit price for the present value form of the government budget constraint. Thus, μ increases with G. We note that

$$\frac{\partial}{\partial \mu}\left(\frac{1+\mu/N}{2\mu+N}\right) = -\frac{1}{(2\mu+N)^2} < 0.$$

Thus, we note that k_t goes down with μ for all t. Likewise,

$$\frac{\partial}{\partial \mu}\left(\frac{\mu/N}{2\mu+N}\right) = \frac{1}{(2\mu+N)^2} > 0.$$

Thus, increases in μ increase the sequence of taxes starting from $t \geq 1$.

Now we solve for k_0, τ_0, θ_0, given a value for μ. For $t \geq 0$,

$$\mu k_t - \frac{\lambda}{d}\frac{1}{1-\lambda L}\theta_t = 0$$

implies that

$$\mu k_0 = \frac{\lambda}{d}\theta_0.$$

We obtain this condition by setting $\theta_{-1}, \theta_{-2}, \ldots = 0$, which is the appropriate set of initial conditions because the social planner takes $k_{-1}, k_{-2}, \ldots$ as given and not as being influenced by his choice of $\{\tau_0, \tau_1, \ldots\}$. We have that conditions (5.46) and (5.48) hold for $t \geq 0$. Thus, operating on both sides of (5.46) by

$$\frac{Nd}{\lambda}(1 - b\lambda L^{-1})$$

and adding the result to (5.48) yields

$$\tau_t = \frac{1}{\mu+N}(1 - b\lambda L^{-1})\theta_t, \quad t \geq 0.$$

Substituting into (5.46) to obtain for $t = 0$ yields

$$\frac{\lambda}{d}\frac{1}{N+\mu}\theta_0 + k_0 - \lambda k_{-1} = \frac{-\lambda}{Nd}\frac{1}{1-b\lambda L^{-1}}s_0.$$

But $\theta_0 = d\mu k_0/\lambda$, which implies

$$k_0 = \left(\frac{N+\mu}{N+2\mu}\right)\lambda k_{-1} - \frac{\lambda}{Nd}\left(\frac{N+\mu}{N+2\mu}\right)\frac{1}{1-b\lambda L^{-1}}s_0. \tag{5.54}$$

Also,

$$\theta_0 = d\mu\left(\frac{N+\mu}{N+2\mu}\right)k_{-1} - \frac{\mu}{N}\left(\frac{N+\mu}{N+2\mu}\right)\frac{1}{1-b\lambda L^{-1}}s_0. \tag{5.55}$$

Finally, using the expression for τ_t, we obtain

$$\tau_0 = \left(\frac{d\mu}{N+2\mu}\right)k_{-1} - \frac{\mu(1+\mu/N)}{(N+\mu)}(N+2\mu)s_0. \tag{5.56}$$

How is μ determined? We know that the present value form of the budget constraint holds:

$$\sum_{t=0}^{\infty} b^t \tau_t k_t = \sum_{t=0}^{\infty} b^t g_t \equiv G.$$

Notice that the solution for τ_t and k_t both depend on μ. First, we note that if $G = 0$, $\mu = 0$ is a solution because $\tau_0 = 0$ and $\tau_t = 0$ for $t \geq 1$. If G is not equal to zero, the value of μ must be found from the PV formula using the solution for the tax sequence and the capital stock sequence. Also, there exist values of G large enough such that no set of τ_t, θ_t, and k_t solves this problem. The reason is that the tax reduces industry output because it reduces industry capital. But if industry capital is reduced sufficiently, the sum of consumer and producer surplus becomes negligible. In that case, the government ends up with nothing.

The time-inconsistency of the government's optimal plans is evident from the fact that different functions are used to determine k_t and τ_t at $t-0$ and $t \geq 1$. In particular, when $G > 0$, $\mu > 0$ and

$$\frac{d\mu}{N+2\mu} > 0.$$

Also let $s_t = s_0$ for all t. Then

$$\tau_0 = \frac{d\mu}{N+2\mu} k_1 - \frac{\mu}{N}\left(\frac{1}{N+2\mu}\right) s_0,$$

but

$$\tau_t - \frac{\mu}{N}\frac{1}{N+2\mu} s_0, \quad t \geq 1.$$

Hence, $\tau_0 > \tau_t$ for $t \geq 1$. This occurs because the government takes $k_{-1}, k_{-2}, \ldots$ as given whereas it realizes that τ_t will affect k_t in periods $t \geq 1$. Another way of understanding the time-inconsistency problem is to solve this problem at time $t = 1$, taking as given the initial conditions k_0 and B_1 from the solution with k_{-1} and $b_0 = 0$. The initial conditions for $\{\theta_t\}_{t=0}^{\infty}$ are $\theta_0 = \theta_{-1} = \ldots = 0$ since the planner now takes k_0 as given and uninfluenced by $\{\tau_1, \tau_2, \ldots\}$. It can be shown that

$$k_1 = \left(\frac{N+\mu}{N+2\mu}\right)\lambda k_0 - \frac{\lambda}{dN}\left(\frac{N+\mu}{N+2\mu}\right)\frac{1}{1-b\lambda L^{-1}} s_1 \quad (5.57)$$

$$\tau_1 = \left(\frac{d\mu}{N+2\mu}\right) k_0 - \left(\frac{\mu/N}{N+2\mu}\right) s_1. \quad (5.58)$$

But this is not identical to the solution that we found for $t = 1$.

Taxation of Initial Capital

The government now taxes initial capital by setting

$$B_0 = -(\tau_{-1} k_{-1}). \quad (5.59)$$

Therefore, the PV form of the government budget constraint becomes

$$\sum_{t=0}^{\infty} b^t \tau_t k_t + \tau_{-1} k_{-1} = \sum_{t=0}^{\infty} b^t g_t \equiv G. \tag{5.60}$$

The Lagrangian becomes

$$\begin{aligned} L = \sum_{t=0}^{\infty} b^t \Big\{ \Big[& A_0 f_0 N k_t - \frac{1}{2} A_1 (f_0^2 N^2 k_t^2) + f_0 U_t k_t \\ & -N J_t (k_t - k_{t-1}) - \frac{1}{2} N d (k_t - k_{t-1})^2 \Big] \\ & +\theta_t \left[\lambda N k_{t-1} - \frac{\lambda}{d} \frac{N}{1 - \lambda b L^{-1}} (J_t - b J_{t+1} - f_0 U_t - A_0 f_0 \tau_t) - k_t \right] \\ & +\mu_t [b B_{t+1} - (g_t + B_t - \tau_t N k_t)] + \mu_{-1} [B_0 + \tau_{-1} k_{-1}] \Big\} \end{aligned} \tag{5.61}$$

The FOC's are found as before for k_t, τ_t, B_{t+1}, θ_t, and μ_t for $t \geq 0$. The first-order conditions for B_0 and τ_{-1} are

$$\mu_{-1} - \mu_0 = 0 \tag{5.62}$$

$$\mu_{-1} k_{-1} = 0. \tag{5.63}$$

But $k_{-1} > 0$, which implies $mu_{-1} = 0 = \mu_t$ for $t \geq 0$. Therefore,

$$\tau_{-1} k_{-1} = \sum_{t=0}^{\infty} b^t g_t, \quad \text{with } \tau_t = 0 \text{ for } t \geq 0.$$

The taxes at $t \geq 0$ are like distorting taxes because they alter the optimal choice of capital whereas the taxes at $t = -1$ are "lump-sum" because k_{-1} is given. Hence, it is optimal to tax everything at time $t = -1$. Also notice

$$\begin{aligned} \tau_{-1} k_{-1} &= \sum_{t=0}^{\infty} b^t g_t \\ B_0 &= -\tau_{-1} k_{-1} < 0 \\ B_1 &= b^{-1} (g_0 - \tau_{-1} k_{-1}) < 0 \text{ since } g_0 < \textstyle\sum_{t=0}^{\infty} b^t g_t = \tau_{-1} k_{-1}, \end{aligned}$$

and so on. This last fact implies that if this problem is solved at time time $t = 1$ with $B_1 = b^{-1}(g_0 - b^{-1} \tau_{-1} k_{-1})$, then the sequence of taxes necessary to finance the remainder of $G - g_0$ will still be given by $\tau_t = 0$ for $t \geq 1$. hence, the solution is time-consistent.

