Local and global rank tests for multivariate varying-coefficient models

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Abstract

In a multivariate varying-coefficient model, the response vectors Y are regressed on known functions $v(X)$ of some explanatory variables X and the coefficients in an unknown regression matrix $\theta(Z)$ depend on another set of explanatory variables Z. We provide statistical tests, called local and global rank tests, which allow to estimate the rank of an unknown regression coefficient matrix $\theta(Z)$ locally at a fixed level of the variable Z or globally as the maximum rank over all levels of Z, respectively. In the case of local rank tests, we do so by applying already available rank tests to a kernel-based estimator of the coefficient matrix $\theta(z)$. Global rank tests are obtained by integrating test statistics used in estimation of local rank tests. We present a simulation study where, focusing on global ranks, we examine small sample properties of the considered statistical tests. We also apply our results to estimate the so-called local and global ranks in a demand system where budget shares are regressed on known functions of total expenditures and the coefficients in a regression matrix depend on prices faced by a consumer.

Keywords: varying-coefficient model, kernel smoothing, matrix rank estimation, demand systems, local and global ranks.

JEL classification: C12, C13, C14, D12.

1 Introduction

1.1 Statement of the problem

Let $(X_i, Z_i) \in \mathbb{R}^p \times \mathbb{R}^q$ be independent variables, $Y_i \in \mathbb{R}^m$ be response variables and $U_i \in \mathbb{R}^m$ be error terms. The focus of this work is on the statistical model

$$Y_i = \Big(\Theta_0(Z_i)\ \Theta(Z_i)\Big) \begin{pmatrix} V_0(X_i) \\ V(X_i) \end{pmatrix} + U_i = \theta(Z_i)v(X_i) + U_i, \quad i = 1, \dots, N, \tag{1.1}$$

where N is the number of observations, $\theta(z)$, $\Theta_0(z)$ and $\Theta(z)$ are unknown $m \times n$, $m \times d_0$ and $m \times d$ matrices of functions of z, respectively, and $v(x)$, $V_0(x)$ and $V(x)$ are known $n \times 1$, $d_0 \times 1$ and $d \times 1$ vectors of functions of x, respectively. We partition the matrix $\theta(z)$ into two submatrices $\Theta_0(z)$ and

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$\Theta(z)$ because we will work with the matrix $\Theta(z)$. When $d_0 = 0$, we suppose by convention that $\Theta_0(z)$ is empty and hence focusing on $\Theta(z)$ is more general than working with the matrix $\theta(z)$. In applications to demand systems below, in particular, we shall suppose $d_0 \neq 0$ and be interested in the matrix $\Theta(z)$ alone. Note also that

$$\Theta(z) = \theta(z) \begin{pmatrix} 0_{d_0 \times d} \\ I_d \end{pmatrix} =: \theta(z)\alpha. \tag{1.2}$$

(The matrix α will be used below.) Important assumptions on the model (1.1) are smoothness of the functions of $\theta(z)$ and non-singularity of the covariance matrix

$$\Sigma = EU_iU_i' \tag{1.3}$$

The rest of the assumptions can be found in Section 2.

The model (1.1) and its slight variants, generally known as *varying-coefficient models*, were considered by Cleveland, Gross and Shyu (1991), Hastie and Tibshirani (1993), Fan and Zhang (1999, 2000), Cai, Fan and Li (2000), Li, Huang, Li and Fu (2002) and others in the context of regression, as well as by many others related to other areas such as longitudinal analysis, nonlinear time series. Several estimation methods for the functions of $\theta(z)$ were proposed. A simple and common choice is a kernel-based estimator

$$\widehat{\theta}(z) = \frac{1}{Nh^q}\sum_{j=1}^{N} Y_j v(X_j)' K\left(\frac{z - Z_j}{h}\right) \left(\frac{1}{Nh^q}\sum_{j=1}^{N} v(X_j)v(X_j)' K\left(\frac{z - Z_j}{h}\right)\right)^{-1}, \tag{1.4}$$

where K is a kernel function and $h > 0$ is a bandwidth.

In this work, we are interested in the model (1.1) with several related objectives in mind. Let $\mathrm{rk}\{A\}$ denote the rank of a matrix A. One of our goals is to address the hypothesis testing problem of $H_0 : \mathrm{rk}\{\Theta(z)\} \le r$ against $H_1 : \mathrm{rk}\{\Theta(z)\} > r$ where r and z are fixed. The problem of testing for the rank of a matrix is well studied. Known and commonly used methods are based on the Lower-Diagonal-Upper triangular decomposition (Gill and Lewbel (1992), Cragg and Donald (1996)), the minimum-χ^2 test statistic (Cragg and Donald (1993, 1996, 1997)) or the idea of the Asymptotic Least Squares (Robin and Smith (1995)). These methods require the asymptotic normality of an estimator of the matrix. As we show, under suitable conditions, the estimator $\widehat{\Theta}(z)$ based on (1.4) is asymptotically normal and hence the aforementioned methods can be applied. Since z is fixed, the statistical tests for the rank of the matrix $\Theta(z)$ will be called *local rank tests*.

We shall also address the problem of *global rank tests*, that is, the hypothesis testing problem of $H_0 : \sup_z \mathrm{rk}\{\Theta(z)\} \le r$ against $H_1 : \sup_z \mathrm{rk}\{\Theta(z)\} > r$ where r is fixed. This problem has not been previously considered to our knowledge. Global rank tests will be based on the statistic obtained by integrating the minimum-χ^2 statistic of the local rank tests over the range of possible values of z. Establishing the asymptotics of the resulting global test statistic is quite involved because the minimum-χ^2 statistic is a nonlinear functional of $\widehat{\Theta}(z)$. In fact, we are not able to find the exact limit of the global test statistic under H_0. We only show that the statistic is asymptotically bounded (stochastically dominated) by the standard normal law. In addition, our present proof works only in the cases $q = \dim(Z) = 1, 2$ or 3. Despite these simplifications, a number of new theoretical difficulties had still to be overcome. Moreover, since we expect the exact limit to be nonstandard (and bounded by the standard limit law), our asymptotic result is quite sufficient from a practical perspective.

1.2 Economic motivation

Our other goal is to apply the obtained local and global rank tests to a demand system. In the context of demand systems, $\widetilde{Y}_i$ are budget shares for j goods, X_i are total expenditures (income, in short) and

Z_i are prices of j goods faced by the ith consumer. The corresponding varying-coefficient model is

$$\widetilde{Y}_i = \widetilde{\theta}(Z_i)v(X_i) + \epsilon_i, \quad i = 1, \ldots, N, \tag{1.5}$$

where, similarly to (1.1), $\widetilde{\theta}(z)$ is a $j \times n$ matrix of unknown functions and $v(x)$ is a $n \times 1$ vector of known functions. The models (1.5) represent the class of deterministic demand systems $\widetilde{y} = f(x,z) = \widetilde{\theta}(z)v(x)$ known as *exactly aggregable demand systems*. These demand systems are important in Economic Theory since they have nice theoretical properties, for example, related to aggregation and representative consumer (Muellbauer (1975, 1976), Gorman (1981) and others), and also since they encompass many well-known examples of demand systems, for example, AIDS, translog, PIGL and others, as their special cases. They have been also widely used in applications (Hausman, Newey and Powel (1995), Banks, Blundell and Lewbel (1997), Nicol (2001) and others).

An important departure from the earlier statistical works on demand systems is that we allow the coefficient matrix $\widetilde{\theta}$ to depend on price variables. Most of the authors exclude variation in prices for simplicity and also because commonly used data sets of demand systems, for example, the Consumer Expenditure Surveys (CEX, in short) data set for the United States, does not contain information on prices. The assumption of constant prices is not realistic. We shall use the CEX data set and assign prices to its households by drawing prices from the American Chamber of Commerce Research Association (ACCRA, in short) data set and by using some location variables in the CEX data set as matching variables. It can be seen from the ACCRA data set that prices are quite different across the United States.

We are interested in estimation of $\mathrm{rk}\{\widetilde{\theta}(z)\}$ and $\sup_z \mathrm{rk}\{\widetilde{\theta}(z)\}$. Following Lewbel (1991), a *local rank at* z of a demand system $\widetilde{y} = f(x,z) = (f_1(x,z) \ldots f_j(x,z))'$ is defined as the dimension of the function space spanned by the coordinate functions of $f(x,z)$ for fixed z. A *global rank* of a demand system is defined as the maximum of local ranks over all possible values of z. It involves simple algebra to see that a local rank $\mathrm{rk}\{f(\cdot,z)\}$ of the exactly aggregable demand system $\widetilde{y} = f(x,z) = \widetilde{\theta}(z)v(x)$ is equal to $\mathrm{rk}\{\widetilde{\theta}(z)\}$ when $v(x)$ consists of linearly independent functions of x (Proposition C.2 below). Some theoretical studies on ranks can be found in Gorman (1981), Lewbel (1991, 1997), and others. In particular, Gorman (1981) showed that exactly aggregable demand systems, when derived through a utility maximization principle, have always rank less than or equal to 3.

To estimate $\mathrm{rk}\{\widetilde{\theta}(z)\}$ and $\sup_z \mathrm{rk}\{\widetilde{\theta}(z)\}$, observe, however, that one cannot readily apply the local and global rank tests under the model (1.1). Since the elements of $\widetilde{Y}_i$ are budget shares, they add up to 1 and hence the covariance matrix of ϵ_i is singular whereas that of U_i in (1.3) is assumed nonsingular. To avoid singularity, one commonly drops one share of goods from the analysis which allows to assume a nonsingular covariance matrix of the reduced error terms. To estimate $\mathrm{rk}\{\widetilde{\theta}(z)\}$, it is then necessary to be able to relate $\mathrm{rk}\{\widetilde{\theta}(z)\}$ to some characteristic of the matrix $\widetilde{\theta}(z)$ with one row eliminated. When $v(x) = (1\ V(x)')'$ as typically assumed in practice, we have that

$$\mathrm{rk}\{\widetilde{\theta}(z)\} = \mathrm{rk}\{\Theta(z)\} + 1, \tag{1.6}$$

where $\Theta(z)$ is the matrix $\widetilde{\theta}(z)$ with an arbitrary row and the first column eliminated (see Proposition C.2 below). In view of this relation, we will therefore eliminate one budget share from $\widetilde{Y}_i$ in the analysis to obtain a vector Y_i, estimate $\mathrm{rk}\{\Theta(z)\}$ and $\sup_z \mathrm{rk}\{\Theta(z)\}$ in the model

$$Y_i = \Big(\Theta_0(Z_i)\ \Theta(Z_i)\Big) \begin{pmatrix} 1 \\ V(X_i) \end{pmatrix} + U_i, \tag{1.7}$$

assuming that the covariance matrix of U_i is nonsingular, and then obtain estimates of $\mathrm{rk}\{\widetilde{\theta}(z)\}$ and $\sup_z \mathrm{rk}\{\widetilde{\theta}(z)\}$ by adding 1. We shall apply this estimation procedure to estimate local and global ranks in the demand system constructed from the CEX and the ACCRA data sets. Related estimation of local and global ranks in a demand system given by a nonparametric model can be found in Fortuna (2004*a*, 2004*b*).

1.3 Outline of the paper

The rest of the paper is organized as follows. In Section 2, we state the assumptions which are used in connection to the model (1.1). In Section 3, we introduce an estimator for the matrix $\Theta(z)$ based on (1.4) and, in particular, state its asymptotic normality result. Local and global rank tests for the matrices $\Theta(z)$ are studied in Sections 4 and 5, respectively. Simulation experiment is presented in Section 6. Estimation of local and global ranks in a demand system can be found in Section 7. Most of technical proofs are postponed till Appendices A, B, and some auxiliary results are given in Appendix C.

