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# A MODEL OF GROWTH WITH INTERTEMPORAL KNOWLEDGE EXTERNALITIES, AUGMENTED WITH CONTEMPORANEOUS KNOWLEDGE EXTERNALITIES

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ABSTRACT

The present model is essentially Romer's (1990) model of endogenous growth with intertemporal knowledge externalities, augmented with contemporaneous knowledge externalities to give a richer explanation of the growth process. Both types of knowledge spillovers seem essential to capturing the features of knowledge in a model of growth. Introducing synchronic complementarities and knowledge externalities across inventive firms immediately creates the possibility of multiple equilibria and threshold effects in the present model. Another advantage of this theoretical formulation is that it allows for an analysis of the effects on steady-state growth of a variety of technology policies relying on changing knowledge complementarities parameters.

KEYWORDS: Endogenous growth, innovation, knowledge complementarities, knowledge externalities, general equilibrium.

## 1. INTRODUCTION

The present model is essentially Romer's (1990) model of endogenous growth with intertemporal knowledge externalities, augmented with contemporaneous knowledge externalities to give a richer explanation of the growth process. Knowledge complementarities are very important in this context. They can be diachronic and synchronic and they can lead to intertemporal and contemporaneous externalities, respectively. Both types of knowledge spillovers seem essential to capturing the features of knowledge in a model of growth. Much of the value to society of any given innovation or discovery is not, without a doubt, captured by the inventor. This implies that any model that missed these spillovers would miss important elements of the endogenous nature of technological change and of the growth process.

Introducing synchronic complementarities and knowledge externalities across inventive firms immediately creates the possibility of multiple equilibria and threshold effects in the present model. Referring to another standard model of endogenous invention, Aghion and Howitt's (1992) model of growth through creative destruction with intertemporal knowledge externalities, Aghion and Howitt (1998) also argue that by allowing contemporaneous technology spillovers in research in their basic model, it will imply that there can be more than one equilibrium growth rate with positive research.

Knowledge complementarities are relevant both diachronically and synchronically. Firms conducting research can benefit from knowledge externalities stemming from the complementarity of innovation activities conducted within the innovation system in the past and at each point in time. Hence, knowledge externalities stemming from knowledge complementarities do not take place through time across individual researchers of different generations alone, but also among contemporaneous inventive agents within an innovation system. The contemporaneous technology spillovers in research are the positive externalities whereby the productivity of any research firm depends on the economy-wide level of research. Individuals of the same generation will benefit from cross-individual spillovers. And the intertemporal spillovers are the positive externalities whereby the knowledge embedded in each innovation can be used by future researchers. Researchers can make use of the accumulated knowledge embodied in the existing designs.

Romer's (1990) specification of the production function of new knowledge does not distinguish one type of knowledge spillover from the other, and so does not allow for the potential interactive effects between the accumulation of knowledge of an individual research unit over time and the contemporaneous spillovers among inventive units, nor for an assessment of the relative contributions of these types of spillovers to the production of new knowledge for that matter. A key feature of models of the knowledge spillover process within a neoclassical theoretical strand, such as Romer (1990) and Aghion and Howitt (1992), is that the public-good aspects of knowledge create economy-wide increasing returns. The positive externalities considered in Romer (1990) are the intertemporal spillovers according to which anyone engaged in research has free access to the total stock of knowledge implicit in previous designs. All researchers in the economy can benefit from the entire stock of knowledge at the same time. As a result, there is a unique stationary equilibrium in the economy. The model exhibits the usual single equilibrium outcome, and therefore it is similar to other typical, standard models of invention.

In our paper, however, we let the accumulation of knowledge of an individual inventive firm over time to be specified separately in the production process of new designs and knowledge from contemporaneous spillovers among inventive firms. The strict discrimination between contemporaneous externalities and intertemporal externalities, as expressed in our formal model through a multiplicative function of a flow of knowledge component and a stock of knowledge component, can naturally lead to a multiplicity of steady states, including a no growth steady state. Allowing for both knowledge complementarities and their external effects creates the possibility of multiple equilibria, with all of the indeterminacies they entail. This multiplicity of steady-state paths captures a threshold effect similar to those obtained in other models with technological complementarities, in particular Young (1993).

The intensity of technological spillovers across innovative firms is dependent upon the degree of technological and/or regional proximity between firms. From the perspective of a firm, the nature of interactions and the efficiency of communication links established between the firm and its more distant technological and/or regional context become plausibly different from those interactions and communication links established within the closer regional or technological proximity of the firm's local innovation system. A local innovation system is characterized by technological knowledge localized in tacit learning processes, and transaction costs supported by each research firm in communicating and receiving new technology. The nature of a local innovation system contrasts with a neoclassical view to technology transfer according to which no connection and absorption attritions take place. Analysis of the conditions and context for effective technological communication to take place is an important subject matter of the innovation system approach (Antonelli, 2001).

Technological knowledge can be stored although subject to knowledge decay and technological obsolescence. The decay or depreciation of the stock of knowledge takes place when people forget or let skills deteriorate. The obsolescence of the stock of knowledge takes place as old ideas are superseded by new, superior discoveries and innovations. Note that these phenomena of knowledge depreciation and technological obsolescence are distinguished from the obsolescence in value represented by creative destruction. Innovations introduce the factor of value obsolescence to the economic system: better products and processes render previous ones obsolete. Obsolescence in value embodies Schumpeter (1942)'s idea of creative destruction. The payoff from innovation in the current period is the prospect of monopoly rents the next period. Those rents will last however until the next innovation occurs, at which time the knowledge underlying the rents will be rendered obsolete.

The range of dynamic, general equilibria outcomes derived in our model naturally result from adopting a particular production function of new knowledge that has constant elasticity of substitution between diachronic complementarities and synchronic complementarities, the well-known Cobb-Douglas case. This sort of imperfect substitution between synchronic complementarities and diachronic complementarities will open the possibility for multiple steady states, including a no growth steady state.

We could arrive at somewhat different results concerning the number and nature of equilibrium outcomes assuming a different pattern of substitutability/complementarity in the production function of new knowledge. In fact the picture becomes different in the case where the production function approaches a fixed-proportion technology, as well as in the case where the production function adopted is linear so that diachronic complementarities and synchronic complementarities are perfect substitutes.

The paper is organized as follows. We describe the structure of the model in Section 2 and study the microeconomic behavior of research units embedded in a general equilibrium setting with technological progress. This section shows how to incorporate intertemporal spillovers in research and contemporaneous spillovers in research in the basic model of Romer (1990). The existence, the multiplicity, and the stability of dynamic general equilibria are analyzed in Section 3. The analysis in this section focuses on stationary equilibria with positive growth. The impact of technology policy acting through changes in knowledge complementarities parameters is dealt with in Section 4. A few concluding remarks are given in Section 5, while some proofs are gathered in a separate Appendix.

## 2. DESCRIPTION OF THE MODEL

Suppose that there are *N* distinct research firms operating in the research sector. Only the finite number *N* of existing research firms in the economy can produce inventive output. Accordingly, the uninteresting case is one in which no quantity of inventive output is produced, either because no potential research unit is attracted into the inventive industry or because no firm can remain in business.

We need to specify the process of accumulation of new designs and knowledge over time of each research unit. The production function of every research unit *i*, $1 \leq i \leq N$, is given by the differential equation (where a dot hereinafter stands for time derivative):

$$\dot{A}_i(t) = \varphi h \left( \int_{-\infty}^{t} \delta \dot{A}_i(\tilde{t}) d\tilde{t} \right)^{\gamma} \left( \sum_{j=1}^{N} \sigma \dot{A}_j(t) \right)^{\mu}, \qquad \text{(F)}$$

where $\varphi$ , $\delta$, $\sigma$, $\gamma$ and $\mu$ are technology parameters, $h$ is firm's $i$ amount of human capital employed in research, and $\dot{A}_i(t)$ and $\dot{A}_j(t)$ are rates of production of new product designs, ideas and knowledge. Last $\varphi$ is a productivity parameter of the research unit such that $0 < \varphi < 1$.

The model assumes that every research unit *i* employs the same research technology. The crucial feature of the specification used here is that technological complementarities in innovation activities enter into the production of new knowledge in two distinct ways. The specific formulation used here separates the diachronic complementarities within an individual research unit from the synchronic complementarities among research units within the innovation system. Hence the equation (F) reflects the existence of both intertemporal spillovers and contemporaneous spillovers in research activities.

Each research firm is assumed to be symmetrical with respect to the intertemporal externalities in research and the contemporaneous externalities in research. Each type of technology spillover effects enters the production function (F) of every research firm in the same, yet distinct way. In addition, we also make the simplifying assumption that all existing research firms are committed to the same amount *h* of human capital, because we are interested in the scale effect of the spillover process in the economy. According to this scale effect, the larger the number *N* of research firms in the economy, the larger the amount of knowledge spillovers and therefore the higher the rate of growth of knowledge. By assuming identical research firms in the model, it allows us to determine the impact of *N* active research firms on the rate of growth of new ideas and designs of the economy, and so on the dynamic efficiency of the economy.

