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A Model of Quality Ladders with Horizontal Entry

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We develop a multi-sector model of R&D-driven endogenous growth that merges the expanding-variety with the quality-ladders mechanism. The mechanism of expanding variety provides the flow of new firms (new product lines), whilst the mechanism of quality ladders provides the accumulation of non-physical capital (technological knowledge). The aim is to explore the view that, from the perspective of the households, wealth can be accumulated either by creating new firms or by accumulating capital, in a setting with no population growth. Differently from the standard expanding-variety literature, we allow for entry as well as exit of product lines from the market, view the creation of new product lines as a product development activity without positive spillovers, and postulate an horizontal entry mechanism that takes explicitly into account dynamic second-order effects. We perform a detailed comparative steady-state analysis and characterise qualitatively the local dynamics properties in a neighbourhood of the interior balanced-growth equilibrium. The model produces specific results with respect to the impact of changes in the entry-cost parameters and the fiscal-policy variables both in the aggregate growth rate and in the market structure and industry dynamics in steady state. We also conclude that the transitional dynamics is characterised by a *catching-up effect*, with an empirically reasonable speed of convergence under standard calibration.

Keywords: endogenous growth, firm dynamics, transitional dynamics

JEL Classification: O41, D92, C62

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1. Introduction

We develop a version of the multi-sector model of R&D-driven endogenous growth, with quality ladders in the intermediate-good sector (e.g., Aghion and Howitt, 1992, and Barro and Sala-i-Martin, 2004, ch. 7). As in the standard model, each quality-adjusted intermediate good is produced by a single-product firm, only (potential) entrants do R&D and innovation arrival follows a Poisson process. However, we merge the quality-ladders with the expanding variety mechanism (e.g., Romer, 1990, and Barro and Sala-i-Martin, 2004, ch. 6) and thus the number of intermediate goods is not necessarily constant over time. Following Dinopoulos and Thompson (1998), Young (1998), Aghion and Howitt (1998, ch. 12) and Howitt (1999),[1] we define the expanding variety mechanism as the addition of new intermediate product lines to the market, which is interpreted as an expansion of the number of firms in the economy. Thus, we build our model on the premise that industrial growth proceeds both along an intensive (increase of product quality) and an extensive margin (introduction of new goods).

The seminal paper in this strand of the literature is Caballero and Jaffe (1993). The authors describe a model where quality rises monotonically over time, so that the newest goods are always the best, and new and old intermediate goods are *imperfect substitutes*. Differently from Caballero and Jaffe (1993), the models of Dinopoulos and Thompson (1998), Young (1998), Aghion and Howitt (1998, ch. 12) and Howitt (1999) interpret the expanding variety mechanism as a process of adding new product lines, such that each product line observes an independent quality-ladders process.[2] In Caballero and Jaffe (1993), the expanding variety mechanism is, in itself, a quality ladder, an approach also adopted in Jones and Williams (2000)'s model of "innovation clusters".

Concerning the formal treatment of the horizontal entry process within the quality-ladder literature, Aghion and Howitt (1998, ch. 12) establish an *exogenous dynamics* for the number of differentiated goods (these grow linearly with the population) justified by an imitation mechanism, whereas Dinopoulos and Thompson (1998) and Young (1998) build in an endogenous entry mechanism regulated by an *exogenous entry cost*, constant over time. *Endogenous non-linear time-varying entry costs* are postulated in Caballero and Jaffe (1993), where the cost is a (negative) function of past knowledge accumulation, and Howitt (1999), where the cost depends on the amount of resources allocated to entry at each moment of time.

Outside the quality-ladders literature, we emphasise the contribution by Peretto (1998, 1999, 2003) and Peretto and Smulders (2002). In these endogenous growth models, incumbents do in-house R&D, whose outcomes improve factor productivity in each incumbent's production function, but entrants bring new products to the market. These models can be seen as "a smooth version (quality being a continuous variable) of the quality-ladder model with an endogenous variety of goods (number of firms)" (Peretto,

[1] There are a number of papers developing on the latter, namely Segerstrom (2000), Cozzi and Spinesi (2002) and Nicoletti (2003).

[2] We observe, however, that the latter two build on the tradition of the models of "competitive capitalism", "tournament" or "patent race", whereas the other two are closer to the "trustified capitalism" approach (see Thompson and Waldo, 1994). Our model follows the former approach.

1999, fn. 3). In Peretto (1998, 1999, 2003) entrants face an *exogenous constant entry cost*, whereas in Peretto and Smulders (2002) they face *endogenous non-linear time-varying entry costs*, as these are a function of past knowledge accumulation. More recently, Funk (2008) emphasises horizontal entry in the context of a vintage-knowledge model, where endogenous growth is induced by old and new knowledge produced by cost-reducing R&D. The number of firms in the R&D sector is endogenously determined, such that, in equilibrium, the return from opening a new R&D-lab (extensive margin) has to match the return of investing the same amount of additional R&D in existing R&D-labs (intensive margin).

The main motivation behind the early models of quality ladders with expanding variety is the removal of the scale effects of population growth. In consequence, the former models predict that the flow of new goods grows at the same (exogenous) rate as the population. However, drawing from Brito and Dixon (2008), our motivation is to explore the view that, from the perspective of the households, wealth can be accumulated either by creating new firms or by accumulating capital, in a setting with no population growth. In this context, the mechanism of quality ladders provides the accumulation of non-physical capital (technological knowledge), whilst the mechanism of expanding variety provides the flow of new firms (new product lines) at an endogenously determined growth rate.

Following Dinopoulos and Thompson (1998), we view the creation of new product lines as an activity without any sort of positive spillovers. Differently from the standard expanding-variety literature, we allow for entry as well as exit of product lines from the market, that is, we do not assume irreversibility of investment in product development. Moreover, we model the quality-ladders mechanism with intertemporal spillovers but no intersectoral spillovers (e.g., Segerstrom and Zolnierek, 1999), consider that the input to R&D and to differentiated-goods production is measured in units of the homogeneous final good instead of labour units (e.g., Segerstrom and Zolnierek, 1999), and postulate an *endogenous time-varying* horizontal-entry mechanism that takes into account dynamic second-order effects. The latter is carried out by means of an entry-cost function that merges the specification suggested by Romer (1990) and Barro and Sala-i-Martin (2004, ch. 6), where the entry cost increases with the number of differentiated goods, with that used by Datta and Dixon (2002) and Brito and Dixon (2008), and implicit in Howitt (1999), such that the entry cost also increases with the number of goods entering (exiting) the market at a given instant.

Within this framework, our model exhibits a steady-state equilibrium where the aggregate growth rate exceeds the growth rate of the number of differentiated goods by an amount corresponding to the growth of intermediate-good quality. All three growth rates are constant and positive in steady state. This implies that the consumption growth rate equals the growth rate of the number of varieties plus the growth rate of intermediate-good quality, a result in line with the view that industrial growth proceeds both along an intensive and an extensive margin, also apparent in, e.g., Dinopoulos and Thompson (1998) and Peretto (1998). Nevertheless, differently from Peretto (1998) and the quality-ladders models with expanding variety already quoted, in our model the growth of the number of varieties is not linked to the (exogenous) population growth rate; it

is sustained by technological-knowledge accumulation resulting from R&D activities. A similar result is obtained by Arnold (1998) and Funke and Strulik (2000), where, however, knowledge accumulation occurs in the form of human-capital production. In our setting, it is not necessarily the larger economy, measured by population size, that produces the greater number of distinct goods, but that with the larger technological-knowledge stock. Hence, the latter emerges as the relevant (endogenous) measure of economic size.

We perform a detailed comparative steady-state analysis and characterise qualitatively the local dynamics properties in a neighbourhood of the interior balanced-growth equilibrium, by studying the solution of the linearised system of properly rescaled variables. The model produces specific results with respect to the impact of changes in the entry-cost parameters and the fiscal-policy variables both in the aggregate growth rate and in the market structure and industry dynamics in steady state. We also conclude that the transitional dynamics is characterised by a *catching-up effect*, that is, our model exhibits the convergence property that applies in the standard Ramsey model.

Mulligan and Sala-i-Martin (1993) lay down some important reasons to study transitional dynamics in endogenous growth models. Our paper contributes to that strand of the literature by building in a new mechanism that produces intermediate-term adjustment with an empirically reasonable speed of convergence under standard calibration. This mechanism hinges on a friction in the horizontal entry, similar to the one that characterises the changes of the physical-capital stock in the literature of firm investment with convex adjustment costs (e.g., Eisner and Strotz, 1963), whereas the aggregate production function exhibits constant returns in the accumulated factor (the technological-knowledge stock).

Hence, our contribution differs from the endogenous growth models with transitional dynamics that feature physical-capital accumulation, either alone (e.g., Jones and Manuelli, 1990) or combined with some form of knowledge/ideas accumulation - human capital production (e.g., Lucas, 1988; Rebelo, 1991; Barro and Sala-i-Martin, 2004, ch. 5),[3]quality ladders (Aghion and Howitt, 1998, ch. 3, ch. 12) or expanding variety (Funke and Strulik, 2000). In these models, transitional dynamics originates ultimately from the decreasing marginal returns of the accumulated factor(s) in the aggregate production function. In particular, in the models with two types of factors, the transition to the steady state is driven by the combination of decreasing marginal returns with imbalances that cannot be instantaneously eliminated in the ratio between those two factors (e.g., due to irreversibility constraints or to different production technologies that make factors imperfect substitutes).

Our model also differs from - but, yet, is closer to - the endogenous growth models without physical capital in which the transitional-dynamics property results, e.g., from time lags in the diffusion of knowledge (Caballero and Jaffe, 1993), decreasing marginal returns in vertical R&D (Dinopoulos and Thompson, 1998; Acemoglu, 1998), positive spillovers in horizontal R&D (Arnold, 1998), or from the flow-stock relationship between

[3]Early detailed studies of the transitional dynamics in this class of models are Mulligan and Sala-i-Martin (1993), Caballé and Santos (1993), Chamley (1993) and Faig (1995). A more recent contribution, which explores a rather general setting of two-sector non-scale growth models, is Eicher and Turnovsky (2001).

private and public R&D (Peretto, 1998).

In particular in Peretto (1998), the mechanism of transition operates through imbalances between total employment and the number of differentiated goods, which is an explicit measure of firm size. In our model, the transition to the steady state is driven by imbalances in the ratio between the technological-knowledge stock and the number of differentiated goods, which we interpret as an alternative measure of firm size. This property allows us to insert this model in the literature that studies the interplay between long-term growth and the factors usually studied in the domain of industrial organization (IO). Whereas some papers emphasise the role of a specific factor within a monopolistic-competition setting - e.g., the number of firms (Peretto and Smulders, 2002), average firm size (Peretto, 1998) or firm size distribution (Thompson, 2001; Klette and Kortum, 2004) -, others explore the interdependence between market structure and growth by focusing on the strategic interaction of firms in an oligopolistic framework (van de Klundert and Smulders, 1997; Peretto, 1999; Aghion, Harris, Howitt, and Vickers, 2001; Aghion, Bloom, Blundell, Griffith, and Howitt, 2005; Minniti, 2006). Our contribution relates more closely to the former set of papers and, in particularly, to Peretto (1998).

The model studied herein is broadly consistent with the well-established empirical evidence on intermediate and long-run firm dynamics (e.g., Jovanovic, 1993; Maddison, 1994; and Laincz and Peretto, 2006), whilst it helps to shed light on the lack of clear-cut empirical results with respect to the link between R&D intensity and both firm size and aggregate growth (e.g., Bassanini, Scarpetta, and Visco, 2000; and Pagano and Schivardi, 2003).

The remaining of the paper is organised as follows. In Section 2, the model is presented, being given detailed account of the production, price and R&D decisions in the intermediate-good sector. In Section 3, we construct the dynamic general equilibrium, analyse the steady-state and local-dynamics properties of the model, and discuss their consistency with the empirical literature. In Section 4, we study the growth and industry-dynamics effects of fiscal policy instruments. Section 5 concludes.

2. The model

Our basic setup is adapted from Segerstrom and Zolnierek (1999) and Barro and Sala-i-Martin (2004, ch. 6; ch. 7). We build a dynamic general equilibrium model of a closed economy where there is a single competitively-produced final good that can be used in consumption, C, production of intermediate goods, X, and R&D activities, R.[4] The final consumption good is produced by a (large) number of firms, indexed by h, each using labour and a continuum of intermediate inputs indexed by ω on the interval $[0, N(t)]$.

The economy is populated by L identical dynastic families, each endowed with one unit of labour that is inelastically supplied to final-good firms. Thus, the total labour

[4] Using Rivera-Batiz and Romer (1991)'s terminology, the assumption that final good is the R&D input means that we adopt the "lab-equipment" version of R&D, instead of the "knowledge-driven" specification, in which labour is the only input.

level is L, which, by assumption, is constant over time.[5] In turn, families invest in firms' equity.

In the intermediate-good sector, firms can devote resources to R&D either to create a new product line (a new industry) or, within an existing industry ω, to improve the quality of its good. Quality is indexed by $j = 0, 1, 2, ...$, where higher values of j denote higher quality products. In particular, when a new quality rung is reached in ω, the jth innovator is the sole producer with the quality level $\lambda^{j(\omega)}$, where the parameter $\lambda > 1$ measures the size of each quality upgrade. Moreover, by improving on the current best quality index j, a successful R&D firm earns monopoly profits from selling the leading-edge $j(\omega) + 1$ quality to final-good firms and, in equilibrium, lower qualities of ω are priced out of business. As each industry leader is driven out of business by further innovation supported by other firm, the duration of the monopoly is finite. Over time, as qualities rise, workers become more productive and thus R&D fuels per capita consumption growth.

2.1. The final-good sector

We consider that each firm h in the final-good sector faces the following production function

$$Y_h(t) = AL_h^{1-\alpha} \int_0^{N(t)} \left[\lambda^{j(\omega,t)} x_h(\omega, t)\right]^{\alpha} d\omega \tag{1}$$

where $A > 0$ is a given scale parameter; L_h is labour input; $(1 - \alpha)$, $0 < \alpha < 1$, is the labour share in production; $x_h(\omega, t)$ is the amount used of the intermediate good ω, weighted by its quality level $\lambda^{j(\omega,t)}$. It is implicit in (1) that: (i) only the highest grade of each $\omega \in [0, N(t)]$ are actually produced and used in equilibrium, meaning $x_h(j, \omega, t) = x_h(\omega, t)$; thus, $N(t) > 0$ is the measure of how many different intermediate goods (i.e., product lines) ω exist at time t; (ii) the varieties of intermediate goods are *imperfect substitutes*, in the sense that the elasticity of substitution with respect to a given pair $(x(\omega_1), x(\omega_2))$, $\forall \omega_1, \omega_2 \in [0, N(t)]$, is $\frac{1}{1-\alpha} < \infty$ for any value of $N(t) > 0$.

Letting final output be the numeraire (that is, setting its price equal to unity), each firm h in the final-good sector seeks to maximise profit by solving

$$\underset{\{x_h(\omega), \omega \in [0, N(t)]\}, L_h}{max} 1 \cdot Y_h(t) - w(t)L_h - \int_0^{N(t)} p(\omega, t) x_h(\omega, t) d\omega \tag{2}$$

where $p(\omega, t)$ is the price of ω relative to the final-good price,[6] and $w(t)$ is the labour wage, also relative to the final-good price, at time t. Both p and w are taken as given by h. In particular, it results from the first-order condition with respect to x

[5] For sake of simplicity, we do not remove explicitly the scale effects associated to L and A (see Barro and Sala-i-Martin, 2004, ch. 6, ch. 7). However, in due time, we normalize L and A to unity at every t, so that the results of the model do not depend on the value of the growth rate of those two variables, either zero or not (see Subsection 3.4, below).

[6] Once again, since only the producer of the highest quality in ω sells goods, $p(j, \omega, t) = p(\omega, t)$. Henceforth, we only use explicitly all arguments (j, ω, t) when they are useful for convenience of exposition.

$$A\alpha L_h^{1-\alpha}\lambda^{j(\omega,t)\alpha}x_h(\omega,t)^{\alpha-1} = p(\omega,t) \tag{3}$$

for each ω and t, where the left-hand side of (3) is the marginal product of ω. Rearranging (3) and noting that $X(j,\omega,t) = X(\omega,t) = \sum_h x_h(\omega,t)$, gives us the aggregate demand of ω

$$X(\omega,t) = L\left(\frac{A\lambda^{j(\omega,t)\alpha}\alpha}{p(\omega,t)}\right)^{\frac{1}{1-\alpha}} \tag{4}$$

2.2. The intermediate-good sector

R&D is carried out in the intermediate-good sector. Therefore, firms face a two-stage decision process. In the first stage, since there is free-entry, they must decide how much to invest in either vertical or horizontal R&D. In the second stage, the successful R&D firms determine the price at which sell their previously invented goods to the producers of final output. We proceed as usual by solving the problem backward.

2.2.1. Production and price decisions

Firms take into account that the government can subsidise the production of all intermediates inputs by paying a fraction s_x of each firm's production costs; this subsidy policy is fully financed by taxes. The intermediate good is nondurable and entails a unit marginal cost of production, measured in terms of final-good output Y.[7] That is, the cost of production is the same for all qualities λ^j. Thus, the latest innovator has an efficiency advantage over the prior innovators in the sector but will eventually be at disadvantage relative to future innovators. We assume that each innovator is a different firm.

