PUBLIC FIRMS IN A DYNAMIC THIRD MARKET MODEL

Sofia B. S. D. Castro[1] António Brandão[2]

Faculdade de Economia do Porto, Rua Dr. Roberto Frias, 4200-464 Porto, Portugal

Abstract

We set the third market model in a dynamic context to decide whether a country can achieve benefits by subsidizing a public firm's exports. We use calculus of variations with the constraint that the welfare is either maximized or grows at constant rate, reflecting the public concern of the firm. We conclude that a subsidy can be a good strategy for the country in some instances, even though only over a finite period of time. The duration of this period depends on the output strategy of the public firm as well as on exogenous factors.

Resumo

Neste trabalho desenvolvemos uma versão dinâmica do modelo do terceiro mercado para decidir se um dado país pode beneficiar de um subsídio à exportação a uma empresa pública. O cálculo de variações é usado com a restrição de um bem-estar máximo ou crescente a uma taxa constante, o que reflecte o carácter público da empresa. É possível concluir que o subsídio à empresa pública é uma boa estratégia em alguns casos, se bem que apenas por períodos de tempo limitados. A duração destes períodos de tempo depende da estratégia de produção da empresa e de factores exógenos.

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[1]email:sdcastro@fep.up.pt
[2]email:abrandao@fep.up.pt

1 Introduction

The literature on strategic trade policy has shown that in certain circumstances a country can achieve benefits subsidizing its firms' exports. This happens, for instance, in the framework of the well-known third market model (see Brander and Spencer (1985)) where firms of two different countries compete on quantities in the market of a third country. The commitment of one of the countries' government to a subsidy to its firm's exports gives an advantage to this firm and enables it to get further profits in the third market. These additional profits can be enough to compensate the social losses associated with the subsidy and the welfare will be enhanced in the country whose firm is subsidized.

This issue has also been addressed as an extension of the Brander and Krugman (1983) "reciprocal dumping model" and in games with different time structure, namely simultaneous, sequential, infinitely repeated and dynamic (for a survey see Brander(1995)).

We can ask if such a welfare enhancing effect of the exports subsidy can be achieved when the domestic firms are public or regulated. To our knowledge, this question has not yet been studied in depth.

Depending on issues such as the structure of the market, the government's policy towards the firm and the firm's degree of autonomy, the behavior of a public firm can vary a lot. De Fraja and Delbono (1989), for instance, consider different forms of behavior for a public firm, including either welfare or profit maximization. They have concluded than in certain circumstances profit maximization is the best strategy. In this work, we consider a public domestic firm which maximizes the profit subject to the restriction that the national welfare must be maximized, or grow at positive constant rate, over time. This seems a more realistic approach than to demand that a public or regulated firm is not concerned with maximizing its profit. The definition of the goal of a public firm in this way requires a dynamic setting for the maximization problem of the firm. We use calculus of variations to construct a dynamic version of the third market model. Note that our description of the public character of the firm is intermediate to the extreme cases treated by De Fraja and Delbono (1989). We stress again that our characterization is not applicable to the non-dynamic case and is, as far as we know, new. We prove that, in some circumstances, a positive subsidy can enhance welfare, although for a finite period of time. The public domestic firm can influence the duration of this finite period of time since it depends on the output strategy of the firm.

We consider that there are two firms each one located in a different country. All the firms' output is exported to a third country where they compete on quantities. The government of one of the countries provides an export subsidy to the domestic firm. This firm is owned by the state or regulated and we have a kind of "mixed duopoly" in the third market.[3]

The interaction between the two firms in the third market will run for an interval of time. Solving for continuous time allows us to use classical and powerful tools of calculus which do not apply to discrete time. We shall use calculus of variations to capture the dynamics of the model. With this method, as we will specify soon, the firm is also engaged with the achievement of the restriction which is what guarantees its public concern.

Calculus of variations is a classical tool (see Bliss (1971) for an interesting historical introduction as well as the basics) used to maximize, over a time interval, the value of an integral which may be subject to constraints. The function to be integrated with respect to time depends on time t and on a finite number of functions of t together with their derivatives with respect to t. The solutions provide optimal paths for the variables as functions of time.