Limited Taxation on Capital

Suppose the government is able to impose only a limited amount of tax on the initial capital stock, i.e.,

$$\tau_{-1} k_{-1} \leq R, \tag{5.64}$$

and, more generally,

$$\tau_t k_t \leq R, \quad t \geq -1. \tag{5.65}$$

The value of R is chosen so that without taxation of initial capital, the taxes implied by the tax collection scheme of the original problem would be less than R for all t. On the other hand, if taxation of initial capital is allowed, then the restriction $\tau_t k_t < R$ becomes binding in the sense that the government cannot raise all of G just by taxing the initial capital.

The Lagrangian is

$$
\begin{aligned}
L = \sum_{t=0}^{\infty} b^t \Big\{ \Big[& A_0 f_0 N k_t - \frac{1}{2} A_1 (f_0^2 N^2 k_t^2) + f_0 U_t k_t \\
& - N J_t (k_t - k_{t-1}) - \frac{1}{2} N d (k_t - k_{t-1})^2 \Big] \\
& + \theta_t \left[\lambda N k_{t-1} - \frac{\lambda}{d} \frac{N}{1 - \lambda b L^{-1}} (J_t - b J_{t+1} - f_0 U_t - A_0 f_0 \tau_t) - k_t \right] \\
& + \mu_t [b B_{t+1} - (g_t + B_t - \tau_t N k_t)] + \mu_{-1} [B_0 + \tau_{-1} k_{-1}] \\
& \phi_t [R - \tau_{t-1} k_{t-1}] + \mu_{-1} [B_0 + \tau_{-1} k_{-1}] \Big\}
\end{aligned}
\tag{5.66}
$$

For τ_t, $t \geq -1$ and B_t, $t \geq 0$, the FOC's are:

$$\frac{\partial L}{\partial \tau_t} = 0 \quad \Leftrightarrow \tag{5.67}$$

$$-\frac{\lambda}{d} \left(\frac{1}{1 - \lambda L} \right) \theta_t + \mu_t k_t - b \phi_{t+1} k_t = 0, \quad t \geq 0 \tag{5.68}$$

$$\frac{\partial L}{\partial \tau_{-1}} = 0 \quad \Leftrightarrow \quad -\phi_0 k_{-1} + \mu_{-1} k_{-1} = 0 \tag{5.69}$$

$$\frac{\partial L}{\partial B_t} = 0 \quad \Leftrightarrow \quad -\mu_t + \mu_{t-1} 0, \quad t \geq 0. \tag{5.70}$$

Also, $\phi_{t+1}(R - \tau_t k_t) = 0$ for $t \geq -1$ so that either $\phi_{t+1} = 0$ or the constraint holds with equality at $t+1$, i.e., $R = \tau_t k_t$, or both. Now R is large relative to $\tilde{\tau}_t \tilde{k}_t$ where $\tilde{\tau}_t \tilde{k}_t$ is the solution to the problem without initial taxation so we argue that $\phi_{t+1} = 0$ for $t \geq 0$. But $R < G$ so that all taxes cannot be collected by taxing initial capital. Thus,

$$\mu_{-1} = \phi + 0 > 0$$

from the first-order condition with respect to τ_t. Then

$$\tau_{-1} = \frac{R}{k_{-1}}.$$

Then from $t = 0$ onwards, the revenues that must be raised are

$$\sum_{t=0}^{\infty} b^r g_t - R.$$

Set $\mu = \mu_{-1}$ so that the solution to the original problem $\{\tau_t k_t\}_{t=0}^{\infty}$ raises the revenues $\sum_{t=0}^{\infty} b^t g_t - R$. The tax rate will be lower at $t = 0$ because taxes have been collected at $t = -1$ already. But the problem is still not time-consistent because at $t = 0$, the government regards k_0 as fixed and given.

6 Consumption Choice

The first intertemporal model of consumer choice problem is due to Irving Fisher. Suppose consumers can borrow and lend all they wish at the constant real interest rate r.

The consumer's choice problem is to solve

$$\max_{\{c_t, B_t\}_{t=0}^{\infty}} V(C_0, C_1, \dots, C_T) = \max_{\{c_t, B_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t u(C_t) \tag{6.1}$$

subject to

$$C_t + B_t \leq Y_t + (1+r)B_{t-1}, \quad t \geq 0, \tag{6.2}$$

given B_{-1} where Y_t is current income. Substituting recursively for B_t from the next period's budget constraint, we obtain the present value form of the budget constraint

$$C_0 + \frac{C_1}{1+R} + \cdots + \frac{C_T}{(1+r)^T} + \frac{B_T}{(1+r)^T} = Y_0 + \frac{Y_1}{(1+r)} + \cdots + \frac{Y_T}{(1+r)^T}.$$

Thus, the problem becomes

$$\max_{\{c_t, B_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t u(C_t) \tag{6.3}$$

subject to

$$C_0 + \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} \leq Y_0 + \sum_{t=1}^{T} \frac{Y_t}{(1+r)^t}, \tag{6.4}$$

with $B_T = 0$. Otherwise, the consumer would like to borrow as much as possible at the end of his life, and he would die in debt.

The FOC's are given by:

$$\beta^t u'(C_t) = \frac{1}{(1+r)^t},$$

or taking the ratio of two consecutive conditions,

$$\beta \frac{u'(C_{t+1})}{u'(C_t)} = \frac{1}{1+r}. \tag{6.5}$$

- If $1/(1+r) = \beta$, then $u'(C_{t+1}) = u'(C_t)$, which implies $C_{t+1} = C_t$.
- If $1/(1+r) < \beta$, then $u'(C_{t+1})/u'(C_t) < 1$, which implies that $C_{t+1} > C_t$. Notice that $\beta = 1/(1+\rho)$, where ρ is the rate of subjective time preference. Thus, $1/(1+\rho) > 1/(1+r)$ implies that $\rho < r$, which says that individuals are not too impatient relative to market opportunities.
- If $1/(1+r) > \beta$, then $u'(C_{t+1})/u'(C_t) > 1$, which implies that $C_{t+1} < C_t$. Notice that $\beta = 1/(1+\rho)$. Thus, $1/(1+\rho) < 1/(1+r)$ implies that $\rho > r$, which says that individuals are more impatient relative to market opportunities.

6.1 Saving Under Certainty

Let the utility function be given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{6.6}$$

and consider the budget constraint

$$A_{t+1} = R_t \left[A_t + y_t - c_t\right], \tag{6.7}$$

A_0 given. Here R_t denotes the time-varying gross real interest rate. Assume that $\{y_t\}_{t=0}^{\infty}$ is a given sequence of exponential order less than $1/\beta$ and $\{R_t\}_{t=0}^{\infty}$ is a known and given sequence of one-period *gross* rates of return on wealth (i.e., we have not dealt with taxation).

$$\begin{aligned} c_t &= \text{consumption} \\ A_t &= \text{non-human wealth} \\ y_t &= \text{labor income.} \end{aligned}$$

Let $y_t = \lambda y_{t-1}$ and $R_t = R$ for all $t \geq 0$ with $R > \lambda$.