The integration of the right-hand side of equation (F) with respect to time yields

$$\dot{A}_i(t) = \varphi h\left(\delta A_i(t)\right)^{\gamma}\left(\sum_{j=1}^{N}\sigma\dot{A}_j(t)\right)^{\mu}, \qquad \text{(F)}$$

where $A_i(t)$ denotes stock of knowledge implicit in previous designs produced by research firm *i*.

We assume that the growth of knowledge is a function of the stock of knowledge $A_i(t)$ of each individual inventive firm, which embodies previous innovations, and also of the current flow of innovations of all inventive firms. In the functional form that we use to describe the research technology, the flow of knowledge component expresses the contribution of synchronic complementarities to the production of new knowledge, whereas the stock of knowledge component expresses the contribution of diachronic complementarities. Note that $\dot{A}_j(t)$, representing flow concepts, are all measured per period of time. On the other hand, $A_i(t)$ are stock concepts, and they indicate quantities in existence at some specific point of time.

The common specification of an individual production function of new designs and ideas (F) contains two multiplicative components of knowledge, representing diachronic technological complementarities and synchronic technological complementarities. Each knowledge component of this production function is indispensable given the hypothesis of imperfect substitution between knowledge sources assumed in the model. If any two inventive firms were completely unconnected in their technological environment, such that any firm could not benefit from knowledge created in other inventive firms, there would be no possibility of growth of knowledge in the economy. On the other hand, if every inventive firm were somehow completely unable of storing the new knowledge it created with its own research effort, there would be no possibility of growth of knowledge as well.

The individual production function (F) is an increasing function of its two knowledge components or arguments. That (F) is increasing in its second argument reflects the existence of positive knowledge spillovers across inventive firms within their innovation system; that (F) is increasing in its first argument reflects the existence of

positive, intertemporal knowledge spillovers within each inventive firm. Thus contemporaneous externalities are an increasing function of $N$ at a given time $t$, whereas intertemporal externalities are increasing in $t$. Overall, the output of new designs and ideas produced by an individual research unit is a deterministic function increasing in time $t$.
The characterization in the present model of the magnitude and impact of diachronic and synchronic complementarities by two single parameters is elegant and simple, but as always this comes at the expense of some realism. Presumably, this convenient analytical formulation has the advantage that it makes little difference on the answering of two important questions: In what direction and to what extent will synchronic complementarities affect the rate of growth of knowledge? And what are the sign and the extent of the impact of diachronic complementarities on the growth rate of the economy?
We realistically allow for the possibility of knowledge depreciation and obsolescence of old ideas in the specification of the process by which knowledge accumulates. Knowledge can be stored although subject to depreciation and to obsolescence. The depreciation of the stock of knowledge takes place when people forget or let skills deteriorate. Technological obsolescence takes place when new knowledge comes along to supersede it. Old knowledge eventually is made obsolete by the emergence of newer, superior knowledge. These new ideas make the knowledge represented by the current stock of ideas less relevant in the production of new knowledge.
Thus the extent of knowledge externalities within research units is determined by the diachronic complementarities parameter $\delta$, which is assumed to be a positive constant strictly less than one.

ASSUMPTION 1: The magnitude and impact of diachronic knowledge complementarities are represented by parameter $\delta$, with $0 < \delta < 1$.

For mathematical convenience, we consider a single, diachronic-complementarities multiplicative constant $\delta$ effectively accounting for the knowledge depreciation and technological obsolescence phenomena. Hence we assume that ideas and designs used at a given time are proportional to existing ideas and designs, so that only the portion $\delta A_i$ of the stock of knowledge $A_i$ implicit in previous designs is actually accumulated within every research unit $i$.
Researchers plausibly either have no free access to all flows of knowledge or are not capable of completely absorbing every new design or knowledge that has been created elsewhere in the economic system. Technological innovation takes place within a particular structure, a specific context of industrial products and production processes. Because technological knowledge is specific to each industry, region or firm, it becomes costly to use elsewhere, increases its appropriability and reduces its spontaneous circulation in the economic system (Antonelli, 1999).
Thus the extent of knowledge spillovers across research units is determined by the synchronic complementarities parameter $\sigma$, which is assumed to be a positive constant strictly less than one.

ASSUMPTION 2: The magnitude and impact of synchronic knowledge complementarities are represented by parameter $\sigma$, with $0 < \sigma < 1$.

For mathematical tractability, we consider only a specific, synchronic-complementarities multiplicative constant $\sigma$ indexing both the conditions and context for effective technological communication. An additional simplifying assumption concerning the flow of knowledge component of the functional form for innovative output is that all knowledge spillover effects among inventive firms are contemporaneous with the current rate of invention. Hence there is production and "instantaneous" diffusion of the fraction $\sigma$ of new knowledge and designs produced by research units within the local innovation system during the same period of time. And the intensity of contemporaneous externalities among $N$ research firms enters the production function of new knowledge as $\sigma \sum_j \dot{A}_j$. Presumably our omission of a relatively short, time lag parameter revealing the delayed nature of other possible spillover effects among inventive firms will not change the basic analysis of synchronic complementarities and technological change at the aggregate level in the model.

Diachronic and synchronic complementarities and their external effects enter the production function of new knowledge of each research unit as two multiplicative, knowledge components. Let parameters $\gamma$ and $\mu$ measure the marginal productivity of each respective component. The extent to which each knowledge externality is relevant and of use to the production of new knowledge is given by each of these respective parameters. They are assumed to be positive constants strictly less than one. There is also an assumption of constant returns to the accumulation of new knowledge in the model: $\gamma + \mu = 1$. Note that $0 < \mu < 1$ and $\mu + \gamma = 1$ implies $0 < \gamma < 1$.

ASSUMPTION 3: Given knowledge productivity parameters $\gamma$ and $\mu$, with $0 < \mu < 1$ and $\mu + \gamma = 1$, the relative weight of synchronic knowledge complementarities over diachronic knowledge complementarities is represented by the parameter transformation $\rho \equiv \mu/\gamma$, so that $0 < \rho < +\infty$.

Because of the assumptions of a flow of knowledge component and a stock of knowledge component contained in the common research technology specification, we are able to isolate and assess the relative importance of synchronic complementarities and diachronic complementarities on economic growth through the single, well-defined parameter transformation $\rho$.

Each research unit views all current innovations and flows of knowledge developed by all research units as equally good technological substitutes for each other in the production of designs and knowledge. All new, different designs and contemporaneous flows of knowledge have essentially the same technological characteristics and are treated symmetrically in the functional form of the research technology of each research unit.

Because of the symmetry in the model, the new ideas and designs of each and every research unit that become available at a given time are supplied at the same rate, henceforth denoted as $\dot{A}_i(t)$. Substituting $\dot{A}_j(t)$ from the symmetric equilibrium condition $\dot{A}_j(t) = \dot{A}_i(t)$ into the individual production function (F) and simplifying yield

$$\dot{A}_i(t) = (\varphi h)^{\rho+1} \delta \sigma^{\rho} N_t^{\rho} A_i(t). \qquad \text{(F)}$$

Let $A(t)$ be the total stock of designs and knowledge in the economy. By definition, we have $A(t) = N_t A_i(t)$. It follows immediately from (F) that the growth rate of $A_i$ and also $A$ is $g = (\varphi h)^{\rho+1} \delta \sigma^{\rho} N_t^{\rho}$.

Let $H$ denote the aggregate level of human capital, $H_{A,\,t}$ denote the amount of human capital employed in research, and $H_{Y,\,t}$ denote the amount of human capital devoted to final output. In Romer (1990) the consumers are endowed with fixed quantities of human capital, which can be used either in research or in the production of final-output goods. Instead Aghion and Howitt (1992) refer the mass of skilled labor, which can be used either in research or in the intermediate sector. In any case, these labor services are assumed to be essential inputs used most intensively in research. Back to Romer (1990), holders of human capital decide whether to work in the research sector or the manufacturing sector. This allocation of human capital between the research sector and the final-goods sector must be consistent with the requirements of a dynamic, general equilibrium for the Romer's model. Nevertheless, the size of individual research firms is indeterminate in equilibrium. This is a common outcome in models of competition with constant returns to scale in human capital. The number and dimensions of research firms are indeterminate in the presence of constant returns to scale in the research lab combined with free entry in the research sector.

Conversely, in the present model with diachronic and synchronic complementarities, we can infer the equilibrium number of research units once the equilibrium level $H_A$ is determined. We have assumed that every research firm employs the same research technology specification, and thus the same amount $h$ of human capital. Accordingly, we can infer that the total number of active firms in the present model is $N = H_A/h$.