2 Assumptions

We shall use the following assumptions on the variables X_i, Z_i and U_i, on the functions θ and v, and on the kernel K. Some of these assumptions are in the spirit of those used by Donald (1997), Fortuna (2004*a*, 2004*b*) in connection to rank estimation for nonparametric models.

Assumption 1: The function K is a symmetric kernel on $\mathbb{R}^q$ of order s, that is, K has a compact support, is bounded and satisfies the following conditions: (i) $\int_{\mathbb{R}^q} K(z)dz = 1$ and (ii) $\int_{\mathbb{R}^q} z^b K(z)dz = 0$ for any $b \in (\mathbb{N}\bigcup\{0\})^q$ satisfying $1 \leq |b| < s$, where $z^b = z_1^{b_1} \dots z_q^{b_q}$ with $z = (z_1, \dots, z_q)$, $b = (b_1, \dots, b_q)$ and $|b| = b_1 + \dots + b_q$. Suppose also that K is Lipschitz.

There are many possible choices for such kernels K. In the simulation experiment and the application below, we use the popular Epanechnikov kernel $K(z) = 3(1 - z^2)/4$, for $|z| \leq 1$, of the order $s = 2$. The kernel $K(z) = 15(7z^4 - 10z^2 + 3)/32$, for $|z| \leq 1$, of the order $s = 4$, is another possibility.

Assumption 2: Suppose that $(X_i, Z_i) \in \mathbb{R}^p \times \mathbb{R}^q$, $i = 1, \dots, N$, are i.i.d. random vectors such that the support of (X_i, Z_i), denoted by $\mathcal{H}_x \times \mathcal{H}_z$, is the Cartesian product of compact intervals $\mathcal{H}_x = [a_1, b_1] \times \dots \times [a_p, b_p]$ and $\mathcal{H}_z = [c_1, d_1] \times \dots \times [c_q, d_q]$, and (X_i, Z_i) are continuously distributed with a density $p(x, z)$. Suppose that the density $p(x, z)$ has an extension to $\mathbb{R}^p \times \mathbb{R}^q$ with $t \geq s$ continuous bounded derivatives in the variable z.

Assumption 3: Suppose that the error terms U_i, $i = 1, \dots, N$, are i.i.d. random vectors, independent of the sequence (X_i, Z_i) and such that $EU_i = 0$ and

$$EU_iU_i' = \Sigma, \tag{2.1}$$

where Σ is a $m \times m$ positive definite matrix. Suppose also that $E|U_i|^u < \infty$ where $u \geq 4$.

Local rank tests can also be obtained under a weaker, heteroscedasticity assumption on U_i, that is, $E(U_iU_i'|X_i = x, Z_i = z) = \Sigma(x, z)$. Under the stronger condition (2.1), the limit covariance matrix in the asymptotic normality result for $\widehat{\theta}(z)$ has a convenient Kronecker product structure. The proof of the global rank tests uses this Kronecker product structure and hence the stronger condition (2.1).

Assumption L4: In the case of local rank tests, the function $\theta : \mathcal{H}_z \to \mathbb{R}^{mn}$ is such that each of its component functions has an extension to $\mathbb{R}^q$ with $t \geq s$ continuous bounded derivatives. The component functions of $v(x)$ have extensions to $\mathbb{R}^p$ which are bounded. **Assumption G4:** In the case of global rank tests, suppose in addition that the component functions of $\theta(z)$ are real analytic on $\mathcal{H}_z$ (see the discussion below).

Assuming smoothness (i.e. continuity of derivatives of some order) of the function $\theta(z)$ is standard for varying-coefficient models (see the references provided in Section 1.1). The assumption of analytic $\theta(z)$ for global rank tests is less common and requires further explanation. According to one possible definition, a function f is analytic if its Taylor series converges to the function f at a neighborhood of each point. We assume analyticity just in order to have smoothness of the eigenvectors of some analytic matrices involving $\theta(z)$. It is well known that smoothness of a matrix is not sufficient to have smooth eigenvectors (see, for example, Kato (1976), Bunse-Gerstner, Byers, Mehrmann and Nichols (1991)).

Assumption L5: The $n \times n$ matrix

$$\psi(z) = p(z)E(v(X_1)v(X_1)'|Z_1 = z) = \int_{\mathbb{R}^p} v(x_1)v(x_1)'p(x_1, z)dx_1 \tag{2.2}$$

is positive definite. **Assumption G5:** In the case of global rank tests, suppose in addition that the components of $\psi(z)$ are real analytic.

In addition to (2.2), we shall also use the $d \times d$ matrix $\Psi(z)$ such that

$$\Psi(z)^{-1} = \alpha'\psi(z)^{-1}\alpha, \tag{2.3}$$

where α is defined in (1.2). Observe that, if the matrix $\psi(z)$ is positive definite, then $\Psi(z)$ is positive definite and, if $\psi(z)$ is real analytic, then $\Psi(z)$ is also real analytic.

Assumption G6: Suppose that $q = 1, 2$ or 3.

In fact, our proof for global rank tests depends on this assumption. We do not expect that the assumption can be removed unless an alternative proof is found.

3 Kernel-based estimator

Let K be a kernel defined in Assumption 1 of Section 2, and set $K_h(z) = h^{-q}K(h^{-1}z)$, where $h > 0$ is a bandwidth. We shall use throughout a kernel-based estimator $\widehat{\theta}(z)$ for the matrix $\theta(z)$ defined by (1.4). The estimator $\widehat{\theta}(z)$ can also be expressed as

$$\widehat{\theta}(z) = YD_zv'(vD_zv')^{-1} =: \frac{1}{N}YD_zv'\widehat{\psi}(z)^{-1}, \tag{3.1}$$

where $Y = (Y_1 \ \dots \ Y_N), v = (v(X_1) \ \dots \ v(X_N))$ and $D_z = \mathrm{diag}\{K_h(z - Z_1), \dots, K_h(z - Z_N)\}$. A more general estimator for $\theta(z)$ can be defined based on the idea of local linear regression (Fan and Gijbels (1996), Fan and Zhang (1999)). We work with the estimator (3.1) for proof simplicity, especially in the context of global rank tests.

Define the estimator $\widehat{\Theta}(z)$ for the submatrix $\Theta(z)$ of $\theta(z)$ by using (1.2) as $\widehat{\Theta}(z) = \widehat{\theta}(z)\alpha$. The following result establishes the asymptotic normality of $\widehat{\Theta}(z)$. The proof can be found in Appendix A. The notation $\|K\|_2^2$ below stands for $\int_{\mathbb{R}^q} |K(z)|^2 dz$.

Theorem 3.1 *Under Assumptions 1–3, L4, L5 of Section 2, we have, for fixed z,*

$$\sqrt{Nh^q}\,\mathrm{vec}(\widehat{\Theta}(z) - \Theta(z)) \xrightarrow{d} \mathcal{N}(0, \Omega(z)), \tag{3.2}$$

as

$$N \to \infty, \quad h \to 0, \quad Nh^q \to \infty \quad \text{and} \quad Nh^{q+2s} \to 0, \tag{3.3}$$

with

$$\Omega(z) = (\Psi(z)^{-1} \otimes \Sigma)\|K\|_2^2, \tag{3.4}$$

where the matrix $\Psi(z)$ is defined by (2.3). Suppose in addition that $N^{1-2/u}h^q/\ln N \to \infty$. Then, the limiting covariance matrix $\Omega(z)$ in (3.4) can be estimated consistently by

$$\widehat{\Omega}(z) = (\widehat{\Psi}(z)^{-1} \otimes \widehat{\Sigma})\|K\|_2^2, \tag{3.5}$$

where $\widehat{\Psi}(z)^{-1} = \alpha'\widehat{\psi}(z)^{-1}\alpha$, *and*

$$\widehat{\Sigma} = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \widehat{\theta}(Z_i)v(X_i))(Y_i - \widehat{\theta}(Z_i)v(X_i))'. \tag{3.6}$$

4 Local rank tests

We consider here the hypothesis testing problem of $H_0 : \mathrm{rk}\{\Theta(z)\} \le r$ against $H_1 : \mathrm{rk}\{\Theta(z)\} > r$, where z and r are fixed. Since $\widehat{\Theta}(z)$ is an asymptotically normal estimator of $\Theta(z)$ and the related covariance matrix $\Omega(z)$ can be consistently estimated by $\widehat{\Omega}(z)$ in (3.5) (Theorem 3.1), this problem can be readily addressed by one of the rank tests available in the literature. Three such tests, mentioned in Section 1.1, are the LDU-based test, the minimum-χ^2 test and the ALS-based test. We recall below the minimum-χ^2 test only because its integrated version will be used for global rank tests.

Minimum-χ^2 rank test: Applied to the matrix $\widehat{\Theta}(z)$ with the covariance matrix $\widehat{\Omega}(z)$, the minimum-χ^2 test is based on the statistic

$$\begin{aligned}\widehat{T}(r,z) &= Nh^q \min_{\mathrm{rk}\{\Theta\}\le r} \mathrm{vec}(\widehat{\Theta}(z) - \Theta)'\widehat{\Omega}(z)^{-1}\mathrm{vec}(\widehat{\Theta}(z) - \Theta) \\ &= Nh^q\|K\|_2^{-2}\sum_{i=1}^{m-r}\widehat{\lambda}_i(z),\end{aligned} \tag{4.1}$$

where $0 \le \widehat{\lambda}_1(z) \le \ldots \le \widehat{\lambda}_m(z)$ are the ordered eigenvalues of the matrix

$$\widehat{\Gamma}(z) = \widehat{\Sigma}^{-1}\widehat{\Theta}(z)\widehat{\Psi}(z)\widehat{\Theta}(z)'. \tag{4.2}$$

The last equality in (4.1) is standard for the covariance matrix $\widehat{\Omega}(z)$ having a Kronecker product structure, and can be proved as, for example, Theorem 3 in Cragg and Donald (1993).

The next result follows from Cragg and Donald (1997), and Robin and Smith (2000). Let $\mathcal{Y}_{a\times b}$ be a $a \times b$ matrix with independent $\mathcal{N}(0,1)$ entries and set $\mathcal{X}_{a,b} = \mathcal{Y}_{a\times b}\mathcal{Y}'_{a\times b}$. Let also $\lambda_1(\mathcal{X}_{a,b}) \le \ldots \le \lambda_a(\mathcal{X}_{a,b})$ be the ordered eigenvalues of the matrix $\mathcal{X}_{a,b}$. The notation $\chi^2(k)$ below stands for a χ^2-distribution with k degrees of freedom, and a stochastic dominance $\xi \le_d \eta$ means that $P(\xi > x) \le P(\eta > x)$ for all $x \in \mathbb{R}$.

Theorem 4.1 *Under the Assumptions of Theorem 3.1, we have:*

(*i*) *when* $r < \mathrm{rk}\{\Theta(z)\}$,

$$\lim(p)\,\widehat{T}(r,z) = +\infty, \tag{4.3}$$

(*ii*) *when* $r \ge \mathrm{rk}\{\Theta(z)\} =: l(z)$,

$$\lim(d)\,\widehat{T}(r,z) = \sum_{i=1}^{m-r}\lambda_i(\mathcal{X}_{m-l(z),d-l(z)}) \overset{d}{\le} \chi^2((m-r)(d-r)), \tag{4.4}$$

where the inequality $\le_d$ *becomes the equality* $=_d$ *for* $r = \mathrm{rk}\{\Theta(z)\}$.

Theorem 4.1 can be used to test for $H_0 : \mathrm{rk}\{\Theta(z)\} \le r$ against $H_1 : \mathrm{rk}\{\Theta(z)\} > r$ in a standard way. The resulting local rank tests can be used to estimate $\mathrm{rk}\{\Theta(z)\}$, for example, by using a sequential procedure (see, for example, Donald (1997), Robin and Smith (2000)).