Since there is a continuum of intermediate goods, one can assume that firms are atomistic and take as given the price of final output (numeraire). Monopolistic competition, therefore, prevails and firms face isoelastic demand curves (4). Leading-edge intermediate input producers choose their prices $p(\omega,t)$ to solve the profit maximization problem

$$\underset{p(\omega,t)}{max} X(\omega,t)\left[p(\omega,t) - (1-s_x)\right] \tag{5}$$

Solving the first-order condition yields the optimal intermediate good price

$$p(\omega,t) \equiv p = \frac{1-s_x}{\alpha} \tag{6}$$

[7] Since, by assumption, intermediate-goods production and R&D activities are financed by the saved resources after consumption of the final good, the simplest hypothesis is to consider that the intermediate-goods production function is identical to that of the final good or, equivalently, the final good is the input in the production of each ω. Thus, the marginal cost of producing ω equals the marginal cost of producing the final good, which, due to perfect competition in the final-good sector, equals the price of the final good (numeraire), that is, unity.

which, since $0 < \alpha < 1$ and $1 - s_x > \alpha$, is the usual monopoly price markup, constant over time and across industries. As in Segerstrom and Zolnierek (1999), we assume that $\frac{1}{\alpha} < \lambda \Leftrightarrow \frac{1}{\alpha\lambda} < 1$, that is, if $\frac{1}{\alpha}$ is the price of the leading-edge good, the price of the next lowest grade, $\frac{1}{\alpha\lambda}$, is less than the unit marginal cost of production. Only in this case are the lower grades of ω unable to provide any effective competition for the leading-edge type, so that its producer can charge the unconstrained monopoly price (6).[8]

Given (4), (5) and (6), the aggregate quantity produced of ω is

$$X(\omega,t) = L\left(\frac{A\lambda^{j(\omega,t)\alpha}\alpha^2}{1-s_x}\right)^{\frac{1}{1-\alpha}} \Rightarrow x_h(\omega,t) = L_h\left(\frac{A\lambda^{j(\omega,t)\alpha}\alpha^2}{1-s_x}\right)^{\frac{1}{1-\alpha}} \tag{7}$$

Using the results above we get the profit accrued by the monopolist in ω

$$\pi(\omega,t) = \bar{\pi}A^{\frac{1}{1-\alpha}}\lambda^{j(\omega,t)\left(\frac{\alpha}{1-\alpha}\right)} \tag{8}$$

where $\bar{\pi} = L\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}}(1-s_x)^{\frac{-\alpha}{1-\alpha}}$. Thus, the profit flow is constant in-between innovations (note that, by assumption, L is constant over time) and jumps every time quality is upgraded: $\pi(j+1,\omega,t) = \pi(j,\omega,t)\cdot\lambda^{\frac{\alpha}{1-\alpha}}$, with $\lambda^{\frac{\alpha}{1-\alpha}} > 1$.

Substituting (7) in (1) and aggregating across final-good firms yields the aggregate output

$$Y(t) = \left(\frac{A^{\frac{1}{\alpha}}\alpha^2}{1-s_x}\right)^{\frac{\alpha}{1-\alpha}} LQ(t) \tag{9}$$

where

$$Q(t) = \int_0^{N(t)} \lambda^{j(\omega,t)\left(\frac{\alpha}{1-\alpha}\right)} d\omega \tag{10}$$

is the intermediate-input aggregate quality index, which can also be interpreted as the *technological-knowledge stock* of the economy, since, by assumption, there are no intersectoral spillovers.[9] The total resources devoted to intermediate input production at time t are also proportional to $Q(t)$

$$X(t) = \int_0^{N(t)} X(\omega,t)d\omega = \left(\frac{A\alpha^2}{1-s_x}\right)^{\frac{1}{1-\alpha}} LQ(t) \tag{11}$$

and the same happens with total profits

$$\Pi(t) = \int_0^{N(t)} \pi(\omega,t)d\omega = \bar{\pi}A^{\frac{1}{1-\alpha}}Q(t) \tag{12}$$

[8] In contrast, if $\frac{1}{\alpha} > \lambda$, the producer of the leading-edge good would have to engage in a limit-pricing strategy in order to drive his competitors out of the market.

[9] It is noteworthy that the aggregate production function is linear in Q, as this is the feature that ultimately allows for endogenous growth, as we show below.

2.2.2. R&D decisions

Vertical R&D free-entry and dynamic arbitrage conditions

As in the standard model of quality ladders, firms decide over their optimal vertical-R&D level, which constitutes the search for new designs (blueprints) that lead to a higher quality of existing intermediate goods. Each new design is granted a patent, meaning that a successful researcher retains exclusive rights over the use of his/her improved intermediate good. In each industry only (potential) entrants can do R&D and innovation arrival follows a Poisson process. There is free entry into each vertical R&D race and each entrant possesses the same R&D technology. Since there is perfect competition among entrants, the individual contribution of any particular entrant to the aggregate R&D expenditures of all entrants is negligible.

Let $I_i(j,\omega,t)$ denote the instantaneous probability of R&D success by potential entrant i in industry ω when the highest quality is j (I is also interpreted as the vertical innovation rate). This probability is independently distributed across firms, industries and over time, and depends on the flow of resources $R_{vi}(j,\omega,t)$ devoted to R&D by entrants in each ω at t (measured in units of final-good output Y). As in Barro and Sala-i-Martin (2004, ch. 7), we assume that each entrant's instantaneous probability of R&D success is given by a relation exhibiting constant returns in R&D expenditures, $I_i(j,\omega,t) = R_{vi}(j,\omega,t)\Phi(j,\omega,t)$, where the function Φ, to be defined below, is the same for every firm in ω and captures the effect of the current technological-knowledge position j. We can either have $\frac{d\Phi(\cdot)}{dj} < 0$, assuming that a congestion effect prevails, or $\frac{d\Phi(\cdot)}{dj} > 0$, considering that a 'standing-upon-the-shoulders-of-giants' effect is more important. In either case, the R&D technology exhibits intertemporal spillovers but no intersectoral spillovers (in contrast with, e.g., Aghion and Howitt, 1998, Dinopoulos and Thompson, 1998, and Howitt, 1999).

Now, let us define

$$\Phi(j,\omega,t) \equiv \frac{1}{\zeta}\lambda^{-(j(\omega,t)+1)\left(\frac{\alpha}{1-\alpha}\right)} \tag{13}$$

where $\zeta > 0$ is a constant that stands for the fixed vertical-R&D cost.[10] By aggregating across firms in ω, we get $R_v(j,\omega,t) = \sum_i R_{vi}(j,\omega,t)$ and $I(j,\omega,t) = \sum_i I_i(j,\omega,t)$, such that

$$I(j,\omega,t) = R_v(j,\omega,t)\frac{1}{\zeta}\lambda^{-(j(\omega,t)+1)\left(\frac{\alpha}{1-\alpha}\right)} \tag{14}$$

From (14), we can aggregate across ω to get the total resources devoted to vertical R&D, $R_v(t)$, for a given $N(t)$,

[10] Thus, we assume that a congestion effect prevails (e.g., Segerstrom and Zolnierek, 1999, and Barro and Sala-i-Martin, 2004, ch. 7). The way Φ depends on j implies that the increasing difficulty of creating new product generations over time exactly offsets the increased rewards from marketing higher quality products; see (13) and (8). This allows for constant probability over time and across industries in balanced-growth path.

$$R_v(t) = \int_0^{N(t)} R_v(j,\omega,t)d\omega = \int_0^{N(t)} \zeta\lambda^{(j(\omega,t)+1)\left(\frac{\alpha}{1-\alpha}\right)} I(j,\omega,t)d\omega \tag{15}$$

Taking $I(j,\omega,t)$ as given, it defines the probability of the incumbent losing his monopoly position. Thus, the present value of an incumbent's profits is a random variable because the terminal date to the monopoly of firm i arrives with probability $I(j,\omega,t)$ per (infinitesimal) increment of time. Let $V(j,\omega,t)$ denote the *expected* discounted value of profits earned by a monopolist when the highest quality in ω is j.[11] V can be interpreted as the market value of the patent or the value of the monopolist firm owned by households. The expected discounted value of profits can be written as $V(j,\omega,t) = \int_t^\infty \pi(j,\omega,t)e^{-\int_t^s (r(v)+I(j,\omega,v))dv}ds$, where r is the equilibrium market real interest rate and $\pi(j,\omega,t)$ is given by (8). The expression $\int_t^s (r(v) + I(j,v))\, dv$ simplifies to $(r+I)(s-t)$ if r and I are constant over time (as expected in a balanced growth equilibrium). The equation above reflects the fact that, if a profit flow can stop when a Poisson event with arrival rate I occurs, then we can calculate the *expected* present value of the stream of profit as if it never stops, but adding I to the discount rate. Thus, we can interpret $r + I$ as an *effective* discount rate. Since $\pi(j,\omega,t)$ is constant in-between innovations, we can further write

$$V(j,\omega,t) = \pi(j,\omega,t)\int_t^\infty e^{-\int_t^s (r(v)+I(j,\omega,v))dv}ds \tag{16}$$

Now, consider the average intermediate-good sector, $\bar{\omega}$, for a given $N(t)$.[12] Average resources devoted to vertical R&D, $R_v(j,\bar{\omega},t) = \frac{R_v(t)}{N(t)}$, can be put into (14) to yield an expression for the probability of vertical innovation for $\bar{\omega}$, $I(j,\bar{\omega},t)$. With free-entry into the vertical R&D business, we have the free-entry condition

$$I(j,\bar{\omega},t)\cdot V(j+1,\bar{\omega},t) = (1-s_r)\cdot R_v(j,\bar{\omega},t) \tag{17}$$

where we have considered that the government can subsidise vertical R&D by paying a fraction s_r of each firm's expenditures.[13] By substituting (16) into (17), we get

$$I(j,\bar{\omega},t)\cdot \pi(j+1,\bar{\omega},t)\int_t^\infty e^{-\int_t^s (r(v)+I(j+1,\bar{\omega},v))dv}ds = (1-s_r)\cdot R_v(j,\bar{\omega},t) \tag{18}$$

whilst time-differentiating (18), bearing in mind Leibniz's rule, yields

[11] We assume that entrants are risk-neutral and, thus, only care about the expected value of the firm.

[12] The usual procedure in the quality-ladders literature is to consider the average intermediate-good sector in order to avoid any jumpiness in quality levels that would occur if the behaviour of an individual sector were contemplated. Among other things, this allows us to characterise the aggregate dynamics of the model by using the standard techniques of deterministic-dynamics analysis (see Section 3, below).

[13] V denotes an *expected* value to the extent that it captures the effect of Poisson *death* on the monopolist's profits; the term $I\cdot V$, in (17), captures the effect of Poisson innovation *arrival* on the entrant's (expected) profits. See Appendix A for derivation.

$$r(t) + I(j+1,\bar{\omega},t) = \frac{\pi(j+1,\bar{\omega},t)\cdot I(j,\bar{\omega},t)}{(1-s_r)\cdot R_v(j,\bar{\omega},t)} -$$

$$- \left(\frac{\dot{\pi}(j+1,\bar{\omega},t)}{\pi(j+1,\bar{\omega},t)} + \frac{\dot{I}(j,\bar{\omega},t)}{I(j,\bar{\omega},t)} - \frac{\dot{R}_v(j,\bar{\omega},t)}{R_v(j,\bar{\omega},t)} \right) \tag{19}$$

This can be interpreted as an arbitrage condition, which *equates the effective rate of return on capital* (i.e., the market rate of return augmented by the Poisson arrival rate) *to the rate of return on vertical R&D, where the latter equals the profit rate earned by setting up now a new firm with an existing intermediate good of improved quality minus the increase in the profit rate due to the next innovation in that intermediate good* (which is accrued to the next innovator).[14]As a result of (8) and (14) applied to $\bar{\omega}$, we have, after time differentiation,

$$\frac{\dot{R}_v(j,\bar{\omega},t)}{R_v(j,\bar{\omega},t)} = \frac{\dot{I}(j,\bar{\omega},t)}{I(j,\bar{\omega},t)} + I(j,\bar{\omega},t)\left[\dot{j}(\bar{\omega},t)\left(\frac{\alpha}{1-\alpha}\right) ln\lambda\right] \tag{20}$$

and

$$\frac{\dot{\pi}(j+1,\bar{\omega},t)}{\pi(j+1,\bar{\omega},t)} = \frac{\dot{\pi}(j,\bar{\omega},t)}{\pi(j,\bar{\omega},t)} = I(j,\bar{\omega},t)\left[\dot{j}(\bar{\omega},t)\left(\frac{\alpha}{1-\alpha}\right) ln\lambda\right] \tag{21}$$

Hence, given (20) and (21), we can rewrite (19) as

$$r(t) = \frac{\pi(j+1,\bar{\omega},t)\cdot I(j,\bar{\omega},t)}{(1-s_r)\cdot R_v(j,\bar{\omega},t)} - I(j+1,\bar{\omega},t) \tag{22}$$

Thus, the market interest rate is equated to a dividend term that captures the expected profit rate from the next innovation, minus an obsolescence term that captures the Schumpeterian concept of creative destruction caused by the next successful innovation (which leads to the obsolescence of the preceding one). Substituting (8) and (14) in the right-hand side of (22), yields

$$r(t) = \frac{\bar{\pi}A^{\frac{1}{1-\alpha}}}{\zeta(1-s_r)} - I(j+1,\bar{\omega},t) \Leftrightarrow r(t) = \frac{\bar{\pi}A^{\frac{1}{1-\alpha}}}{\zeta(1-s_r)} - I(t) \tag{23}$$

According to (23), the relationship between r and I is independent of t, ω, and j, implying $I(t) \equiv I(j+1,\bar{\omega},t)$. Thus, if I is constant over time, then r is also constant.

Horizontal R&D free-entry and dynamic arbitrage conditions

Variety expansion results from R&D aimed at creating a new intermediate-good line, corresponding to a new firm, at a cost of η units of final output. In particular, we view

[14]In (19), $\frac{\dot{\pi}}{\pi}$ (as well as $\frac{\dot{R}_v}{R_v}$ and $\frac{\dot{I}}{I}$) must be interpreted in *expected* terms, since it reflects the stochastic process of innovation arrival.

the creation of new product lines as a product development activity without positive spillovers (e.g., Dinopoulos and Thompson, 1998) and allow for entry as well as exit of product lines from the market - that is, we do not assume irreversibility of investment in product development.

After a new product is launched, an initial quality level is observed, drawn at random from the distribution of quality indexes matching the existing product lines. Let $q(j,\omega,t) \equiv \lambda^{j(\omega,t)\left(\frac{\alpha}{1-\alpha}\right)}$ be an alternative measure of product quality, as, e.g., in Segerstrom (2007). Then, from (10), we have

$$Q(t) = \int_0^{N(t)} q(j,\omega,t)d\omega = q(j,\bar{\omega},t)N(t) \tag{24}$$

where $q(j,\bar{\omega},t) \equiv E_\omega(q)$ is the average of q over industries.[15] Taking into account that the government can subsidise horizontal R&D by paying a fraction s_n of each firm's expenditures, entry costs $(1-s_n)\eta$ and generates value $V(q(j,\bar{\omega},t)) \equiv V(j,\bar{\omega},t)$. A free-entry equilibrium requires that new product lines are created (or destroyed) at a rate $\dot{N}$ (henceforth, the dot denotes time derivative) necessary to ensure that

$$V(j,\bar{\omega},t) = (1-s_n)\eta \tag{25}$$

such that the total flow of resources devoted to horizontal R&D is

$$R_n(t) = \eta\dot{N}(t) \tag{26}$$

Henceforth, we explore the case where η is time-varying.[16] In particular, let

$$\eta \equiv \eta(N,R_n) = \varphi_1(N)\cdot\varphi_2(R_n) \tag{27}$$

where $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ are positive, invertible functions.[17] This specification of the entry cost function merges the one suggested by Romer (1990) and Barro and Sala-i-Martin (2004, ch. 6), $\varphi_1(N)$, where the entry cost increases with the number of differentiated goods in the market,[18]with the one implicit in Howitt (1999), $\varphi_2(R_n)$, where the aggregate function of horizontal innovation exhibits *decreasing marginal returns* in total horizontal R&D expenditures.[19] Let $\varphi_1(N) = \psi N(t)^{\nu_1}$ and $\varphi_2(R_n) = R_n(t)^{\nu_2}$, where the parameter $\psi > 0$ stands for a fixed horizontal-R&D cost, $\nu_1 > 0$ measures the negative spillover effect related to the accumulation of intermediate-good varieties, whilst

[15]Hence, from (7), (8), (11) and (12), we have $X(q(j,\bar{\omega},t)) \equiv X(j,\bar{\omega},t)$, $\pi(q(j,\bar{\omega},t)) \equiv \pi(j,\bar{\omega},t)$, $X(t) = X(j,\bar{\omega},t)N(t)$ and $\Pi(t) = \pi(j,\bar{\omega},t)N(t)$.

[16]The case where η is a fixed-entry cost, constant over time, is analysed in Appendix B.

[17]Appendix B also explores alternative specifications of time-varying entry costs.

[18]The basic specification in Romer (1990) is similar to the one used by Barro and Sala-i-Martin (2004, ch. 6) in that it implies the entry cost, measured in wage labour units in Romer (1990), raises with N, since an increase in the latter raises the marginal product of labour and, thus, the real wage rate.

[19]See the arbitrage condition (10) in that paper

$0 < \nu_2 < 1$ measures the degree of the increasing returns of the *marginal* horizontal innovation function, $\varphi_2(R_n)^{-1}$. By substituting (26) in (27), we arrive at

$$\eta(\cdot) = \psi N(t)^{\nu_1} \left(\eta(\cdot)\dot{N}(t)\right)^{\nu_2} \Leftrightarrow \eta(\cdot) = \psi^{\frac{1}{1-\nu_2}} N(t)^{\frac{\nu_1}{1-\nu_2}} \dot{N}(t)^{\frac{\nu_2}{1-\nu_2}} = \phi N(t)^{\sigma} \dot{N}(t)^{\gamma} \quad (28)$$

where $\phi \equiv \psi^{\frac{1}{1-\nu_2}} > 0$, $\sigma \equiv \frac{\nu_1}{1-\nu_2} > 0$ and $\gamma \equiv \frac{\nu_2}{1-\nu_2} > 0$. Equation (28) shows the link between our specification of the horizontal entry-cost function with respect to R_n and that used by Datta and Dixon (2002) and Brito and Dixon (2008), where the entry cost increases with the number of goods entering the market at a given instant, $\dot{N}$. This mechanism, which introduces dynamic second-order effects in entry,is also similar to the one that characterises the changes of the physical-capital stock in the literature of firm investment with convex adjustment costs, where the cost of installing (dismantling) capital increases with the amount of investment (disinvestment) at a given instant (e.g., Eisner and Strotz, 1963).