Earlier in this section, while describing the model with synchronic and diachronic complementarities, we took that any alternative number of research units is indivisible or "lumpy". All the possible numbers of research units are thus all the positive integers and the number zero. Henceforth, the positive number $N$ of research firms is assumed to be a continuous variable instead of a discrete one. Assuming that $N \geq 1$ becomes continuously variable, however, there must exist a continuum of numbers of research units – a different $N$ for each alternative set of values $H_A$ and $h$ which satisfies the equation $N = H_A/h$. The remaining case is again one in which the number of research firms is zero, i.e., $N = 0$.

Now we change variable to characterize the behavior of the research sector in the model with diachronic and synchronic complementarities. The aggregate variable $H_A$ is replaced by the optimization variable $N$. Sufficient conditions for a stationary, balance growth solution are derived below with respect to this new control variable. The formal specification used here emphasizes the importance of the sources of invention in the research process: the research firms. The number of distinct research firms producing different types of new designs and ideas is an important determinant of the equilibrium outcome and the rate of growth in the present model.

Consequently, the appropriate measure of scale of the research sector is the number $N$ of active research firms. Without loss of generality, set from this time on the common level $h = 1$. Following from this relevant index for the scale of operation of the innovation network, the highest possible level of activity in the research sector is given by the maximum number of potential research units in the economy $H$, whereas the lowest possible level of activity in the research sector is, of course, simply $N = 0$.

Following Aghion and Howitt (1998), it is argued that the model of endogenous growth through technological innovation in the present paper is completely described by both an arbitrage condition and a labor market clearing condition. The equilibrium condition for the research sector is called the arbitrage condition (a). It is assumed in Romer (1990) a competitive research sector, with any individual being free to engage in research activities. As a result the value of an hour of manufacturing must also be the wage rate paid to skilled workers in research. Reflecting this free allocation feature and conditional on the value of an hour in research, the arbitrage condition determines the amount of labor devoted to research activities, and so the number of research units.

Let $w_t$ be the rental rate per unit of human capital or the wage rate of skilled labor, and $P_A$ be the price of new designs. Using the aggregate production function of new designs, Romer (1990) derives the equilibrium relation between $w_t$ and $P_A$: $w_t = P_A \varphi A(t)$. Every research unit in the model with diachronic and synchronic knowledge complementarities faces the same research technology, or individual research function, and so the behavior of the research sector can be described in terms of a single research firm. Because of the symmetry in this model, the arbitrage equation in the present model is

$$w_t = P_A \varphi^{\rho+1} \delta \sigma^{\rho} N_t^{\rho} A_i(t). \qquad \text{(a)}$$

Define the "productivity-adjusted (or growth-adjusted) wage rate" as $\omega_t = w_t/A(t)$. Thus the arbitrage equation can now be restated after dividing both sides of equation (a) by $N_t A_i(t)$ as

$$\omega_t = P_A \varphi^{\rho+1} \delta \sigma^{\rho} N_t^{\rho-1}. \qquad \text{(a)}$$

It is assumed in Romer (1990) that the final (or consumption) good sector is competitive, and is described by a single, aggregate, price-taking firm. The wage for human capital in the final-output sector is its marginal product $\alpha H_{Y,t}^{\alpha-1} L^{\beta} A(t) \bar{x}_t^{1-\alpha-\beta}$, where $\alpha$ and $\beta$ are parameters of the aggregate production function, and $\bar{x}_t$ is the aggregate demand for durables, which is a function of $H_{Y,t}$ (and fixed quantity $L$) and is stationary in equilibrium. Define the "marginal product function" in the model with diachronic and synchronic knowledge complementarities as $\omega_t = \tilde{\omega}(H_{Y,t})$. The marginal product function in this model is therefore $\omega_t = \alpha H_{Y,t}^{\alpha-1} L^{\beta} \bar{x}_t^{1-\alpha-\beta}$.

The equilibrium condition for the labor market is called the labor market condition (l). Society's fixed stock of human capital or skilled labor $H$ has two competing uses. Persons can devote human capital to either the research sector or the final-output sector. That is,

$$H = H_{A,t} + H_{Y,t} = N_t + \tilde{H}_Y(\omega_t), \qquad \text{(l)}$$

where $\tilde{H}_Y$ is the demand function for manufacturing labor. It is assumed a perfectly informed and flexible world in which the price variable rapidly equilibrates the labor market. Reflecting this frictionless feature, the labor market clearing condition determines the productivity-adjusted wage rate $\omega_t$ as function of the residual supply of manufacturing labor $H - N_t$, $\omega_t = \tilde{H}_Y^{-1}(H - N_t)$, where $\tilde{H}_Y^{-1}$ is the above function $\tilde{\omega}$. Thus condition (l) can now be re-expressed as

$$\omega_t = \alpha (H - N_t)^{\alpha-1} L^{\beta} \bar{x}_t^{1-\alpha-\beta}. \qquad \text{(l)}$$

A stationary (or steady-state) equilibrium exhibits balanced growth, in the sense that allocation of skilled labor between research and manufacturing remains unchanged with each innovation. Given that the endogenous growth model is fully characterized by both conditions (a) and (l), a stationary equilibrium is simply defined as a stationary solution to system (a) and (l). Accordingly, a stationary equilibrium corresponds to both $N_t \equiv N$ and $\omega_t \equiv \omega$.

Romer (1990)'s equation (8) says that the price $P_A$ of new designs must equal the present value of the net revenue that a monopolist producer of each specialized durable can extract, which is given by the ratio between a constant flow of profit and the interest rate $r$. After substituting $P_A$ from equation (8) into equation (a), the arbitrage and labor market clearing equations of the model in a steady-state just become

$$\omega = \frac{\alpha+\beta}{r}(1-\alpha-\beta)(H-N)^{\alpha} L^{\beta}\bar{x}^{1-\alpha-\beta}\varphi^{\rho+1}\delta\sigma^{\rho}N^{\rho-1}, \tag{A}$$

$$\omega = \alpha(H-N)^{\alpha-1}L^{\beta}\bar{x}^{1-\alpha-\beta}. \tag{L}$$

The equilibrium condition of the model determining the allocation of human capital between research (that is, the number of research units) and manufacturing states that the productivity-adjusted wage rate $\omega$ must be the same in both sectors. Then we easily see that equations (A) and (L) can be combined as $1 = \Psi(N)$:

$$1 = \delta\sigma^{\rho}N^{\rho-1}(H-N)\varphi^{\rho+1}\frac{(1-\alpha-\beta)(\alpha+\beta)}{\alpha r} \equiv \Psi(N), \text{ for } N \geq 1.$$

In a steady state the equilibrium research level $N$, whenever positive must satisfy the equation $1 = \Psi(N)$. The $\Psi(N)$ function is assumed to be continuous and to have continuous derivatives in the interval $[1, H]$, and so we refer to it as a continuously differentiable function in that interval of its domain. The point $N = 0$ is also in the domain of the function: $\Psi(0) = 0$.

To close the model, an assumption is required to ensure the existence of equilibrium with positive growth in the economy, while at the same time allowing for the possibility of multiple equilibria. That is, it remains to impose the relation between the $\Psi(N)$ curve and the horizontal drawn at 1 for (neighboring values of) $N = 1$, with both $\delta$ and $\sigma$ being set at the limit value 1. The possibility of dynamic, general equilibrium or equilibria with positive research will only be present in our model when the size of the economy is considerably large. The existence of stationary equilibrium or equilibria with positive growth also crucially depends on the size of the economy in Romer (1990), Murphy, Shleifer and Vishny (1989), and Young (1993).

ASSUMPTION 4: The size of the economy as measured by the total amount of human capital or skilled labor $H$ is very large. This means that the society's fixed stock of human capital or skilled labor $H$ satisfies both of the following conditions:

*(i)* $H - 1 > \dfrac{1}{\varphi^{\rho+1}}\dfrac{\alpha r}{(1-\alpha-\beta)(\alpha+\beta)} \equiv G(\rho)$;

*(ii)* $H > \dfrac{\rho}{\rho-1}$, for $\rho > 1$.

After Assumption 4, the model with diachronic and synchronic knowledge complementarities has the property that, for very large market sizes, the existence of at least one interior solution with positive research is possible. For an interior solution to

actually exist, it is sufficient to impose restrictions on the parameters $\delta$, $\sigma$, and $\rho$. Assumption 4(i) allow us to impose sufficient conditions on the product $\delta\sigma$, whereas Assumption 4(ii) allow us to establish sufficiency requirements on $\rho$.
The model of endogenous economic growth is now entirely specified. This section has shown how a stationary equilibrium is determined in the economy as well. Next section answers the question of whether or not, or more precisely, under what conditions positive economic growth can be sustained in a framework of knowledge complementarities.