The budget constraint shows how non-human wealth evolves. To rule out infinite wealth obtained by arbitrarily large amounts of borrowing, let's solve

$$c_t + \frac{A_{t+1}}{R_t} = A_t + y_t.$$

But

$$A_{t+i} = c_{t+i} - y_{t+i} + \frac{A_{t+i+1}}{R_{t+i}}.$$

Therefore, we have

$$c_t + \frac{c_{t+1} - y_{t+1}}{R_t} + \frac{A_{t+2}}{R_t R_{t+1}} = A_t + y_t,$$

or, more generally,

$$c_t + \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right) c_{t+j} \leq y_t + \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right) y_{t+j} + A_t, \tag{6.8}$$

where

$$\lim_{h\to\infty} \left(\Pi_{k=0}^{h-1} R_{t+k}^{-1}\right) A_{t+h} \to 0 \tag{6.9}$$

is a transversality condition which rules out arbitrarily large wealth. We don't have a terminal condition because the problem is an infinite horizon problem. Let's make the specialization that $u(c_t) = \ln(c_t)$. Then the problem becomes

$$\max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \ln(c_t), \tag{6.10}$$

subject to

$$c_t + \frac{A_{t+1}}{R_t} \leq y_t + A_t. \tag{6.11}$$

The FOC's are:

$$\frac{\beta^t}{c_t} = \lambda_t, \tag{6.12}$$

$$\frac{\lambda_t}{R_t} = \lambda_{t+1}, \tag{6.13}$$

where λ_t is the Lagrange multiplier associated with the one-period budget constraint. Notice that

$$\lambda_t = R_t \lambda_{t+1}, \tag{6.14}$$

which implies that

$$\begin{aligned} \frac{\beta^t}{c_t} &= R_t \lambda_{t+1} \\ &= R_t R_{t+1} \lambda_{t+2} \\ &= \left(\Pi_{k=0}^{j-1} R_{t+k} \right) \lambda_{t+k}. \end{aligned} \tag{6.15}$$

But

$$\lambda_{t+j} = \frac{\beta^{t+j}}{c_{t+j}}, \tag{6.16}$$

which implies that

$$c_{t+j} = \beta^j \left(\Pi_{k=0}^{j-1} R_{t+k} \right) c_t. \tag{6.17}$$

Substituting this solution into the budget constraint yields

$$c_t + \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1} \right) \beta^j \left(\Pi_{k=0}^{j-1} R_{t+k} \right) = y_t + \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t,$$

which implies

$$c_t \sum_{j=0}^{\infty} \beta^j = y_t + \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t,$$

or

$$c_t = (1-\beta)\left[y_t + \sum_{j=1}^{\infty}\left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right) y_{t+j} + A_t\right]. \tag{6.18}$$

Thus, we find that consumption is proportional to human and non-human wealth. Human wealth is just the discounted stream of income earned by the individual over his lifetime. Let us specialize this expression a bit more.

$$\begin{aligned} c_t &= (1-\beta)\left[y_t + R_t^{-1} y_{t+1} + R_t^{-1} R_{t+1}^{-1} y_{t+2} + \cdots + A_t\right] \\ &= (1-\beta)\left[(1 + R^{-1}\lambda + R^{-2}\lambda^2 + \cdots) y_t + A_t\right] \\ &= (1-\beta)\left[A_t + y_t/(1-\lambda R^{-1})\right], \end{aligned} \tag{6.19}$$

with $R^{-1}\lambda < 1$, which implies that wealth grows at a rate less than the growth rate of income. Thus, the consumption derived from optimizing behavior varies inversely with the real interest rate and directly with current income and the growth rate of income.

Suppose we have an example in which the gross rate of interest on assets held between periods t and $t+1$ is random and becomes known only at the beginning of period $t+1$, after a decision about consumption at t must be made. When the time t decisions are made, the consumer knows A_t, y_t, and $R_{t-1}, R_{t-2}, \ldots$.

Now the problem becomes

$$\max_{\{c_{t+j}\}_{j=0}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \tag{6.20}$$

s.t.

$$c_t + E_t \sum_{j=1}^{\infty}\left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right) c_{t+j} \le y_t + E_t \sum_{j=1}^{\infty}\left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right) y_{t+j} + A_t. \tag{6.21}$$

We further assume that $u(c_t) = \ln(c_t)$, $\{R_t\}_{t=}^{\infty}$ is an i.i.d process with $1 \le ER_t \le 1/\beta^2$, and that $\{y_t\}_{t=0}^{\infty}$ is an i.i.d. process of mean exponential order less than $1/\beta$ that is also independent of $\{R_t\}_{t=0}^{\infty}$.

The FOC's are:

$$\frac{\beta^t}{c_t} = \lambda \tag{6.22}$$

$$E_t\left[\frac{\beta^{t+j}}{c_{t+j}} - \lambda\left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right)\right] = 0, \quad j \ge 1. \tag{6.23}$$

These conditions imply that

$$\beta^j c_t = E_t\left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right) c_{t+j}.$$

Substitute this into the budget constraint:

$$c_t + \sum_{j=1}^{\infty} \beta^j = y_t + E_t \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + A_t,$$

which implies that

$$\begin{aligned} c_t &= (1-\beta) \left[y_t + \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} E_t R_{t+k}^{-1} \right) E_t y_{t+j} + A_t \right] \\ &= (1-\beta) \left[y_t + \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} \bar{R}^{-1} \right) \bar{y} + A_t \right] \\ &= (1-\beta) \left[A_t + y_t + \bar{y}/(1-\bar{R}^{-1}) \right]. \end{aligned} \tag{6.24}$$

But the assumption that $\{R_t\}$ and $\{y_t\}$ are i.i.d and independent is arbitrary!

Suppose that we make no assumptions about $\{R_t\}$ and $\{y_t\}$, which allow us to simplify as we have done before. Then,

$$c_t = (1-\beta) \left[A_t + y_t + \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j} + u_t \right], \tag{6.25}$$

where $H_{t+1} = \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1} \right) y_{t+j}$ and $u_t = H_{t+1} - E_t H_{t+1}$. We know that

$$E(u_t | A_t, y_t, R_{t-1}, R_{t-2}, \ldots) = 0. \tag{6.26}$$

Thus, we can write

$$c_t = (1-\beta) \left[A_t + y_t + H_{t+1} \right] + v_t, \tag{6.27}$$

where $v_t = (1-\beta) u_t$. Then we might be able to test this theory by using generalized instrumental variables estimation.