5 Global rank tests

Let $\widehat{T}(r,z)$ be the minimum-χ^2 statistic defined by (4.1) and used in the local rank tests. Consider the statistic

$$\widehat{T}_{glb}(r) = \frac{\int_{\mathcal{H}_z} \widehat{T}(r,z)dz - |\mathcal{H}_z|(m-r)(d-r)}{h^{q/2}|\mathcal{H}_z|^{1/2}\sqrt{(m-r)(d-r)}} \frac{\|K\|_2^2}{\|\overline{K}\|_2}, \tag{5.1}$$

where $|\mathcal{H}_z| = \int_{\mathcal{H}_z} dz$ and $\overline{K}(z) = \int K(w)K(z+w)dw$ is the convolution kernel derived from K.

The next result establishes the asymptotics of the statistic $\widehat{T}_{glb}(r)$. We write $\limsup(d)\,\widehat{\xi} \leq_d \eta$ if $\limsup P(\widehat{\xi} > x) \leq P(\eta > x)$ for all $x \in \mathbb{R}$.

Theorem 5.1 *Under the Assumptions 1–3, G4, G5 and G6 of Section 2, and when*

$$Nh^{q/2} \to \infty, \quad N^{1-2/u}h^{q+2}/\ln N \to \infty, \quad Nh^{q+2s} \to 0, \quad h^{2-q/2}\ln N \to 0,$$

we have:

(*i*) *when* $r < \sup_{z\in\mathcal{H}_z} \mathrm{rk}\{\Theta(z)\}$,

$$\lim(p)\,\widehat{T}_{glb}(r) = +\infty, \tag{5.2}$$

(*ii*) *when* $r \geq \sup_{z\in\mathcal{H}_z} \mathrm{rk}\{\Theta(z)\}$,

$$\limsup(d)\,\widehat{T}_{glb}(r) \overset{d}{\leq} \mathcal{N}(0,1), \tag{5.3}$$

where $\limsup$ *becomes* $\lim$ *and the inequality* $\leq_d$ *becomes the equality* $=_d$ *when* $r = \mathrm{rk}\{\Theta(z)\}$ *for all* $z \in \mathcal{H}_z$.

Theorem 5.1 is proved in Appendix B. It can be used to test for $H_0 : \sup_{z\in\mathcal{H}_z} \mathrm{rk}\{\Theta(z)\} \leq r$ against $H_1 : \sup_{z\in\mathcal{H}_z} \mathrm{rk}\{\Theta(z)\} > r$ in a standard way. (In practice, the integral over z in (5.1) is approximated by a sum and $\mathcal{H}_z$ is replaced by the support of the data Z_i.) Several remarks regarding Theorem 5.1 are in place.

Remark 5.1 Observe that the term $|\mathcal{H}_z|$ appearing in (5.1) twice is consistent with a linear transformation of the data Z_i. For example, when $m = 1$ and $\mathcal{H}_z = [a,b]$, we have

$$\int_a^b \widehat{T}(L,z)dz = (b-a)\int_0^1 \widehat{T}^*(L,w)dw, \tag{5.4}$$

where $\widehat{T}^*(L,w)$ is defined as $T(L,z)$ by using the data $W_i = (Z_i - a)/(b-a)$ and the bandwidth $h/(b-a)$. Then, in view of (5.1), $\widehat{T}_{glb}(L) = \widehat{T}^*_{glb}(L)$, where the latter is defined by (5.1) using the data $W_i = (Z_i - a)/(b-a)$ and the bandwidth $h/(b-a)$.

Remark 5.2 Observe also that the centering term $|\mathcal{H}_z|(m-r)(d-r)$ and the normalization $h^{q/2}|\mathcal{H}_z|^{1/2}\sqrt{(m-r)(d-r)}$ used in (5.1) are meaningful. Suppose for instance, that $m = 1$ and $\mathcal{H}_z = [0,1]$. The statistic $\widehat{T}(r,z)$ can be thought as independent $\chi^2_k((m-r)(d-r))$ over disjoint intervals $[(k-1)h, kh)$, and constant within each of the intervals. Hence,

$$\int_0^1 \widehat{T}(r,z)dz \overset{d}{\approx} h\sum_{k=1}^{h^{-1}} \chi^2_k((m-r)(d-r)) \overset{d}{=} h\,\chi^2((m-r)(d-r)h^{-1}), \tag{5.5}$$

which has mean $(m-r)(d-r)$ and variance $h(m-r)(d-r)$.

6 Simulation study

Using Monte Carlo simulations, we examine here the performance of our proposed rank tests. For shortness sake and since usual rank tests (in particular, local rank tests) have already been studied, we shall focus only on global rank tests. We shall use the model of the type (1.1) given by

$$Y_i = \delta\Theta(Z_i)V(X_i) + U_i, \quad i = 1, \dots, N, \tag{6.1}$$

where we suppose, for simplicity, $\theta(z) = \Theta(z)$ (or $n = d$) in (1.1). The sequence (X_i, Z_i) consists of independent random vectors with X_i and Z_i being independent and uniformly distributed on $[0, 1]$ and $[-1, 1]$, respectively. The sequence U_i consists of independent $\mathcal{N}(0, 1)$ random variables. $\delta = 1/2$ or $\delta = 1/4$ is the signal-to-noise ratio. The sample size is $N = 500$ or $N = 1000$.

The coefficient matrix $\Theta(z)$ and the vector of regressors $V(x)$ are given by

$$\Theta(z) = \begin{pmatrix} 1 & 1+z^2 & z & 1 \\ 0 & \frac{1}{2} - z & \frac{1}{2} - z & z(2z-1) \\ 0 & 0 & z(1-2z) & z(\frac{1}{2} - z) \\ 0 & 0 & 0 & 0 \end{pmatrix} D, \quad V(x) = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}. \tag{6.2}$$

The symmetric, positive-definite matrix $D = \Psi^{-1/2} \equiv \Psi(z)^{-1/2}$ is such that $D\Psi(z)D = I_4$ with $\Psi(z)$ given by (2.3) (or (2.2) in our case). Using such matrix D is standard in simulation studies for rank tests. Its role can be explained as follows. The matrix D ensures that the non-zero eigenvalues of the matrix $\Gamma(z)$ given by (4.2) without the hats, are not too close to zero. From another perspective, it ensures that, for fixed z, coordinate functions of $\Theta(z)V(x)$ appear quite different when plotted as functions of x. (See Figure 1 below.)

We are interested in testing for $\sup_z \text{rk}\{\Theta(z)\}$ through the global rank tests. Observe that $\sup_z \text{rk}\{\Theta(z)\} = 3$ and, in particular, $\text{rk}\{\Theta(1/2)\} = 1$, $\text{rk}\{\Theta(0)\} = 2$ and $\text{rk}\{\Theta(-1/2)\} = 3$. In Figure 1, we plot the coordinate functions of $\Theta(z)V(x)$ at $z = -1/2$, 0 and 1/2. Adding the noise U_i and taking into account the signal-to-noise ratio δ, the reader may easily visualize how much noise is present in data. Note that $\delta = 1/2$ (1/4, resp.) corresponds to a moderate (large, resp.) amount of noise. Note also that, when the noise is added, one can informally think of the local ranks as changing from 1 to 3 when z moves from 1 to -1.

We shall now examine the performance of the global rank tests through a number of PP-plots in Figures 2-3. These plots have probability $p \in (0, 1)$ on the vertical axis against $\alpha_r(p) = P(\widehat{T}_{glb}(r) > c_r(p))$ on the horizontal axis. Throughout the simulation study, the probability $\alpha_r(p)$ is computed

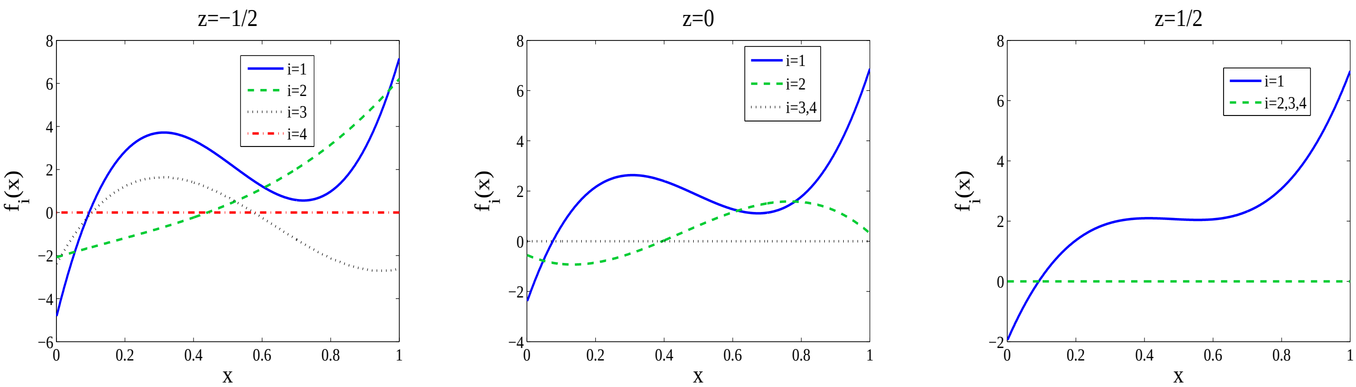

Figure 1: Coordinate functions $f_i(x)$ in $f(x) = (f_i(x)) = \Theta(z)V(x)$ for $z = -1/2$, 0 and 1.

based on 500 Monte Carlo replications, and $c_r(p)$ is the nominal critical value such that $P(\mathcal{N}(0,1) > c_r(p)) = 1-p$. In kernel smoothing, we use a popular Epanechnikov kernel, and choose the smoothing parameter $h = 0.1$, 0.2 or 0.4. If $\widehat{T}_{glb}(r)$ has the limiting distribution $\mathcal{N}(0,1)$, then $\alpha_r(p) = p$. The case $\alpha_r(p) > p$ ($\alpha_r(p) < p$, resp.) corresponds to $\widehat{T}_{glb}(r) \geq_d \mathcal{N}(0,1)$ ($\widehat{T}_{glb}(r) \leq_d \mathcal{N}(0,1)$, resp.).

In Figure 2, we provide the PP-plots for various combinations of the considered values of the parameters N, δ, h and r. In particular, the PP-plots in the first column correspond to testing for the rank $r = 2$, and those in the second column correspond to testing for $r = 3$. (Testing for $r = 4$ is meaningless since matrices $\Theta(z)$ are 4×4. The PP-plots for the cases $r = 1$ and 0 would appear as the corresponding ones for $r = 2$ with the graphs for the 3 values of h stretching even more along the bottom-right corner.) Several basic observations can be made at this point. The plots in the second column (see the bottom-left corner) show that the global rank tests are undersized, even for a large sample size such as $N = 1000$. This is perhaps not surprising in view of our limiting result in Theorem 5.1, (ii), showing that $\widehat{T}_{glb}(r)$ is asymptotically dominated by (and not convergent to) $\mathcal{N}(0,1)$. Observe also that, as the signal-to-noise ratio δ decreases (more noise), the test is obviously more likely to accept a global rank which is too low. Note also that, when h decreases, accepting too low a rank is more likely as well. For fixed z, smaller h means averaging over fewer data points, leading to a smaller value of a test statistic. Smaller test values over z lead to smaller averaged global test value, leading to a smaller global rank.

In terms of size, larger h (oversmoothing) performs better in all plots of Figure 1. Focusing on the first column, oversmoothing also leads to greater power. Though observe that these are not size-adjusted powers. Taking size-adjusting into account would improve the power for smaller h. If the size of test is better when oversmoothing, and the power is better or about the same, we should perhaps use larger h in global rank tests, corresponding to oversmoothing. Note, however, that this may also be the result of our particular model where the local rank changes relatively slowly as z moves from -1 to 1.