## 3. MAIN RESULTS

One primary issue arising in economic analysis after the introduction of external effects is the potential for multiple equilibria, with all of the indeterminacies they involve. Typically externalities give rise to multiple equilibria with little guarantee that the inventive market will choose the best one. The equilibrium attained by the economy depends upon the expectations of inventors.
Both multiple equilibria and indeterminacy exhibited in our model are created by the assumption of two types of knowledge complementarities: the diachronic complementarities and the synchronic complementarities. Multiple equilibria in the economy can only be revealed for values of the relative weight of synchronic complementarities over diachronic complementarities that are large enough, i.e., $\rho > 1$, which in turn means that the synchronic knowledge complementarities must be relatively more important than the diachronic knowledge complementarities: $\mu > 0.5$.
The next proposition states sufficient conditions under which there is more than one equilibrium outcome with positive research in the economy. Define $N_C(\sigma)$ to be the critical size of the inventive network given $\sigma$. Then it satisfies the condition $\varphi\sigma N = 1$. Hence the $\Psi(N)$ curve shifts downward with increases in $\rho$ whenever $N < N_C(\sigma)$, and shifts upward whenever $N > N_C(\sigma)$. Moreover the upper limit $\overline{\delta\sigma}(\rho)$ of product $\delta\sigma$ and the limit $\tilde{\rho}(\delta\sigma)$ of $\rho$, together with the critical value $N_M(\delta\sigma)$ of research units are all defined in the proof of this proposition (see Appendix). We can write the $\Psi(N)$ curve as $\Psi(N;\ \delta,\ \sigma,\ \rho)$, which has four arguments instead of one: one endogenous variable and three exogenous variables. Then it can be shown that the $\Psi(N;\ \delta,\ \sigma,\ \rho)$ curve achieves a maximum when $N = N_M(\delta\sigma)$ and $\rho = \tilde{\rho}(\delta\sigma)$.

PROPOSITION I: Existence of a Dynamic, General Equilibrium with Positive Growth: Multiplicity of Equilibria.
There exist two general equilibria (interior solutions) in the economy if the following conditions hold:
$\delta\sigma \leq \overline{\delta\sigma}(\rho) < 1$,
and
$\tilde{\rho}(\delta\sigma) \geq \rho > 1$ whenever $N_M(\delta\sigma) \leq N_C(\sigma)$,
$\rho > \tilde{\rho}(\delta\sigma) > 1$ whenever $N_M(\delta\sigma) > N_C(\sigma)$.

PROOF: See Appendix.

Figure 1 illustrates some of the different types of equilibria that might arise in the model. Given the inverted U-shape of the $\Psi(N)$ curve, we then get a multiplicity of steady-state growth paths. For sufficiently small values of the knowledge complementarities parameters $\delta$ and $\sigma$, the $\Psi(N)$ curve cuts the horizontal line drawn at 1 twice, and two steady states with innovation exist: a low growth steady state, and a high growth steady state.

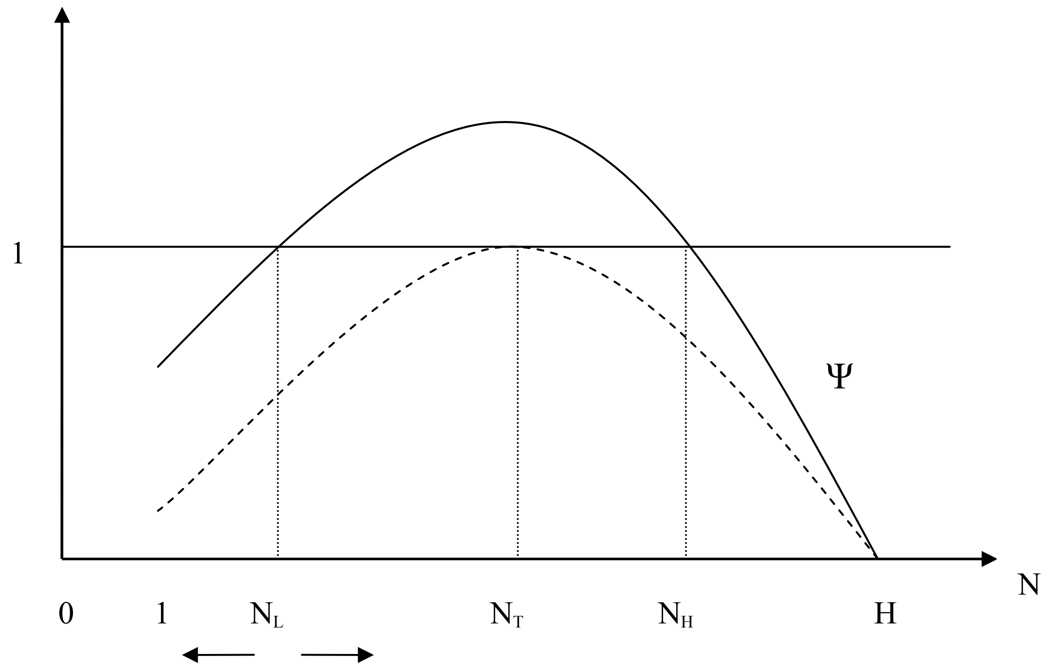


FIGURE 1: Illustration of the multiple general equilibria ($N_L$) and ($N_H$), the possibility of a threshold effect ($N_T$), and the issue of equilibrium stability (direction of arrows).

There are two equilibrium outcomes with positive research, $N_L$ and $N_H$. While $N_L$ is labeled the low-research equilibrium, $N_H$ is the high-research equilibrium. Note that this figure also depicts a stagnant, no-growth equilibrium outcome. For sufficiently small values of $\delta$ and $\sigma$, given $\rho > 1$, the model possesses three steady states: a stagnant, no-research trap, and those two interior solutions already referred to. The selection of an equilibrium is then crucially dependent upon the coordination of expectations.
For intermediate levels of diachronic and synchronic knowledge complementarities, the model reveals important discontinuities in its parameters, as long as the relative weight of synchronic complementarities over diachronic complementarities is greater than one. In particular, a threshold effect exists in which small parameters changes may drastically transform the opportunity set of a stagnant economy by creating an additional, high growth equilibrium.

COROLLARY I: Equilibrium Discontinuity and a Threshold Effect.
Suppose that the relative weight of synchronic complementarities over diachronic complementarities is $\rho > 1$. When the $\Psi(N)$ curve lies slightly below the horizontal line at 1, small increases in the diachronic and/or synchronic knowledge complementarities parameters $\delta$ and $\sigma$, respectively, can discontinuously enlarge the economy's opportunity set of a stagnant economy by allowing the $\Psi(N)$ curve to attain a tangency with the horizontal line drawn at 1 and, given the appropriate modification of expectations, leading to a drastic transition from economic stagnation to rapid growth. Hence the sudden emergency of this high growth equilibrium generates an important threshold effect.

When $\delta$ and $\sigma$ are sufficiently large to allow the $\Psi(N)$ curve to attain a maximum at 1, i.e., at the equilibrium number of research units $N_T$ in the figure 1 above, a positive growth rate suddenly appears. This is because knowledge externalities may generate situations in which research firms find it profitable to undertake an investment in new technology only when other research firms do so too. For sufficiently large values of $\delta$ and $\sigma$ given $\rho > 1$, the apex of the $\Psi(N)$ curve in figure 1 attains a tangency with horizontal line drawn at 1, which, along with the stagnant steady state (where $N = 0$), constitutes a potential steady-state equilibrium. The economy's equilibrium rate of growth will then depend upon whether inventors are pessimistic or optimistic about future rates of inventive activity.
We will now restrict attention to cases in which there is a unique steady-state equilibrium. This model also exhibits the usual single equilibrium outcome for a given set of parameters. Traditional models of endogenous growth and invention, such as Romer (1990), where only one steady-state equilibrium exists, correspond to this special situation. Then there is no indeterminacy in the model and the economy´s growth rate is not dependent upon any coordination of expectations.
The model has the property that for very large values of the diachronic and synchronic knowledge complementarities parameters $\delta$ and $\sigma$, respectively, given any value taken by the relative weight of synchronic complementarities over diachronic complementarities $\rho$, only one, unique general equilibrium with innovation exists. The following proposition states sufficient conditions under which there exists such an equilibrium outcome in the economy.

PROPOSITION II: Existence of a Dynamic, General Equilibrium with Positive Growth: Uniqueness of Equilibrium.
There exists a unique, general equilibrium (interior solution) in the economy if the following conditions hold:

$1 > \delta\sigma \geq \underline{\delta\sigma}(\rho)$

and

$0 < \rho \leq \overline{\rho}(\delta\sigma).$

PROOF: See Appendix.

For very large values of $\delta$ and $\sigma$, it is clear from examination of figure 2 that $\Psi(1)$ is greater than one, and consequently, $\Psi(N)$ cuts the horizontal line at 1 only once, from above. In this case, where each research firm can easily recoup the cost of invention, there is only one unique steady-state equilibrium. This is an equilibrium in which the $\Psi(N)$ curve depicted in figure 2 cuts the horizontal line at 1 from above, because the former curve has a negative slope given our assumption that $\rho < 1$.