In our model we want to maximize the accumulated discounted profit of the domestic firm over an interval of time $[0, T]$ subject to the first order condition for maximizing, or guaranteeing growth, of the discounted welfare. The theory of calculus of variations states that each variable of the integrating function is chosen so that the restriction holds. This poses the following problem: using the form in Brander (1995) for the profit of the domestic firm (dependent on the output of the competing foreign firm) means that the foreign firm is contributing to keeping the welfare of the other country. This does not make sense and we must overcome this problem. We do so by considering the following interaction between the firms.[4]

The foreign firm, say Firm 2, observes the output of the domestic firm, say Firm 1, as it appears in the third market and responds to that output as in a static game. This allows us to compute the value of the output of Firm 2 as a function of the output of Firm 1 using the first order condition for the maximization of Firm 2's profit. This function-value of the output of Firm 2 is observed by Firm 1 as it appears in the third market and is used by Firm 1 in its play. In so doing, Firm 1 is replacing the output of Firm 2, as it appears in the profit of Firm 1, by its expression as a function of the output of Firm 1. Hence, we may write the profit of Firm 1 as a function of the output of Firm 1 and the subsidy alone. In this way, the setting of calculus of variations states that both the domestic firm and the government are using the output and the subsidy, respectively, to maximize the profit

[3] Work on "mixed oligopolies" can be found in Cremer, Marchand, and Thisse (1989), De Fraja and Delbono (1989, 1990).

[4] This reminds us of a Stackelberg interaction.

of the domestic firm over time while ensuring that the welfare is maximized or growing. Simultaneously, the foreign firm is maximizing its profit. This process takes place in continuous time. The competition between the firms is reflected in the way the output of Firm 2 depends on the output of Firm 1. Note that the maximization of the profit by Firm 2 is done at every instant of time and, being dependent on the output of the domestic firm, reflects the dynamics of the problem.

In the next section, we state the problem in its mathematical form taking into account a discount factor and the most generic form for the welfare. We find necessary conditions for the output and subsidy to be solutions to our problem without specifying any particular form for the profit function. In Section 3, we present some qualitative results on the behaviour of the shadow price for a particular profit function. These include the study of the variation of the shadow price with the rate of growth of the welfare. The following two sections are dedicated to the qualitative analysis of the possible solutions to the problem for particular values of some of the many parameters involved. These sections contain results on when and for how long a subsidy should be given, focussing on the perspective of the government. The first of these sections considers the problem of maximizing the welfare in a discounted setting and the second considers the non-discounted case with a growing welfare. We summarize our conclusions in the last section. The Appendix is technical and proves that, despite the non-standard form of the function describing our problem, the Euler equations are the standard ones.

2 Description of the problem using calculus of variations

In this section, we present a mathematical statement of the problem described previously. We start with the most general model since it is easier to do the calculations with a general model than with a particular one. It also has the advantage of providing a solution independent from the particular expressions of profit and welfare. At the end of this section, we introduce particular functions to describe the profit and welfare which we shall use throughout this work.

Let x and π_1 be, respectively, the output and profit of the domestic Firm 1, ; y and π_2 be, respectively, the output and profit of the foreign Firm 2; s the subsidy and W the welfare. Welfare is represented so as to include Neary's (1994) remark that "the social cost of public funds exceeds unity" (Neary, 1994, p. 197). Hence, $W(x, y, s) = \pi_1(x, y, s) - \delta xs$, with $\delta \geq 1$. In sections 4 and 5, we consider the limit case of $\delta = 1$.

As explained above, Firm 2 chooses its output y so that $(\partial\pi_2)/(\partial y) = 0$. The solution to this equation expresses the output of Firm 2 as a function of the output x of Firm 1, say $y \equiv y(x)$. We use this form of y to transform the profit of the domestic Firm 1 and the welfare into functions of x and s alone, thus describing the problem of profit maximization of Firm 1 via the mathematical model

$$\max \int_0^T \pi(x,s)e^{-rt}dt$$

subject to

$$\frac{d}{dt}(W(x,s)e^{-rt}) = k, \quad k \geq 0,$$

where $\pi(x,s) = \pi_1(x, y(x), s)$, $W \equiv W(x, y(x), s)$ and r is the interest rate. In this model the domestic firm maximizes its profit subject to the growth of the welfare, assuming that the foreign firm is maximizing its own profit. We consider that both the welfare and the profit are discounted over time. The restriction determines that the discounted welfare grows at rate k; when $k = 0$, the restriction becomes the first order condition for the welfare to be optimal. Note that, mathematically, the problem is now one of two variables (x and s) only. This is a major simplification which does not neglect the point of view of the foreign Firm 2 since its output will be determined through x.