Consider now Figure 3 where, in the same PP-plots, we compare the results for $N = 500$ and 1000. Interestingly, note that the plots for the two sample sizes are close together, especially for $h = 0.4$. This supports our previous conjecture that the global test statistic $\widehat{T}_{glb}(r)$ has a nonstandard limit (dominated by $\mathcal{N}(0,1)$ according to Theorem 5.1, (ii)).

In conclusion to this section, we find the simulation results positively surprising. Observe from Figure 1 that the test does quite well even in the "worst" case $N = 500$, $\delta = 1/4$. Despite intrinsic mathematical proofs, the test therefore seems quite practical, at least for the types of models (6.1)–(6.2) considered here.

7 Application to demand system

We apply here the introduced global rank test to estimate the global rank of a demand system. See Section 1.2 for motivation, discussion and relevant notation. In the data set used here, budget shares $\widetilde{Y}_i$ and income X_i are taken from the U.S. CEX micro data of the first quarter of 2000.[1] We consider only those households which contain married couples, whose tenure status is renter household or homeowner with or without mortgage, whose age of the head is between 25 and 60, and whose total income is between \$3,000 and \$75,000. (We also only consider households in the so-called metropolitan statistical areas because we can associate prices only to these households.) The total number of households which met these criteria was $N = 897$ (out of 7860 in the CEX data set). The number of the budget shares

[1]U.S. Dep. of Labor, Bureau of Labor Statistics. Consumer Expenditure Survey, 1999: Interview Survey and Detailed Expenditure Files [Computer file]. Washington, DC: U.S. Dept. of Labor, Bureau of Labor Statistics [producer], 2001. Ann Arbor, MI: Inter-University Consortium for Political and Social Research [distributor], 2001.

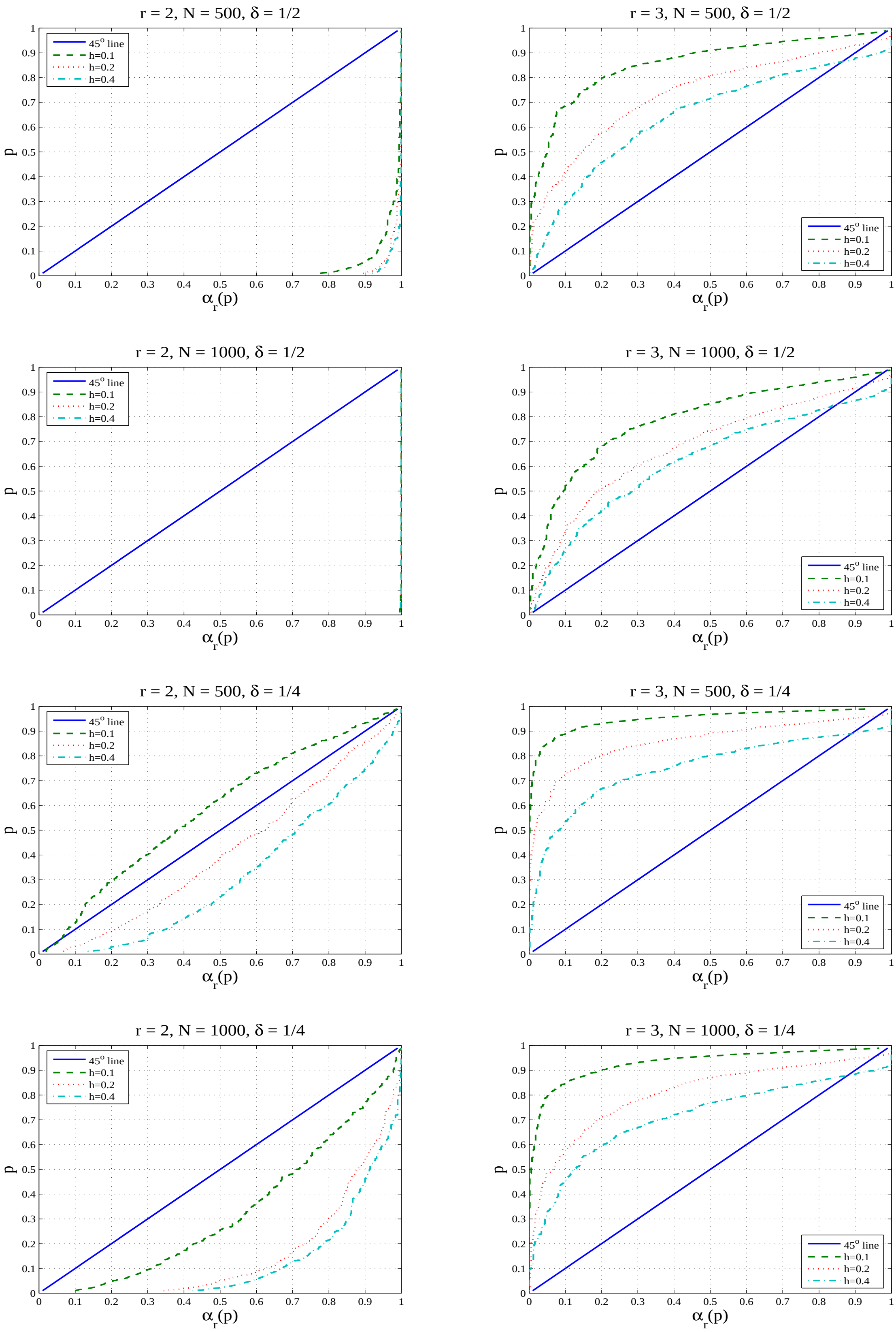

Figure 2: PP-plots in a simulation study.

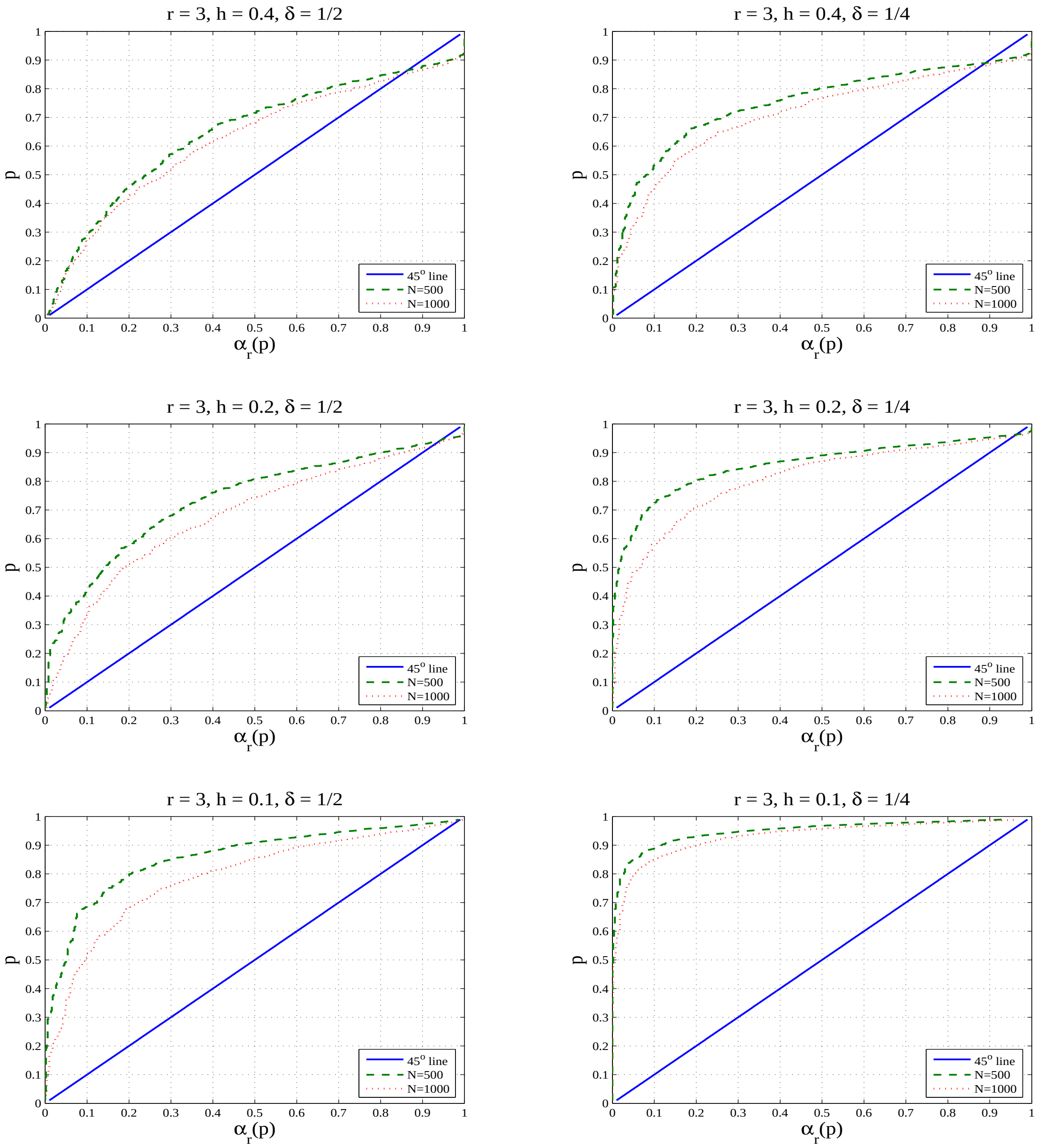

Figure 3: PP-plots in a simulation study.

considered is $j = 6$. They are expenditures on food, health care, transportation, household, apparel (clothing) and miscellaneous goods.

The prices Z_i are drawn from the ACCRA data set.[2] ACCRA provides a composite price index and also price indices for 6 different categories of goods for various cities across the U.S. We use throughout only a composite price index and therefore have $\dim(Z_i) = q = 1$. The prices Z_i are assigned to (Y_i, X_i) by using some location variables in the CEX data set as matching variables, and also some confidential information kindly provided by the Bureau of Labor Statistics. (Details on matching procedure are available from the authors upon request.) In the data set constructed, the values of Z_i range from $z_{\min} = 0.911$ to $z_{\max} = 1.251$.

We estimate the global rank in the considered demand system as described at the end of Section 1.2. First, one budget share is eliminated from the data set, leading to vectors Y_i consisting of $6-1 = 5$ budget shares. (The test statistic is invariant to which share is eliminated.) Then, the model (1.7) is assumed, where we take

$$V(x) = \begin{pmatrix} 1 \\ \ln x \\ (\ln x)^2 \\ (\ln x)^3 \end{pmatrix}. \tag{7.1}$$

The global rank of the original (full) demand system is estimated by adding 1 (see (1.6)) to the estimated global rank of the model (1.7).

The results of estimation for the original demand system are reported in Table 1. We use the data interval $[z_{\min}, z_{\max}] = [0.911, 1.251]$ for the range of z, and try several values $h = 0.05$, 0.10 and 0.15 for the smoothing parameter h.

Global rank estimation				
h	$r = 1$	$r = 2$	$r = 3$	$r = 4$
0.05	70.1211	11.3627	-2.7714	-2.5609
0.10	88.7294	18.1172	-1.9534	-1.8108
0.15	100.6408	22.4453	-1.9972	-1.4785

Table 1: Values of test statistic $T_{glb}(r)$ in global rank estimation for a demand system.

Note from Table 1 that the estimated global rank is 3 for all considered values of h. This perhaps should not be surprising. The same rank can also be found in other work on ranks in demand systems: Lewbel (1991), Donald (1997) where variation in prices is ignored, and Fortuna (2004*b*) where local ranks are considered.

Global rank estimation				
h	$r = 1$	$r = 2$	$r = 3$	$r = 4$
0.05	17.9940	0.7217	-1.9894	-1.8108
0.10	30.2089	6.2713	-1.1757	-1.2805
0.15	42.4361	10.2029	-1.2834	-1.0455

Table 2: Values of test statistic $T_{glb}(r)$ in global rank estimation for a demand system.