By substituting (16) into (25), where $\eta \equiv \eta(\cdot)$ is a time-varying function, we have

$$\pi(j,\bar{\omega},t) \int_t^{\infty} e^{-\int_t^s (r(v)+I(j,\bar{\omega},v))dv} ds = (1-s_n)\cdot\eta(\cdot) \quad (29)$$

If we time-differentiate (29), assuming η is differentiable with respect to time, we get

$$r(t) + I(j,\bar{\omega},t) = \frac{\pi(j,\bar{\omega},t)}{(1-s_n)\eta(\cdot)} - \left(\frac{\dot{\pi}(j,\bar{\omega},t)}{\pi(j,\bar{\omega},t)} - \frac{\dot{\eta}(\cdot)}{\eta(\cdot)}\right) \quad (30)$$

This is another arbitrage equation, according to which *the effective rate of return on capital equals the rate of return on horizontal R&D, where the latter equals the profit rate earned by setting up now a new firm with a new product line minus the increase in the profit rate due to the next innovation in that intermediate good* (which is accrued to the next vertical innovator).[20]

Consistency arbitrage condition

Finally, a consistency condition between vertical and horizontal arbitrage conditions is needed. First, we find an expression for $R_v(j-1,\bar{\omega},t)$, by applying (15) to $j-1$ and combining it with (24), for a given $N(t)$,

$$R_v(j-1,\bar{\omega},t) = \frac{\int_0^{N(t)} R_v(j-1,\omega,t)d\omega}{N(t)} = \frac{I(t)\zeta Q(t)}{N(t)} = I(t)\zeta q(j,\bar{\omega},t) \quad (31)$$

where we used $I(t) \equiv I(j-1,\bar{\omega},t)$. Then, from the vertical free-entry condition, (17), solved in order to V, we get $V(j+1,\bar{\omega},t) = (1-s_r)\frac{R_v(j,\bar{\omega},t)}{I(j,\bar{\omega},t)} \Rightarrow V(j,\bar{\omega},t) = (1-s_r)\frac{R_v(j-1,\bar{\omega},t)}{I(j-1,\bar{\omega},t)}$. Together with (31), we have

$$V(j,\bar{\omega},t) = (1-s_r)\zeta q(j,\bar{\omega},t) \quad (32)$$

[20]Remember that $\frac{\dot{\pi}}{\pi}$, in (30), must be interpreted in *expected* terms.

At last, equating (32) and the horizontal free-entry condition, (25), yields

$$q(j,\bar{\omega},t) = \frac{Q(t)}{N(t)} = \left(\frac{1-s_n}{1-s_r}\right)\frac{\eta(\cdot)}{\zeta} \tag{33}$$

As one can see, the consistency condition (33) ties up the average quality to the ratio of the cost of horizontal entry, $\eta(\cdot)$, to the fixed cost of vertical R&D, ζ.[21]

2.3. The consumer sector

The economy consists of L identical dynastic families who consume and collect income (dividends) from investments in financial assets (equity) and labour income. They choose the path of final-good aggregate consumption $\{C(t), t \geq 0\}$ to maximise discounted lifetime utility

$$U = \int_0^{\infty} \left(\frac{C(t)^{1-\theta}-1}{1-\theta}\right) e^{-\rho t} dt \tag{34}$$

where $\rho > 0$ is the subjective discount rate and $\theta > 0$ is the constant elasticity of marginal utility with respect to consumption. We assume consumers have perfect foresight concerning the aggregate rate of technological change over time and choose their expenditure paths accordingly to maximise their discounted utilities, dispensing with the time expectations operator, $E(.)$, in (34).[22]

Intertemporal utility is maximised subject to the flow budget constraint

$$\dot{a}(t) = (1-\tau_a)\, r(t) a(t) + (1-\tau_w)\, w(t) L - C(t) \tag{35}$$

where a stands for households' financial assets (equity) holdings, measured in terms of final-good output Y. Households take the real rate of return on financial assets, r (that is, dividend payments in units of asset price corrected by the Poisson death rate, $r = \frac{\pi}{V} - I$[23]) and the real labour wage, w, as given. The initial level of wealth $a(0)$ is also given, whereas the condition $lim_{t\to\infty} e^{-\int_0^t r(s)ds} a(t) = 0$ is imposed in order to prevent Ponzi schemes. We assume the households can be subject to government-imposed income taxes, at a rate τ_w on labour income and τ_a on assets income, in case they are needed for a balanced government budget.

[21] Note that if we equate the two arbitrage conditions (30) and (23), each solved in order to $r + I$, and use (33) to simplify, one obtains $\frac{\dot{\eta}}{\eta} = \frac{\dot{\pi}}{\pi} = \frac{\dot{Q}}{Q} - \frac{\dot{N}}{N}$. But this result does not imply a further constraint on η since it also obtains from direct time-differentiation of (33). Thus, the latter can indeed be interpreted as an arbitrage consistency condition. In any case, we are implicitly assuming that both the real rate of return to vertical R&D and to horizontal R&D equal r for every t, meaning that the (competitive) capital market is always willing to finance both activities.

[22] The uncertainty associated with R&D at the industry level creates jumpiness in microeconomic outcomes. However, as the probabilities of successful R&D across industries are independent and there is a continuum of industries this jumpiness is not transmitted to macroeconomic variables.

[23] This equation can be read as an arbitrage condition for investors, which requires that the real interest rate equals the dividend rate, $\frac{\pi}{V}$, plus the rate of capital gain, $-I$. This condition can be derived, e.g., by solving (17) in order to V and substituting the result in (22).

The optimal path of consumption satisfies the well-known differential Euler equation

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}\left[(1-\tau_a)\, r(t) - \rho\right] \tag{36}$$

as well as the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} C(t)^{-\theta} a(t) = 0 \tag{37}$$

2.4. The government budget

We assume that the government budget is balanced at each point in time. The government can support subsidies for intermediate goods production, s_x, for vertical R&D, s_r, and horizontal R&D, s_n, paid to firms, and can collect tax revenue from assets income, τ_a, and from labour income, τ_w, both supported by households. Thus,

$$\tau_w w(t) L + \tau_a r(t) a(t) - s_x X(t) - s_r R_v(t) - s_n R_n(t) = 0 \tag{38}$$

holds for every t.

3. General equilibrium

In this section, we construct the general equilibrium, discuss the comparative statics of the interior steady state and characterise the local dynamics in its neighbourhood. For sake of simplicity, we abstract from then on from government intervention (i.e., we set $s_r = s_n = s_x = \tau_a = 0$), saving the analysis of its effects for Section 4 and Appendix J.

3.1. The aggregate resource constraint

The balance sheet of households equates the value of equity holdings to the market value of firms, that is

$$a(t) = V(j, \bar{\omega}, t) N(t) = \eta(\cdot) N(t) \tag{39}$$

Hence, we can characterise the change in the value of equity as

$$\dot{a}(t) = \eta(\cdot)\dot{N}(t) + \dot{\eta}(\cdot) N(t) \tag{40}$$

Substituting (39), (35) and (30) (the latter solved in order to $\dot{\eta}$) in (40), yields, after some algebric manipulation,

$$Y(t) = X(t) + C(t) + R_n(t) + R_v(t) \tag{41}$$

Equation (41) tells us that total final-good output, Y, is allocated among total consumption, C, total production of intermediate goods, X, total vertical R&D expenditures,

R_v, and total horizontal R&D expenditures, R_n, thus being a *product market equilibrium* equation. The results above also imply that households' gross savings equal total investment at every t, meaning that

$$\dot{a}(t) = R_n(t) + R_v(t) - I(t)a(t) \tag{42}$$

which is the accumulation equation for a.[24] The first two terms on the right-hand side of (42) represent the *gross investment* in technological knowledge at time t, whereas the third term represents the *depreciation* (obsolescence) of the existing technological-knowledge stock due to the stochastic arrival of vertical innovations (i.e., as j jumps to $j+1$). Equation (42) has obvious similarities to the accumulation equation of physical capital in the standard Ramsey model. However, the depreciation rate of the technological-knowledge stock displayed by our model is not exogenous, but rather an endogenous function of vertical R&D activity, in line with the notion of "endogenous obsolescence" explored by Caballero and Jaffe (1993). Our concept of technological-knowledge stock can be linked to the measure of knowledge stock proposed by Griliches (1979) and analysed recently by Klette and Kortum (2004).[25]

According to (42), the dynamics of a depends on the dynamics of Y, C, R_n and R_v, which in turn depend ultimately on the dynamics of Q. To see this, first note that

$$R_v(t) = \int_0^{N(t)} \Phi(j,\omega,t)^{-1} I(j,\omega,t) d\omega = I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}} Q(t) \tag{43}$$

obtained by using $I(t) \equiv I(j,\omega,t)$ in (15).[26] Second, recall the consistency condition (33), from Section 2.2.2; since, in balanced-growth path, $g_N \equiv \frac{\dot{N}}{N} \Rightarrow R_n = \eta\dot{N} = \eta g_N N$, R_n grows with N and η; in turn, from (33), η grows with $\frac{Q}{N}$. Therefore, R_n grows at the same rate as Q in balanced-growth equilibrium. Finally, recall from (9) and (11) that

[24] Using (39) together with (33) (with $s_r = s_n = 0$), we get $a = \zeta Q$, from which we conclude that (42) is equivalent to $\dot{Q} = \frac{1}{\zeta}(R_n + R_v) - IQ$. See Appendix C for a detailed derivation of (41) and (42), considering government intervention.

[25] For Griliches, the stock of "technical knowledge" is the discounted sum of *past* R&D, which he denotes by K. Klette and Kortum (2004) propose a measure of the knowledge stock conditional on past R&D expenditures, R, considering that the appropriate discount rate on past R&D is the intensity of creative destruction. In our model, this is the Poisson arrival rate I, which may be time variable. Hence, we have

$$K(t) = \int_{t_0}^{t} e^{-\int_s^t I(\tau)d\tau} R(s) ds$$

where t_0 is the date on which the first intermediate-good line was born. If we time-differentiate this expression, we get $\dot{K}(t) = R(t) - I(t)K(t)$, which is similar to (42). This result should apply to the whole class of lab-equipment quality-ladders models, as discussed in Gil and Afonso (2008). (For an alternative measure of knowledge stock, set within a model that takes into account obsolescence and diffusion effects, see Caballero and Jaffe, 1993).

[26] From (41), we also know that $R_v = \left(\frac{1-s_n}{1-s_r}\right) \cdot \left(I\eta N + \frac{\dot{\pi}}{\pi}\eta N\right)$ (see Appendix C), which means that we must have $\left(\frac{1-s_n}{1-s_r}\right) \cdot \left(I(t)\eta N + \frac{\dot{\pi}}{\pi}\eta N\right) = I\zeta\lambda^{\frac{\alpha}{1-\alpha}} Q$. Using the fact that $\frac{\dot{\pi}}{\pi} = I\left[\dot{j}\left(\frac{\alpha}{1-\alpha}\right) ln\lambda\right] = I\left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right)$ for $\dot{j} = 1$ and small λ, the above implies, for $I > 0$, $\frac{Q}{N} = \left(\frac{1-s_n}{1-s_r}\right)\frac{\eta}{\zeta}$, which is the consistency condition (33).

Y and X exhibit direct linear dependence on Q. Thus, in balanced growth equilibrium, when I is constant, (41) holds with Y, X, R_v, R_n and C growing at the same rate.

However, in the medium term (i.e., during the transition to the balanced-growth equilibrium), only Y and X grow at the same rate (the growth rate of Q), and the growth rate of C is determined residually after the dynamics of R_v and R_n. Using (9), (11) (with $s_x = 0$) and (43) in (41) yields

$$H_Y Q(t) = H_X Q(t) + C(t) + \eta(\cdot)\dot{N}(t) + I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}}Q(t) \tag{44}$$

where $H_Y \equiv L\left(A^{\frac{1}{\alpha}}\alpha^2\right)^{\frac{\alpha}{1-\alpha}}$ and $H_X \equiv L\left(A\alpha^2\right)^{\frac{1}{1-\alpha}}$.

3.2. The dynamic system

The general equilibrium is defined by the system of six equations: the Euler equation for consumption (36); the households' transversality condition (37); the horizontal arbitrage condition (30); the vertical arbitrage condition (23); the arbitrage consistency condition (33); the product market equilibrium equation (44), plus the necessary initial conditions.

We substitute (28) in the dynamic horizontal arbitrage equation (30) and take into account (23) (with $s_r = s_x = 0$) and (24) in order to get

$$\frac{\dot{Q}(t)}{Q(t)} = I(t)\left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right)\left(\frac{1+\sigma}{\sigma}\right) +$$

$$+\left[\bar{\pi}A^{\frac{1}{1-\alpha}}\left(\frac{\eta(\cdot)}{\zeta} - \frac{Q(t)}{N(t)}\right) - \gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\eta(\cdot)\right]\frac{1}{\eta(\cdot)}\left(\frac{1}{\sigma}\right) \tag{45}$$

and

$$\frac{\dot{N}(t)}{N(t)} = I(t)\left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right)\left(\frac{1}{\sigma}\right) +$$

$$+\left[\bar{\pi}A^{\frac{1}{1-\alpha}}\left(\frac{\eta(\cdot)}{\zeta} - \frac{Q(t)}{N(t)}\right) - \gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\eta(\cdot)\right]\frac{1}{\eta(\cdot)}\left(\frac{1}{\sigma}\right) \tag{46}$$

where we have considered that the number of sectors, N, is large enough to treat Q as time-differentiable and non-stochastic (two dots denote a second-order time derivative).[27]By subtracting (46) to (45), we can see that

$$\frac{\dot{Q}(t)}{Q(t)} - \frac{\dot{N}(t)}{N(t)} = I(t)\left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right) \tag{47}$$

which tells us that the gap between the growth rate of Q and the growth rate of N is always positive, provided there is vertical R&D activity, i.e., $I > 0$. Moreover, as we show below, this gap is constant and positive along the balanced-growth path.

[27]See Appendix D for a detailed derivation, considering government intervention.

Equations (45) and (46) define a system of non-linear ordinary differential equations whose solution, together with the initial conditions $N(0)$ and $Q(0)$, gives us the time-paths of N and Q (and thus of Y, X, R_n and R_v) as a function of I. On the other hand, those two differential equations are of second-order in N. To obtain a fully workable dynamic system we need to add (36) and (44) to the system, reduce the order of (45) and (46) and simplify across using (33). Firstly, we solve (33) (with $s_r = s_n = 0$) in order to $\dot{N}$, to obtain an ordinary differential equation (ODE) in N. Secondly, we solve (46) in order to $\ddot{N}$, replace in (45) and use (33) to simplify, so that we have an ODE in Q. Together with the Euler equation for consumption, (36) (with $\tau_a = 0$), the dynamic system reads

$$\dot{N}(t) = x(Q,N) \cdot N(t) \tag{48}$$

$$\dot{Q}(t) = \left(I(Q,N,C) \cdot \Xi + x(Q,N)\right) Q(t) \tag{49}$$

$$\dot{C}(t) = \left(\mu - \frac{1}{\theta} I(Q,N,C)\right) C(t) \tag{50}$$

where $\Xi \equiv \left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right)$, $\mu \equiv \frac{1}{\theta}\left(\frac{\bar{\pi} A^{\frac{1}{1-\alpha}}}{\zeta} - \rho\right)$, the latter obtained by replacing (23) in (36), and

$$x(Q,N) = \left(\frac{\zeta}{\phi}\right)^{\frac{1}{\gamma}} Q(t)^{\frac{1}{\gamma}} N(t)^{-\left(\frac{\sigma+\gamma+1}{\gamma}\right)} \tag{51}$$

$$I(Q,N,C) = \frac{1}{\zeta \lambda^{\frac{\alpha}{1-\alpha}}} \left(H_Y - H_X - \frac{C(t)}{Q(t)} - \zeta \cdot x(Q,N)\right) \tag{52}$$

Equation (52) results from solving (44) in order to I and using (33) and (48) to simplify, such that $I \equiv I(Q,N,C)$. Thus, we are able to define a system of three ODE's $\dot{Q} = F_Q(Q,N,C)$, $\dot{N} = F_N(Q,N)$, and $\dot{C} = F_C(Q,N,C)$.

3.3. The steady state

3.3.1. Steady-state equilibrium

Now, we derive and characterise the interior steady-state equilibrium. First, it is convenient to find a transformation of the system (48)-(50) such that we can work with an equivalent system whose equilibria are fixed points. The stability and unicity of the interior steady-state equilibrium are shown within this framework.

Proposition 1. Let $g_y \equiv \dot{y}/y$, the growth rate of a variable y along the balanced-growth path. In this model, steady-state equilibria have the following characteristics: (i) $g_C = g_Q = g$; (ii) $g_I = 0$; and (iii) $\frac{g_Q}{g_N} = (\sigma + \gamma + 1)$, $x \neq 0$.

Proof: See Appendix E.