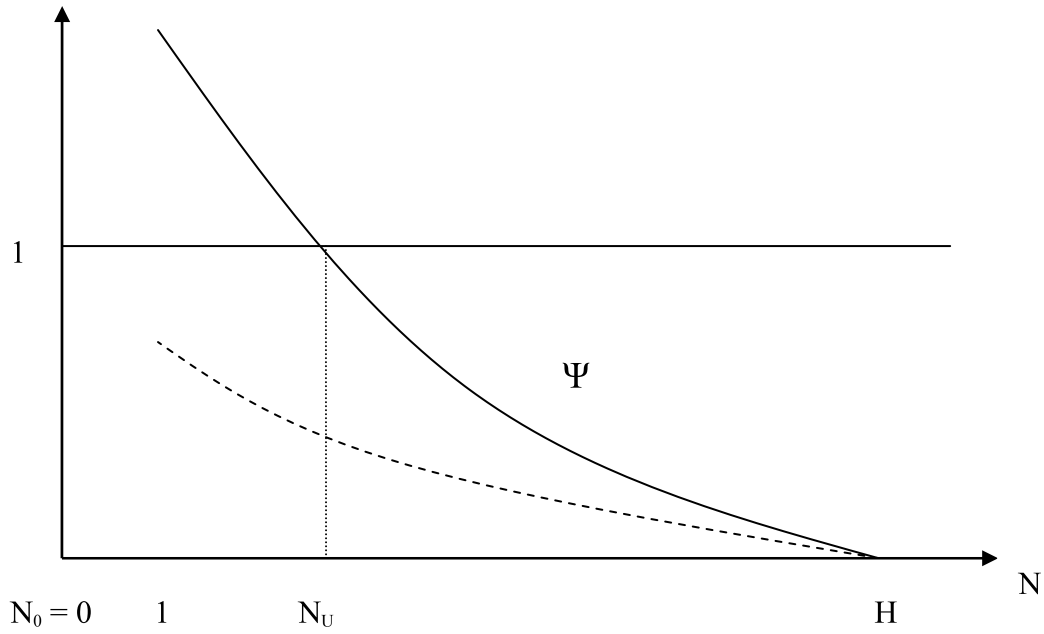


FIGURE 2: Illustration of the unique general equilibrium ($N_U$), given $\rho < 1$, and the stagnant, zero-growth equilibrium ($N_0 = 0$).

The danger of no interior solution for the economy was ruled out once large enough knowledge complementarities have been assumed to exist in the first place. In this case of a unique stationary equilibrium, economic growth is positive because innovations arrive at a positive rate as $N_U > 0$.
However, there is also a real possibility of complete market collapse in which no research unit produces positive inventive output. The potential problem is that no research unit would be willing to spend time and effort necessary to produce inventions unless it was sure that there was going to be sufficient rewards to be reaped. Particularly, in an inventive market exhibiting rather weak knowledge complementarities and external effects, potential research units are understandably reluctant to join the innovation network because there is no assurance that there will be any number at all of active research units operating in the market. The possibility that the particular equilibrium outcome is one in which no one has joined the research network is formally stated in the next corollary.

COROLLARY II: Equilibrium (Interior Solution) Inexistence and the Zero-Growth Equilibrium Outcome.
Let the relative weight of synchronic complementarities over diachronic complementarities take any value $\rho$. When the $\Psi(N)$ curve lies everywhere below the horizontal line drawn at 1, the profit flow accruing to innovators is insufficient to justify their research cost regardless of the number $N$ of research units and thereby the rate of innovation in the economy. In this case the only steady state is the stagnant equilibrium with no innovation $N_0 = 0$. That is, no general equilibrium with a positive number of research units exists, and consequently there will be no growth in the economy.

For very small values of $\delta$ and $\sigma$, the $\Psi(N)$ curve lies below the horizontal line at 1 for all positive number $N$, as depicted in figure 2 above for a given $\rho < 1$. In the absence of extensive knowledge complementarities, the inventive market expected to be faced by each research firm is too small to cover the cost of invention. The inventive market will simply fail and the innovative network will not be viable. If no interior solution exists, then the only equilibrium level of research activity is zero. Hence, for either very small or very large levels of $\delta$ and $\sigma$, the model behaves exactly like a standard model of invention and endogenous growth. In both cases, there exists a unique steady-state growth rate in the economy.

The first proposition above makes clear that there might exist multiple steady state equilibria. Which of these general equilibria might we expect to occur in the economy? The next proposition shows that, although the no-research trap and the high research steady state are both stable, the intermediate, low research steady state is unstable. We argue that the high research or $N_H$ equilibrium is stable, and hence we are lead to conclude that it is the most likely candidate for a long-run equilibrium with innovation. The low-research or $N_L$ equilibrium is unstable and so there is no guarantee that the inventive market will choose the right one, i.e., the high research steady state.

The problem of equilibrium selection is an issue that models of endogenous growth which are built around external effects or externalities must surely confront. Allowing for knowledge externalities creates the possibility of multiple equilibria; the equilibrium indeterminacy is therefore created by the knowledge complementarities of the model.

However, equilibrium selection is a nontrivial problem. The analysis of the stability of different equilibria, both with complete and incomplete information, has been used to develop opinions and draw conclusions as to the equilibrium most likely to prevail in practice.

PROPOSITION III: Equilibrium Indeterminacy and Instability, and Not Reaching Critical Mass.

In a multiple equilibrium setting, the stagnant, no growth equilibrium ($N_0 = 0$) and the general equilibrium $N_H$ are both stable, whereas the general equilibrium $N_L$ is unstable.

The general equilibrium $N_L$ is unstable and so, whenever the installed base of current research units is below the critical mass for the network, $N_L$, the alternative and worse equilibrium outcome that arises is one in which there is no research, i.e., $N_0 = 0$. The other alternative outcome, which is a real possibility for the invention market whenever more than $N_L$ research units have joined the innovation network, is the high-research, or $N_H$ equilibrium.

PROOF: See Appendix.

Any analysis of dynamics will help select an equilibrium so long as it can be used to determine the stability and instability of different steady states. The implications drawn from an analysis of "dynamics" with full information, self-fulfilling expectations and positive feedback as a means of choosing among multiple equilibria in our model are shown below in the proof of this proposition.

The dynamics of the process of knowledge accumulation could take the following form. If the current number of research firms has insufficiently invested in R&D, and therefore $N < N_L$, investing in R&D tends to become unattractive for (potential) research firms next period as well, hence the possibility of a low-growth path where in all successive periods research firms invest too little in R&D. Along this low-growth path, insufficient investment in R&D in the past discourages further knowledge acquisition and thereby future growth.

Positive feedback makes the strong get stronger, and so as the number of research units grow, more and more research units find joining the innovation network worthwhile.

But there is a flip side of this positive feedback force in case there is not a large enough installed base of research units. In this danger zone consisting of those numbers of research units below the critical mass, “the strong gets stronger” is replaced by “the weak gets weaker” and the virtuous cycle of growth is changed to a vicious cycle of collapse.

Going back to figure 1 above, it is suggested there through arrows pointing in opposite directions but always starting from the critical mass for the innovation network $N_L$ where the network will probably grow to attain the high value of research units $N_H$, and where the network will probably fail.

We should note, however, that these results about the stability and instability of the different steady states of the model are extraordinary dependent upon the selection of the production function of new knowledge. For example, with CES production functions with an infinite elasticity of substitution, in a multiple equilibria scenario, the conclusions are apparently the reverse of those shown above: the low-growth equilibrium becomes stable, whereas the high-growth equilibrium is the instable one.

We explore further the implications of knowledge complementarities for general equilibrium (equilibria) and economic growth by conducting now a thorough comparative-static exercise. Note that only qualitative conclusions will be drawn from changing exogenous parameters and parameter transformations in this analysis.

PROPOSITION IV: Comparative-Static Analysis.

An infinitesimal change in every parameter and parameter transformation $\delta$, $\sigma$, and $\rho$ whenever an equilibrium number of research units is greater than $N_C(\sigma)$, associated respectively to diachronic knowledge complementarities, synchronic knowledge complementarities, and the relative weight of synchronic complementarities over diachronic complementarities, will change equilibrium values (concerning numbers of research units):

(i) $N_H$ and $N_U$ all in the same direction, which is that of the parameter’s change;

(ii) $N_L$ in the opposite direction, with $N_L$ changing in the opposite direction of the parameter’s change.

However, an infinitesimal change in parameter transformation $\rho$ whenever an equilibrium number of research units is smaller than $N_C(\sigma)$ will have reversed effects. In this case, a small change in the relative weight of synchronic complementarities over diachronic complementarities will change equilibrium values:

(iii) $N_L$ in the same direction, which is that of the parameter’s change;

(iv) $N_H$ and $N_U$ all in the opposite direction, with $N_H$ and $N_U$ changing in the opposite direction of the parameter’s change.

PROOF: See Appendix.

The next figure illustrates through a minor shift upward of the $\Psi(N)$ curve, and so of the curve corresponding to condition (A), what has just been stated regarding knowledge complementarities effects or impact upon multiple general equilibria values. Note that the relationship between conditions (A) and (L) which fully characterize the model of endogenous growth with innovation can in turn be depicted in a $(N, \omega)$ plane.