Note that Table 1 above is based on the interval $[z_{\min}, z_{\max}] = [0.911, 1.251]$ spanning the whole range of the values of Z_i. It may be interesting to see whether global rank remains the same over

[2]ACCRA Cost of Living Index, Data for First Quarter 2000, ACCRA, July 2000, **33**(1). For more information, see http://www.accra.org

smaller subintervals of $[z_{\min}, z_{\max}]$. Experimenting with several possibilities of such subintervals, we generally found the global rank to be the same (equal to 3). A potential, interesting exception to the rule, however, seems to be the case of larger values of z. Table 2 presents global rank estimation results over the interval $[1.081, 1.251]$. Observe that the global rank is estimated as 2 for the smallest considered value of h.

The statement above is also supported by local rank tests at larger values of z. For example, Table 3 gives the P-values for local rank tests at $z = 1.2$. Note that, except $h = 0.15$, the test points to the local rank 2 at $z = 1.2$. Local rank smaller than 3 for larger values of z was also found in the same data set but using nonparametric model by Fortuna (2004*b*). To complement Table 1, Table 4 presents the P-values for the local rank test at $z = 1$. Tests for all the considered values of h suggest the local rank 3 at $z = 1$.

Local rank estimation at $z = 1.2$				
h	$r = 1$	$r = 2$	$r = 3$	$r = 4$
0.05	0	0.4645	0.5522	1
0.10	0	0.1706	0.8072	1
0.15	0	0.0067	0.9671	1

Table 3: P-values for local rank test at $z = 1.2$.

Local rank estimation at $z = 1$				
h	$r = 1$	$r = 2$	$r = 3$	$r = 4$
0.05	0	0	0.876	1
0.10	0	0	0.9938	1
0.15	0	0	0.9943	1

Table 4: P-values for local rank test at $z = 1$.

A Technical proofs for asymptotic normality

PROOF OF THEOREM 3.1: It is enough to prove the theorem for $d_0 = 0$, that is, $\Theta(z) = \theta(z)$, $\Psi(z) = \psi(z)$, $\Omega(z) = w(z) = (\psi(z)^{-1} \otimes \Sigma)\|K\|^2$ and similar expressions with the hats. The proof is straightforward but we outline it for completeness. Observe that

$$\widehat{\theta}(z) = \theta(z) + (\widehat{\Delta}_1(z) + \widehat{\Delta}_2(z))\ \widehat{\psi}(z)^{-1}, \tag{A.1}$$

where

$$\widehat{\Delta}_1(z) = \frac{1}{N}\sum_{i=1}^{N}(\theta(Z_i) - \theta(z))v(X_i)v(X_i)'K_h(z - Z_i), \quad \widehat{\Delta}_2(z) = \frac{1}{N}\sum_{i=1}^{N} U_i v(X_i)'K_h(z - Z_i).$$

To prove the theorem, it is enough to show that

$$\widehat{\psi}(z) \xrightarrow{p} \psi(z) \tag{A.2}$$

$$\widehat{\Delta}_1(z) = o_p\Big((Nh^q)^{-1/2}\Big), \tag{A.3}$$

$$(Nh^q)^{1/2}\widehat{\Delta}_2(z) \xrightarrow{d} \mathcal{N}(0, w_0(z)), \tag{A.4}$$

$$\widehat{\Sigma} \xrightarrow{p} \Sigma, \tag{A.5}$$

where $w_0(z) = (\psi(z) \otimes \Sigma)\|K\|^2$. The convergence (A.5) follows from Proposition C.4 below.

The convergence (A.2) is standard. Letting $M^2 = MM'$ for a matrix M, consider

$$E(\widehat{\psi}(z) - \psi(z))^2 = E\widehat{\psi}(z)^2 - E\widehat{\psi}(z)\psi(z)' - \psi(z)E\widehat{\psi}(z)' + \psi(z)^2.$$

Since $E\widehat{\psi}(z) = \int_{\mathrm{IR}^n} \{\int_{\mathrm{IR}^m} v(x_1)v(x_1)'p(x_1, z_1)K_h(z - z_1)dz_1\}\, dx_1$, by applying Proposition C.1 to the integral in the braces, we obtain that $E\widehat{\psi}(z) = \int_{\mathrm{IR}^n} v(x_1)v(x_1)'p(x_1, z)dx_1 + O(h^s) = \psi(z) + O(h^s)$. As for $E\widehat{\psi}(z)^2$, by using independence of (X_i, Z_i) and (X_j, Z_j) for $i \neq j$, we have

$$E\widehat{\psi}(z)^2 = \frac{\|K\|_2^2}{Nh^q} E\Big((v(X_1)v(X_1)')^2 K_{2,h}(z - Z_1)\Big) + \frac{N-1}{N}\Big(E(v(X_1)v(X_1)'K_h(z - Z_1)\Big)^2.$$

with the kernel $K_2(z) = K(z)^2/\|K\|_2^2$. By using Proposition C.1, the first term above is of the order $O((Nh^q)^{-1})$. The order of the second term is that of $(E\widehat{\psi}(z))^2 = \psi(z)^2 + O(h^s)$. Combining all asymptotic relations above yields $\widehat{\psi}(z) = \psi(z) + O_p(h^s + (Nh^q)^{-1/2})$.

To show (A.3), suppose for simplicity that $m = n = 1$. By using Proposition C.1, $E\widehat{\Delta}_1(z) = O(h^s)$. Similarly,

$$\mathrm{Var}(\widehat{\Delta}_1(z)) = \|K\|_2^2(Nh^q)^{-1}E\Big((\widehat{\theta}(Z_i) - \theta(z))^2 v(X_i)^2 K_{2,h}(z - Z_i)\Big) = O\Big((Nh^q)^{-1}h^2\Big).$$

Hence, $E(\widehat{\Delta}_1(z))^2 = O((Nh^q)^{-1}h^2 + h^{2s}) = o((Nh^q)^{-1})$.

To show (A.4), write $(Nh^q)^{1/2}\mathrm{vec}(\widehat{\Delta}_2(z))) = N^{-1/2}\sum_{i=1}^N \xi_{N,i}$ with $\xi_{N,i} = h^{q/2}(v(X_i)\otimes I_G)U_i K_h(z - Z_i)$. By the Cramér-Wold theorem, we need to show that

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N \eta_{N,i} \xrightarrow{d} \mathcal{N}(0, \omega_1(z)), \tag{A.6}$$

where $\eta_{N,i} = \lambda'\xi_{N,i}$, $\lambda \in \mathrm{IR}^{mn} \setminus \{0\}$ is an arbitrary vector and $\omega_1(z) = \lambda'\omega_0(z)\lambda$. By using the Lyapunov's version of Central Limit Theorem for triangular arrays and since $E\eta_{N,i} = 0$, this follows from

$$\frac{E\eta_{N,1}^4}{N(E\eta_{N,1}^2)^4} \to 0, \tag{A.7}$$

$$E(\eta_{N,1})^2 \to \omega_1(z). \tag{A.8}$$

The convergence (A.8) follows from $E\xi_{N,1}^2 \to w_0(z)$. For this, observe that $E\xi_{N,1}^2 = \|K\|_2^2 E(((v(X_1)\otimes I_G)U_1)^2 K_{2,h}(z - Z_1)) = \|K\|_2^2 E((v(X_1)v(X_1)'\otimes\Sigma)K_{2,h}(z - Z_1)) = \omega_0(z) + O(h^2)$. For the convergence (A.7), observe that $E\eta_{N,1}^4 \leq \mathrm{const}\, E|\xi_{N,1}|^4 \leq \mathrm{const}\, h^{-q}E|v(X_1)|^4|U_1|^4 K_{4,h}(z - Z_1)$ with the kernel $K_4(z) = K(z)^4/\|K\|_4^4$. By using Proposition C.1, we conclude that $E\eta_{N,1}^4 = O(h^{-q})$. By using (A.8), we deduce the convergence (A.7). □

B Technical proofs for global rank tests

Notation and its simplification: We suppose for simplicity that $d_0 = 0$, that is, $\Theta(z) = \theta(z)$, $\Psi(z) = \psi(z)$ and similar expressions with the hats. We shall write

$$g(z) = \theta(z)\psi(z).$$

Observe that we have $\Gamma(z) = \Sigma^{-1}g(z)\psi(z)^{-1}g(z)'$ and a similar expression with the hats. To simplify notation further, we shall drop throughout the proofs dependence on z. We shall write, in particular,

$g = g(z)$, $\psi = \psi(z)$, $\Gamma = \Gamma(z)$, $\lambda_i = \lambda_i(z)$, with similar expressions with the hats, and $l = \mathrm{rk}\{\Gamma\}$, $\sup = \sup_{z\in\mathcal{H}_z}$. We shall also write

$$\widehat{\xi} = O_{p,\sup}(a_N)$$

for $\xi = \xi(z)$ when $a_N^{-1}\sup|\widehat{\xi}(z)| = O_p(1)$.

Let also $C_0 = (\widetilde{C}\ C)$ be a $m\times m$ matrix consisting of the eigenvectors of $\Sigma^{-1}g\psi^{-1}g'$ such that a $m\times l$ matrix $\widetilde{C}$ corresponds to l nonzero eigenvalues, a $m\times(m-l)$ matrix C corresponds to $m-l$ zero eigenvalues, and

$$C_0'\Sigma C_0 = I_m. \tag{B.1}$$

Similarly, let $D_0 = (\widetilde{D}\ D)$ be a $n\times n$ matrix consisting of the eigenvectors of $\psi^{-1}g'\Sigma^{-1}g$ such that a $n\times l$ matrix $\widetilde{D}$ corresponds to l nonzero eigenvalues, a $n\times(n-l)$ matrix D corresponds to $n-l$ zero eigenvalues, and

$$D_0'\psi D_0 = I_n. \tag{B.2}$$

The next result shows that, under our assumptions, the eigenvectors in C_0, D_0, can be chosen smooth.

Lemma B.1 *Under Assumptions G4 and G5, the matrices C_0 and D_0 above can be chosen analytic.*

PROOF: By Assumptions G4 and G5, the matrix $\Sigma^{-1/2}g\psi^{-1/2}$ is analytic. By using the analytic Singular Value Decomposition (Bunse-Gerstner et al. (1991)), there are $m\times m$, $m\times n$ and $n\times n$ analytic matrices U, T and V, respectively, such that $\Sigma^{-1/2}g\psi^{-1/2} = UTV'$, where $T = \mathrm{diag}(t_1,\dots,t_k)$ with $k = \min(m,n)$, $t_1 \geq \dots \geq t_k$ are the singular values, and orthogonal matrices U and V consist of the eigenvectors of $\Sigma^{-1/2}g\psi^{-1}g'\Sigma^{-1/2}$ and $\psi^{-1/2}g'\Sigma^{-1}g\psi^{-1/2}$, respectively. Now take $C_0 = \Sigma^{1/2}U$. Then, C_0 is analytic, satisfies $g\psi^{-1}g'\Sigma^{-1}C_0 = C_0T^2$ and $C_0\Sigma^{-1}C_0 = I_m$. The case of the matrix D_0 can be considered similarly. $\Box$

The next result allows to replace $\widehat{\lambda}_i$ by the eigenvalues that are easier to work with asymptotically. The result is standard for fixed z.