We do not specify initial or terminal conditions. This is non-standard and will provide information about when (in terms of initial and final values of the output) it will make sense for the government to subsidize the domestic firm. In fact, this allows more room for choice in government intervention. The government will be able to decide, given an initial value for the output, whether to attribute a subsidy or not. On the other hand, the firm can establish an initial value and/or estimate a final value for the output to induce a subsidy from the government.

Note that the first order condition for maximizing the welfare is contained in the one above by making $k = 0$. We shall for the moment concentrate on maximizing the welfare.

In what follows we make explicit the dependence of the output of the domestic firm on the subsidy by writing

$$\dot{x} = \frac{dx}{ds}\dot{s}, \tag{1}$$

where the dot means differentiation with respect to time. We make the further assumption that the variation of x with s does not depend on time so that the derivative with respect to time of the quocient (dx/ds) is zero. This seems reasonable since we expect the firm to respond in the same way to the same variation in subsidy, regardless of the moment in time when the latter occurs.

Note that $\dot{s} = 0$ corresponds to a constant subsidy which causes a constant production and we are not interested in this case.

The following theorem presents the system of differential equations whose solutions will describe the behavior of the agents in our problem. A further explanation of the variable θ is provided in the next section.

Theorem 1 *If the functions $x(t)$ and $s(t)$ are solutions to the problem stated above then there exists a function $\theta(t)$ such that x, s and θ satisfy the following system of differential equations*

$$\pi_x + \dot{\theta} W_x = 0 \tag{2}$$

$$\pi_s \dot{s} + W_s \dot{\theta} \dot{s} + (W_x \dot{\theta} + \theta \dot{W}_x - r\theta W_x)\dot{x} = 0 \tag{3}$$

$$W_x \dot{x} + W_s \dot{s} - rW = 0 \tag{4}$$

where F_z stands for the partial derivative of the function F with respect to the variable z.

Proof: Solutions to a problem in calculus of variations, must satisfy the Euler equations for the problem (see Chiang (1992)). In the case of a constrained problem, we write the Euler equations using the Lagrangian function

$$\mathcal{L}(x, s, \theta) = \pi(x, s)e^{-rt} - \theta(t)\frac{d}{dt}W(x, s)e^{-rt}.$$

The first order Euler equations (see the Appendix for a derivation of the Euler equations in this particular non-standard case) are

$$\frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} = 0,$$

for $z = x, s, \theta$. We suppose $\dot{s} \neq 0$. We can use Equation (1) and the fact that $\dot{W}_z = W_{zz}\dot{z} + W_{zw}\dot{w}$ for $z, w = x$ or s, to simplify the Euler equations and finish the proof. *Q.E.D.*

An important remark is that the system of differential equations could be made independent of the discount factor only because the restriction is that of maximizing the welfare, that is, the right-hand side of the constraining equality is zero. For the problem of a growing welfare the Euler equations cannot be simplified in this way and become far too complicated for any information to be retrieved from them. We do not pursue this further.

In the following three sections we proceed with the qualitative study of Equations (2), (3) and (4) in some particular instances. We consider the functions as in Brander (1995)

$$\begin{aligned}
\pi_1(x,y,s) &= xp(x+y) - cx + sx - F; \\
\pi_2(x,y) &= yp(x+y) - c^* y - F^*; \\
W(x,y,s) &= xp(x+y) - cx - F + (1-\delta)sx.
\end{aligned}$$

where c and c^* represent the marginal costs, and F and F^* the fixed costs for the domestic and the foreign firm, respectively.