Having the above in mind, and following, e.g., Mulligan and Sala-i-Martin (1993), we transform the system (48)-(50) into a system of rescaled variables. Recall (51) and let

$$z(t) = \frac{C(t)}{Q(t)} \tag{53}$$

with the property that, in the steady state, $\dot{x} = \dot{z} = 0$. After time-differentiating (51) and (53), and substituting with (48), (49) and (50) where necessary, we get the system

$$\dot{x}(t) = \left[\frac{1}{\gamma}\frac{\dot{Q}(t)}{Q(t)} - \left(\frac{\sigma+\gamma+1}{\gamma}\right)\frac{\dot{N}(t)}{N(t)}\right]x(t) = I(x,z)\cdot\Xi\cdot\frac{1}{\gamma}x(t) - \left(\frac{\sigma}{\gamma}+1\right)x(t)^2 \tag{54}$$

$$\dot{z}(t) = \left(\frac{\dot{C}(t)}{C(t)} - \frac{\dot{Q}(t)}{Q(t)}\right)z(t) = \mu z(t) - \left(\frac{1}{\theta}+\Xi\right)I(x,z)z(t) - x(t)z(t) \tag{55}$$

where $I(x,z) \equiv I(Q,N,C)$ (see (52)). Thus, we find a system of rescaled variables equivalent to (48)-(50) which comprises two ODE's, $\dot{x} = F_x(x,z)x$ and $\dot{z} = F_z(x,z)z$. Notably, (54) and (55) are quasi-linear ODE's, whereas I is a linear function of x and z. Equations (54) and (55), plus the transversality condition and the initial condition $x(0)$, describe the transitional dynamics and the steady state of the model, by jointly determining the variables $(x(t), z(t))$. From these we can determine the original variables $N(t)$ and $C(t)$, for a given $Q(t)$. That is, the system is undetermined in $Q(t)$.[28]

As usual, the fixed points of the system are found by equating $\dot{x} = 0$ and $\dot{z} = 0$. It is straightforward to show that there are four fixed points, but only is an interior equilibrium, i.e., $x^* \neq 0 \ \wedge \ z^* \neq 0$, where $*$ indicates steady-state value. Let $I(x,z) = I_0 + I_1 z + I_2 x$, where $I_0 \equiv \frac{1}{\zeta\lambda^{\frac{\alpha}{1-\alpha}}}(H_Y - H_X)$, $I_1 \equiv -\frac{1}{\zeta\lambda^{\frac{\alpha}{1-\alpha}}}$ and $I_2 \equiv -\frac{1}{\lambda^{\frac{\alpha}{1-\alpha}}}$. Then, we have, for the interior steady-state,

$$z^* = \left(-I_0 - I_2 x^* + \frac{\sigma+\gamma}{\Xi}x^*\right)\frac{1}{I_1} \tag{56}$$

$$x^* = \frac{\mu\Xi}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(\sigma+\gamma)} \tag{57}$$

From (48) and (51), we find that

$$g_N^* = x^* \tag{58}$$

and, from Proposition 1-(iii),

[28] This means that one of the three eigenvalues associated to the original dynamic system equals zero. Thus, the Jacobian of the original system has a null determinant, which guarantees the existence of an equilibrium with non-null growth rates of Q and N. Caballé and Santos (1993) give a detailed analysis of this sort of indeterminacy in the context of an endogenous growth model with both physical and human-capital accumulation.

$$g^* = \frac{\mu\Xi\left(\sigma+\gamma+1\right)}{\Xi\left(\sigma+\gamma+1\right)+\frac{1}{\theta}\left(\sigma+\gamma\right)} \tag{59}$$

Finally, using (56) and the definition of I_0, I_1 and I_2, we find

$$I^* = \frac{\sigma+\gamma}{\Xi}x^* = \frac{\mu\left(\sigma+\gamma\right)}{\Xi\left(\sigma+\gamma+1\right)+\frac{1}{\theta}\left(\sigma+\gamma\right)} \tag{60}$$

and

$$z^* = H_Y - H_X - \zeta\left(1+\frac{\sigma+\gamma}{\Xi}\right)\frac{\mu\Xi}{\Xi\left(\sigma+\gamma+1\right)+\frac{1}{\theta}\left(\sigma+\gamma\right)} \tag{61}$$

The condition $H_Y - H_X > \zeta\left(1+\frac{\sigma+\gamma}{\Xi}\right)\frac{\mu\Xi}{\Xi(\sigma+\gamma+1)+\frac{1}{\theta}(\sigma+\gamma)}$ is required in order to have $z^* > 0$.[29]We conclude that the fundamental determinants of vertical innovation intensity, I, are the technological parameters (i.e., the production parameters A and α, the vertical-innovation parameters λ and ζ, and the horizontal-entry parameters γ and σ), and the preferences parameters (ρ and θ). The link to the preferences side of the model runs primarily through the relationship between I and r represented by the arbitrage equation (23). This relation is in line with the Schumpeterian early view that the market real interest rate must evolve in tandem with the process of creative destruction. Note also that, as one should expect,

$$\lim_{\sigma\to\infty} g^* = \lim_{\gamma\to\infty} g^* = g^*_{no-entry}$$

Let $\dot{N} = E$, where E stands for the *instantaneous rate of entry*. The steady-state values of N, E and C are derived from (51) and (53), given Q. Thus,

$$N^* = \left(\frac{\zeta}{\phi}\right)^{\frac{1}{\sigma+\gamma+1}}\left(x^*\right)^{-\left(\frac{\gamma}{\sigma+\gamma+1}\right)}\left(Q^*\right)^{\frac{1}{\sigma+\gamma+1}} \tag{62}$$

$$E^* = x^*N^* \tag{63}$$

$$C^* = z^*Q^* \tag{64}$$

The results above can be summarized in the following proposition.

Proposition 2. There is a unique interior steady-state equilibrium, as defined by equations (57)-(64).

Proof: See derivations above and Appendix F.

[29]Yet, in our numerical simulations this constraint was not binding even for a wide range of parameter values.

Finally, $g^* > 0$ requires $\mu \equiv \frac{1}{\theta}\left(\frac{\bar{\pi}A^{\frac{1}{1-\alpha}}}{\zeta} - \rho\right) > 0$. Since, from (36), $g = g_C = \frac{1}{\theta}(r-\rho)$, then $r > \rho$ must occur. This condition also guarantees $g_N^* > 0$. On the other hand, according to the transversality condition, (37), together with (39) and (33), we have

$$\lim_{t\to\infty} e^{-\rho t}C(t)^{-\theta}\zeta Q(t) = \lim_{t\to\infty} e^{-\rho t}\left(\frac{C(t)}{Q(t)}\right)^{-\theta}\zeta Q(t)^{1-\theta} = 0 \tag{65}$$

where $\frac{C}{Q}$ is stationary in steady-state, as shown above. Let $Q = \hat{Q}e^{gt}$, where $\hat{Q}$ denotes detrended Q (thus stationary in steady-state), and substitute in (65), to see that the transversality condition implies $\rho \geq (1-\theta)g$; using again $g = \frac{1}{\theta}(r-\rho)$, the latter condition can be written alternatively as $r > g$. As it happens, this condition also guarantees that attainable utility is bounded, i.e., the integral (34) converges to infinity.

Thus, our model predicts, under a sufficiently productive technology, a steady-state equilibrium with constant positive g and g_N, where the former exceeds the latter by an amount corresponding to the growth of intermediate-good quality, driven by vertical innovation (see (47)). This implies that the consumption growth rate equals the growth rate of the number of varieties plus the growth rate of intermediate-good quality, in line with the view that industrial growth proceeds both along an intensive and an extensive margin. A similar result can be found, e.g., in Dinopoulos and Thompson (1998), Peretto (1998) or Arnold (1998).[30]

But differently from Peretto (1998) and the already quoted quality-ladders models with expanding variety, in our model the growth of the number of varieties is not linked to the (exogenous) population growth rate. The negative spillover effect in (28), de per se, determines a constant number of varieties in balanced-growth equilibrium(see Barro and Sala-i-Martin, 2004, ch. 6); however, variety expansion is sustained by endogenous technological-knowledge accumulation (independently of population growth), as the expected growth of intermediate-good quality due to vertical R&D makes it attractive for potential entrants to always put up an entry cost, in spite of its increase with N.[31] In this sense, our model predicts that it is not necessarily the larger economy, measured by population size, that produces the greater number of distinct goods, but that with the larger technological-knowledge stock, which thus emerges as the relevant *endogenous* measure of economic size. The positive steady-state relation between N and Q, for a given x, is made clear by (62), above.

Arnold (1998) and Funke and Strulik (2000) also obtain a positive growth rate of the number of varieties in the steady state that is solely driven by knowledge accumulation. However, in their models, this occurs in the form of human-capital production, with the latter counterbalancing the increasing entry cost due to rising real wages caused by the positive impact of a growing N in labour marginal productivity. The monotonic positive

[30] In Peretto (1998)'s model of endogenous growth with cost-reducing R&D, the intensive margin is due to productivity growth, whilst in Arnold (1998) it reflects human-capital accumulation.

[31] In fact, as shown in Appendix B, the dependence of η on N is necessary to eschew the explosive balanced-growth path that would occur in our model if η were constant over time (or depended solely on $\dot{N}$). This is not the case in Barro and Sala-i-Martin (2004, ch. 6)'s basic model of pure expanding variety.

relationship between the steady-state values of g and g_N observed in our model (see Proposition 1-(iii), above) is qualitatively similar to the one in Funke and Strulik (2000) (see their Proposition 1).

We interpret the technological-knowledge stock per firm, $\frac{Q}{N}$, as a measure of average firm size.[32] According to (47), it expands at the growth rate of intermediate-good quality in steady state. Alternative measures of firm size such as production (or sales) per firm, $\frac{X}{N} = H_x \frac{Q}{N}$ (see (44)), or financial assets per firm, $\frac{a}{N} = \eta = \zeta \frac{Q}{N}$ (see (39) and (33)) produce the same result.[33] Peretto (1998) takes employment per firm as an explicit measure of firm size, while physical capital per firm in efficiency units and human capital per firm can be interpreted as measures of firm size in Aghion and Howitt (1998, ch. 12) and Funke and Strulik (2000) and Arnold (1998), respectively. All three ratios are constant in the steady state.[34]

These results are broadly supported by historical empirical evidence. The increase of sales per firm over time is referred, e.g., by Jovanovic (1993) for the US, whilst Ehrlich (1985) finds a "relative stability of establishment sizes", measured as employment per establishment, in a long time-series data base for the US, Japan and eight European countries. The increase of the number of firms and establishments over the long run is reported, e.g., by Maddison (1994).[35] The last two empirical regularities are confirmed by Laincz and Peretto (2006), who analyse more recent data for the US.[36]

3.3.2. Comparative steady-state analysis

Now, we discuss the comparative statics of the interior steady-state. The following proposition summarizes the main results with respect to the structural parameters.

[32] Since our model exhibits a symmetric equilibrum, the number of firms determines the two dimensions of market structure that have deserved the bulk of attention in IO literature, which are firm size (relative to market size) and concentration. This property allows us to focus on only one variable to describe market structure, which is firm size. Note also that our general-equilibrium framework makes explicit the endogenous determination of market structure, in line with the more recent literature that develops the "Schumpeterian hypotheses" by recognising the feedbacks between market structure and economic performance (see Peretto, 1999).

[33] The conditions of interchangeability among alternative measures of firm size at the empirical level are discussed by, e.g., Smyth, Boyes, and Peseau (1975), Shalit and Sankar (1977) and Jackson and Dunlevy (1982).

[34] To see the parallel with the results of the standard growth model, suppose that the number of firms increases at the (exogenous) population growth rate, total sales increase at the output growth rate and technological progress is exogenous. Hence, in steady state, sales per firm grow at the rate of technological progress, whereas employment per firm and the capital stock per firm in efficiency units are constant.

[35] See Gil (2008) for a discussion of the empirical literature relating firm dynamics with long-run economic growth.

[36] As regards the relationship between economic size and the number of firms, Sherer and Ross (1990, ch. 3) state the robustness of the cross-sectional evidence that shows that large countries (measured by population size) tend to have a larger number of firms (and lower concentration rates) than small countries. However, they also note that the relationship is far less clear when the comparison stands between intermediate and large-size countries. An explanation for this result may be that population size is not the best measure of economic size. As already said, our model suggests that one should rather focus on *endogenous* measures of economic size, such as the technological-knowledge stock.

Proposition 3. The aggregate growth rate, g, and the growth rate of the number of varieties, g_N, are *decreasing* in the fixed cost of vertical R&D, ζ, and in the elasticities of the horizontal entry cost function, σ and γ, are *increasing* in the size of quality upgrade, λ, and *do not depend* on the fixed cost of horizontal entry, ϕ. For a given Q, the number of firms in the differentiated-good sector, N, (alternatively, firm size as measured by the technological-knowledge stock per firm, $\frac{Q}{N}$) is *decreasing* (*increasing*) in σ, ϕ and λ, and is *increasing* (*decreasing*) in γ and ζ.[37] Instantaneous entry, E, for a given Q, is *decreasing* in σ, ϕ and ζ, and is *increasing* in γ and λ.

Proof: Differentiate (57), (59), (62) and (63) with respect to the relevant parameters.

See Table 1, in Appendix G, for the complete set of qualitative results concerning g, g_N, I, N, and E. Below, we comment the main results.

The lack of relationship between the steady-state growth rate, g, and the fixed cost of horizontal entry, ϕ, is noteworthy. Intuitively, it results from the dominant effect exerted by the vertical-innovation mechanism (the intensive margin) over the horizontal entry dynamics (the extensive margin). Given the postulated horizontal entry technology, a steady state with positive net entry occurs ultimately because entrants expect incumbency value to grow propelled by quality-enhancing R&D. However, ϕ influences the steady-state levels of the industry variables N and E, reducing both - and thus increasing firm size - for a given value of Q, due to its impact on the arbitrage consistency condition (33). Arnold (1998) and Funke and Strulik (2000) obtain a similar result with respect to both the aggregate growth rate and firm size, while Peretto (1998) predicts a similar relationship with firm size but he instead establishes a positive relationship with the aggregate growth rate.[38]

With respect to the elasticities of the horizontal entry cost function, both changes in σ and in γ have a negative impact on g, as intuition suggests, but their effects contrast when it comes to the impact on N and E. The parameter σ exerts a first-order effect on N and E, qualitatively similar to the impact due to changes in ϕ, whereas γ exerts a second-order effect. These results suggest that industry policies aiming at reducing the *variable* costs faced by entrants may have opposing outcomes with respect to their impact on the market structure and industry dynamics, depending on whether they target σ or γ.

For a given Q, the correlation between the number of firms and the growth rate of the number of varieties tends to be *negative* in steady state. This conforms with some of the empirical evidence on the rate of entry and the number of firms reported, e.g.,

[37] In some particular cases, the alternative measures of firm size $\frac{X}{N} = H_x \frac{Q}{N}$ and $\frac{a}{N} = \zeta \frac{Q}{N}$ yield different comparative static results than those in the text. For example, $\frac{a}{N}$ is *increasing* in ζ, whilst $\frac{Q}{N}$ and $\frac{X}{N}$ are both decreasing in that parameter.

[38] In the three models, the fixed cost of horizontal entry is represented by the reciprocal of the efficiency parameter in the production function of new varieties. However, in Arnold (1998) and Funke and Strulik (2000) new varieties are produced with human capital, which can be accumulated, whereas in Peretto (1998) they are produced with labour, which follows an exogenous growth process. Thus, in the latter, an increase in the cost of entry diverts resources from the production of new varieties to cost-reducing R&D activities, which enhances the aggregate growth rate.

by Sherer and Ross (1990, ch. 3). Since the same relationship applies to the aggregate growth rate, our model also predicts that economies with higher long-run growth rates tend to have a market structure characterised by larger firms, in line with Peretto (1998) and Aghion and Howitt (1998, ch. 12).[39] A positive empirical relation between firm size (employment per firm) and the aggregate growth rate is found by Pagano and Schivardi (2003).

Our model offers mixed results with respect to the steady-state correlation between the rate of vertical innovation and both the aggregate growth rate and firm size. Shifts in the steady state due to changes in the elasticities of the horizontal entry cost function, σ and γ, yield a *negative* relationship between the two variables, whilst changes in the preferences parameters or in ζ give rise to a *positive* correlation. The impact of changes in σ and γ is explained as follows: an increase in those two parameters induces a shift of resources from horizontal entry to vertical R&D, but the effect of the increment in the latter is dominated by the decrease in the former, thus yielding a lower g.The same results apply to the relationship between the aggregate growth rate (or firm size) and the aggregate investment rate, measured as $\frac{R_v+R_n}{Y-X}$.This ratio is also known as "R&D intensity" in IO literature.[40]

According to Bassanini, Scarpetta, and Visco (2000), empirical studies in general find a strong positive relationship between R&D intensity and growth at the sectoral and firm level, but a clear link is usually difficult to establish at the aggregate (cross-country) level. On the other hand, Pagano and Schivardi (2003) note that empirical firm-level studies on the relationship between firm size and innovation have failed to reach a clear conclusion. Yet, those authors conduct a cross-section study at the aggregate level and find that average firm size matters for growth through its effects on R&D intensity, thus implying a positive correlation between firm size and innovation intensity. The literature often places the emphasis on the several conceptual and measurement problems that still afflict empirical analysis in this field to explain the lack of robust results. In contrast, by producing mixed theoretical results with this respect, our model lends theoretical support to the lack of clear-cut empirical findings.

By performing a simple numerical exercise, we conclude that the impact of a change in λ on g is the higher among the parameters analysed in Proposition 3, followed closely by ζ. Changes in σ and γ have a similar, relatively small, impact on g.[41]

[39] A positive correlation between firm size and the aggregate growth rate is the general case in our model. However, that correlation is negative when a shift in the steady-state equilibrium is due to a variation in σ. Likewise, in Aghion and Howitt (1998, ch. 12), a change in the exogenous labour growth rate (implying a change in the rate of creation of differentiated goods, an effect qualitatively similar to a change in σ in our model) also implies a negative relationship between those two variables.

[40] The negative correlation associated with changes in σ and γ reflects the negative correlation between g and R_v (and thus the vertical-innovation rate), whose effect overweights the positive relation between g and R_n.

[41] Whenever we perform numerical exercises, we use the baseline parameters in Section 3.4, below.