Lemma B.2 *Under Assumptions 1, 2, 3, G4 and G5, we have, for $i = 1,\dots,m-l$,*

$$\sup|\widehat{\lambda}_i - \widehat{\eta}_i| = O_p\Big((Nh^q)^{-3/2}(\ln N)^{1/2}\Big), \tag{B.3}$$

where $\widehat{\eta}_1 \leq \dots \leq \widehat{\eta}_{m-l}$ are the ordered eigenvalues of the matrix

$$C'(\widehat{g}-g)DD'(\widehat{g}-g)'C. \tag{B.4}$$

PROOF: Let $T = Nh^q$. As on p. 173 of Robin and Smith (2000), $\widehat{\lambda}_i$ satisfy

$$0 = \det\Big(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma}\Big) = \det\Big((\widetilde{C}\ \ T^{1/2}C)'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})(\widetilde{C}\ \ T^{1/2}C)\Big)$$

$$= \det\begin{pmatrix} \widetilde{C}'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})\widetilde{C} & \widetilde{C}'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})T^{1/2}C \\ T^{1/2}C'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})\widetilde{C} & TC'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})C \end{pmatrix}.$$

By using the relation $\det((A\ C; B\ D)) = \det(A)\det(D - BA^{-1}C)$, we further obtain that

$$0 = \det(\widehat{S})\det\Big(\widehat{W} - T\widehat{\lambda}_i\widehat{V}^{-1}\Big), \tag{B.5}$$

where $\widehat{S} = \widetilde{C}'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})\widetilde{C}$, $\widehat{V}^{-1} = C'\widehat{\Sigma}C$, and

$$\widehat{W} = TC'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'C - TC'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'\widetilde{C}\,\widehat{S}^{-1}\,\widetilde{C}'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'C$$

$$+T\widehat{\lambda}_i C'\widehat{\Sigma}\widetilde{C}\,\widehat{S}^{-1}\,\widetilde{C}'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})C + T\widehat{\lambda}_i C'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})\widetilde{C}\,\widehat{S}^{-1}\,\widetilde{C}'\widehat{\Sigma}C. \tag{B.6}$$

By using Propositions C.3–C.5 and the smoothness of $\widetilde{C}$ by Lemma B.1, observe that

$$\widehat{S} = \widetilde{C}'g\psi^{-1}g'\widetilde{C} + O_{p,\sup}\left((Nh^q/\ln N)^{-1/2}\right). \tag{B.7}$$

As in the proof of Lemma A.1 of Robin and Smith (2000), observe also that

$$\widetilde{C}'g = \mathrm{diag}(\lambda_m^{1/2}, \ldots, \lambda_{m-l+1}^{1/2})\,\widetilde{D}_*', \tag{B.8}$$

where $D_0^{-1} = (\widetilde{D}_*\ D_*)'$ is the inverse of D_0 with a $n \times (n-l)$ matrix $\widetilde{D}_*$. Since $\widetilde{D}_*'\psi^{-1}\widetilde{D}_* = I_{n-l}$ by (B.2), we obtain from (B.7) and (B.8) that

$$\widehat{S} = \mathrm{diag}(\lambda_m, \ldots, \lambda_{m-l+1}) + O_{p,\sup}\left((Nh^q/\ln N)^{-1/2}\right). \tag{B.9}$$

Relation (B.9) shows that, asymptotically, $\det(\widehat{S}) > 0$. Hence, in view of (B.5), we may suppose without loss of generality that $\det(\widehat{W} - T\widehat{\lambda}_i\widehat{V}^{-1}) = 0$, that is, $T\widehat{\lambda}_i$ are the eigenvalues of the matrix $\widehat{V}^{1/2}\widehat{W}\widehat{V}^{1/2}$. Since this matrix is symmetric, applying the Wielandt-Hoffman theorem (Golub and Van Loan (1996), Stewart and Sun (1990)), we obtain that

$$\sup|T\widehat{\lambda}_i - T\widehat{\eta}_i|^2 \le \sup\sum_{i=1}^{m-l}|T\widehat{\lambda}_i - T\widehat{\eta}_i|^2 \le \sup\left|\widehat{V}^{1/2}\widehat{W}\widehat{V}^{1/2} - TC'(\widehat{g}-g)DD'(\widehat{g}-g)'C\right|^2. \tag{B.10}$$

Finally, we bound the right-hand side of (B.10) by examining the terms of the matrix $\widehat{W}$ in (B.6). By using $C'g = 0$ and Proposition C.3, we have

$$TC'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'C = TC'(\widehat{g}-g)\psi^{-1}(\widehat{g}-g)'C + O_{p,\sup}\left(T(Nh^q/\ln N)^{-1/2}\right). \tag{B.11}$$

Similarly, by using (B.8), (B.9) and the relation $\psi^{-1}\widetilde{D}_* = \widetilde{D}$, we obtain that

$$TC'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'\widetilde{C}\widehat{S}^{-1}\widetilde{C}'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'C = TC'(\widehat{g}-g)\widetilde{D}\widetilde{D}'(\widehat{g}-g)'C + O_{p,\sup}\left(T(Nh^q/\ln N)^{-1/2}\right). \tag{B.12}$$

The last two terms of $\widehat{W}$ are also $O_{p,\sup}((Nh^q/\ln N)^{-1/2})$. By using $\psi^{-1} - \widetilde{D}\widetilde{D}' = DD'$, we conclude from (B.11) and (B.12) that

$$\widehat{W} = TC'(\widehat{g}-g)DD'(\widehat{g}-g)'C + O_{p,\sup}\left(T(Nh^q/\ln N)^{-1/2}\right). \tag{B.13}$$

By using Proposition C.4 and the fact $C'\Sigma C = I_{m-l}$, we have $\widehat{V}^{1/2} = I_{m-l} + O_{p,\sup}((Nh^q/\ln N)^{-1/2})$. Hence, in view of (B.13), the right-hand side of (B.10) is $O_{p,\sup}(T^2(Nh^q/\ln N)^{-1})$. This implies the result (B.3). □

Suppose now that $\sup l \le r$. Lemma B.1 will be used to replace $\widehat{\lambda}_i$ in $\widehat{T}_{glb}(r)$ by $\widehat{\eta}_i$. To establish a limit, we bound $\widehat{\eta}_i$ in the lemma below. Let

$$C_1 = C\begin{pmatrix} 0_{r-l\times m-r} \\ I_{m-r} \end{pmatrix}, \quad D_1 = D\begin{pmatrix} 0_{r-l\times n-r} \\ I_{n-r} \end{pmatrix}, \tag{B.14}$$

be the last $m-r$ and $n-r$ columns of C and D, respectively. Observe from (B.1) and (B.2) that

$$C_1'\Sigma C_1 = I_{m-r}, \quad D_1'\psi D_1 = I_{n-r}. \tag{B.15}$$

Let also $0 \leq \widehat{\xi}_1 \leq \ldots \leq \widehat{\xi}_{m-r}$ be the ordered eigenvalues of the matrix

$$C_1'(\widehat{g}-g)D_1D_1'(\widehat{g}-g)'C_1. \tag{B.16}$$

Lemma B.3 *We have* $\widehat{\eta}_i \leq \widehat{\xi}_i$, $i = 1,\ldots,m-r$.

Proof: Applying the Poincaré separation theorem (Magnus and Neudecker (1999), p. 209, or Rao (1973), p. 65), we have $\widehat{\eta}_i \leq \widehat{\zeta}_i$, where $\widehat{\zeta}_i$ are the ordered eigenvalues of the matrix $C_1'(\widehat{g}-g)DD'(\widehat{g}-g)'C_1$. These are also the eigenvalues of the matrix $D'(\widehat{g}-g)'C_1C_1'(\widehat{g}-g)D$. Applying the Poincaré separation theorem again, we further obtain that $\widehat{\eta}_i \leq \widehat{\xi}_i$, where $\widehat{\xi}_i$ are the eigenvalues of $D_1'(\widehat{g}-g)'C_1C_1'(\widehat{g}-g)D_1$. These are also the eigenvalues of the matrix in (B.16). □

Define the local and global rank test statistic through the eigenvalues $\widehat{\xi}_i$ of (B.16) as

$$\widehat{S}(r) = Nh^q\|K\|_2^{-2}\sum_{i=1}^{m-r}\widehat{\xi}_i, \quad \widehat{S}_{glb}(r) = \frac{\int_{\mathcal{H}_z}\widehat{S}(r)dz - |\mathcal{H}_z|(m-r)(n-r)}{h^{q/2}|\mathcal{H}_z|^{1/2}\sqrt{(m-r)(n-r)}}\,\frac{\|K\|_2^2}{\|\overline{K}\|_2}. \tag{B.17}$$

Combining Lemmas B.1 and B.3, we obtain the following result.

Corollary B.1 *We have*

$$\limsup(d)\,\widehat{T}_{glb}(r) \overset{d}{\leq} \limsup(d)\,\widehat{S}_{glb}(r). \tag{B.18}$$

We will finally show that

$$\widehat{S}_{glb}(r) \overset{d}{\to} \mathcal{N}(0,1). \tag{B.19}$$

Observe that

$$\|K\|_2^2\,\widehat{S}(r) = Nh^q\,\mathrm{tr}\{C_1'(\widehat{g}-g)D_1D_1'(\widehat{g}-g)'C_1\} = Nh^q\,\mathrm{tr}\{C_1'\widehat{g}D_1D_1'\widehat{g}'C_1\}, \tag{B.20}$$

where the equality before holds since $C_1'g = 0$. In order to integrate $\widehat{S}(r)$ over z, we shall replace $D_1 = D_1(z)$ and $C_1 = C_1(z)$ in (B.20) by

$$D_{1,i} = D_1(Z_i), \quad C_{1,i} = C_1(Z_i).$$

Lemma B.4 *Under Assumptions 1, 2, 3, L4, and when* $N^{1-2/u}h^{q+2}/\ln N \to \infty$, *we have*

$$\|K\|_2^2\,\widehat{S}(r) = \frac{Nh^q}{N^2}\sum_{i,j} v(X_i)'D_{1,i}D_{1,j}'v(X_j)'\,U_i'C_{1,i}C_{1,j}'U_j\,K_h(z-Z_i)K_h(z-Z_j) + O_{p,\sup}(h^2\ln N). \tag{B.21}$$

Proof: Observe from Proposition C.6 below that

$$\sup\left|C_1'\widehat{g}D_1 - \frac{1}{N}\sum_{i=1}^{N}C_{1,i}'Y_iv(X_i)D_{1,i}K_h(z-Z_i)\right| = O_p\left(h(Nh^q/\ln N)^{-1/2}\right). \tag{B.22}$$

Relations (B.20) and (B.22) imply (B.21) where the sum on the right-hand side is

$$\frac{Nh^q}{N^2}\sum_{i,j}\mathrm{tr}\Big\{C_{1,i}'Y_iv(X_i)'D_{1,i}D_{1,j}'(Y_jv(X_j)')'C_{1,j}\Big\}K_h(z-Z_i)K_h(z-Z_j).$$

Finally observe that $C_{1,i}'Y_i = C_{1,i}'U_i$ since $C_1'\theta = 0$, and use the fact $\mathrm{tr}\{AA'\} = \mathrm{vec}(A)'\mathrm{vec}(A)$. □

Lemma B.4 implies that

$$\|K\|_2^2 \int_{\mathcal{H}_z} \widehat{S}(r)dz = \frac{\|K\|_2^2}{N} \sum_i v(X_i)'D_{1,i}D_{1,i}'v(X_i)'\, U_i'C_{1,i}C_{1,i}'U_i$$

$$+\frac{Nh^q}{N^2} \sum_{i\neq j} v(X_i)'D_{1,i}D_{1,j}'v(X_j)\, U_i'C_{1,i}C_{1,j}'U_j\, \overline{K}_h(Z_i - Z_j) + O_p\Big(h^2 \ln N\Big)$$

$$=: \|K\|_2^2\, \widehat{S}_{1,glb}(r) + \widehat{S}_{2,glb}(r) + O_p\Big(h^2 \ln N\Big). \tag{B.23}$$

The convergence (B.19) follows from the next two results. Observe that Assumption G6 is used here to have $h^{2-q/2} \ln N \to 0$.