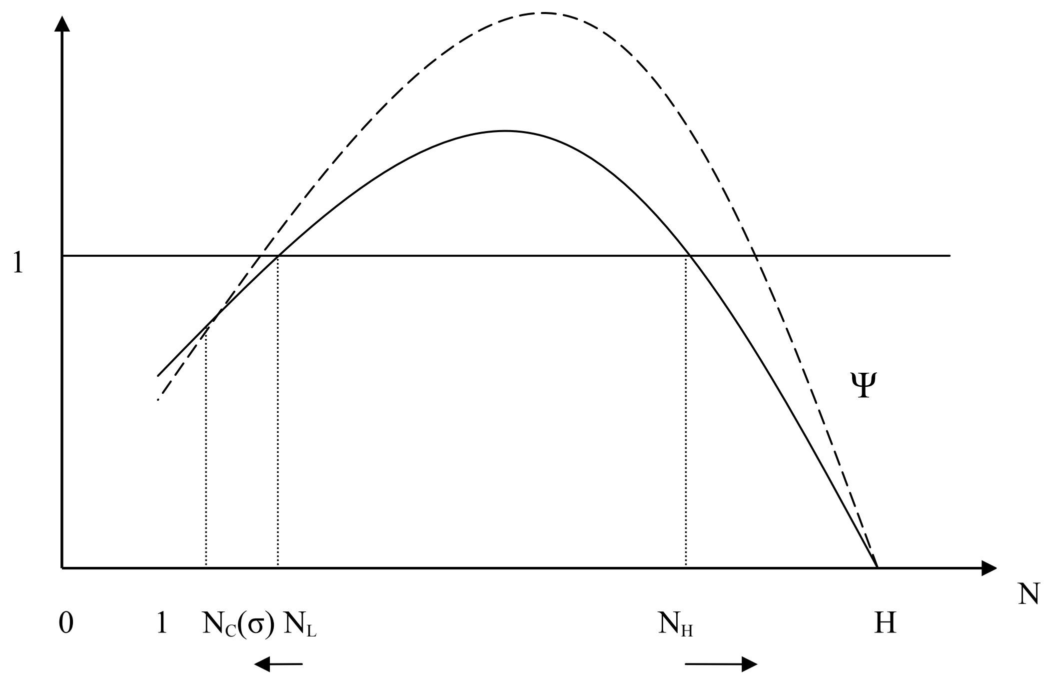


FIGURE 3: Illustration of the impact of small increases in the extent of knowledge complementarities and their relative weight, with $N_C(\sigma) < N_L < N_H$, on the multiple general equilibria $N_L$ and $N_H$.

Using this figure, we easily see that the equilibrium level of research activity $N_L$ will be lowered by a higher $\delta$, a higher $\sigma$, and a higher $\rho$ if $N_H > N_C(\sigma)$. Moreover, the larger the diachronic knowledge complementarities and the synchronic knowledge complementarities, the larger the equilibrium level of research $N_H$ and therefore the larger the growth rate in the economy.

Interestingly, in the multiple equilibrium case, the two steady states display completely opposite comparative properties. A small increase in $\delta$, $\sigma$, or $\rho$ if $N_H > N_C(\sigma)$, raises the number of research units in the high growth steady state and therefore the value of the high growth rate, while lowering the number of research units in the low growth steady state and the associated value of the low growth rate. These comparative-statics results are all intuitive to a great extent. If one raises the return to an endogenous activity like innovation in a situation in which the payoff to economic agents is locally decreasing in the level of research activity, then a return to equilibrium will require an increase in their level of activity. However, if the payoff to the economic agents is locally increasing in the level of research activity $N$, as is the case near the low growth steady state, then a return to equilibrium requires a paradoxical reduction in their level of activity. Therefore there is a perverse comparative static implication of the low growth steady state. Additionally, note that the contrasting comparative statics properties of the different equilibria of this model highlight the importance of equilibrium selection.

## 4. POLICY IMPLICATIONS

The previous section has pointed out that knowledge externalities can be a source of market imperfections and have the potential effect of discouraging R&D investments. This in turn suggests a motivation and role for public intervention in the R&D sector. Governments will increase the level of inventive activity and the rate of innovation in the economy not only by actively financing R&D activities but also through innovative policies designed to enhance knowledge complementarities within local innovation systems.

Innovative policies can have a significant impact upon knowledge spillovers and economic growth by affecting the relevant contextual conditions for technological transfer. Real world innovation systems are characterized by imperfect knowledge spillovers. Some of the important channels of technological knowledge communication among firms are patent disclosure, movements of personnel between firms, professional meetings, input suppliers and customers. With imperfect knowledge spillovers, the communication of ideas and inventions, their development and the final comparative advantage of innovative firms become endogenous to the local innovation system. That is, the channels and networks through which ideas and inventions circulate are embedded in a political, social and institutional background that facilitates or constrains innovation. Therefore any deliberate, publicly designed improvements of those external institutions which are an integral part of innovation systems will necessarily influence innovative activities of firms.

Note that another important instrument of policy intervention in the R&D sector is subsidizing R&D activities. Government subsidies to R&D will increase the profitability of R&D activities, and thereby speed up technological progress. We think this is another important policy prescription that might emerge from our model which is built upon the representative inventive agent assumption.

The model of endogenous growth presented in this paper possesses an interesting range of equilibria, including threshold effects, where small policies might have large consequences. The steady state growth rates of the model are discontinuous in its parameters. Hence, small policy interventions, such as those designed to increase knowledge complementarities within a local innovation system, can have small, positive effects on the equilibrium growth rate as well as dramatically enlarge the inventive opportunity set of the economy and lead to a dramatic transition from no-growth to rapid growth.

Last section has established the possibility of multiple steady states with innovation, the low research equilibrium and the high research equilibrium. Since each of these equilibria display different comparative static properties, the implications of small parameters changes or policy actions are dependent upon which equilibrium is selected. On this issue of equilibrium selection, however, last section has also established that the high research equilibrium is stable while the low research equilibrium is actually unstable. Moreover other stable equilibrium exists in this multiple equilibrium setting: the no-growth equilibrium with no innovation.

An important question is whether and how the technology and the innovative network can reach the critical mass and get started. On this regard, the latter diagram suggests through its arrows pointing in opposite directions why knowledge complementarities have a growth-enhancing value. Greater positive economic growth rates are more likely to take place in inventive markets exhibiting stronger knowledge externalities and in research networks enjoying more extensive knowledge complementarities.

COROLLARY III: Knowledge Complementarities Effects on General Equilibria and Getting the Network Started.

In a multiple equilibria (interior solution) setting, there are two sorts of positive effects on initial dynamic general equilibria $N_L$ and $N_H$, and therefore on the rate of economic growth, of diachronic knowledge complementarities, synchronic knowledge complementarities, and the relative weight of synchronic complementarities over

diachronic complementarities whenever an equilibrium number of research units is greater than $N_C(\sigma)$:
(i) Direct effects: by increasing $N_H$, increases in $\delta$, $\sigma$, and $\rho$ raise the level of innovative activity which speeds up growth.
(ii) Indirect effects: by lowering $N_L$, increases in $\delta$, $\sigma$, and $\rho$ work as a tool that makes it more likely to ignite the positive feedback which translates into rapid growth.
However, the relative weight of synchronic complementarities over diachronic complementarities whenever an equilibrium number of research units is smaller than $N_C(\sigma)$ also has reversed, negative effects on the ultimate level of research activity and economic growth:
(iii) Direct effects: by decreasing $N_H$, increases in $\rho$ lower the level of innovative activity which reduces growth.
(iv) Indirect effects: by increasing $N_L$, increases in $\rho$ make it less likely to ignite the positive feedback which would translate into rapid growth.