3.4. Aggregate transitional dynamics

Next, we characterise qualitatively the local dynamics properties in a neighbourhood of the interior steady state, by studying the solution of the linearised system obtained from (54) and (55)

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} x(t) - x^* \\ z(t) - z^* \end{pmatrix} \tag{66}$$

given the initial condition $x(0)$ and the transversality condition (65). The elements of the Jacobian matrix in (66), denoted by J, are

$$J_{11} = -\left[\left(\frac{\sigma+\gamma}{\gamma}\right) + \frac{\Xi}{\gamma\left(\Xi+1\right)}\right] x^* \tag{67}$$

$$J_{12} = -\frac{\Xi}{\gamma\zeta\left(\Xi+1\right)} x^* \tag{68}$$

$$J_{21} = -\left[-\left(\frac{1}{\theta}+\Xi\right)\frac{1}{\left(\Xi+1\right)}+1\right] c^* \tag{69}$$

$$J_{22} = \mu + \left(\frac{1}{\theta}+\Xi\right)\frac{1}{\zeta\left(\Xi+1\right)}\left(H_Y - H_X\right) -$$

$$-\left\{\left(\frac{1}{\theta}+\Xi\right)\frac{1}{\left(\Xi+1\right)}\left[1+2\frac{\left(\Xi+1\right)}{\Xi}\left(\sigma+\gamma\right)\right]+1\right\} x^* \tag{70}$$

It is clear that $J_{11}, J_{12} < 0$, whereas $J_{21} < 0$ iff $\theta > 1$. It can also be shown that $J_{22} > 0$ provided $\mu > 0 (\Rightarrow g > 0)$. This last condition ensures that $det(J) \equiv J_{11}J_{22} - J_{12}J_{21} < 0, \forall_{\theta>0}$, such that J has two distinct real eigenvalues with opposite signs

$$\delta_1 = \frac{1}{2}\left(tr(J) - \Delta^{\frac{1}{2}}\right) < 0 \tag{71}$$

$$\delta_2 = \frac{1}{2}\left(tr(J) + \Delta^{\frac{1}{2}}\right) > 0 \tag{72}$$

where $\Delta \equiv tr(J)^2 - 4 \cdot det(J)$ and $tr(J) \equiv J_{11} + J_{22}$. Therefore, the dynamics are saddle-path stable, where δ_1 determines the dynamics for the transversality condition to hold. Having solved (66), the dynamics of I, g and r are derived from (52), (49) and (23), whereas the dynamics of E, C and N are obtained from (51) and (53).

We focus on the empirically relevant case of $\theta > 1$. In Appendix I, Figure 2 shows the phase diagram with the linearised isoclines for x and z, whereas Figure 3 illustrates the transitional dynamics by considering deviations of x above its steady-state value, i.e.,

$x(0) = x_0 > x^*$.[42] This would be the case of an economy with a N too low relatively to Q and, thus, firm size too large (e.g., an emergent-market economy, displaying a shallow market of differentiated goods, in contrast to a mature-market economy, which is expected to boast a deep market of differentiated goods). Thus, we have

Proposition 4. Consider an economy initially endowed with $N(0)$ varieties, such that $x(0) > x^*$, and $\theta > 1$. The transitional dynamics is characterised by a *catching-up effect*. The economy experiences a decreasing z *and* $x\,(= g_N)$; this implies that more resources become available to R_v, boosting I and reducing r; consequently, g falls due to the downward movement of g_N but less so due to the effect of accelerated vertical innovation, reflecting the increase in I. In stationarized terms, both N and Q grow along the transition path,[43] but the former grows more than the latter, implying a falling firm size; also, C grows less than Q, whereas E decreases.

Proof: See Appendix H.

According to Proposition 4, our model exhibits the convergence property that applies in the standard Ramsey model (falling aggregate growth rate and real interest rate towards the steady state), although through a distinct mechanism - one which introduces dynamic second-order effects in horizontal entry (see (28)),similarly to the mechanism that regulates the changes of the physical-capital stock in the literature of firm investment with convex adjustment costs.[44]An economy with too few varieties relatively to the technological-knowledge stock (i.e., a too high firm size) starts with a smaller I, a higher

[42]We use the following set of baseline parameter values: $\gamma = 1$, $\sigma = 1$, $\phi = 1$, $\zeta = 0.9$, $\lambda = 2.5$, $\rho = 0.02$, $\theta = 1.5$, $\alpha = 0.4$, $A = 1$, $L = 1$. The values for λ, θ, ρ and α where set in line with previous work on growth and guided either by empirical findings or by theoretical specification. The values of the remaining parameters were chosen in order to calibrate the steady-state aggregate growth rate around 2.5 percent/year. Note that the obtained Poisson arrival rate is of nearly 3 percent in steady state, corresponding to the average of the estimates provided by Caballero and Jaffe (1993) for the creative-destruction rate in the US, whilst the real interest rate value of around 6 percent conforms more with the long-run average real rate of return on the stock market than with the risk-free rate on treasury bills (see Mehra and Prescott, 1985). On the other hand, the choice of values for σ and γ implies that, in steady state, the entry-cost function $\eta(\cdot)$ takes a quadratic form: $\dot{N} = g_N N \Rightarrow \eta(N, \dot{N}) = \phi N \dot{N} = \phi \frac{1}{g_N} (\dot{N})^2$. This is a widely used specification for the adjustment-cost function in the literature of firm investment with convex adjustment costs of the capital stock (e.g., Eisner and Strotz, 1963). Finally, the normalization of A and L to unity at every t implies that the results do not depend on the value of the growth rate of those two variables (either zero or not), and also that all aggregate magnitudes can be interpreted as per capita magnitudes.

[43]In order to compute stationarized Q, Q_{stat}, let $Q = Q_0 e^{g_Q(t)t}$ (with $Q_0 > 0$) and $Q_{stat} = Qe^{-g^*t}$; hence, $Q_{stat} = Q_0 e^{(g_Q(t) - g^*)t}$. Stationarized N, C and E are computed from Q_{stat}, after numerical integration.

[44]Barro and Sala-i-Martin (2004, ch. 6)'s model of expanding variety is able to produce transitional dynamics based on an entry-cost function with N as the single argument; however, in the steady state, N is constant. By repeating the steps used to derive (45) and (46), it can be shown that with such an entry-cost function our model displays *notransitional dynamics*, but N exhibits positive steady-state growth (see Appendix B): in short, under the arbitrage consistency condition (33), the trajectory in the phase space collapses into a singular point (the steady-state equilibrium). Finally, as shown above, we are able to reintroduce transitional dynamics in our model when we specify an entry cost function with N and $\dot{N}$ as arguments.

g_N, a higher g and a higher r than a mature market economy. A lower I (which is also the obsolescence hazard rate associated to vertical R&D activities) implies a higher r (see (23)) and a lower R_v (see (14)), which secure the larger resources allocated to R_n. Note also that the higher g is solely justified by the higher g_N. Along the transition path, g and g_N decrease, whereas part of the resources allocated to R_n is gradually re-targeted to R_v. The consequent increase in I implies a falling r.

Thus, in our model, the aggregate growth rate and the vertical-innovation rate are *negatively* related during the transitional dynamics. The same applies to the relation between the aggregate growth rate and the aggregate investment rate, $\frac{R_v+R_n}{Y-X}$. The quality-ladder model with expanding variety by Dinopoulos and Thompson (1998) obtains a similar re-balancing effect between R_v and R_n; however, in their model, the vertical innovation rate falls in parallel with g and r along the transition path. On the contrary, the medium-run negative relationship between the vertical-innovation rate and the aggregate growth rate is also apparent in the quality-ladders model by Aghion and Howitt (1998, ch. 3), but only for a specific set of parameter values.[45]

The latter results may give theoretical support to the lack of positive correlation between innovation intensity (measured as R&D intensity) and economic growth found recently (e.g., OECD, 2006), particularly in countries situated below the technological frontier. In the context of our model, these would be countries approaching steady-state equilibrium from above, and exhibiting a low speed of transition.[46]

With respect to firm size (technological-knowledge stock per firm) and the aggregate growth rate, they are *positively* related during transitional dynamics. In Aghion and Howitt (1998, ch. 12), firm size, measured as the physical-capital stock per firm in efficiency units, is commanded by the physical-capital stock along the transition path.[47] As long as the model exhibits the convergence property, it produces a positive relationship between firm size and the aggregate growth rate. The models by Peretto (1998), Arnold (1998) and Funke and Strulik (2000) also generate a positive relationship along the transition path between the aggregate growth rate and firm size, measured as employment per firm in the former and the human-capital stock per firm in the other two.[48] The

[45] These authors develop a quality-ladders model with intersectoral spillovers and a constant number of differentiated products, combined with physical-capital accumulation. The latter is a direct input to intermediate-good production and, indirectly, an input to R&D. Innovation is stimulated by a rise in capital intensity, whilst diminishing marginal returns to physical capital per capita imply, for some parameter values, a decrease in aggregate growth rate as the capital stock approaches its steady-state level from below. If this is the case, economic growth and innovation move in opposite directions along the transition path.

[46] Available empirical evidence suggests slow transitions are the case in general. See Proposition 5, below, and ensuing discussion.

[47] This model has a structure similar to Aghion and Howitt (1998, ch. 3) (see fn. 45), but where the number of differentiated goods varies over time. More precisely, the number of goods follows an exogenous growth process according to which it increases linearly with labour.

[48] In Dinopoulos and Thompson (1998), there is no assumption with respect to the span of firms, meaning that there can be both single and multi-product firms. Nevertheless, their model also displays a positive relationship between the aggregate growth rate and employment per differentiated good (which is a measure of firm size, if we postulate that the number of goods per firm is constant across the economy).

results described above imply that, in our model, the rate of vertical innovation (or the aggregate investment rate) and firm size display a *negative* relationship along the transition path, although their correlation may be either positive or negative in steady state (see Subsection 3.3.2, above).

Empirical evidence of a positive correlation between the aggregate growth rate and firm size (employment per firm) in the intermediate-run is provided by Jovanovic (1993) and Laincz and Peretto (2006), whilst Campbell (1998) reports evidence on the positive correlation between the aggregate growth rate and the rate of entry (corresponding, in our model, to the growth rate of the number of varieties). With respect to the latter relationship, it is noteworthy that the value of less than unity predicted by our model is clearly matched by the empirical findings in Campbell (1998).[49]

To give account of the dynamics of households' gross savings, let us define the gross saving rate as $sv = \frac{Y-X-C}{Y-X}$. Using the relationship between Y, X and Q obtained above (see (44)), we get

$$sv(t) = \frac{(H_y - H_x)\,Q(t) - C(t)}{(H_y - H_x)\,Q(t)} = 1 - \frac{z(t)}{H_y - H_x} \tag{73}$$

Consider again a deviation of x above its steady-state value for the case $\theta > 1$.[50] Since z decreases along the transition path (see Proposition 4), then, according to (73), sv *increases* towards the steady state. Hence, our model predicts that a shallow-market economy is inequivocally characterised by a too low gross saving rate (and, consequently, a too low investment rate), but which gradually raises during transition, as income per capita increases. Since the real interest rate, r, decreases during transition, we conclude that the income effect from the interest rate dominates.[51] This accords with the available empirical evidence regarding transition (Barro and Sala-i-Martin, 2004, ch. 12).[52] The standard Ramsey model is also capable of predicting a moderately increasing gross saving rate along the transition path in an economy that starts with scarcity of physical capital per capita, but only for a particular combination of parameter values (in particular, for θ *sufficiently* above 1; e.g., Barro and Sala-i-Martin, 2004, ch. 2).

Finally, in practice, one should be interested not only in the sign of the effects of some change in the parameters of the model, but also in how rapidly those effects occur, as emphasized, e.g., by Mulligan and Sala-i-Martin (1993). We highlight the following comparative statics results with respect to the speed of convergence.

Proposition 5. The speed of convergence (measured by the modulus of the stable eigen-

[49] Gil (2008) presents a discussion of the empirical literature on the relationship between firm dynamics and growth over the short and intermediate run.

[50] We see from (61) that, in steady-state, the gross saving rate is given by $sv^* = \frac{1}{H_y - H_x}\zeta\left(1 + \frac{\sigma+\gamma}{\Xi}\right)\frac{\mu\Xi}{\Xi(\sigma+\gamma+1)+\frac{1}{\theta}(\sigma+\gamma)}$. Hence, $sv^* > 0$ trivially requires $H_y - H_x > 0 \Leftrightarrow Y - X > 0$, besides the already imposed $\mu > 0$.

[51] It results from Appendix H that, with $\theta < 1$, we have $v_{11} < 0$ (v_{11} is the first element of the stable eigenvector). In this case, z *increases* towards the steady state, and, thus, both sv and r *decrease* along the transition path, meaning that the substitution effect from the interest rate dominates.

[52] However, in our model, the relevant dynamics for Y is defined in stationarized terms. As already shown, Y dynamics follows directly from Q. See Proposition 4, above.

value, δ_1) depends *positively* on the elasticity of the horizontal entry cost function, σ, and the vertical innovation step, λ. It depends *negatively* on the elasticity of marginal utility, θ, the elasticity of the horizontal entry cost function, γ, and the fixed cost of vertical R&D, ζ. It *does not depend* on the fixed cost of horizontal entry, ϕ.

Proof: Taking into account (67)-(70), differentiate (71) with respect to the relevant parameters.

In particular, it is noteworthy that σ and γ exhibit effects of opposite signs on the speed of convergence to the steady state, with the latter having a rather higher magnitude than the former. The impact of a change in λ on δ_1 is the highest among the parameters analysed in Proposition 5, followed at some distance by ζ and γ (the impact of the former doubles, in absolute terms, the impact of the last two).

With the chosen set of baseline parameters (see fn. 42), the speed of convergence to the steady state is 2.2 percent/year, implying an half-life of roughly thirty-two years, which is within the range of estimates given by Barro and Sala-i-Martin (2004, ch. 11) for the US, Japan and several European countries. This result contrasts with the rather higher values obtained with the standard Ramsey model calibrated with reasonable parameter values. As shown by Ortigueira and Santos (1997) in a two-sector endogenous growth model with physical and human capital, "the presence of a simple adjustment costs technology reasonably can reduce the rate of convergence without altering substantially some other relevant predictions" (Ortigueira and Santos, 1997, p. 384). In the class of models studied therein, the adjustment costs originate from the distinct technologies used to accumulate human and physical capital, whereas, in ours, the adjustment costs relate directly to the parameter γ in the horizontal entry-cost function.

4. The effects of fiscal-policy instruments

We now study the role of taxes on assets income and of subsidies to intermediate-goods production and to vertical and horizontal R&D, under the assumption that the government budget is balanced at each point in time.[53] The steady-state values of g, g_N, z, I and N are given by the expressions in Appendix J. There we also present the expressions for the alternative measures of firm size (production, or sales, per firm and financial assets per firm) and for the aggregate investment rate.

The steady-state growth and level effects of fiscal policy instruments are described by the following proposition.

Proposition 6. The aggregate growth rate, g, is *decreasing* in the tax rate on assets income, τ_a,[54] is *increasing* in the subsidies granted to intermediate goods produc-

[53] By assumption, increases in subsidies and reductions in taxes on assets income are financed with nondistortionary taxes, which, in our model, are the labour income taxes, τ_w.

[54] It is easily shown that the tax on households' assets income is equivalent to a tax on firms' profits gross of R&D expenditures (i.e., operational profits, Π) and net of the effect of depreciation of non-physical capital (i.e., obsolescence of technological knowledge, Ia): $\tau_a ra = \tau_a(\Pi - Ia)$.

tion, s_x, and to vertical R&D, s_r, and does not depend on the subsidies granted to horizontal entry, s_n. For a given Q, the number of firms in the differentiated good sector, N, (alternatively, the average firm size, $\frac{Q}{N}$) is *decreasing* (*increasing*) in s_x and s_r and is *increasing* (*decreasing*) in τ_a and s_n.[55] Instantaneous entry, E, for a given Q, is *decreasing* in τ_a and *increasing* in s_n, s_r and s_x.

Proof: See Appendix J(a).

Peretto (1998) carries out an analysis very similar to ours. In order to contrast his results to our own, see that, according to Proposition 6, our model predicts that (i) in response to a subsidy granted to vertical R&D, the economy converges to a steady-state with a larger firm size in the differentiated good sector and faster aggregate growth (this result is similar to Peretto's "Result 4");[56] (ii) in response to a subsidy granted to horizontal entry, the economy converges to a steady-state with smaller firm size, but unchanged aggregate growth (Peretto's "Result 5" establishes a slower aggregate growth); and (iii) in response to a tax on firms' operational profits (net of capital depreciation)[57], the economy converges to a steady-state with smaller firm size and slower aggregate growth (Peretto's "Result 8" establishes a larger firm size and unchanged aggregate growth). The results in (iii) are, in qualitative terms, the reverse of (i). As far as the growth rate of the number of differentiated goods (which Peretto refers to as "entry") is concerned, its steady-state value is pegged to the population growth rate in Peretto (1998) - hence, unchanged across (i)-(iii) -, whilst in our case it is pegged to the aggregate growth rate.

From above, we emphasise the lack of relationship between g and s_n, in contrast to the effect of the latter on firm size. In order to gain further insight, notice that s_n increases the *aggregate investment rate*, $\frac{R_v+R_n}{Y-(1-s_x)X}$, while it decreases *investment per firm*, measured as $\frac{R_v+R_n}{N}$, for a given Q. Yet, changes in s_r, s_x (τ_a) increase (decrease) both ratios. The corollary is that investment per firm is the relevant investment rate as far as the effect of fiscal variables on long run growth is concerned. This theoretical result is strongly underlined by Peretto (2003),[58] and corroborated by Laincz and Peretto (2006)'s empirical findings.

By performing a numerical exercise, we also conclude that the effect of s_r, s_x and τ_a on g are very close in absolute magnitude, although somewhat higher when s_r is the chosen policy variable.

We add to Proposition 6 the following second-order results.

[55] The alternative measures of average firm size pointed out in Appendix J(b) yield different comparative static results in some particular cases. For example, $\frac{a}{N}$ is *decreasing* in s_r, whereas $\frac{Q}{N}$ and $\frac{X}{N}$ are both increasing in that parameter. With respect to changes in s_x, the sign of variation is the same for all three measures of firm size, but a numerical exercise shows that $\frac{X}{N}$ exhibits a rather higher sensitivity, in relative terms, than the other two. Regarding changes in s_n and τ_a, both sign and relative magnitude of variation are identical for all three measures of firm size.

[56] The aggregate growth rate, g, in our model is comparable with the consumption growth rate in Peretto (1998).

[57] This effect is absent from Peretto (1998).

[58] Peretto (2003) conducts an extensive analysis of the effect of fiscal variables on long-run growth by augmenting Peretto (1998)'s model with endogenous labour and public goods supply.

Proposition 7. The positive effect of s_r and s_x on g diminishes rapidly as s_r and s_x increase, that is, the marginal returns to granting subsidies to intermediate good production and vertical R&D have a steep negative slope. On the other hand, the higher the elasticities in the entry cost function, σ and γ, the lower is the impact of s_r and s_x on g.

Proof: See Appendix J(a).