Lemma B.5 *Under Assumptions 2,3, we have*

$$\widehat{S}_{1,glb}(r) = |\mathcal{H}_z|(m-r)(n-r) + O_p(N^{-1/2}). \tag{B.24}$$

Proof: The result follows by using (B.15) from

$$E\widehat{S}_{1,glb}(r) = E\left(v(X_i)'D_1(Z_i)D_1(Z_i)'v(X_i)'E\Big((C_1(z)'U_i)'C_1(z)'U_i\Big)\Big|_{z=Z_i}\right)$$

$$= (m-r)E\Big(v(X_i)'D_1(Z_i)D_1(Z_i)'v(X_i)'\Big)$$

$$= (m-r)\int_{\mathcal{H}_z} p(z)E\Big((D_1(z)'v(X_i))'(D_1(z)'v(X_i))\Big|Z_i = z\Big)dz$$

$$= (m-r)(n-r)\int_{\mathcal{H}_z} dz = (m-r)(n-r)|\mathcal{H}_z|. \quad \square$$

Lemma B.6 *Under Assumptions 2,3, we have*

$$\lim(d)\, \frac{\widehat{S}_{2,glb}(r)}{h^{q/2}|\mathcal{H}_z|^{1/2}\, \|\overline{K}\|_2\sqrt{(m-r)(n-r)}} \stackrel{d}{=} \mathcal{N}\Big(0,1\Big).$$

Proof: Observe that $\widehat{S}_{2,glb}(r) = \sum_{i<j} W_{ij}$ with

$$W_{ij} = W((X_i,Z_i,U_i),(X_i,Z_i,U_i)) = \frac{2h^q}{N} v(X_i)'D_{1,i}D_{1,j}'v(X_j)\, U_i'C_{1,i}C_{1,j}'U_j\, \overline{K}_h(Z_i - Z_j),$$

and that $E(W_{ij}|X_i,Z_i,U_i) = 0$, $i<j$. Hence, by applying Proposition 3.2 in de Jong (1987), it is enough to prove that

$$\frac{E(\widehat{S}_{2,glb}(r))^2}{h^q|\mathcal{H}_z|\, \|\overline{K}\|_2^2(m-r)(n-r)} \to 1 \tag{B.25}$$

and

$$\sum_{i<j} EW_{ij}^4 = o(h^{2q}), \tag{B.26}$$

$$\sum_{i<j<k} \Big(EW_{ij}^2W_{ik}^2 + EW_{ji}^2W_{jk}^2 + EW_{ki}^2W_{kj}^2\Big) = o(h^{2q}), \tag{B.27}$$

$$\sum_{i<j<k<l}\Big(EW_{ij}W_{ik}W_{lj}W_{lk}+EW_{ij}W_{il}W_{kj}W_{kl}+EW_{ik}W_{il}W_{jk}W_{jl}\Big)=o(h^{2q}). \tag{B.28}$$

The convergence (B.25) follows from

$$\frac{h^{-q}N}{\|\overline{K}\|_2^2(N-1)}E(\widehat{S}_{2,glb}(r))^2$$

$$\begin{aligned}
&=\frac{h^q}{\|\overline{K}\|_2^2N(N-1)}\sum_{i\neq j}E\Big((v(X_i)'D_{1,i}D'_{1,j}v(X_j))^2\,(U_i'C_{1,i}C'_{1,j}U_j)^2\,(\overline{K}_h(Z_i-Z_j))^2\Big)\\
&=E\Big((v(X_i)'D_{1,i}D'_{1,j}v(X_j))^2\,U_i'C_{1,i}\Big(C'_{1,j}U_jU_j'C_{1,j}\Big)C'_{1,i}U_i\,\overline{K}_{2,h}(Z_i-Z_j)\Big)\\
&=E\Big((v(X_i)'D_{1,i}D'_{1,j}v(X_j))^2\,U_i'C_{1,i}\Big(C'_{1,j}\Sigma C_{1,j}\Big)C'_{1,i}U_i\,\overline{K}_{2,h}(Z_i-Z_j)\Big)\\
&=E\Big((v(X_i)'D_{1,i}D'_{1,j}v(X_j))^2\,U_i'C_{1,i}C'_{1,i}U_i\,\overline{K}_{2,h}(Z_i-Z_j)\Big)\\
&=(m-r)E\Big((v(X_i)'D_{1,i}D'_{1,j}v(X_j))^2\,\overline{K}_{2,h}(Z_i-Z_j)\Big)\\
&=(m-r)E\Big(v(X_i)'D_{1,i}\Big(D'_{1,j}v(X_j)v(X_j)'D_{1,j}\Big)D'_{1,i}v(X_i)\,\overline{K}_{2,h}(Z_i-Z_j)\Big)\\
&=(m-r)E\Big(v(X_i)'D_{1,i}\Big(D'_{1,j}\psi(Z_j)D_{1,j}\Big)p(Z_j)^{-1}D'_{1,i}v(X_i)\,\overline{K}_{2,h}(Z_i-Z_j)\Big)\\
&=(m-r)E\Big(v(X_i)'D_{1,i}D'_{1,i}v(X_i)\,p(Z_j)^{-1}\overline{K}_{2,h}(Z_i-Z_j)\Big)\\
&=(m-r)E\Big(v(X_i)'D_{1,i}D'_{1,i}v(X_i)\Big)+o(h^2)=(m-r)(n-r)|\mathcal{H}_z|+o(h^2).
\end{aligned}$$

For the relation (B.26), the sum is $O(N^{-2})=O((Nh^q)^{-2}h^{2q})=o(h^{2q})$. Consider now the relation (B.27). For the first term in the sum, observe that

$$\begin{aligned}
\frac{EW_{ij}^2W_{ik}^2}{16\|\overline{K}\|_2^4}&=\frac{h^{2q}}{N^4}E\Big\{\Big(v(X_i)'D_{1,i}D'_{1,j}v(X_j)\Big)^2\Big(U_i'C_{1,i}C'_{1,j}U_j\Big)^2\overline{K}_{2,h}(Z_i-Z_j)\cdot\\
&\qquad\cdot\Big(v(X_i)'D_{1,i}D'_{1,k}v(X_k)\Big)^2\Big(U_i'C_{1,i}C'_{1,k}U_k\Big)^2\overline{K}_{2,h}(Z_i-Z_k)\Big\}\\
&=\frac{h^{2q}}{N^4}E\Big\{\Big(v(X_i)'D_{1,i}D'_{1,j}v(X_j)\Big)^2\Big(v(X_i)'D_{1,i}D'_{1,k}v(X_k)\Big)^2\Big(U_i'C_{1,i}C'_{1,i}U_i\Big)^2\overline{K}_{2,h}(Z_i-Z_j)\overline{K}_{2,h}(Z_i-Z_k)\Big\}\\
&=\frac{h^{2q}}{N^4}E\Big\{\Big(v(X_i)'D_{1,i}D'_{1,i}v(X_i)\Big)^2\Big(U_i'C_{1,i}C'_{1,i}U_i\Big)^2p(Z_j)^{-1}p(Z_k)^{-1}\overline{K}_{2,h}(Z_i-Z_j)\overline{K}_{2,h}(Z_i-Z_k)\Big\}\\
&=\frac{h^{2q}}{N^4}\Big\{E\Big(v(X_i)'D_{1,i}D'_{1,i}v(X_i)\Big)^2\Big(U_i'C_{1,i}C'_{1,i}U_i\Big)^2+o(h^2)\Big\}=O\left(\frac{h^{2q}}{N^4}\right).
\end{aligned}$$

The other two terms of the sum in (B.27) can be considered similarly. This yields (B.27).

For the first term in the sum of (B.28), one may similarly show that

$$\frac{N^4}{16h^{4q}}EW_{ij}W_{ik}W_{lj}W_{lk}$$

$$=(m-r)(n-r)E\left\{\overline{K}_h(Z_i-Z_j)\overline{K}_h(Z_i-Z_k)\overline{K}_h(Z_l-Z_j)\overline{K}_h(Z_l-Z_k)\right\}=O\Big(h^{-2q}\Big).$$

Other terms in the sum can be considered similarly. This establishes (B.28). □

Suppose now that $r < \sup l(z)$. Then, by smoothness of $\theta(z)$, there is an interval $\mathcal{H}_0 \subset \mathcal{H}_z$ such that $r < l(z)$ for all $z \in \mathcal{H}_0$. By using Proposition C.3, it follows that

$$\sup_{z\in\mathcal{H}_0} \left|(Nh^q)^{-1}\widehat{T}(r,z) - T_0(r,z)\right| \xrightarrow{d} 0,$$

where

$$T_0(r,z) = \min_{\mathrm{rk}\{\theta\}\leq r} \mathrm{vec}(\theta(z)-\theta)' w(z)^{-1} \mathrm{vec}(\theta(z)-\theta)$$

and

$$\sup_{z\in\mathcal{H}_0} T_0(r,z) > 0.$$

Hence,

$$\widehat{T}_{glb}(r) \geq Nh^{q/2} \frac{\int_{\mathcal{H}_0}(Nh^q)^{-1}\widehat{T}(r,z)dz - (Nh^q)^{-1}|\mathcal{H}_z|(m-r)(n-r)}{|\mathcal{H}_z|^{1/2}\sqrt{(m-r)(n-r)}} \xrightarrow{d} +\infty,$$

which implies (5.2).

C Auxiliary proofs

The following localization property of kernel functions can be easily proved by using Taylor expansions and the definition of the order of a kernel function. We omit its proof for shortness sake.

Proposition C.1 *Let K be a kernel on $\mathbb{R}^q$ of order $r \in \mathbb{N}$. Suppose that a function $g : \mathbb{R}^q \to \mathbb{R}$ is r-times continuously differentiable in a neighborhood of $z_0 \in \mathbb{R}^q$. Then, as $h \to 0$,*

$$\int_{\mathbb{R}^q} g(z)K_h(z-z_0)dz = g(z_0) + O(h^r). \tag{C.1}$$

Moreover, if the function g has its r-order derivatives bounded on $\mathbb{R}^q$, then the term $O(h^r)$ in (C.1) does not depend on z_0.

The next elementary result allows us to determine a local rank in a demand system.

Proposition C.2 *Let $\widetilde{y} = f(x,z) = \widetilde{\theta}(z)v(x)$ be a demand system with a $j \times n$ matrix $\widetilde{\theta}(z)$ and a $n \times 1$ vector $v(x) = (1\ V(x)')'$ of n linearly independent functions of x. Then, for fixed z,*

$$\mathrm{rk}\{f(\cdot,z)\} = \mathrm{rk}\{\widetilde{\theta}(z)\} = \mathrm{rk}\{\Theta(z)\} + 1, \tag{C.2}$$

where $\Theta(z)$ is the matrix $\widetilde{\theta}(z)$ with an arbitrary row and the first column eliminated. The second equality in (C.2) holds for any matrix $\widetilde{\theta}(z)$ where the entries in the first column add up to 1 and those in the other columns add up to 0.