The inverse demand is $p(x+y) = a - x - y$. The first order condition for the foreign firm to maximize its profit is given by $(\partial \pi_2)/(\partial y) = 0$, which has as solution $y(x) = (a - x - c^*)/2$. We substitute this function $y(x)$ in the profit function for the domestic firm and obtain

$$\pi(x,s) = x(\alpha - \frac{x}{2}) + sx - F, \tag{5}$$

with $\alpha = (a + c^* - 2c)/2$.

By the same process, the welfare function becomes

$$W(x,s) = x(\alpha - \frac{x}{2}) - F + (1-\delta)sx. \tag{6}$$

We remark that $\alpha = 3x_0/2$, where x_0 is the Cournot ouput of a one stage simultaneous game without state intervention.

3 The shadow price

Even with particular forms for the profit and the welfare, Equations (3) and (4) are far too complicated to consider. In this section we concentrate on Equation (2) which provides information on the variation of the shadow price θ. We can solve this equation for $\dot{\theta}$ to obtain $\dot{\theta} = -(\pi_x/W_x)$. Thus we see that the shadow price decreases in time if profit and welfare either both increase or both decrease with the output x. If the profit and the welfare have opposite behavior with respect to x then the shadow price increases in time.

Differentiating Equations (5) and (6), we have $\pi_x = \alpha - x + s$ and $W_x = \alpha - x + (1-\delta)s$. Supposing that the subsidy is always positive (that is, not a tariff) we have the following possible scenarios, for $\delta > 1$:

- if $\pi_x < 0$ then $W_x < 0$;

- if $W_x > 0$ then $\pi_x > 0$;
- if $\pi_x > 0$ then W_x can be either positive or negative.

To better understand the significance of the shadow price, consider the general restriction of growing welfare. The Euler equations are the same except for the last, which is precisely the restriction in the problem. The Lagrangian in this case is

$$\mathcal{L} = \pi(x,s)e^{-rt} + \theta(t)(k - \frac{d}{dt}W(x,s)e^{-rt}).$$

The shadow price θ may be obtained by partial differentiation of $\mathcal{L}$ with respect to k. Hence, we see that θ is a measure of the sensitivity of the problem to the restriction in the sense that the variation of $\mathcal{L}$ caused by that of a unit in k is given by the shadow price. See the two volumes by Chiang (1984, 1992) for a detailed explanation in this and other contexts.

For our problem, only when $\pi_x > 0$ can the shadow price grow. This occurs for $\alpha - x + (1-\delta)s < 0$, that is $W_x < 0$. Note that this means that the growth of profit with the output is accompanied by a decrease of welfare with output. This, of course, does not mean that the welfare cannot grow or be maximized but it means that its growth or maximization depends on government intervention.

4 Results with $\delta = 1$ and $F = 0$

We begin this section with an explanation of why we choose $F = 0$. For $F \neq 0$, Equation (4) can still be solved analytically, although with more intricate calculations. If $\alpha^2 - 2F \geq 0$ the solution for x will depend exponentially on time as when $F = 0$. Otherwise, the solution for x will depend on the tangent of time as well. This is an unnecessary complication. So, we from now on assume $F = 0$.

Equation (4) can be solved analytically for the output x and the solution is given by

$$x^2 - 2\alpha x + 2\beta e^{rt} = 0, \tag{7}$$

where β is a positive constant arising in the integration related to the initial value of the ouput.

Theorem 2 *For values of time such that $rt > \ln(\alpha^2/(2\beta))$, equation (7) has no solution. Otherwise stated, in the long run, there will be no output x and no subsidy s that will maximize the profit and the welfare and therefore, it will not pay for the government to subsidize Firm 1.*

PROOF: The discriminant of equation (7) is $\Delta = \alpha^2 - 2\beta \exp(rt)$ which is positive if and only if $rt \leq \ln(\alpha^2/(2\beta))$. *Q.E.D.*

Note that the interval of time for which a solution exists depends on the values of r, α and β. A low interest rate r will, under some circumstances, produce a longer interval of time in which subsidizing Firm 1 is a good policy for the government. We return to this issue in Lemma 2 below.

For values of time for which a solution exists, we proceed with the study and begin by noting that equation (7) represents two solutions. Which solution to this equation will be a solution to our problem depends on the initial conditions chosen.