6.2 The Consumption Function

The early Keynesian consumption function was postulated of the form

$$C_t = a + bY_t. \tag{6.28}$$

But from time series data, it became apparent that consumption is a much smoother series than income so this relationship did not fit very well in terms of the time series data. Researchers tried to fit consumption to distributed lags of income, i.e.,

$$C_t = b(L) Y_t = b_1 Y_t + b_2 Y_{t-1} + \ldots, \tag{6.29}$$

so consumption seemed a smoothed version of income. Theories were sought to interpret this distributed lag. This led to the idea of permanent income, i.e., that individuals did not choose their consumption to depend on current income but on some measure of income expected over the lifetime. This led researchers to examine Fisher's theory of intertemporal consumption allocation to reconcile the data with an optimizing model of consumption choice over the individual's lifetime. Up to this point, we derived the consumption function

$$c_t = (1-\beta)\left[y_t + E_t \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right) y_{t+j} + A_t\right]. \tag{6.30}$$

How can we get consumption to be a distributed lag of income, i.e., how can we get current and past values of income to affect current consumption? Suppose

$$y_t = a_1 y_{t-1} + a_2 y_{t-2}. \tag{6.31}$$

Now

$$\begin{aligned} y_t + \sum_{j=1}^{\infty} R^{-j} y_{t+j} &= y_t + R^{-1}(a_1 y_t + a_2 y_{t-1}) + R^{-2}(a_1 y_{t+1} + a_2 y_t) \\ &\quad + R^{-3}(a_1 y_{t+2} + a_2 y_{t+1}) + \ldots \\ &= (1 + R^{-1} a_1 + R^{-2} a_2) y_t + R^{-1} a_2 y_{t-1} + R^{-2} a_1 (a_1 y_t + a_2 y_{t-1}) \\ &\quad + R^{-3}[a_1(a_1 y_{t+1} + a_2 y_t) + a_2(a_1 y_t + a_2 y_{t-1})] + \ldots \\ &= \left[1 + R^{-1} a_1 + R^{-2}(a_2 + a_1^2) + 2R^{-3} a_1 a_2\right] y_t \\ &\quad + R^{-1} a_2 (1 + R^{-1} a_1 + R^{-2} a_2) + R^{-3} a_1^2 y_{t+1} + \ldots \\ &= \frac{(1 + a_2 R^{-1} L) y_t}{1 - a_1 R^{-1} - a_2 R^{-2}} \end{aligned} \tag{6.32}$$

Therefore,

$$c_t = (1-\beta)\left[A_t + \frac{(y_t + a_2 R^{-1} y_{t-1})}{1 - a_1 R^{-1} - a_2 R^{-2}}\right]. \tag{6.33}$$

Now suppose we assume that $\{y_t\}_{t=0}^{\infty}$ and $\{R_t\}_{t=0}^{\infty}$ are arbitrary stochastic processes. Define

$$H_t = y_t + E_t H_{t+1} = y_t + E_t \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right) y_{t+j}. \tag{6.34}$$

Likewise,

$$H_{t-1} = y_{t-1} + E_{t-1} H_t = y_{t-1} + E_{t-1} \sum_{j=1}^{\infty} \left(\Pi_{k=0}^{j-1} R_{t+k}^{-1}\right) y_{t+j-1}. \tag{6.35}$$

Suppose $R_t = R$. Then $H_t = y_t + E_t \sum_{j=1}^{\infty} R^{-j} y_{t+j} = E_t \sum_{j=0}^{\infty} R^{-j} y_{t+j}$. Likewise, $H_{t-1} = y_{t-1} + E_{t-1} \sum_{j=1}^{\infty} R^{-j} y_{t+j-1}$. Also, we have that

$$RH_{t-1} = Ry_{t-1} + E_{t-1} \sum_{j=1}^{\infty} R^{-j+1} y_{t+j-1} = Ry_{t-1} + E_{t-1} \sum_{j=0}^{\infty} R^{-j} y_{t+j}.$$

Combining these results yields

$$H_t = R(H_{t-1} - y_{t-1}) + E_t \sum_{j=0}^{\infty} R^{-j} y_{t+j} - E_{t-1} \sum_{j=0}^{\infty} R^{-j} y_{t+j}, \tag{6.36}$$

which implies that

$$H_t = R(H_{t-1} - y_{t-1}) + \epsilon_t, \tag{6.37}$$

where

$$\epsilon_t = \sum_{j=0}^{\infty} R^{-j} (E_t y_{t+j} - E_{t-1} y_{t+j}), \tag{6.38}$$

i.e., the discounted value of the revision in income forecasts based on additional information available at time t. Notice that $E_{t-1}\epsilon_t = 0$. Now $c_t = (1-\beta)[A_t + H_t]$, or $(1-\beta)^{-1} c_t - A_t = H_t$ and $(1-\beta)^{-1} c_{t-1} - A_{t-1} = H_{t-1}$. Therefore, we can write consumption as

$$c_t = Rc_{t-1} + (1-\beta)\left[A_t - RA_{t-1} - Ry_{t-1}\right] + \epsilon_t. \tag{6.39}$$

Thus, we obtain the result that the only variables useful for predicting current consumption are current wealth, past consumption, and past income.

6.3 The Random Walk Model of Consumption

Hall takes an alternative approach in that he derives restrictions on the behavior of consumption that do not involves stochastic processes for income.

Consider the problem of solving

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{6.40}$$

subject to

$$c_0 + \sum_{t=1}^{\infty} R^{-t} c_t \leq y_0 + \sum_{t=1}^{\infty} R^{-t} y_t. \tag{6.41}$$

The FOC's are:

$$\begin{aligned} \beta^t u'(c_t) &= \lambda R^{-t} && (6.42) \\ \beta^{t+1} E_t u'(c_{t+1}) &= \lambda R^{-(t+1)}. && (6.43) \end{aligned}$$

Solving for the Lagrange multiplier yields

$$\lambda = (\beta R)^{t+1} E_t u'(c_{t+1}). \tag{6.44}$$

Substituting it into the first condition yields

$$u'(c_t) = \beta R E_t u'(c_{t+1}), \tag{6.45}$$

or

$$E_t u'(c_{t+1}) = (\beta R)^{-1} u'(c_t). \tag{6.46}$$

This condition implies that no other information besides current consumption should help to predict future consumption.

Equivalently, we have that

$$u'(c_{t+1}) = (\beta R)^{-1} u'(c_t) + \epsilon_{t+1},$$

where $E_t \epsilon_{t+1} = 0$.

Suppose $u(c_t) = (1/2)(c_t - \bar{c})^2$, where $\bar{c}$ is the bliss level of consumption. Then $u'(c_t) = \bar{c} - c_t$ and

$$c_{t+1} = (\beta R)^{-1} c_t + \left[1 - (\beta R)^{-1}\right] \bar{c} + \epsilon_{t+1}, \tag{6.47}$$

where $E_t \epsilon_{t+1} = 0$. Hall assumes further that $\beta = 1/R = 1/(1+r)$, or equivalently, that the rate of subjective time preference equals the real interest rate. Then we obtain the *random walk of consumption*:

$$c_{t+1} = c_t + \left[1 - (\beta R)^{-1}\right] \bar{c} + \epsilon_{t+1}. \tag{6.48}$$

Suppose that we regress current consumption against current income. Then the random walk model of consumption developed by Hall states that the coefficient on income should be zero. ϵ_{t+1} summarizes new information that becomes available at time t about the consumer's future marginal utility of consumption.

How may be the permanent income hypothesis be rejected? Suppose that a set of consumers is constrained by their current incomes. For example, they may not be able to borrow. Then their consumption will follow

$$c_t' = \mu y_t'. \tag{6.49}$$

The remainder behave optimally so that

$$c_t'' = \lambda c_{t-1}'' + \epsilon_t. \tag{6.50}$$

Aggregate consumption follows

$$c_t = c_t' + c_t'' = \mu y_t' + \lambda c_{t-1}'' + \epsilon_t. \tag{6.51}$$

Suppose that income follows an $AR(2)$ process, i.e., $y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t$.

$$\begin{aligned} E(c_t|c_{t-1}, y_{t-1}, y_{t-2}) &= E(c'_t|c_{t-1}, y_{t-1}, y_{t-2}) + E(c''_t|c_{t-1}, y_{t-1}, y_{t-2}) \\ &= \mu E(y'_t|y_{t-1}, y_{t-2}) + \lambda(c_t - \mu y_{t-1}) \\ &= \mu\rho_1 y_{t-1} + \mu\rho_2 y_{t-2} - \mu\lambda y_{t-1} + \lambda c_{t-1} \\ &= \mu(\rho_1 - \lambda) y_{t-1} + \mu\rho_2 y_{t-2} + \lambda c_{t-1}. \end{aligned}$$

The permanent income hypothesis will be rejected unless $\rho_1 = \lambda$ or $\rho_2 = 0$.