Proof: For notational simplicity, we omit z throughout the proof. We shall prove the first equality in (C.2). Let $r = \mathrm{rk}\{f(\cdot)\}$ and $l = \mathrm{rk}\{\widetilde{\theta}\}$. By the definition of $\mathrm{rk}\{f(\cdot)\}$, there are $j-r$ elements of a vector $f(x) = \widetilde{\theta}\, v(x)$ that can be expressed as linear combinations of the rest r elements of $f(x)$. Supposing without loss of generality that these are the last $j-r$ elements of $f(x)$, we obtain that, for $i = r+1, \ldots, j$,

$$\widetilde{\theta}_{i1}\, v_1(x) + \cdots + \widetilde{\theta}_{in}\, v_n(x) = \sum_{k=1}^{r} c_{ik}(\widetilde{\theta}_{k1}\, v_1(x) + \cdots + \widetilde{\theta}_{kn}\, v_n(x)), \tag{C.3}$$

where $\widetilde{\theta} = (\widetilde{\theta}_{ik})$, $v(x) = (v_l(x))$ and c_{ik} are some vectors. Since the functions $v_1(x), \ldots, v_n(x)$ are linearly independent by the assumption, relation (C.3) implies that $\theta_{ik} = c_{i1}\widetilde{\theta}_{1k} + \cdots + c_{ir}\widetilde{\theta}_{rk}$, for $i = r+1, \ldots, j$ and $k = 1, \ldots, n$. This shows that $l \leq r$ since $j - r$ rows of the matrix $\widetilde{\theta}$ can be expressed as linear combinations of the other r rows. To obtain the converse inequality $r \leq l$, observe that $\widetilde{\theta} = \theta_1\theta_2$ where θ_1 is a $j \times l$ matrix and θ_2 is a $l \times n$ matrix. Then, $\widetilde{y} = f(x) = \widetilde{\theta}\, v(x) = \theta_1(\theta_2\, v(x))$. Since $\theta_2\, v(x)$ is a $l \times 1$ vector, we obtain from the definition of local rank that $r \leq l$. Hence, $l = r$ which concludes the first part of the proof.

We will now prove the second equality in (C.2). Set $\widetilde{\theta} = (\widetilde{\theta}_1\ \widetilde{\theta}_2)$ where $\widetilde{\theta}_1$ is a $j \times 1$ vector and $\widetilde{\theta}_2$ is a $j \times (n-1)$ matrix. Since the functions $v_k(x), k = 1, \ldots, n$ are linearly independent and $v_1(x) = 1$ by assumption, and since the j budget shares add up to 1, we obtain that the elements of the first column of the matrix $\widetilde{\theta}$ add up to 1 and those in the other columns add up to 0. We want to show first that $\text{rk}\{\widetilde{\theta}\} = \text{rk}\{\widetilde{\theta}_2\} + 1$. If the first column of $\widetilde{\theta}$ can be written as a linear combination of the other columns, that is, for $k = 1, \ldots, n$, $\widetilde{\theta}_{1k} = \lambda_2\widetilde{\theta}_{2k} + \ldots + \lambda_n\widetilde{\theta}_{nk}$, $\lambda_2, \ldots, \lambda_n \in \mathbb{R}$, by summing up all the equations over k, we get

$$\sum_{k=1}^{n} \widetilde{\theta}_{1k} = \lambda_2 \sum_{k=1}^{n} \widetilde{\theta}_{2k} + \ldots + \lambda_n \sum_{k=1}^{n} \widetilde{\theta}_{nk}. \tag{C.4}$$

We know that $\sum_{k=1}^{n} \widetilde{\theta}_{1k} = 1$ and, for $i = 2, \ldots, j$, $\sum_{k=1}^{n} \widetilde{\theta}_{ik} = 0$. Therefore (C.4) is not true and hence the first column of $\widetilde{\theta}$ is linearly independent of the other columns. This implies that $\text{rk}\{\widetilde{\theta}\} = \text{rk}\{\widetilde{\theta}_2\} + 1$. Finally, it is enough to show that $\text{rk}\{\widetilde{\theta}_2\} = \text{rk}\{\Theta\}$. This holds since the rows of $\widetilde{\theta}_2$ add up to 0 and hence the last row of $\widetilde{\theta}_2$ is a linear combination of the other rows. □

The next three lemmas were used in the proof of global rank tests.

Proposition C.3 *Under Assumptions 1, 2, 3, L4 and L5, and $N^{1-2/u}h^q/\ln N \to \infty$, $Nh^{q+2s} \to 0$, we have*

$$\sup_{z \in \mathcal{H}_z} \left| (\widehat{\psi}(z))^k - (\psi(z))^k \right| = O_p\left((Nh^q/\ln N)^{-1/2} \right), \quad k = -1, 1, \tag{C.5}$$

and

$$\sup_{z \in \mathcal{H}_z} |\widehat{\theta}(z) - \theta(z)| = O_p\left((Nh^q/\ln N)^{-1/2} \right). \tag{C.6}$$

Proof: To prove (C.5), we consider only the case $k = 1$. Under the assumptions of the proposition and by using Lemma B.1 in Newey (1994) (see also Lemma 1 in Fan and Zhang (1999)), we have

$$\sup_{z \in \mathcal{H}_z} |\widehat{\psi}(z) - E\widehat{\psi}(z)| = O_p\left((Nh^q/\ln N)^{-1/2} \right).$$

By using Proposition C.1 above and the assumptions,

$$\sup_{z \in \mathcal{H}_z} |E\widehat{\psi}(z) - \psi(z)| = O\left(h^s \right).$$

This implies (C.5) with $k = 1$. Relation (C.6) follows similarly by using (C.5). □

Proposition C.4 *Under the assumptions of Proposition C.3 above, we have*

$$\widehat{\Sigma} - \Sigma = O_p\left((Nh^q/\ln N)^{-1/2} \right).$$

Proof: Write

$$\begin{aligned}\widehat{\Sigma} &= \frac{1}{N}\sum_{i=1}^{N} U_iU_i' + \frac{1}{N}\sum_{i=1}^{N}(\theta(Z_i)-\widehat{\theta}(Z_i))v(X_i)v(X_i)'(\theta(Z_i)-\widehat{\theta}(Z_i))' \\ &\quad + \frac{1}{N}\sum_{i=1}^{N} U_iv(X_i)'(\theta(Z_i)-\widehat{\theta}(Z_i))' + \frac{1}{N}\sum_{i=1}^{N}(\theta(Z_i)-\widehat{\theta}(Z_i))v(X_i)U_i'.\end{aligned}$$

The first term on the right-hand side is $\Sigma + O_p(N^{-1/2})$. By using Proposition C.3, the second term is $O_p((Nh^q/\ln N)^{-1})$. Similarly, the third and fourth terms are $O_p((Nh^q/\ln N)^{-1/2})$ since $N^{-1}\sum_{i=1}^{N}|U_i| = O_p(1)$. □

Proposition C.5 *Let $l(z) = \mathrm{rk}\{\Gamma(z)\}$. Under the assumptions of Proposition C.3 above, for $i = 1,\ldots,m$,*

$$\sup_{z\in\mathcal{H}_z} |\widehat{\lambda}_i(z)-\lambda_i(z)| = O_p\left((Nh^q/\ln N)^{-1/2}\right).$$

Proof: Observe by the Wielandt-Hoffman theorem (Golub and Van Loan (1996), Stewart and Sun (1990)) that

$$\begin{aligned}&\sup_{z\in\mathcal{H}_z}\sum_{i=1}^{m}|\widehat{\lambda}_i(z)-\lambda_i(z)|^2 \\ &\le \sup_{z\in\mathcal{H}_z}\left|\widehat{\Sigma}^{-1/2}\widehat{\Theta}(z)\widehat{\Psi}(z)\widehat{\Theta}(z)'\widehat{\Sigma}^{-1/2} - \Sigma^{-1/2}\Theta(z)\Psi(z)\Theta(z)'\Sigma^{-1/2}\right|^2,\end{aligned}$$

which is $O_p((Nh^q/\ln N)^{-1})$ by Propositions C.3 and C.4. □

Proposition C.6 *Let $f(z)$ be a bounded function, $g(z)$ be a Lipschitz function on $z\in\mathcal{H}_z$ and set*

$$\widehat{G}(z) = \frac{1}{N}\sum_{i=1}^{N} Y_iv(X_i)'f(Z_i)\Big(g(z)-g(Z_i)\Big)K_h(z-Z_i).$$

Then, under Assumptions 1, 2, 3, L4, and when $N^{1-2/u}h^{q+2}/\ln N\to\infty$, we have

$$\sup_{z\in\mathcal{H}_z}|\widehat{G}(z)| = O_p\left(h(Nh^q/\ln N)^{-1/2}\right).$$

Proof: The result can be proved by adapting the proof of Lemma B.1 in Newey (1994). Let

$$\widetilde{G}(z) = \frac{1}{N}\sum_{i=1}^{N} Y_{iN}v(X_i)'f(Z_i)\Big(g(z)-g(Z_i)\Big)K_h(z-Z_i),$$

where $Y_{iN} = \mathrm{sign}(Y_i)(|Y_i|\wedge LN^{1/u})$ with a constant L. Set also $\delta = h(Nh^q/\ln N)^{-1/2}$. As in (B.2) of Newey (1994),

$$P\Big(\widehat{G}(z)\neq\widetilde{G}(z) \text{ for some } z\Big) \le L^{-u}E|Y_i|^u. \tag{C.7}$$

Since g and K are Lipschitz, we have (with C denoting a generic constant throughout)

$$\sup_{|z_1-z_2|\le N^{-3}}\left|\widetilde{G}(z_1)-\widetilde{G}(z_2)\right| \le C\,h^{-q-1}N^{1/u-3}. \tag{C.8}$$

Since $\mathcal{H}_z$ is a compact, it can be covered by the balls $B(z_j, N^{-3})$, $j = 1, \ldots, CN^{3k}$, at the centers z_j and of radius N^{-3}. For $z \in \mathcal{H}_z$, let $z_j(z)$ be such that $z \in B(z_j(z), N^{-3})$. Then, by (C.8),

$$\sup_{z\in\mathcal{H}_z} \left|\widetilde{G}(z)\right| \leq \mathrm{C}\, h^{-q-1}N^{1/u-3} + \sup_j \left|\widetilde{G}(z_j)\right| \tag{C.9}$$

Note that, for large M, N,

$$2M\delta - C\,h^{-q-1}N^{1/u-3} = 2M\delta\left(1 - \frac{C}{(M^2N^{5-2/u}h^{q+4}\ln N)^{1/2}}\right)$$

$$\geq 2M\delta\left(1 - \frac{C}{(M^2N^{1-2/u}h^{q+2}(Nh^q)^4\ln N)^{1/2}}\right) \geq M\delta. \tag{C.10}$$

Then, by (C.9) and (C.10), for large M, N,

$$P\Big(\sup_{z\in\mathcal{H}_z}|\widehat{G}(z)| > 2M\delta\Big) \leq P\Big(\sup_j|\widehat{G}(z_j)| > M\delta\Big) \leq \sum_{j=1}^{CN^{3q}} P\Big(|\widehat{G}(z_j)| > M\delta\Big). \tag{C.11}$$

Observe now that

$$E|Y_{iN}v(X_i)'f(Z_i)(g(z)-g(Z_i))K_h(z-Z_i)| \leq E|Y_{iN}v(X_i)'f(Z_i)(g(z)-g(Z_i))K_h(z-Z_i)| \leq C\,h^{-q+2},$$

and $|Y_{iN}v(X_i)'f(Z_i)\Big(g(z)-g(Z_i)\Big)K_h(z-Z_i)| \leq C\,N^{1/u}h^{-q}$. Then, by the Bernstein inequality, for large M, N,

$$P\Big(|\widehat{G}(z_j)| > M\delta\Big) \leq 2\exp\left\{-C\frac{NM\delta^2}{h^{-q+2}+N^{1/u}h^{-q}\delta}\right\}$$

$$\leq 2\exp\left\{-C\frac{M\ln N}{1+(N^{1-2/u}h^{q+2}/\ln N)^{-1/2}}\right\} \leq 2\exp\Big\{-C\,M\ln N\Big\}. \tag{C.12}$$

By (C.11) and (C.12), for large M, N,

$$P\Big(\sup_{z\in\mathcal{H}_z}|\widehat{G}(z)| > 2M\delta\Big) \leq C\exp\Big\{-C(M-3q)\ln N\Big\}.$$

This shows that $\sup_{z\in\mathcal{H}_z}|\widetilde{G}(z)| = O_p(\delta)$. By using (C.7), one may conclude as in Newey (1994) that $\sup_{z\in\mathcal{H}_z}|\widehat{G}(z)| = O_p(\delta)$ as well. □

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