These results suggest that industry policies favouring the reduction of the variable costs faced by entrants - through a reduction of either σ or γ, or both - may have an indirect positive effect on g by alleviating the decreasing marginal returns associated to s_r and s_x. This effect reinforces the direct effect described in Proposition 6, above.

Finally, we present the comparative statics results with respect to the speed of convergence.

Proposition 8. The speed of convergence (measured by the modulus of the stable eigenvalue, δ_1) depends *positively* on s_r, s_n and s_x, and depends *negatively* on τ_a.

Proof: See Appendix J(c).

In particular, it is noteworthy that, in contrast to ϕ (see Proposition 5), s_n has a positive (although small) impact on the speed of convergence, in spite of the fact that both parameters have no influence on the steady-state value of g.

5. Conclusion

In this paper, we develop a version of the multi-sector model of R&D-driven endogenous growth, with quality ladders in the intermediate-good sector (e.g., Aghion and Howitt, 1998, ch. 3, and Barro and Sala-i-Martin, 2004, ch. 7). In particular, we merge the expanding variety (e.g., Romer, 1990, and Barro and Sala-i-Martin, 2004, ch. 6) with the quality-ladders mechanism and thus the number of intermediate goods is not necessarily constant over time. Within our framework, the assessment of the effects of R&D on economic growth and on firm dynamics comprised two analytical stages: balanced-growth path and transitional dynamics.

Our paper contributes to the literature of endogenous growth models with transitional dynamics by building in a new mechanism that produces intermediate-term adjustment with an empirically reasonable speed of convergence, without relying on an aggregate production function with decreasing marginal returns in the accumulated factors. Owing to this mechanism, transition is driven by imbalances in the ratio between the technological-knowledge stock and the number of differentiated goods, which is our measure of average firm size. This property allows us to insert our model in the literature that studies the interplay between long-term growth and the factors usually studied in the domain of IO. We conduct this study in the context of monopolistic competition,[59]using average firm size as the pivotal IO variable.

[59]Note that monopolistic competition is the limiting market structure for Bertrand oligopoly as the number of firms becomes very large.

The model predicts, under a sufficiently productive technology, a steady-state equilibrium with constant positive growth rates, and where the consumption growth rate equals the growth rate of the number of varieties plus the growth rate of intermediate-good quality, in line with the general view that industrial growth proceeds both along an intensive and an extensive margin. The growth of the number of varieties is sustained by (endogenous) technological-knowledge accumulation, as the expected growth of intermediate-good quality makes it attractive for potential entrants to always put up an entry cost, in spite of its upward trend. In this setting, it is not necessarily the larger economies, measured by population size, that produce the greater number of distinct goods. Instead, the relevant distinction herein is based on an endogenous measure of economic size, such as the technological-knowledge stock.

We obtain specific results with respect to the impact of changes in the entry-cost parameters and the fiscal policy variables both in the aggregate growth rate and in the market structure in steady state. We emphasise (i) the lack of relationship between economic growth and the fixed cost of horizontal entry (government subsidy to horizontal entry), but the positive (negative) relation between the latter and firm size, (ii) the negative impact of the government tax on firms' operational profits on both economic growth and firm size, (iii) the relevance of investment per firm, instead of aggregate investment rate, to assess the effect of the fiscal variables on economic growth, (iv) the contrasting effect of changes in the two elasticity parameters of the entry cost function in firm size, and (v) the positive correlation between firm size and economic growth. The model offers a mixed result with respect to the steady-state relation between R&D intensity and both the aggregate growth rate and firm size.

We also conclude that the model exhibits the convergence property that applies in the standard Ramsey model, although based on a rather distinct mechanism. The model produces results that differ from (or expand the results from) the early models of quality ladders with expanding variety. In particular, we obtain as a general result that medium-term economic growth and firm size are positively correlated, whereas R&D intensity and both medium-term economic growth and firm size move in opposite directions. The former result adds to the theoretical predictions already found in the literature of positive correlation between economic growth and firm size measured either as employment per firm, human-capital stock per firm or physical-capital stock per firm in efficiency units, and which have had wide empirical support. The latter result - together with the mixed steady-state comparative result mentioned above -, on the contrary, helps to shed light on the lack of clear-cut findings with this respect at the empirical level.

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Appendix

A. Derivation of equation (17)

With respect to the incumbent in industry ω, and bearing in mind that the monopoly profit has Poisson death, we apply the Bellman Principle to get the Hamilton-Jacobi-Bellman equation (henceforth, we omitt the dependence on ω and t for sake of simplification and take $j(\omega, t)$ as the relevant state variable)

$$rV(j) = \pi(j) + \frac{1}{dt} E\left\{dV(j)\right\} \Leftrightarrow$$

$$\Leftrightarrow rV(j) = \pi(j) - I(j)V(j) \Leftrightarrow V(j) = \frac{\pi(j)}{r + I(j)} \tag{74}$$

where r is the equilibrium market real interest rate, to be determined in general equilibrium.[60] In the first line of (74), the second term on the right-hand side of the equation reflects the Poisson process followed by j.[61]

[60] This is a recursive formulation that depends on the constancy of r and I over time. This is a suitable assumption when analysing a balanced growth equilibrium.

[61] Here we used the version of Itô's Lemma for Poisson jump processes (e.g., Dixit and Pindyck, 1994).

On the other hand, an entrant i chooses the flow of resources R_{vi} devoted to vertical R&D in order to maximise expected discounted profits, taking into account that the government subsidises vertical R&D by paying a fraction s_r of each firm's total vertical R&D expenditures. Let $V_{fi}(j)$ denote the expected discounted profits earned by an entrant i when the highest quality in ω is $j(\omega, t)$. The relevant Hamilton-Jacobi-Bellman equation in this case is

$$rV_{fi}(j) = \max_{R_{vi}(j)\geq 0} I_i(j)\left[V(j+1) - V_{fi}(j)\right] - (1-s_r)\cdot R_{vi}(j) +$$

$$+I_{-i}(j)\left[V_{fi}(j+1) - V_{fi}(j)\right] \tag{75}$$

where I_{-i} is the instantaneous probability that all the other entrant firms in ω combined innovate.

Now, we compute the first-order condition for profit maximization for entrant i in ω by equating the derivative of the right-hand side of (75) with respect to R_{vi} (or I_i) to zero, with $I_i(j) = R_{vi}(j)\Phi(j)$ in mind, such that

$$\Phi(j)\left\{V(j+1) - V_{fi}(j)\right\} - (1-s_r) = 0 \tag{76}$$

With free-entry into the vertical R&D business, and assuming that the individual contribution of any particular entrant i to the aggregate innovation rate of all entrants, I, is negligible, we have

$$V_{fi}(j) = 0 \tag{77}$$

that is, the market value of each entrant firm equals zero at each point in time. Therefore, using (77) in (76) and aggregating across firms in ω yields

$$\Phi(j)\cdot V(j+1) - (1-s_r) = 0 \Leftrightarrow I(j)\cdot V(j+1) = (1-s_r)\cdot R_v(j)$$

which is (17) applied to ω. Note that, since I is a linear function of R_v, the result above could be obtained by using directly (77) in (75), without going through the computation of the first-order condition for profit maximization.[62]

B. Behaviour of the aggregate quality index and the number of firms

In this appendix, we study the impact of alternative specifications of the horizontal entry cost function, η, in the behaviour of Q and N along the balanced growth path. For the sake of simplicity, we abstract from government intervention.

[62] This explains why Barro and Sala-i-Martin (2004, ch. 7) consider (17) without any derivation.

Constant fixed entry cost

Consider the free-entry condition

$$V(\bar{q}) = \eta \tag{78}$$

where η is a fixed-entry cost, constant over time, and $\bar{q}(t) \equiv q(j, \bar{\omega}, t) = \frac{Q(t)}{N(t)}$. This is the basic specification in Barro and Sala-i-Martin (2004, ch. 6). In a balanced growth path, with r and I also constant over time, $\bar{q}$ must be constant so that (78) is verified. From (24), this is only possible if $\frac{\dot{Q}(t)}{Q(t)} = \frac{\dot{N}(t)}{N(t)} = g$, where g is the growth rate of Q in balanced-growth path. In this case, $R_N(t) = \eta \dot{N}(t) = \eta g N(t)$ and thus $\dot{R}_N(t) = \eta g \dot{N}(t) \Rightarrow \frac{\dot{R}_N(t)}{R_N(t)} = \frac{\dot{N}(t)}{N(t)} = g$, as expected in a balanced-growth path.

Next, recall that

$$Q(t) = \int_0^{N(t)} q(\omega, t) d\omega$$

We assume that the number of sectors, N, is large enough to treat Q as time-differentiable. Thus, by Leibniz's rule, we have

$$\dot{Q}(t) = \int_0^{N(t)} \dot{q}(\omega, t) d\omega + q(N, t) \dot{N}(t) \Leftrightarrow$$

$$\Leftrightarrow \dot{Q}(t) = \int_0^{N(t)} \dot{q}(\omega, t) d\omega \left(\frac{\bar{q}(t)}{\bar{q}(t) - q(N, t)} \right) \tag{79}$$

where we used $q(j, \bar{\omega}, t) \equiv \bar{q}(t) = \frac{Q(t)}{N(t)}$ and $\dot{N}(t) = \dot{Q}(t) \frac{N(t)}{Q(t)} = \dot{Q}(t) \frac{1}{\bar{q}(t)}$, in balanced growth path.

On the other hand, the probability per unit of time of R&D success in an industry is the Poisson rate I. As the quality index q evolves over time according to j, we cannot assume that q is time-differentiable. The expected change in q per unit of time is, in fact, given by

$$E(\Delta q) = I \left(\lambda^{(j+1)\left(\frac{\alpha}{1-\alpha}\right)} - \lambda^{j\left(\frac{\alpha}{1-\alpha}\right)} \right) = I \left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1 \right) q(\omega, t) \tag{80}$$

Assuming that the number of sectors, N, is large enough to treat Q as time-differentiable, from (79) and (80) we have

$$E(\dot{Q}(t)) = I \left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1 \right) \cdot \int_0^{N(t)} q(\omega, t) d\omega \cdot E \left(\frac{\bar{q}(t)}{\bar{q}(t) - q(N, t)} \right) \Leftrightarrow$$

$$\Leftrightarrow \frac{E(\dot{Q}(t))}{Q(t)} = I \left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1 \right) \cdot \left[E \left(1 - \frac{q(N, t)}{\bar{q}(t)} \right) \right]^{-1} \tag{81}$$

Therefore, our model of quality ladders with constant entry cost η is characterised by $\frac{E(\dot{Q}(t))}{Q(t)} = \infty$, whatever N, since $E(\bar{q}(t)) = E(q(N,t))$.[63]Moreover, if N is very large and the time interval dt is very small we can treat $\dot{Q}(t)$ as nonstochastic; hence, we conclude also that $\frac{\dot{Q}(t)}{Q(t)} = \infty$.[64]

This result should be contrasted with the ones obtained by Dinopoulos and Thompson (1998), Aghion and Howitt (1998, ch. 12) and Howitt (1999). The last two works explore a model of quality ladders with expanding variety and vertical intersectoral spillovers, where the aggregate growth rate is determined by the technological knowledge (quality index) of the most advanced industry. In this model, the authors posite that as the economy develops an increased number of specialised goods, an innovation of a given size with respect to any given good will have a smaller impact on the aggregate economy. This yields an aggregate rate of technological-knowledge progress finite and independent of N, thus echewing any possible explosive feedback running from the expanding-variety mechanism to the equilibrium aggregate growth rate. In addition, entry cost per each new good grows with the size of the economy in this model.

Dinopoulos and Thompson (1998) also study a model of quality ladders with expanding variety and vertical intersectoral spillovers, but where the aggregate growth rate is determined by the *average* technological knowledge over industries. In this model, there is a constant entry cost measured in labour cost units. However, in free-entry equilibrium, this entry cost must equal the value of the incumbent firm in terms of the *relative* quality of its product, which follows a stationary process. Both models described above (similarly to the model by Young, 1998, and others) generate a steady state with a finite positive aggregate growth rate and in which the flow of new goods grows at the same rate as the population.

In our model with constant entry costs, such dampening mechanisms are absent. As in Segerstrom and Zolnierek (1999) and Barro and Sala-i-Martin (2004, ch. 7), entry costs are measured in output units against the value of the incumbent firm in terms of the *absolute* quality of its good, whereas the aggregate growth rate is determined by the proportional growth in Q, which reflects the impact of independent innovations over N industries over time, without any intersectoral spillover effects. As the value of incumbency grows due to vertical innovations, the entry costs become smaller in relative terms, imposing an ever decreasing constraint on entry, which, in turn, propels Q and the aggregate growth rate.

[63]These are the unconditional expectations of stochastic variables governed by the same stochastic process.

[64]There is another solution to (79), but it is trivial. If $\int_0^{N(t)} \dot{q}(\omega,t)d\omega = 0$ then there are no quality ladders and $\frac{\dot{Q}(t)}{Q(t)} = \frac{\dot{N}(t)}{N(t)} < \infty$.

Time-varying entry cost

Entry-cost function $\eta(N)$

Consider now the free-entry condition, in line with, e.g., Romer (1990) and Barro and Sala-i-Martin (2004, ch. 6)

$$V(\bar{q}) = \eta(N) \tag{82}$$

where $\bar{q}(t) \equiv q(j, \bar{\omega}, t) = \frac{Q(t)}{N(t)}$. Let $\eta(N) = \phi N(t)^{\sigma}$, with $\phi > 0$ and $\sigma > 0$ exogenous constants. In a balanced growth path, with r and I constant over time, (82) implies that

$$\dot{V}(\cdot) = \dot{\eta}(\cdot) \Leftrightarrow V(\cdot)\left(\frac{\dot{Q}(t)}{Q(t)} - \frac{\dot{N}(t)}{N(t)}\right) = \eta(\cdot)\sigma\frac{\dot{N}(t)}{N(t)} \Leftrightarrow$$

$$\Leftrightarrow \dot{N}(t) = \left(\frac{1}{1+\sigma}\right)\frac{\dot{Q}(t)}{Q(t)}N(t) \tag{83}$$

On the other hand, we have

$$\dot{Q}(t) = \int_0^{N(t)} \dot{q}(\omega, t)d\omega + q(N,t)\dot{N}(t) \Leftrightarrow$$

$$\Leftrightarrow \dot{Q}(t) = \int_0^{N(t)} \dot{q}(\omega, t)d\omega \cdot \left[1 - \frac{q(N,t)}{\bar{q}(t)}\left(\frac{1}{1+\sigma}\right)\right]^{-1} \tag{84}$$

where we used the result in (83). In expected terms, we have

$$E\left(\frac{\dot{Q}(t)}{Q(t)}\right) = I\left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1\right) \cdot \left[E\left(1 - \frac{q(N,t)}{\bar{q}(t)}\left(\frac{1}{1+\sigma}\right)\right)\right]^{-1} \tag{85}$$

Since $E(\bar{q}(t)) = E(q(N,t))$, equation (85) becomes

$$\frac{E(\dot{Q}(t))}{Q(t)} = I\left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1\right) \cdot \left(\frac{1+\sigma}{\sigma}\right)$$

which implies that

$$\frac{\dot{Q}(t)}{Q(t)} = I\left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1\right) \cdot \left(\frac{1+\sigma}{\sigma}\right) \tag{86}$$

for N large enough. From (83) and (86), we see also that

$$\frac{\dot{N}(t)}{N(t)} = I\left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1\right) \cdot \frac{1}{\sigma} \tag{87}$$

The growth rates of Q and N are *constants* if I is constant over time. Thus, *this model produces a balanced-growth path with ever increasing Q and N.* This result contrasts with Barro and Sala-i-Martin (2004, ch. 6)'s model of expanding variety in which the cost

of horizontal R&D is an increasing function of the number of product lines previously introduced and the long-run equilibrium implies a *constant* number of products (as long as L is constant, as in our model).

Entry-cost function $\eta(Q, N)$

Inspired in Peretto and Smulders (2002), we also consider the free-entry condition

$$V(\bar{q}) = \eta(Q, N) \tag{88}$$

Let $\eta(Q, N) = \phi Q(t)^{\sigma_1} N(t)^{-\sigma_2}$, with $\phi > 0$, $\gamma > 0$ and $\sigma_1 > 0$ exogenous constants. In a balanced growth path, with r and I constant over time, (88) implies that

$$\dot{V}(\cdot) = \dot{\eta}(\cdot) \Leftrightarrow$$

$$\Leftrightarrow \dot{N}(t) = \left(\frac{1-\sigma_1}{1-\sigma_2}\right) \frac{\dot{Q}(t)}{Q(t)} N(t) \tag{89}$$

Following the same steps as before, we get

$$\frac{\dot{Q}(t)}{Q(t)} = I\left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1\right) \cdot \left(\frac{1-\sigma_2}{\sigma_1 - \sigma_2}\right) \tag{90}$$

and

$$\frac{\dot{N}(t)}{N(t)} = I\left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1\right) \cdot \left(\frac{1-\sigma_1}{\sigma_1 - \sigma_2}\right) \tag{91}$$

for N large enough. Again, the growth rates of Q and N are *constants* if I is constant over time. This model yields a balanced-growth path characterised by positive growth rates if $\sigma_1 > \sigma_2 \wedge \sigma_1 < 1 \wedge \sigma_2 < 1$ or $\sigma_1 < \sigma_2 \wedge \sigma_1 > 1 \wedge \sigma_2 > 1$.

In the case $\sigma_1 < \sigma_2$, the entry-cost fuction can be written as $\eta(Q, N) = \phi Q(t)^{\sigma_1} N(t)^{-\sigma_2} = \phi\left(\frac{\bar{q}(t)^{\sigma_2}}{Q(t)^{\sigma_2 - \sigma_1}}\right)$, which means that the entry cost increases with the productivity level at which a new firm is established, $\bar{q}$, and decreases with the (intertemporal) spillovers available post-entry to each firm, Q, closely resemblying the entry-cost function posited by Peretto and Smulders (2002).