Alternatively, suppose consumers use a non-optimal distributed lag to form expectations of permanent income, i.e.,

$$c_t = \alpha \sum_{t=0}^{\infty} \delta^i y_{t-i}, \tag{6.52}$$

or

$$c_t = \alpha y_t + \delta \sum_{i=0}^{\infty} \delta^i y_{t-1-i},$$

which implies that

$$c_t = \alpha y_t + \delta c_{t-1}. \tag{6.53}$$

In that case,

$$\begin{aligned} E(c_t|c_{t-1}, y_{t-1}, y_{t-2}) &= \delta c_{t-1} + \alpha E(y_t|y_{t-1}, y_{t-2}) \\ &= \delta c_{t-1} + \alpha(\rho_1 y_{t-1} + \rho_2 y_{t-2}). \end{aligned} \tag{6.54}$$

Hence, we find that past income is useful for predicting current consumption.

7 Monetary Economies

So far we have considered only real models of the aggregate economy and individual behavior. We have not talked about the existence of some medium of exchange, as money is usually termed. Alternatively, any good can serve as a numeraire or unit of account in our model.

Now we will consider economies where there exist money. Money can take on different forms. For example, there are commodity monies (gold and the gold standard). There are partially backed monies. There are also fiat money systems. Fiat monies are unbacked currencies. They contain no clause providing for convertibility into another good. We begin with the study of a very simple fiat money economy. A crucial question arises when we consider fiat money: it has no value in itself. It cannot be fashioned into gold bracelets. It yields no real rate of return. So if there exist other assets or goods which do have value in themselves or which yield a real non-zero rate of return stream, why should fiat money be held?

One reason is that money yields utility in itself. Money-in-the-utility-function models exploit this rationale. But this is a less than satisfactory explanation. So let us step back and consider a slightly better explanation, i.e., that money yields services just like any other asset but its services are in the form of liquidity services. But this still seems to beg the question in that why are liquidity services required. They are required because there exists some restriction which says that goods purchases must be made with money. This is the famous Clower constraint. It is a simple but in some sense arbitrary way of bestowing value of money. There are more theoretically satisfactory ways of bestowing value of money, some of which we will consider.

7.1 A Cash-in-Advance Model

We consider a closed economy monetary economy in which output is stochastically determined and non-storable. The supply of money is also random and determined according to the law o motion

$$\bar{M}_{t_1} = \omega_t \bar{M}_t, \tag{7.1}$$

where ω_t is the gross rate of monetary expansion and is stochastic. Let $s_t(y_t, \omega_t)$ be the state in period t. Then assume s_{t+1} follows a Markov process with probability distribution $Prob(s_{t+1} \leq s' | s_t = s) = F(s', s)$.

The representative consumer has preferences of the form

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau). \tag{7.2}$$

Which markets exist in this economy?

1. Claims to the output of the single firm may be traded, i.e., equity shares.

2. Goods market.

How do trades take place?

Consumers enter period t with holdings of money M_t and shares z_t of claims to the firm's output. The consumer learns the state of the economy s_t and may purchase consumption goods with money at prices $P(s_t, \bar{M}_t)$. His purchases must obey the *liquidity constraint* or *cash-in-advance constraint*:

$$P(s_t, \bar{M}_t) c_t \leq M_t. \tag{7.3}$$

After the goods market is closed, at the end of period t, the consumer receives *in cash* his share of dividends, $P(s_t, \bar{M}_t) y_t z_t$ as well as some lump-sum money transfers $(\omega_t - 1)\bar{M}_t$. The consumer receives the money transfer after the goods market closes. Hence, he cannot use the money transfer to buy consumption goods within the same period. At the end of the period, shares and money are traded.

The timing of trades is as follows:

$$\text{time } t \left| \begin{array}{l} \text{given } M_t, z_t, \text{buy goods} \\ \\ \text{receive money transfers} \\ \\ \text{trade money and shares} \end{array} \right.$$

The budget constraint when trading shares and determining how much money to carry over into the next period is defined by

$$\begin{aligned} M_{t+1} + Q(s_t, \bar{M}_t) z_{t+1} &\leq [M_t - P(s_t, \bar{M}_t) c_t] \\ &+ [Q(s_t, \bar{M}_t) + P(s_t, \bar{M}_t) y_t] z_t + (\omega_t - 1) \bar{M}_t. \end{aligned}$$

where

M_{t+1}, z_{t+1} :	holdings of money and shares to be carried over into the next period
$Q(s_t, \bar{M}_t)$:	share price determined in money.

Let $\pi_t = 1/p_t$ be the real price of money or its purchasing power, and $q_t = Q_t/p_t$ the real price of shares. Re-write out the budget constraint above using primed variables for the future:

$$\begin{aligned} c + \pi M' + q z' &\leq \pi M + (q + y) z + \pi(\omega - 1)\bar{M} \equiv w && (7.4) \\ c &\leq \pi M, && (7.5) \end{aligned}$$

where w is defined as real wealth in period t. Wealth in period $t+1$, w', is defined as

$$w' \equiv \pi' M' + (q' + y') z' + \pi'(\omega' - 1)\bar{M}'.$$

The consumer takes as given the prices $q(s, \bar{M})$ and $\pi(s, \bar{M})$ and chooses allocations c, z', and M' which will attain the value function

$$u(c) + \beta \int v(w', M', s', \bar{M}') dF(s', s), \tag{7.6}$$

provided there exists a bounded, continuous solution to the functional equation defined by

$$v(w, M, s, \bar{M}) = \max_{c, M', z'} \left\{ u(c) + \beta \int v(w', M', s', \bar{M}') dF(s's) \right\} \tag{7.7}$$

s.t.

$$c + \pi M' + q z' \leq \pi M + (q + y) z + \pi(\omega - 1)\bar{M} \tag{7.8}$$

$$c \leq \pi M. \tag{7.9}$$

We wish the solution to be bounded and continuous. Let λ and μ be the Lagrange multipliers associated with the constraints. In equilibrium, $c = y$, $M = \bar{M}$, $M' = \bar{M}' = \omega \bar{M}$, and $z' = z = 1$.

The FOC's are:

$$u'(c) = \lambda + \mu \tag{7.10}$$

$$\pi\lambda = \beta \int \left[\frac{\partial v(w', M', s', \bar{M}')}{\partial M'} + \frac{\partial v(w', M', s', \bar{M}')}{\partial w'} \frac{\partial w'}{\partial M'} \right] dF(s', s) \tag{7.11}$$

$$\lambda q = \beta \int \frac{\partial v(w', M', s', \bar{M}')}{\partial w'} \frac{\partial w'}{\partial z'} dF(s's,) \tag{7.12}$$

$$c + \pi M' + qz' = w \;\text{ for } \lambda > 0 \tag{7.13}$$

$$c \leq \pi M \;\text{ for } \mu \geq 0 \tag{7.14}$$

Notice that

$$\frac{\partial w'}{\partial M'} = \pi' \tag{7.15}$$

and

$$\frac{\partial w'}{\partial z'} = q' + y'. \tag{7.16}$$

To show that

$$\frac{\partial v(w, M, s, \bar{M})}{\partial w} = \lambda \tag{7.17}$$

and

$$\frac{\partial v(w, M, s, \bar{M})}{\partial M} = \pi\mu, \tag{7.18}$$

let us use the approach in Lucas (1978).