Knowledge complementarities can have both so-called direct and indirect effects. As to direct effects to start with, increases in the equilibrium value of research units $N_H$ due to stronger diachronic or synchronic technological complementarities directly affect the speed of growth in an economy, as the growth rate is a strictly increasing function of research units.
The policy implications related to these so-called direct effects of knowledge complementarities and drawn from the earlier comparative-statics exercise are clear. Technology policy by means of strengthening technological complementarities can conceivably be effective in inducing higher innovative performance in the local innovation system and promoting higher sustained growth in the economy. Small policy interventions directed to strengthen technological complementarities, i.e., increasing $\delta$ and/or $\sigma$, can have small and continuous effects on the equilibrium growth rate of the economy.
Corollary III also highlights so-called indirect effects and how the assumption of self-fulfilling beliefs can be useful. Assume that the current number of firms $N$ is smaller than the critical mass $N_L$. Under such conditions, getting $N_H$ research units to operate in the inventive market is likely to be difficult unless the critical mass is reached by $N$. The key challenge facing society here is therefore to somehow obtain more easily critical mass necessary to get an innovative network started. Knowledge complementarities can actually act as a mechanism that enables an economy to lower the critical mass itself. That is, a way for the economy to achieve strong and sustainable growth is through stronger knowledge complementarities which in turn will lower its critical mass.
Therefore, small policy interventions designed to raise knowledge complementarities, i.e., to increase $\delta$ and/or $\sigma$, can conceivably reduce the critical mass of the innovation network as well. Thus for a given value of $N$, the current number of research units, the low research equilibrium will eventually occur to the left of $N$. In this case, the behavior of research units is changed because the positive feedback associated with network externalities change the expectations of potential research units regarding the value to them of joining the innovative network. And self-fulfilling expectations are one manifestation of positive-feedback economics. Hence, due to the multiplicity of stationary equilibria of the model, a technology policy enlarging knowledge complementarities can have positive effects on economic growth by affecting not only the high research equilibrium but also the low research equilibrium.
Finally, the effects of small changes in the relative weight of synchronic complementarities over diachronic complementarities $\rho$ on innovation and growth are

ambiguous. That is so because the direction of the impact of, say, increasing $\rho$ on innovative and economic performance depends on the underlying structure of the economy – specifically, on the equilibrium size of the inventive network relative to its critical size $N_C(\sigma)$. Indeed, increases (or, for that matter, decreases) in $\rho$ will have reversed effects on technological change and growth depending on the magnitude of the inventive network relative to its critical size. Until $N_C(\sigma)$ is reached, further increases in $\rho$ will reduce the growth rate, but beyond that point it will raise the growth rate.

The model has an important policy implication regarding small changes in the relative weight of synchronic complementarities over diachronic complementarities. There is not a simple, clear-cut technology policy designed to change $\rho$ in a unique direction that always enables innovation authorities to push the high research equilibrium $N_H$ forward and to force the economy back to the high research equilibrium whenever it finds itself below the critical mass $N_L$. In particular, the effects of network policies designed to increase the relative productivity of synchronic complementarities and networks over synchronic complementarities and firms, as measured by $\rho$, whenever an equilibrium number of research units in the economy is smaller than the critical size of the network, are decidedly perverse.

The following figure depicts the worst-case-scenario for any government policy aimed at encouraging innovation and boosting growth by relying heavily on raising the relative productivity $\rho$. Such a policy will turn out to be counterproductive because both equilibrium research values $N_L$ and $N_H$ are smaller than the critical size of the network $N_C(\sigma)$.

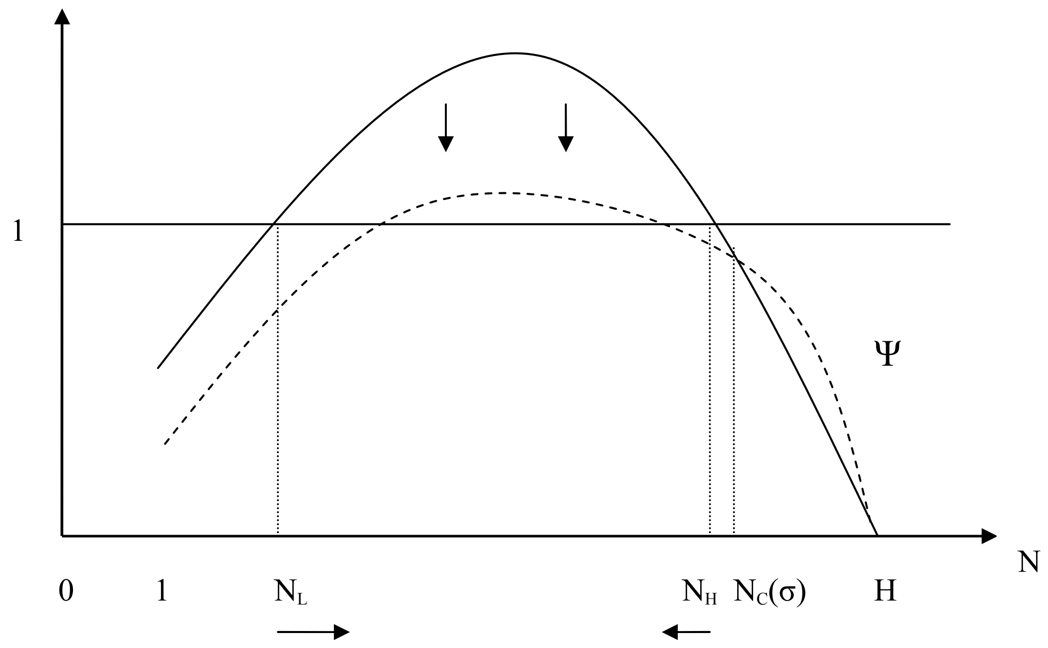


FIGURE 4: Illustration of the impact of small increases in the relative weight of knowledge complementarities, with $N_L < N_H < N_C(\sigma)$, on the multiple general equilibria $N_L$ and $N_H$.

Contrary to the scenario depicted in figure 3, where a higher relative productivity of networks $\rho$ has a boosting effect on innovation and thereby on economic growth, a higher $\rho$, for example induced by network policy, may also slow down innovation and growth. This is because the rate of growth depends upon both networks and firms and, as depicted in figure 4 above, the equilibrium level of research activity is small compared to the critical size of the network. Under constant returns to scale in the production of new technology, the level of productivity of networks $\mu$ can only be pushed at the expense of the productivity of firms $\gamma$, as by definition $\gamma = 1 - \mu$. Having too little firms operating in the inventive network can make such productivity growth sterile if not counterproductive. Such kind of policy gives little chance or benefit of inventive entrepreneurs to learn from diachronic knowledge complementarities. In this case, supporting networks at the expense of firms will reduce the steady state growth.

## 5. CONCLUSIONS

The present extension of Romer (1990) emphasizes knowledge spillovers across inventive firms as important elements of the growth process. It must be acknowledge that both intertemporal knowledge externalities and contemporaneous knowledge externalities allow for a much richer characterization of the growth process. It is the introduction of contemporaneous knowledge externalities that creates the multiple equilibria and threshold effects which are inherent to this model. Another advantage of this theoretical formulation incorporating some degree of contemporaneous spillovers is that it allows for an analysis of the effects on steady-state growth of a variety of technology policies relying on changing knowledge complementarities parameters.

## APPENDIX:

PROOF OF PROPOSITION I:

Define $\widetilde{\Psi}(N;\delta,\sigma,\rho) = (\delta\sigma)^{\rho}\,\Psi(N; 1, 1, \rho)$. By construction, $\widetilde{\Psi}(N;\delta,\sigma,\rho) < \Psi(N; \delta, \sigma, \rho)$ as $(\delta\sigma)^{\rho} < \delta\sigma^{\rho}$, given that $\delta < 1$ (by Assumption 1) so long as $\rho > 1$ (possible by Assumption 3). Let $N^* = \arg\max \Psi(N; 1, 1, \rho)$. Hence $N^*(\rho) = [(\rho - 1)/\rho]H$, which is positive if $\rho > 1$, and, by Assumption 4(ii), is actually greater than one.

There may exist a solution $\widetilde{\rho} > 1$ to the equation $1 = \widetilde{\Psi}(N^*(\rho);\delta,\sigma,\rho)$, where the function representing its right-hand side is defined for given product $\delta\sigma$. Suppose that such an equation does indeed define the implicit function $\widetilde{\rho}\,(\delta\sigma)$. This implicit function can in turn be used to define $N_M(\delta\sigma) = N^*(\widetilde{\rho}\,(\delta\sigma))$. Then $1 < \Psi(N_M(\delta\sigma); \delta, \sigma, \widetilde{\rho}\,(\delta\sigma))$. Hence $1 < \Psi(N_M(\delta\sigma); \delta, \sigma, \rho)$ for $\rho \leq \widetilde{\rho}\,(\delta\sigma)$ whenever $N_M(\delta\sigma) \leq N_C(\sigma)$ and for $\rho > \widetilde{\rho}\,(\delta\sigma)$ whenever $N_M(\delta\sigma) > N_C(\sigma)$. This is because $\Psi(N; \delta, \sigma, \rho)$ shifts upward in the vicinity of $N$ with decreases (increases) in $\rho$ whenever $\varphi\sigma N <(>) 1$, i.e, whenever $N <(>) N_C(\sigma)$. And $\Psi(N; \delta, \sigma, \rho)$ does not move at $N = N_C(\sigma)$.

Moreover, we have $1 > \Psi(H; \delta, \sigma, \rho) = 0$ for every parameter $\delta$, $\sigma$, and $\rho$ as $H - N = 0$, where $N = H > N_M(\delta\sigma)$. Now we need to show that, for a sufficiently low product $\delta\sigma$ given $\rho$, $1 > \Psi(1; \delta, \sigma, \rho)$. Define $\overline{\delta\sigma}\,(\rho) = 1/\Psi(1; 1, 1, \rho)$. That is, $\overline{\delta\sigma}\,(\rho) = G(\rho)/(H - 1)$ which, by Assumption 4(i), is strictly less than one. By construction, $\overline{\delta\sigma}\,(\rho)\ \Psi(1; 1, 1, \rho) > \delta\sigma^{\rho}\ \Psi(1; 1, 1, \rho)$ as $\overline{\delta\sigma}\,(\rho) > \delta\sigma^{\rho}$, given that $\sigma < 1$ (by Assumption 2) and $\rho > 1$ so long as $\delta\sigma = \overline{\delta\sigma}\,(\rho)$. Hence $1 > \Psi(1; \delta, \sigma, \rho)$ for $\delta\sigma \leq \overline{\delta\sigma}\,(\rho)$, where $N = 1$.