Entry-cost function $\eta(R_n)$

Finally, consider the free-entry condition

$$V(\bar{q}) = \eta(R_n) \tag{92}$$

where R_n denotes total horizontal R&D expenditures. Let $\eta(R_n) = \psi R_n(t)^{\nu}$, with $\psi > 0$ and $0 < \nu < 1$ exogenous constants. The rational to this entry cost function can be found, e.g., in the aggregate function of horizontal innovation in Howitt (1999), which

exhibits *decreasing marginal returns* in total horizontal R&D expenditures.[65] But since $R_n = \eta\dot{N}$, then

$$\eta(\cdot) = \psi\eta(\cdot)^{\nu}(\dot{N})^{\nu} \tag{93}$$

Consecutive substitution of η in the right-hand side of (93) yields

$$\eta(\cdot) = \psi^{\left(1+\nu+\nu^2+...\right)}\eta^{\nu^{(3+\cdots)}}(\dot{N})^{\left(\nu+\nu^2+\nu^3+...\right)} \tag{94}$$

The exponents in (94) are geometric series. Since $\nu < 1$, we have $\nu^{\infty} = 0$, $\sum_{i=0}^{\infty}\nu^i = \frac{1}{1-\nu}$ and $\sum_{i=1}^{\infty}\nu^i = \frac{\nu}{1-\nu}$, which means that $\eta(R_N)$ has only dependence of infinitesimal order on η itself, and thus

$$\eta(\cdot) = \psi^{\frac{1}{1-\nu}}\dot{N}(t)^{\frac{\nu}{1-\nu}} = \phi\dot{N}(t)^{\gamma} \tag{95}$$

where $\phi \equiv \psi^{\frac{1}{1-\nu}} > 0$ and $\gamma \equiv \frac{\nu}{1-\nu} > 0$. The result above shows that $\eta(R_n) \underset{\nu<1}{\rightarrow} \eta(\dot{N})$, offering a rational to the entry technology implicit in the entry-cost function $\eta(\dot{N})$, postulated, e.g., in Datta and Dixon (2002) and Brito and Dixon (2008). In this light, $\eta(\dot{N})^{-1}$ can be interpreted, through $\eta(R_n)^{-1}$, as a *marginal* horizontal-innovation function with *increasing* returns, where ν measures a third-order effect. Notice that the elasticity of R_n in the marginal innovation function $\eta(R_n)^{-1}$determines the elasticity of $\dot{N}$ in $\eta(\dot{N})^{-1}$. For example, $\eta(\dot{N}) = \phi\dot{N}$, as in Brito and Dixon (2008), defines implicitly an elasticity of 0.5 in the marginal innovation function.

In a balanced-growth path, with r and I constant over time, $V(\cdot) = \eta(\cdot)$, together with (95), implies that

$$\dot{V}(\cdot) = \dot{\eta}(\cdot) \Leftrightarrow V(\cdot)\left(\frac{\dot{Q}(t)}{Q(t)} - \frac{\dot{N}(t)}{N(t)}\right) = \eta(\cdot)\gamma\frac{\ddot{N}(t)}{\dot{N}(t)} \Leftrightarrow$$

$$\Leftrightarrow \dot{N}(t) = \frac{\dot{Q}(t)}{Q(t)}N(t) - \gamma\frac{\ddot{N}(t)}{\dot{N}(t)}N(t) \tag{96}$$

On the other hand, we have

$$\dot{Q}(t) = \int_0^{N(t)}\dot{q}(\omega,t)d\omega + q(N,t)\dot{N}(t) \Leftrightarrow$$

$$\Leftrightarrow \dot{Q}(t) = \left[\int_0^{N(t)}\dot{q}(\omega,t)d\omega - \frac{q(N,t)}{\bar{q}(t)}\gamma\frac{\ddot{N}(t)}{\dot{N}(t)}Q(t)\right]\left(1 - \frac{q(N,t)}{\bar{q}(t)}\right)^{-1} \tag{97}$$

where we used the result in (96). In expected terms, we get

$$\frac{E(\dot{Q}(t))}{Q(t)} = \left[I\left(\lambda^{\left(\frac{\alpha}{1-\alpha}\right)} - 1\right) - E\left(\frac{q(N,t)}{\bar{q}(t)}\right)\gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\right]\cdot\left[E\left(1 - \frac{q(N,t)}{\bar{q}(t)}\right)\right]^{-1} \tag{98}$$

[65] See the arbitrage condition (10) in that paper

Therefore, our model of quality ladders with an entry-cost function $\eta(\dot{N})$ is characterised by $\frac{E(\dot{Q}(t))}{Q(t)} = \infty$, whatever N, since $E(\bar{q}(t)) = E(q(N,t))$. With a large enough N, we can treat $\dot{Q}(t)$ as nonstochastic, and thus $\frac{\dot{Q}(t)}{Q(t)} = \infty$, a result similar to the model with a constant entry cost η.

C. General equilibrium: derivation of equations (41) and (42)

Consider the balance sheet of households in a setting with government intervention

$$a(t) = V(\bar{q})N(t) = (1 - s_n) \cdot \eta(\cdot)N(t) \tag{99}$$

where $\bar{q}(t) \equiv q(j, \bar{\omega}, t) = \frac{Q(t)}{N(t)}$. Hence, we can characterise the change in the value of equity as

$$\dot{a}(t) = (1 - s_n) \left(\eta(\cdot)\dot{N}(t) + \dot{\eta}(\cdot)N(t) \right) \tag{100}$$

Solve (30), in the text, in order to $\dot{\eta}$ and, together with (99) and (35), substitute in (100) to get

$$(1 - \tau_a)\, r(t)a(t) + (1 - \tau_w)\, w(t)L - C(t) = (1 - s_n)\, \eta(\cdot)\dot{N}(t) - \pi(\bar{q})N(t) +$$

$$+ (1 - s_n)\, (r + I(\bar{q}))\, \eta(\cdot)N(t) + (1 - s_n)\, \frac{\dot{\pi}(\bar{q})}{\pi(\bar{q})}\eta(\cdot)N(t) = 0 \Leftrightarrow$$

$$\Leftrightarrow (w(t)L + \pi(\bar{q})N(t)) - C(t) - (1 - s_n)\, \eta(\cdot)\dot{N}(t) - (1 - s_n) \left(I(\bar{q})\eta(\cdot)N(t) + \frac{\dot{\pi}(\bar{q})}{\pi(\bar{q})}\eta(\cdot)N(t) \right) -$$

$$- \tau_w w(t)L - \tau_a r(t)a(t) = 0 \Leftrightarrow$$

$$\Leftrightarrow Y(t) - (1 - s_x)\, X(t) - C(t) - (1 - s_n)\, \eta(\cdot)\dot{N}(t) - (1 - s_n) \left(I(\bar{q})\eta(\cdot)N(t) + \frac{\dot{\pi}(\bar{q})}{\pi(\bar{q})}\eta(\cdot)N(t) \right) -$$

$$- \tau_w w(t)L - \tau_a r(t)a(t) = 0 \tag{101}$$

Using $Y = wL + pX \Leftrightarrow Y - (1 - s_x)\, X = wL + \pi N$, $(1 - s_n)\, R_n = (1 - s_n)\, \eta\dot{N}$ and $(1 - s_r)\, R_v = (1 - s_n) \left(I\eta N + \frac{\dot{\pi}}{\pi}\eta N \right)$ and simplifying with (38), (101) reads[66]

[66] Having in mind (1), (6), (9) and (11) and (12), and that, in equilibrium, w and p are equated to the marginal product of labour and the marginal product of intermediate goods, respectively, it is easily shown that $wL = (1 - \alpha)Y$, $X = \alpha^2 \left(\frac{1}{1 - s_x} \right) Y$, $pX = \alpha Y$ and total profits $\Pi = X \cdot [p - (1 - s_x)] = \alpha Y - \alpha^2 Y$. Also, have in mind that, by definition of $\bar{q}(t) \equiv q(j, \bar{\omega}, t)$, total profits can be represented as $\Pi = \pi(\bar{q})N$.

$$Y(t)-(1-s_x)\,X(t)-C(t)-(1-s_n)\,R_n(t)-(1-s_r)\,R_v(t)-\tau_w w(t)L-\tau_a r(t)a(t)=0 \Leftrightarrow$$

$$\Leftrightarrow Y(t)=X(t)+C(t)+R_n(t)+R_v(t)$$

which is (41). Note also that since the real interest rate r consists of dividend payments in units of asset price minus the Poisson death rate, i.e., $r=\frac{\pi}{V}-I$, for each t, then $a=VN\Rightarrow \pi N=(r+I)a$. From here, and using $Y-(1-s_x)\,X=wL+\pi N$ together with (35) and (41), in the text, the aggregate resource constraint reads

$$Y(t)-(1-s_x)\,X(t)-C(t)=w(t)L+\pi(\bar{q})N(t)-C(t)\Leftrightarrow$$

$$\Leftrightarrow Y(t)-(1-s_x)\,X(t)-C(t)=\dot{a}(t)+I(t)a(t)+\tau_w w(t)L+\tau_a r(t)a(t) \tag{102}$$

Moreover, since $Y(t)-(1-s_x)\,X(t)-C(t)-(1-s_n)\,R_n(t)-(1-s_r)\,R_v(t)-\tau_w w(t)L-\tau_a r(t)a(t)=0$, as shown above, (102) becomes

$$\dot{a}(t)=(1-s_n)R_n(t)+(1-s_r)R_v(t)-I(t)a(t)$$

which is equivalent to (42) with $s_n=s_r=0$.

D. Derivation of equations (45) and (46)

Take the horizontal free-entry condition in a setting with government intervention

$$V(\bar{q})=(1-s_n)\cdot\eta(\cdot),\ \ \eta(\cdot)=\phi N(t)^{\sigma}\dot{N}(t)^{\gamma} \tag{103}$$

where $\bar{q}(t)\equiv q(j,\bar{\omega},t)=\frac{Q(t)}{N(t)}$. By substituting in the dynamic horizontal arbitrage equation (30), we have

$$r(t)+I(\bar{q})=\frac{\bar{\pi}A^{\frac{1}{1-\alpha}}\bar{q}(t)}{(1-s_n)\,\eta(\cdot)}+\frac{\dot{\eta}(\cdot)}{\eta(\cdot)}-\frac{\dot{\pi}(\bar{q})}{\pi(\bar{q})} \tag{104}$$

Then, recall the dynamic vertical arbitrage equation (23), with $I(t)\equiv I(\bar{q})$, and take into account that, from (8), $\pi(\bar{q})=\bar{\pi}A^{\frac{1}{1-\alpha}}\bar{q}(t)\Rightarrow\frac{\dot{\pi}(\bar{q})}{\pi(\bar{q})}=\frac{\dot{Q}(t)}{Q(t)}-\frac{\dot{N}(t)}{N(t)}$, and that, from (103), $\dot{\eta}=\eta(\cdot)\left(\sigma\frac{\dot{N}(t)}{N(t)}+\gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\right)$. Thus, after rearranging, we rewrite (104) as

$$\bar{\pi}A^{\frac{1}{1-\alpha}}\left[\frac{\eta(\cdot)}{\zeta\,(1-s_r)}-\frac{Q(t)}{(1-s_n)\,N(t)}\right]=-\eta(\cdot)\frac{\dot{Q}(t)}{Q(t)}+\eta(\cdot)\frac{\dot{N}(t)}{N(t)}\,(\sigma+1)+\eta(\cdot)\gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\Leftrightarrow$$

$$\Leftrightarrow \frac{\dot{N}(t)}{N(t)} = \frac{\dot{Q}(t)}{Q(t)}\frac{1}{\sigma+1} + \left\{\bar{\pi}A^{\frac{1}{1-\alpha}}\left[\frac{\eta(\cdot)}{\zeta\left(1-s_r\right)} - \frac{Q(t)}{\left(1-s_n\right)N(t)}\right] - \gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\eta(\cdot)\right\}\frac{1}{\eta(\cdot)}\frac{1}{\sigma+1} \tag{105}$$

On the other hand, recall, from (24), that

$$Q(t) = \int_0^{N(t)} q(\omega, t)d\omega$$

We assume that the number of sectors, N, is large enough to treat Q as time-differentiable. Thus, by Leibniz's rule, we have

$$\dot{Q}(t) = \int_0^{N(t)} \dot{q}(\omega, t)d\omega + q(N, t)\dot{N}(t)$$

which, with (105) above, yields

$$\dot{Q}(t) = \int_0^{N(t)} \dot{q}(\omega, t)d\omega + q(N, t)\cdot$$

$$\cdot\left\{\frac{\dot{Q}(t)}{Q(t)}\frac{1}{\sigma+1}N(t) + \left\{\bar{\pi}A^{\frac{1}{1-\alpha}}\left[\frac{\eta(\cdot)}{\zeta\left(1-s_r\right)} - \frac{Q(t)}{\left(1-s_n\right)N(t)}\right] - \gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\eta(\cdot)\right\}\frac{1}{\eta(\cdot)}\frac{1}{\sigma+1}\right\} \Leftrightarrow$$

$$\Leftrightarrow \dot{Q}(t) = \int_0^{N(t)} \dot{q}(\omega, t)d\omega\left(1 - \frac{q(N,t)}{\bar{q}(t)}\frac{1}{\sigma+1}\right)^{-1} +$$

$$+\frac{q(N,t)}{\bar{q}(t)}\left\{\bar{\pi}A^{\frac{1}{1-\alpha}}\left[\frac{\eta(\cdot)}{\zeta\left(1-s_r\right)} - \frac{Q(t)}{\left(1-s_n\right)N(t)}\right] - \gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\eta(\cdot)\right\}\frac{Q(t)}{\eta(\cdot)}\left(1 - \frac{q(N,t)}{\bar{q}(t)}\frac{1}{\sigma+1}\right)^{-1}$$

Taking expectations with respect to $\dot{Q}$ and $\dot{q}$, the equation above reads[67]

$$\frac{E(\dot{Q}(t))}{Q(t)} = I(t)\left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right)\left(\frac{1+\sigma}{\sigma}\right) +$$

$$+\left\{\bar{\pi}A^{\frac{1}{1-\alpha}}\left[\frac{\eta(\cdot)}{\zeta\left(1-s_r\right)} - \frac{Q(t)}{\left(1-s_n\right)N(t)}\right] - \gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\eta(\cdot)\right\}\frac{1}{\eta(\cdot)}\left(\frac{1}{\sigma}\right) \tag{106}$$

Moreover, if N is very large and the time interval dt is very small we can treat $\dot{Q}(t)$ as nonstochastic; hence, (106) becomes

$$\frac{\dot{Q}(t)}{Q(t)} = I(t)\left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right)\left(\frac{1+\sigma}{\sigma}\right) +$$

[67] Note that the derivation below requires that I is constant across industries but not necessarily constant over time (see Appendix B, above).

$$+\left\{\bar{\pi}A^{\frac{1}{1-\alpha}}\left[\frac{\eta(\cdot)}{\zeta\left(1-s_r\right)}-\frac{Q(t)}{\left(1-s_n\right)N(t)}\right]-\gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\eta(\cdot)\right\}\frac{1}{\eta(\cdot)}\left(\frac{1}{\sigma}\right) \tag{107}$$

From (105) and (107), we obtain

$$\frac{\dot{N}(t)}{N(t)}=I(t)\left(\lambda^{\frac{\alpha}{1-\alpha}}-1\right)\left(\frac{1}{\sigma}\right)+$$

$$+\left\{\bar{\pi}A^{\frac{1}{1-\alpha}}\left[\frac{\eta(\cdot)}{\zeta\left(1-s_r\right)}-\frac{Q(t)}{\left(1-s_n\right)N(t)}\right]-\gamma\frac{\ddot{N}(t)}{\dot{N}(t)}\eta(\cdot)\right\}\frac{1}{\eta(\cdot)}\left(\frac{1}{\sigma}\right) \tag{108}$$

Equations (107) and (108) are equivalent to, respectively, (45) and (46) with $s_x=s_n=s_r=0$.

E. Proof of Proposition 1

We give the proof of (i) and (ii) in a setting with government intervention. From (44), using (33), and the fact that, in balanced-growth path, $\dot{N}=g_N N$, we have

$$H_Y Q(t)=H_X Q(t)+C(t)+\zeta\left(\frac{1-s_r}{1-s_n}\right)g_N+I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}}Q(t) \tag{109}$$

where $H_Y\equiv L\left(\frac{A^{\frac{1}{\alpha}}\alpha^2}{1-s_x}\right)^{\frac{\alpha}{1-\alpha}}$ and $H_X\equiv L\left(\frac{A\alpha^2}{1-s_x}\right)^{\frac{1}{1-\alpha}}$. Time-differentiating (109) and solving in order to $\dot{I}$, yields

$$\dot{I}(t)=\frac{1}{\zeta\lambda^{\frac{1}{1-\alpha}}Q(t)}\left\{\dot{Q}(t)\left[H_Y-H_X-\zeta\left(\frac{1-s_r}{1-s_n}\right)g_N-I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}}\right]-\dot{C}(t)\right\}\Leftrightarrow$$

$$\Leftrightarrow g_I=\frac{1}{\zeta\lambda^{\frac{\alpha}{1-\alpha}}I(t)}\left\{g_Q\left[H_Y-H_X-\zeta\left(\frac{1-s_r}{1-s_n}\right)g_N-I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}}\right]-g_C\frac{C(t)}{Q(t)}\right\} \tag{110}$$

In balanced growth path, g_I, g_Q and g_C must be constant. Consider three scenarios:

(a) Suppose $g_I,g_Q,g_C\neq 0$ and $g_I\neq g_Q\neq g_C$. From (110), this implies that g_I changes as I and $\frac{C}{Q}$ change over time, which is a contradiction.

(b) Suppose $g_I,g_Q,g_C\neq 0$ and $g_Q=g_C=g$, then we have, from (44) and (110),

$$g_I=\frac{1}{\zeta\lambda^{\frac{\alpha}{1-\alpha}}I(t)}\left\{g\left[H_Y-H_X-\zeta\left(\frac{1-s_r}{1-s_n}\right)g_N-I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}}-\frac{C(t)}{Q(t)}\right]\right\}\Leftrightarrow g_I=0$$

which is a contradiction.

(c) Suppose $g_I=0$ and $g_Q\neq g_C$, then we have, from (110),

$$g_C = \frac{Q}{C} g_Q \left[H_Y - H_X - \zeta \left(\frac{1-s_r}{1-s_n} \right) g_N - I(t) \zeta \lambda^{\frac{\alpha}{1-\alpha}} \right]$$

This implies that g_C changes as $\frac{Q}{C}$ changes over time, which is also a contradiction.