Proof. Assume the existence of a bounded, continuous $v(w, M, s, \bar{M})$ which is attained by $c > 0$. Define $f : \Re^+ \times \Re^+ \to \Re^+$ by

$$f(w, \tilde{M}) = \max_{c, w', M'} \left\{ u(c) + \beta \int v(w', \tilde{M}', s', \bar{M}') dF(s', s) \right\}$$

$$\text{s.t.}$$

$$c + \pi M' + qz' \leq \pi M + (q + y)z + \pi(\omega - 1)\bar{M}$$

$$c \leq \pi M \equiv \tilde{M}$$

$$c \geq 0, z' \geq 0, M' \geq 0.$$

For each w, $f(w, \tilde{M})$ is attained by $c(w, \tilde{M})$, $w'(w, \tilde{M})$ and $M(w, \tilde{M})$. It can be shown that the maximand is strictly concave in c and consequently, that $c(w, \tilde{M})$ is unique and varies continuously with w and $\tilde{M}$. If $c(w, \tilde{M}) > 0$ and if h is sufficiently small, then $c(w, \tilde{M}) + h$ is feasible for wealth $w + h$

and money balances $\tilde{W}$, and $c(w+h, \tilde{M}+h)-h)$ is feasible for wealth $w+h$ and money balances $\tilde{M}+h$. Therefore,

$$\begin{aligned} f(w+h,\tilde{M}+h) &\geq u(c(w,\tilde{M})+h) \\ &+\beta\int v(w(w,\tilde{M}),M(w,\tilde{W}),s',\bar{M}')dF(s',s) \\ &= u(c(w,\tilde{M})+h)-u(c(w,\tilde{M}))+f(w,\tilde{M}) \end{aligned}$$

and

$$\begin{aligned} f(w,\tilde{M}) &\geq u(c(w+h,\tilde{M}+h)-h) \\ &+\beta\int v(w(w+h,\tilde{M}+h),M(w+h,\tilde{W}+h),s',\bar{M}')dF(s',s) \\ &= u(c(w+h,\tilde{M}+h)-h)-u(c(w+h,\tilde{M}+h))+f(w+h,\tilde{M}+h). \end{aligned}$$

Together we obtain

$$\begin{aligned} u(c(w,\tilde{M})+h)-u(c(w,\tilde{M})) &\leq f(w+h,\tilde{M}+h)-f(w,\tilde{M}) \\ &\leq u(c(w+h),\tilde{M}+h)-u(c(w+h,\tilde{M}+h)-h). \end{aligned}$$

As a consequence,

$$\begin{aligned} u(c(w,\tilde{M})+h)-u(c(w,\tilde{M})) &\leq f(w+h,\tilde{M}+h)-f(w+h,\tilde{M}) \\ +f(w+h,\tilde{M})-f(w,\tilde{M}) &\leq u(c(w+h),\tilde{M}+h)-u(c(w+h,\tilde{M}+h)-h). \end{aligned}$$

Divide by h, let $h\to 0$, and use the continuity of $c(w,M)$ with respect to w and M to obtain

$$\frac{\partial f(w,\tilde{M})}{\partial\tilde{M}}+\frac{\partial f(w,\tilde{W})}{\partial w}=u'(c(w,\tilde{M})).$$

Therefore,

$$\frac{\partial f(w,\tilde{M})}{\partial\tilde{M}}\frac{\partial\tilde{M}}{\partial M}=\frac{\partial v(w,M,s,\bar{M})}{\partial M},$$

and

$$\frac{\partial f(w,\tilde{M})}{\partial w}=\frac{\partial v(w,M,s,\bar{M})}{\partial w}.$$

Therefore, we have that

$$\frac{\partial v(w,M,s,\bar{M})}{\partial M}\frac{1}{\pi}+\frac{\partial v(w,M,s,\bar{M})}{\partial w}=u'(c(w,M)).$$

■

Here λ is defined as the marginal utility of wealth and μ is the marginal utility of real balances, $\pi\mu$ being the marginal utility of nominal money. Notice that if $m=\pi\bar{M}$ is defined, then $\bar{M}_t$ does not enter as a separate exogenous variable so that the equilibrium relationships may be expressed only as a function of $s = (y, \omega)$.

$$\begin{aligned} y &\leq m(s) \text{ for } \mu(s) \geq 0 && (7.19) \\ \lambda(s) + \mu(s) &= u'(y) && (7.20) \\ \pi(s)\lambda(s) &= \beta E\left[\pi(s')\mu(s') + \lambda(s')\pi(s')|s\right] && (7.21) \\ \lambda(s)q(s) &= \beta E\left[\lambda(s)(q(s') + y')|s\right]. && (7.22) \end{aligned}$$

The last two constraints may be simplified by noting that $\pi(s) = m(s)/\bar{M}$, $\pi(s') = m(s')/\bar{M} = m(s')/\omega\bar{M}$ and using the second condition to substitute out for $\lambda(s') + \mu(s')$:

$$\frac{m(s)}{\bar{M}}\lambda(s) = \beta\left[\frac{m(s')}{\omega\bar{M}}(\mu(s') + \lambda(s'))|s\right], \tag{7.23}$$

which implies that

$$m(s)\lambda(s) = \beta E\left[\frac{m(s')u'(y')}{\omega} \mid s\right]. \tag{7.24}$$

These conditions mat be interpreted as follows: the liquidity constraint says that real balances m must equal or exceed output. Since μ is the marginal utility of real balances, the first condition says that if the liquidity constraint is not binding, then the marginal utility of real balances is zero and if the marginal utility of real balances is positive, then the liquidity constraint is binding.

We have that

$$\lambda + \mu = u_c, \tag{7.25}$$

i.e., the sum of the marginal utility of real balances and of wealth equals the marginal utility of consumption. Because real wealth cannot buy consumption, the marginal utility of wealth does not equal the marginal utility of consumption.

The fourth condition above gives the standard relation for the pricing of assets:

$$\lambda(s)q(s) = \beta E[\lambda(s')(q(s') + y)|s].$$

Solving this condition froward and using the transversality condition which comes out of the consumer's problem yields

$$q_t = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t}\frac{E(\lambda_\tau y_\tau|s_t)}{\lambda_\tau}. \tag{7.26}$$

The transversality condition is given by

$$\lim_{h\to\infty} \beta^{\tau-t}\frac{E(\lambda_h q_h|s_t)}{\lambda_t} \to 0. \tag{7.27}$$

This formula is very similar to the formula for the share price that we derived in Lucas except that in his model, the marginal utility of wealth always equals the marginal utility of consumption. In this case, we have a divergence between the two because of the liquidity constraint. So the existence of a distortion in monetary transactions affects the real side of the economy as well.

Now consider the third condition above:

$$\lambda(s)\pi(s) = \beta E[(\lambda(s') + \mu(s'))\pi(s')|s]. \tag{7.28}$$

Solving this forward, we obtain

$$\begin{aligned} \pi_t &= \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \frac{E(\mu_t \pi_t | s_t)}{\lambda_t} \\ &= \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \frac{E(v_{M_\tau} | s_t)}{\lambda_t}. \end{aligned} \tag{7.29}$$

Imposing the transversality condition yields

$$\lim_{h\to\infty} \beta^{\tau-t} E[\mu_h \pi_h M_{h+1}] \to 0. \tag{7.30}$$

Thus, money is priced just like other assets, once its direct return has been appropriately defined. Specifically, its direct return is the value of liquidity services provided by money. But notice that unlike other assets, the real return on money is not observable. So if we were to test any of these relationships, we would have to somehow measure these shadow prices.

Nominal and Real Interest Rates.