Finally, by appealing to the continuity of the $\Psi(N)$ function in the interval $[1, H]$ once all the above sufficiency requirements on $\delta\sigma$ and $\rho$ are met, it should suffice to establish that the graph of the $\Psi(N)$ function cuts the horizontal line at 1 twice (therefore, implying the existence of two interior solutions with positive research).

PROOF OF PROPOSITION II:

Define $\overline{\Psi}(1;\delta,\sigma,\rho)$ to be $(\delta\sigma)\,\Psi(1;\,1,\,1,\,\rho)$ for $\rho \leq 1$ (possible by Assumption 3), and $(\delta\sigma)^{\rho}\,\Psi(1;\,1,\,1,\,\rho)$ for $\rho > 1$ (also possible by Assumption 3). Note that the $\overline{\Psi}(1;\delta,\sigma,\rho)$ function, which has two branches, is continuous at $\rho = 1$. By construction, $\overline{\Psi}(1;\delta,\sigma,\rho)$ $\leq \Psi(1;\,\delta,\,\sigma,\,\rho)$ as $\delta\sigma \leq \delta\sigma^{\rho}$, given that $\sigma < 1$ (by Assumption 2) and $\rho \leq 1$, and $\overline{\Psi}(1;\delta,\sigma,\rho) < \Psi(1;\,\delta,\,\sigma,\,\rho)$ as $(\delta\sigma)^{\rho} \leq \delta\sigma^{\rho}$, given that $\delta < 1$ (by Assumption 1) and $\rho > 1$.

An implicit function $\overline{\rho}\,(\delta\sigma)$ may be implied by the equation $1 = \overline{\Psi}(1;\delta,\sigma,\rho)$. Suppose that such an equation does indeed define that implicit function for given product $\delta\sigma$. Then $1 \leq \Psi(1;\,\delta,\,\sigma,\,\overline{\rho}\,(\delta\sigma))$. Note that $\partial\Psi(1;\,\delta,\,\sigma,\,\rho)/\partial\rho < 0$ as $N = 1 < N_C(\sigma) = 1/(\varphi\sigma)$, given that $0 < \varphi < 1$ and, by Assumption 2, $0 < \sigma < 1$. Hence $1 \leq \Psi(1;\,\delta,\,\sigma,\,\rho)$ for $\rho \leq \overline{\rho}\,(\delta\sigma)$.

Define $\underline{\delta\sigma}\,(\rho)$ to be $1/\Psi(1;\,1,\,1,\,\rho)$ for $\rho \leq 1$, and $[1/\Psi(1;\,1,\,1,\,\rho)]^{1/\rho}$ for $\rho > 1$. That is, $\underline{\delta\sigma}\,(\rho) = G(\rho)/(H-1)$ for $\rho \leq 1$, and $\underline{\delta\sigma}\,(\rho) = [G(\rho)/(H-1)]^{1/\rho}$ for $\rho > 1$ which, by Assumption 4(i), are strictly less than one. Note that the $\underline{\delta\sigma}\,(\rho)$ function with two branches is continuous at $\rho = 1$. By construction, for $\rho \leq 1$, $\underline{\delta\sigma}\,(\rho)\;\Psi(1;\,1,\,1,\,\rho) \leq \delta\sigma^{\rho}$ $\Psi(1;\,1,\,1,\,\rho)$ as $\underline{\delta\sigma}\,(\rho) \leq \delta\sigma^{\rho}$, given that $\sigma < 1$ so long as $\delta\sigma = \underline{\delta\sigma}\,(\rho)$, and for $\rho > 1$, $[\underline{\delta\sigma}\,(\rho)]^{\rho}\;\Psi(1;\,1,\,1,\,\rho) < \delta\sigma^{\rho}\;\Psi(1;\,1,\,1,\,\rho)$ as $[\underline{\delta\sigma}\,(\rho)]^{\rho} < \delta\sigma^{\rho}$, given that $\delta < 1$ so long as $\delta\sigma = \underline{\delta\sigma}\,(\rho)$. Hence $1 \leq \Psi(1;\,\delta,\,\sigma,\,\rho)$ for $\delta\sigma \geq \underline{\delta\sigma}\,(\rho)$.

We also have $1 > \Psi(H;\,\delta,\,\sigma,\,\rho) = 0$ for every parameter $\delta$, $\sigma$, and $\rho$. The same inequality also holds true at least for other numbers of research units in the neighborhood of $N = H$.

Under the sufficiency conditions imposed above on $\delta\sigma$ and $\rho$, together with the continuity property of the $\Psi(N)$ function in the interval $[1,\,H]$, we can be sure that the graph of the $\Psi(N)$ function cuts the horizontal line at 1 only once (therefore, implying the existence of one interior solution with positive research).

PROOF OF PROPOSITION III:

Starting from a general equilibrium with a small number of research units as $N_L$, consider for example the effect of a small loss in the number of current research units on the dynamic behavior of the model. As a few research units leave the innovation network, the expected value of being part of the technological system for those remaining in the network is reduced. In fact, this loss would decrease the value of the innovation network below the opportunity cost of joining the network of all research units in the interval $[1,\,N_L)$, as the curve corresponding to condition (A) lies below the curve corresponding to condition (L) and, accordingly, the $\Psi(N)$ curve lies below the horizontal line at 1 in Figure 1. The eventual outcome is that all research units leave the market (that is to say, $N$ goes down to zero) and the network fails. Starting again from the low-research equilibrium, consider now the effect of a small increase in the number of research units. This would increase the expected value of the network above the opportunity cost for all research units in the interval $(N_L,\,N_H)$. The positive feedback associated with network externalities change the expectations regarding the value of joining the innovation network. This would in turn certainly lead to the establishment of the high-research, or $N_H$ equilibrium.

PROOF OF PROPOSITION IV:

The equilibrium position of the economy is defined by the general equilibrium condition $1 = \Psi(N; \delta, \sigma, \rho)$ which, upon rearrangement, can be expressed by $1 - \Psi(N; \delta, \sigma, \rho) = 0$.

To find how an infinitesimal change in the parameter $\sigma$ will affect the single equilibrium value $N_U$, one has to differentiate partially the general equilibrium condition with respect to $\sigma$ and $N$. The comparative-static derivative we are looking for is

$$\frac{\partial N_U}{\partial \sigma} = -\frac{\partial \Psi / \partial \sigma}{\partial \Psi / \partial N}.$$

The sign of this partial derivative is determined by the sign of its denominator, as by virtue of the assumptions of the model the first derivative of the curve corresponding to condition (A) w.r.t. $\sigma$ is strictly positive and, accordingly, $\partial\Psi/\partial\sigma > 0$. Hence, as this is an equilibrium in which the $\Psi(N; \delta, \sigma, \rho)$ curve cuts the horizontal line at 1 from above, $\frac{\partial N_U}{\partial \sigma} > 0$. Note that only qualitative conclusions will be drawn here. Likewise, in a multiple equilibrium setting at the high growth equilibrium $N_H$, as the $\Psi(N; \delta, \sigma, \rho)$ curve also intercepts the horizontal line at 1 from above, $\frac{\partial N_H}{\partial \sigma} > 0$. However, taking into account that the $\Psi(N; \delta, \sigma, \rho)$ curve might also intercept the horizontal line at 1 from below, as it does at the equilibrium number of firms $N_L$, these same arguments lead us to conclude instead that $\frac{\partial N_L}{\partial \sigma} < 0$.

Similar arguments to those above can be developed to draw additional qualitative conclusions: the first derivative of the curve corresponding to condition (A) w.r.t. $\delta$ is strictly positive and, accordingly, $\partial\Psi/\partial\delta > 0$; hence $\frac{\partial N_U}{\partial \delta} > 0$, $\frac{\partial N_H}{\partial \delta} > 0$, and $\frac{\partial N_L}{\partial \delta} < 0$. Furthermore, the first derivative of the curve corresponding to condition (A) w.r.t. $\rho$ is strictly positive and $\partial\Psi/\partial\rho > 0$ whenever $\varphi\sigma N > 1$, which happens by definition of critical size of an inventive network if and only if $N > N_C(\sigma)$; hence $\frac{\partial N_U}{\partial \rho} > 0$, $\frac{\partial N_H}{\partial \rho} > 0$, and $\frac{\partial N_L}{\partial \rho} < 0$. However, the first derivative of the arbitrage curve w.r.t. $\rho$ is strictly negative whenever $N < N_C(\sigma)$, hence reversing the sign of every such comparative-static derivative $\partial N/\partial\rho$.

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