Then, it must be true that, in balanced growth path, $g_Q = g_C = g$ and $g_I = 0$. Q.E.D.

The proof of (iii) is as follows. From (48), using (33), we have

$$\frac{\dot{N}(t)}{N(t)} = g_N(t) = \left[\frac{\zeta}{\phi} \left(\frac{1-s_r}{1-s_n} \right) \right]^{\frac{1}{\gamma}} Q^{\frac{1}{\gamma}} N^{-\left(\frac{\sigma+\gamma+1}{\gamma} \right)} \equiv x(Q,N) \tag{111}$$

Since g_N must be constant in balanced growth path, then, by time-differentiating the equation above and equating to zero, we find

$$x \left[\frac{1}{\gamma} \frac{\dot{Q}(t)}{Q(t)} - \left(\frac{\sigma+\gamma+1}{\gamma} \right) \frac{\dot{N}(t)}{N(t)} \right] = 0 \Leftrightarrow \frac{g_Q}{g_N} = (\sigma+\gamma+1)$$

provided $x \neq 0$. Q.E.D.

F. Derivation of the system of rescaled variables (54)-(55)

The fixed points of the system of rescaled variables are found by equating $\dot{x} = 0$ and $\dot{z} = 0$. Thus, we get

$$\begin{cases} x^* = 0 \quad \vee \quad z^* = \left[-I_0 - I_2 x^* + \frac{\sigma+\gamma}{\Xi} x^* \right] \frac{1}{I_1} \\ z^* = 0 \quad \vee \quad x^* = \frac{\mu \Xi}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(\sigma+\gamma)} \end{cases}$$

where $I_0 \equiv \frac{1}{\zeta \lambda^{\frac{\alpha}{1-\alpha}}} (H_Y - H_X)$, $I_1 \equiv -\frac{1}{\zeta \lambda^{\frac{\alpha}{1-\alpha}}}$ and $I_2 \equiv -\frac{1}{\lambda^{\frac{\alpha}{1-\alpha}}}$. Thus, the dynamic system has four fixed points: $(0,0)$, $(0, z_1^*)$, $(x_1^*, 0)$ and (x_2^*, z_2^*), where $x^* \neq 0 \wedge z^* \neq 0$ and $x_1^* \neq x_2^*$, $z_1^* \neq z_2^*$. The unique interior steady-state (x_2^*, z_2^*) is found by following the steps (56)-(61) described in the text.

G. Comparative steady-state analysis

The table below presents the qualitive results of the comparative steady-state analysis with respect to g, g_N, I, N, and E. The results concerning N and E are obtained by normalising Q to unity in every steady state (see, e.g., Caballé and Santos, 1993).

Table 1. Comparative steady-state analysis of g, g_N, I, N and E

	∂g^*,∂g_N^*	∂I^*	$\partial N^* \vert Q^* = 1$	$\partial E^* \vert Q^* = 1$
$\partial\sigma$	-	+	-	-
$\partial\gamma$	-	+	+	+
$\partial\phi$	o	o	-	-
$\partial\lambda$	+	-	-	+
$\partial\zeta$	-	-	+	-
$\partial\rho$	-	-	+	-
$\partial\theta$	-	-	+	-
$\partial\alpha$	+|-	+|-	-|+	+|-

The results in the first column, rows λ to α, are already known from the literature of quality-ladders models. The remaining results are specific to our model. The notation in the last row is to be interpreted as follows: up to a certain critical value of α, $\bar{\alpha}$, an increase in this parameter induces an increase in the steady-state value of g, a decrease in N and an increase in E; the opposite is true for values of α above $\bar{\alpha}$. The threshold $\bar{\alpha}$ is a function of the remaining set of parameters of the model. The non-monotonic behaviour induced by changes in α reflects two effects of opposite signs: as α grows, (i) the vertical innovation step, $\lambda^{\frac{\alpha}{1-\alpha}}$, increases and (ii) the intermediate good mark-up, $\frac{1}{\alpha}$, decreases. If $\alpha < \bar{\alpha}$, (i) dominates (ii), and vice-versa if $\alpha > \bar{\alpha}$.

H. Proof of Proposition 4

We focus on the first part of Proposition 4, which states the dynamics of x and z towards steady state with $x(0) > x^*$ and $\theta > 1$. The solution to the non-homogeneous linearised system (66) under the transversality condition is given by

$$x(t) = v_{11} k_1 e^{\delta_1 t} + x^* \tag{112}$$

$$z(t) = v_{12} k_1 e^{\delta_1 t} + z^* \tag{113}$$

where k_1 is an arbitrary constant of integration and v_{11} and v_{12} are the elements of the eigenvector corresponding to the stable eigenvalue, $\delta_1 < 0$. Equations (112) and (113) were derived by considering that the transversality condition requires that the constant of integration associated to the unstable eigenvalue, $\delta_2 > 0$, is set to zero, i.e., $k_2 = 0$. To determine k_1 we use the information about the initial value of the state variable. Consider, as in the text, $x(0) = x_0 > x^*$. Then, solving (112) for $t = 0$, we get $v_{11} k_1 = x_0 - x^* > 0$ and, thus, $v_{11} > (<) 0 \Rightarrow k_1 = \frac{x_0 - x^*}{v_{11}} > (<) 0$. In turn, through (113) also solved for $t = 0$, we determine $z(0) = v_{12} k_1 + z^*$, the coordinate along the z axis that places the economy on the sadlle path for a given $x(0) = x_0$. In p articular, $v_{12} k_1 = z(0) - z^* > 0$ requires $k_1 > (<) 0 \Rightarrow v_{12} > (<) 0$. That is, an economy that

starts with $x_0 > x^*$ experiences a transition towards steady state with a *falling x and z* - and thus an *increasing sv* (see (73)) - iff $v_{11}, v_{12} > (<) 0$. Having in mind (66)-(70), in the text, we obtain the eigenvector matrix $V = [V_1 V_2]$ associated to J in the usual way, such that

$$V_1 = [v_{11} v_{12}]^T = \left[-\frac{\frac{1}{2}\left(-J_{11} + J_{22} + \Delta^{\frac{1}{2}}\right)}{J_{21}}, 1 \right]^T \tag{114}$$

is the eigenvector corresponding to δ_1 and

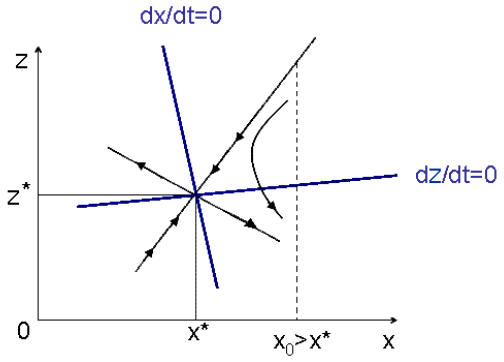
$$V_2 = [v_{21} v_{22}]^T = \left[-\frac{\frac{1}{2}\left(-J_{11} + J_{22} - \Delta^{\frac{1}{2}}\right)}{J_{21}}, 1 \right]^T \tag{115}$$

is the eigenvector corresponding to δ_2, where $\Delta \equiv tr(J)^2 - 4 \cdot det(J)$, $det(J) \equiv J_{11}J_{22} - J_{12}J_{21}$ and $tr(J) \equiv J_{11} + J_{22}$. From (114), we see that $v_{12} > 0$; thus, $z(0) - z^* > 0$ requires $v_{11} > 0$. Given that $J_{11} < 0$, $J_{22} > 0$ and $\Delta > 0$, $v_{11} > 0$ requires $J_{21} < 0$, which is true iff $\theta > 1$. Without government intervention, the latter is trivially proved by showing $-\frac{\left(\frac{1}{\theta} + \Xi\right)}{(\Xi + 1)} + 1 > 0 \Leftrightarrow \theta > 1$. QED

Appendix J(c) analyses the necessary conditions to $J_{21} < 0$ in case of government intervention.

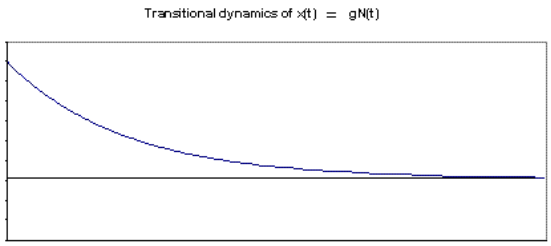
I. Transitional dynamics

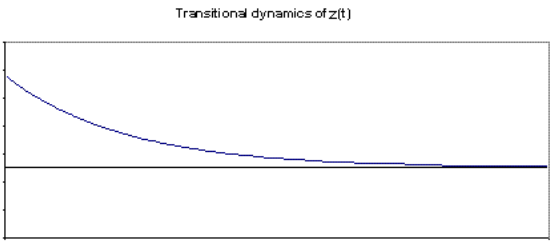
Figure 2 - Phase diagram in space (x, z) (linearised isoclines).

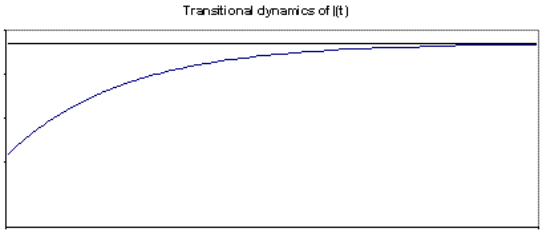


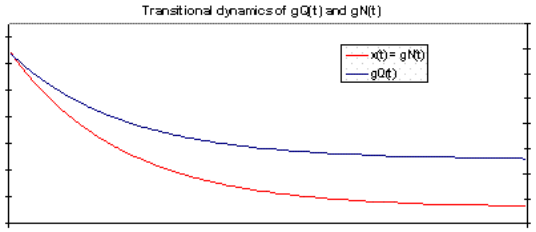
Paths above the stable manifold eventually yield zero technological-knowledge stock with a positive consumption, forcing c to a downward jump. This outcome violates the Euler equation. Paths below the stable manifold eventually yield zero consumption, with a growing technological-knowledge stock. This violates the transversality condition. Given initial conditions, the only choice that satisfies all first-order conditions is to jump to the saddle path and converge to the steady state.

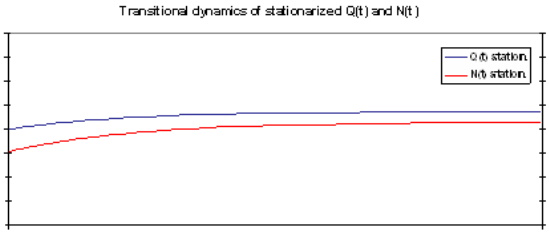
Figure 3 - Transitional dynamics $(x_0 > x^*)$

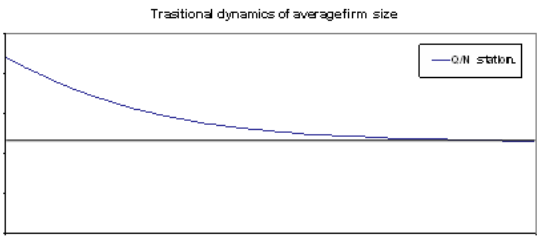












J. The model with government intervention

In this appendix, we present the main analytical expressions referred to in the text, herein derived in a setting with government intervention (subsidies for intermediate goods production, s_x, for vertical R&D activities, s_r, and horizontal R&D activities, s_n, and taxes on assets income, τ_a). We also describe the proof to Propositions 6, 7 and 8 and re-state the proof to Proposition 4.

(a) Let us rewrite (111) and (109), first presented in Appendix E,

$$x(Q,N) = \left[\frac{\zeta}{\phi}\left(\frac{1-s_r}{1-s_n}\right)\right]^{\frac{1}{\gamma}} Q^{\frac{1}{\gamma}} N^{-\left(\frac{\sigma+\gamma+1}{\gamma}\right)}$$

$$I(Q,N,C) = \frac{1}{\zeta\lambda^{\frac{\alpha}{1-\alpha}}}\left(H_Y - H_X - \frac{C(t)}{Q(t)} - \zeta\left(\frac{1-s_r}{1-s_n}\right)\cdot x(Q,N)\right)$$

where $H_Y \equiv L\left(\frac{A^{\frac{1}{\alpha}}\alpha^2}{1-s_x}\right)^{\frac{\alpha}{1-\alpha}}$ and $H_X \equiv L\left(\frac{A\alpha^2}{1-s_x}\right)^{\frac{1}{1-\alpha}}$. With (23) replaced in (36), the ODE for consumption is

$$\dot{C}(t) = \left[\mu - (1-\tau_a)\frac{1}{\theta}I(Q,N,C)\right]C(t) \tag{116}$$

where $\mu \equiv \frac{1}{\theta}\left(\frac{\bar{\pi}A^{\frac{1}{1-\alpha}}(1-\tau_a)}{\zeta(1-s_r)} - \rho\right)$. After using (109) and (111) together with (48), (49) and (116), in order to transform the dynamic system in space (N,Q,C) into a dynamic system of rescaled variables in space (x,z), we obtain the following (interior) steady-state expressions for x, g, I, z, N and E

$$x^* = g_N^* = \frac{\mu\Xi}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(1-\tau_a)(\sigma+\gamma)} \tag{117}$$

$$g^* = \frac{\mu\Xi(\sigma+\gamma+1)}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(1-\tau_a)(\sigma+\gamma)} \tag{118}$$

$$I^* = \frac{\sigma+\gamma}{\Xi}x^* = \frac{\mu(\sigma+\gamma)}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(1-\tau_a)(\sigma+\gamma)} \tag{119}$$

$$z^* = H_Y - H_X - \zeta\left[\left(\frac{1-s_r}{1-s_n}\right) + \frac{\sigma+\gamma}{\Xi}\right]\frac{\mu\Xi}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(1-\tau_a)(\sigma+\gamma)} \tag{120}$$

$$N^* = \left[\frac{\zeta}{\phi}\left(\frac{1-s_r}{1-s_n}\right)\right]^{\frac{1}{\sigma+\gamma+1}} (x^*)^{-\left(\frac{\gamma}{\sigma+\gamma+1}\right)} (Q^*)^{\frac{1}{\sigma+\gamma+1}} \tag{121}$$

$$E^* = x^*N^* \tag{122}$$

The proof of Proposition 6 is carried out by differentiating (117), (118), (121) and (122) with respect to the relevant parameters. The proof of Proposition 7 follows from the second-order differentiation of (118) with respect to the relevant parameters.

(b) The alternative measures of firm size are $\frac{X}{N} = H_x \frac{Q}{N}$ (production, or sales, per firm), where $H_X \equiv L\left(\frac{A\alpha^2}{1-s_x}\right)^{\frac{1}{1-\alpha}}$, and $\frac{a}{N} = \eta = \zeta(1-s_r)\frac{Q}{N}$ (financial assets per firm). The aggregate investment rate, or R&D intensity, is measured as $\frac{R_v+R_n}{Y-(1-s_x)X}$ and the gross saving rate as $sv = \frac{Y-(1-s_x)X-C}{Y-(1-s_x)X}$.

(c) The elements of the Jacobian matrix, J, associated to the linearised system in space (x, z) are

$$J_{11} = -\left\{\left(\frac{\sigma+\gamma}{\gamma}\right) + \frac{\Xi}{\gamma(\Xi+1)}\left(\frac{1-s_r}{1-s_n}\right)\right\}x^* \tag{123}$$

$$J_{12} = -\frac{\Xi}{\gamma\zeta(\Xi+1)}x^* \tag{124}$$

$$J_{21} = -\left\{-\left[\frac{1}{\theta}(1-\tau_a)+\Xi\right]\frac{1}{(\Xi+1)}\left(\frac{1-s_r}{1-s_n}\right)+1\right\}c^* \tag{125}$$

$$J_{22} = \mu + \left[\frac{1}{\theta}(1-\tau_a)+\Xi\right]\frac{1}{\zeta(\Xi+1)}(H_Y - H_X) -$$

$$-\left\{\left[\frac{1}{\theta}(1-\tau_a)+\Xi\right]\frac{1}{(\Xi+1)}\left[\left(\frac{1-s_r}{1-s_n}\right)+2\frac{(\Xi+1)}{\Xi}(\sigma+\gamma)\right]+1\right\}x^* \tag{126}$$

The proof of Proposition 8 is done by using (123)-(126) in (71) and differentiating the latter with respect to the relevant parameters. As for Proposition 4, with government intervention, $\theta > 1$ may no longer be a necessary or sufficient condition to $J_{21} < 0$, that is, for the model to display a transition with both x and z decreasing when $x(0) > x^*$ (Cf. Appendix H, above). From (125), we find that $J_{21} < 0$ requires

$$\frac{-\left[\frac{1}{\theta}(1-\tau_a)+\Xi\right]}{(\Xi+1)}\left(\frac{1-s_r}{1-s_n}\right)+1>0 \Leftrightarrow \theta > \frac{1-\tau_a}{\Xi\left(\frac{1-s_n}{1-s_r}-1\right)+\left(\frac{1-s_n}{1-s_r}\right)} \tag{127}$$

In particular, if $1-\tau_a > \Xi\left(\frac{1-s_n}{1-s_r}-1\right)+\left(\frac{1-s_n}{1-s_r}\right)$, then (127) implies $\theta > \epsilon$, where $\epsilon > 1$. A necessary (though possibly not sufficient) condition for the former is $\left(\frac{1-s_n}{1-s_r}\right) < 1 \Leftrightarrow s_n > s_r$, for a given $\tau_a \geq 0$. A numerical illustration of this result may be as follows: let $\Xi = 0.6$ (which corresponds to $\lambda = 2$); $\tau_a = 0.1$; and $\left(\frac{1-s_n}{1-s_r}\right) = 0.8$ (which corresponds, e.g., to $s_n = 0.28$, if $s_r = 0.1$). With these values, $\theta > 1.32$ is a necessary condition to $J_{21} < 0$. Yet, most empirical estimates of θ in the literature are above this threshold, whereas the value for the ratio between the two subsidies needed to generate it seems too large when compared with the empirical evidence.

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