The nominal interest rate is the rate of return on a nominal bond which pays one sure unit of money in the next period. The bond is bought at the *end* of period t, and pays one unit of money at the end of period $t+1$.

The nominal budget constraint is:

$$\begin{aligned} p_t c_t + \tilde{Q}_t b_{t+1} + Q_t z_{t+1} + M_{t+1} \leq \; & M_t + b_t \\ & +(Q_t + y_t)z_t + (\omega_t - 1)\bar{M}_t. \end{aligned}$$

The real budget constraint (after dividing by p_t) is:

$$\begin{aligned} c_t + \tilde{q}_t b_{t+1} + q_t z_{t+1} + \pi_t M_{t+1} \leq \; & \pi_t M_t + \pi_t b_t \\ & +(q_t + \pi_t y_t)z_t + \pi_t(\omega_t - 1)\bar{M}_t. \end{aligned}$$

The optimality condition for the choice of bonds is

$$\lambda_t \tilde{q}_t = \beta E_t[\lambda_{t+1}\pi_{t+1}], \tag{7.31}$$

which says that the marginal (utility) cost of investing in nominal bonds must be equal to the marginal benefit of investing in them. Thus, the real present value at the end of period t of this bond or the nominal present value deflated by the price level is

$$\tilde{q}_t = \frac{\beta E_1[\lambda_{t+1}\pi_{t+1}]}{\lambda_t}. \tag{7.32}$$

By contrast, the nominal present value of this bond is

$$\tilde{Q}_t = \frac{\beta E_1[\lambda_{t+1}\pi_{t+1}]}{\pi_t\lambda_t}, \tag{7.33}$$

and the nominal interest rate is

$$\frac{1}{1+i_t} = \frac{\beta E_1[\lambda_{t+1}\pi_{t+1}]}{\pi_t\lambda_t}, \tag{7.34}$$

or

$$1 + i_t = \frac{\lambda_t\pi_t}{\beta E_t[\lambda_{t+1}\pi_{t+1}]},$$

which implies

$$\begin{aligned} i_t &= \frac{\lambda_t\pi_t - E_t[\lambda_{t+1}\pi_{t+1}]}{E_t[\lambda_{t+1}\pi_{t+1}]} \\ &= \frac{E_t[\mu_{t+1}\pi_{t+1}]}{E_t[\lambda_{t+1}\pi_{t+1}]}. \end{aligned} \tag{7.35}$$

How would we price a bond which pays one unit of real output for sure in period $t+1$? We may think of this as an indexed bond. Consider the real budget constraint

$$c_t + q_t z_{t+1} + \pi_t M_{t+1} + \tilde{p}_t B_T \leq B_t + \ldots$$

The FOC is given by

$$\tilde{p}_t\lambda_t = \beta E_t\lambda_{t+1}, \tag{7.36}$$

which implies the pricing relation

$$\tilde{p}_t = \beta E_t\left[\frac{\lambda_{t+1}}{\lambda_t}\right], \tag{7.37}$$

and the real rate of return

$$\frac{1}{1+\rho_t} = \beta E_t\left[\frac{\lambda_{t+1}}{\lambda_t}\right]. \tag{7.38}$$

Note that this bond pays off $1/\pi_{t+1}$ units of cash in period $t+1$.

Prices, Real Balances, and Interest Rates.

One issue that has interested monetary economists is the effect of output and the quantity of money (or its rate of expansion) on prices, real balances, and interest rates. We can use the equations

$$y \leq m(s) \tag{7.39}$$

$$\lambda(s)+\mu(s) = u_c(y) \tag{7.40}$$

$$\lambda(s)m(s) = \beta E\left[\frac{u_c(y')m(s')|s}{\omega}\right] \tag{7.41}$$

to solve for $\lambda(s), m(s), \omega(s)$ and

$$\pi(s,\bar{M}) = \frac{m(s)}{\bar{M}}. \tag{7.42}$$

We note that $m(s)$ are real balances and together with the remaining valuation criteria such as $\lambda(s)$, $\mu(s)$ and $\rho(s)$ are independent of the quantity of money. On the other hand, $\pi(s,\bar{M})$ and the nominal interest rate will depend on the quantity of money. Notice that the equation $\pi(s,\bar{M}) = m(s)/\bar{M}$ may be interpreted as the demand price for money so we are about to derive a proper demand function for money.

As a simplifying device, we will assume that the probability distribution of $s' = (y'\omega')$ is independent of the realization of s, i.e., $F(s',s) = F(s')$. Under this interpretation, $E[u_c(y')m(s')|s] = A$, a constant that is independent of s. Then there will be two regions in which the marginal utility of real balances is zero and the liquidity constraint is not binding ($m > y$ and $\mu(s) = 0$) , and one in which it is ($\mu(s) > 0$ and $m(s) = y$). The border line between the two regions is given by the set y and ω such that

$$y = m(s) \tag{7.43}$$

$$\mu(s) = 0. \tag{7.44}$$

These conditions allow us to determine the value of ω as

$$\omega = \frac{\beta A}{u_c(y)y} \equiv \tilde{\omega}(y). \tag{7.45}$$

When $\omega < \tilde{\omega}(y)$, the liquidity constraint will not bind, whereas it will be binding for $\omega \geq \tilde{\omega}(y)$. To show this, suppose $\omega = \tilde{\omega}(y)$ where $m = y$,$\mu = 0$, and $\lambda = u_c$. (i) Let ω increase for constant y. Suppose μ remains equal to zero, and hence, λ remains equal to u_c. Then, by (7.43), we have that $m(s) = \beta A/u_c(y)y$ is falling, which implies $m < y$, which is a contradiction. Hence, μ must increase and be positive when $\omega > \tilde{\omega}(y)$, (ii) Let ω be less that $\tilde{\omega}(y)$. Assume $m = y$. By (7.44), $\lambda = \beta A/y\omega$ must rise. But $\mu \geq 0$ implies that $\lambda(s) \leq u_c(y)$, which is a contradiction. Hence, $m < y$ for $\omega < \tilde{\omega}(y)$.

Consider the relative risk aversion utility function

$$u(c) = \frac{c^{1-r}}{1-r}. \tag{7.46}$$

We have three cases:

$$r < 1 \quad \Rightarrow \quad \omega = \frac{\beta A}{y^{1-r}} \tag{7.47}$$

$$r = 1 \quad \Rightarrow \quad \omega = \beta A \;\; \forall y \tag{7.48}$$

$$r > 1 \quad \Rightarrow \quad \omega = \beta A y^{r-1}. \tag{7.49}$$

Hence, the borderline between the two regions has a negative, zero, and positive slope depending on whether relative risk aversion is lower, equal to, or greater than unity.

More generally, the solution when $\omega < \tilde{\omega}(y)$ is given by

$$m(s) = \frac{\beta A}{u_c(y)\omega} > y, \; \lambda(s) = u_c(y), \; \mu(s) = 0, \tag{7.50}$$

and when $\omega \geq \tilde{\omega}(y)$, the solution is

$$m(s) = y, \; \lambda(s) = \frac{\beta A}{y\omega}, \; \mu(s) = u_c(y) - \frac{\beta A}{\omega} \geq 0. \tag{7.51}$$

The intuition for this borderline is that a high ω at the beginning of the period implies that more money will be distributed at the end of the period and that the future value of money $\pi' = m/\bar{M}'$ will be lower in all future states since $\bar{M}' = m'\bar{M}$ is higher. Therefore, it will be less attractive to spend money on consumption, which will bid up the money price of consumption goods in the beginning of the current period. This lowers the value of money and current real balances. So if ω is large enough, then real balances will fall to hit the liquidity constraint.

The income velocity of money is y/m. In this model, $y/m \leq 1$.

- When $\omega > \tilde{\omega}(y)$, then $y/m = \omega y^{r-1}/\beta A < 1$ since $m = \beta A/\omega y^{-r}$,
- when $\omega \geq \tilde{\omega}(y)$, $y/m = 1$ since $m = y$.

How velocity varies with income depends on the degree of relative risk aversion r. For the constant relative risk aversion utility function, we have that

$$-\frac{yu''(y)}{u'(y)} = \frac{yry^{-r-1}}{y^{-r}} = r < 1. \tag{7.52}$$

Therefore, the marginal utility of consumption and wealth decreases less than proportionately with income. Since $\lambda(s)m(s) = \beta A/\omega$, real balances vary inversely with the marginal utility of wealth. Hence, real balances increase less than proportionately with income so velocity falls.

The price level is given by

$$P(s, \bar{M}) = \frac{1}{\pi(s, \bar{M}} = \frac{\bar{M}}{m(s)} \tag{7.53}$$

varies inversely with real balances. For $r < 1$, $P(s, \bar{M})$ decreases less than proportionately with output or income. Inflation fulfills the relation

$$\frac{P'}{P} = \frac{\pi}{\pi'} = \frac{\omega m(s)}{m(s')}, \tag{7.54}$$

where $\pi' = m'/\bar{M}' = \bar{m}/\omega\bar{M}$. Hence, inflation varies directly with end-of-period balances. Also expected inflation satisfies

$$E\left(\frac{P'}{P}\right) = E\left(\frac{\pi}{\pi'}\right) = \omega m E\left(\frac{1}{m'}\right), \tag{7.55}$$

which is known and non-random since $E(1/m')$ is a constant.

- When $\mu = 0$, $\omega m = \beta A/u_c(y)$. Hence, expected inflation varies with $1/u_c(y)$ and is increasing in income.
- When $\mu > 0$ (so that the liquidity constraint is binding), $E(\pi/\pi') = \omega y E(1/m')$ and so it depends on both income and the gross rate of monetary expansion. A temporary increase in income temporarily lowers the current price level which increases inflation: $P(s, \bar{M}) = \bar{M}/m(s) = \bar{M}/y$. A temporary increase in monetary expansion increases expected inflation because next period's price increases while the current price level and current real balances are independent of the current monetary expansion when the liquidity constraint is binding.

Why is expected inflation independent of monetary expansion when the liquidity constraint is not binding? Current real balances falls proportionately and current real balances rise proportionately to monetary expansion. So the current price rises proportionately to next period's price level and expected inflation remains unaffected.

In summary, a temporary increase in monetary expansion leads implies that real balances fall and the price level rises when the liquidity constraint is not binding. Expected inflation rises when the liquidity constraint binds, the nominal interest rate remains constant because

$$i(s) = i = \frac{E(\mu' m')}{E(\lambda' m')} = \text{constant, indepdendent of monetary expansion.}$$

The real interest falls with monetary expansion because

$$\rho = \frac{\lambda}{\beta E(\lambda')} - 1. \tag{7.56}$$

When the liquidity constraint is not binding, λ equals the marginal utility of consumption and hence is independent of monetary expansion. When the liquidity constraint binds, $\lambda(s) = \beta A/y\omega$ so the real interest rate falls. Hence, we obtain a lower interest rate on real loans when monetary expansion is high.

The Fisher Relation and the Risk Premium on Nominal Bonds

Consider the ratio

$$\begin{aligned}
\frac{1+i}{1+\rho} &= \left[1+\frac{E(\mu'\pi')}{E\lambda'\pi')}\right] / \left[\frac{\lambda}{\beta E(\lambda')}\right] \\
&= \frac{\beta E[(\lambda'+\mu')\pi'^{"}_{-}E(\lambda')}{\lambda E(\lambda'\pi')} \\
&= \frac{\lambda\pi E(\lambda')}{\lambda E(\lambda'\pi')} = \frac{1}{E(\lambda'\pi')/E(\lambda')}. \qquad (7.57)
\end{aligned}$$

To interpret this result, recall that π'/π is the gross rate of appreciation of money. $(\pi'/\pi)\lambda'$ denotes the version that is weighted by the marginal utility of wealth. Dividing the result by the expected marginal utility of wealth yields the result.

If the simple Fisher relation were to hold,

$$\begin{aligned}
\frac{1+i}{1+\rho} &= E\left(\frac{P'}{P}\right) \text{ expected gross inflation rate} \\
&= E\left(\frac{\pi}{\pi'}\right). \qquad (7.58)
\end{aligned}$$

Therefore the simple Fisher relation does not hold. Let us rewrite

$$\begin{aligned}
\frac{1+i}{1+\rho} &= \left[\frac{E(\pi'/\pi)E(\lambda')}{E(\lambda')} + \frac{Cov(\lambda',\pi'/\pi)}{E(\lambda')}\right]^{[} - 1) \\
&= \left[E(\pi'/\pi) + \frac{Cov(\lambda',\pi'/pi)}{E(\lambda')}\right]^{-1}. \qquad (7.59)
\end{aligned}$$

Recall that

$$\frac{1}{PE(1/P')} \neq \frac{E(P')}{P} = \frac{\pi}{E(\pi')} = \frac{1}{E(\pi')/\pi}, \qquad (7.60)$$

which is the expected gross real rate of monetary appreciation. Thus, the sign of the covariance between the marginal utility of wealth and the gross real rate of appreciation of money determines whether $(1+i)/(1+\rho)$ is less or greater than one over the expected real rate of appreciation of money.

Another parity condition has to do with the expected real return on nominal bonds and the real rate of interest on indexed bonds. Consider first the expected real return on nominal bonds:

$$R = (1+i)E\left(\frac{\pi'}{\pi}\right) - 1. \qquad (7.61)$$

To understand this condition, consider investing one unit of money in nominal bonds at t. This yields $1+i$ units of money at $t+1$. The real value of money at $t+1$ is π' so $(1+i)\pi'$ is the real value of the investment in nominal bonds. The real value of money in period t is π. Hence the expected real return on this investment is defined as in (7.61).

There exists a risk premium on nominal bonds if the expected real rate of return on nominal bonds exceeds the real interest rate on indexed bonds:

$$\begin{aligned}
\frac{1+R}{1+\rho} &= \frac{(1+i)E(\pi'/\pi)}{\lambda/\beta E(\lambda')} \\
&= \frac{\beta\left[E(\mu'\pi')+E(\lambda'\pi')\right]}{E(\lambda'\pi')}\frac{E(\pi')E(\lambda')}{\pi\lambda} \\
&= \frac{E(\pi')E(\lambda')}{E(\lambda'\pi')}. \qquad (7.62)
\end{aligned}$$

Thus,

$$1+R=(1+\rho)\frac{E(\lambda')E(\pi')}{E(\lambda'\pi')},$$

or

$$\begin{aligned}
R-\rho &= (1+\rho)\frac{E(\lambda')E(\pi')}{E(\lambda'\pi')}-1-\rho \\
&= (1+\rho)\frac{E(\lambda')E(\pi')}{E(\lambda'\pi')}-\frac{(1+\rho)E(\lambda'\pi')}{E(\lambda'\pi')} \\
&= -(1+\rho)\frac{Cov(\lambda',\pi'/\pi)}{E(\lambda'\pi'/\pi)}. \qquad (7.63)
\end{aligned}$$

Therefore, the risk premium on nominal bonds depends on the covariance between the rate of appreciation of money and the marginal utility of wealth. If the correlation is negative, then nominal bonds are less attractive than indexed bonds which in equilibrium, requires a higher nominal interest rate relative to the real rate of interest.

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