Lemma 1 *For each choice of α and β such that $\alpha^2 \geq 2\beta$ there are two choices for the initial value of the output. These are $x_0 = \alpha - \sqrt{\alpha^2 - 2\beta}$ and $x_0 = \alpha + \sqrt{\alpha^2 - 2\beta}$.*

PROOF: The initial value x_0 of x is any which satisfies (7) for $t = 0$. If we solve this equation we find that it has the pair of solutions in the above statement provided that the discriminant is positive, that is, $\alpha^2 \geq 2\beta$. It now suffices to prove that both solutions are positive. This is so because, since $\beta > 0$, the value of the square root is smaller than that of α. *Q.E.D.*

Note also that for $\alpha^2 = 2\beta e^{rt}$, we have $x = \alpha$, which corresponds to the static case $\dot{x} = 0$.

Lemma 1 means that, for initial values other than those given, there is no solution to the problem of calculus of variations. Hence, there will be no point in subsidizing Firm 1. On the other hand, since the initial conditions vary with parameters α and β, it is possible for Firm 1 to adjust its production in order to induce a subsidy from the government.

When a solution exists, it is then chosen according to whether the initial condition x_0 for the output is greater or less than α.

In Figure 1 we depict solutions to equation (7) for different values of r, α and β. It illustrates the results stated in the lemma below.

PLEASE, INSERT FIGURE 1 HERE.

Lemma 2 *The interval of time over which a solution can be found*

- ***a**. decreases with r if $\alpha^2 > 2\beta$;*
- ***b**. increases with α;*
- ***c**. decreases with β.*

Note that the case $\alpha^2 < 2\beta$ does not correspond to a solution.

PROOF: We differentiate $T(r,\alpha,\beta) = [\ln(\alpha^2/(2\beta))]/r$ with respect to r, α and β, respectively, to obtain

$$\frac{\partial T}{\partial r} = -\frac{1}{r^2}\ln(\frac{\alpha^2}{2\beta}); \quad \frac{\partial T}{\partial \alpha} = \frac{2}{r\alpha} \text{ and } \frac{\partial T}{\partial \beta} = -\frac{1}{r\beta}.$$

Q.E.D.

The results in Lemma 2 can be used both by the government and the firm. Recall that the interest rate r is exogenous, as is α, which depends on the demand and the foreign and domestic marginal costs. The parameter β is related to the initial value of the output and can be adjusted by the domestic firm. It can be used, if decreased, to extend the interval over which a solution exists (if α is constant, since a solution exists only for $\alpha^2 > 2\beta$, a decrease in β will give the domestic firm a better chance of being subsidized). This can be seen both from **a** and **c**. Note that this means that a high initial output is not a good strategy for the domestic firm, if the Cournot output is not so high. From **b**, we see that a high Cournot output will extend the time of subsidy. It is interesting to see that α increases with the demand and the foreign marginal cost, whereas it decreases with the domestic marginal cost. Recall that an increase in foreign marginal cost, enhances the relative position of the domestic firm in the third market, assuming its own marginal cost is fixed. On the other hand, a growth in domestic marginal cost is discouraging for state intervention.

From Figure 1, we see that for values of x_0 greater than α the output decreases with time until it reaches the value of α, after which there is no longer a solution to the problem. If $x_0 < \alpha$ then the output increases until it reaches the value of α.

The dependence between the output of the national firm and the subsidy can be studied using Equation (3). The equation however is too complicated for qualitative analysis.

5 Results with $\delta = 1$, $F = 0$ and $r = 0$

Since the choice of subsidy is a core issue in this paper, we set r to zero so that we can obtain some information on the variation of the subsidy. This corresponds to a setting without discounts which is justifiable if the interval of time $[0,T]$ we are considering is not large.

Lemma 3 *If $r = 0$ then either both the output and subsidy which maximize profit and welfare are constant or the output is equal to zero.*

PROOF: In this case, the Euler equation for x is $(\alpha - x)\dot{x} = 0$, which has only constant solutions for x. To these corresponds either a constant subsidy or a zero output, as we can see by replacing $\dot{x}$ in Equation (3). *Q.E.D.*

This case corresponds to a static problem or, in the limit, to the absence of production.

We therefore replace the restriction of maximizing the welfare by that of keeping a growing welfare at rate k, $dW/dt = k;\ k > 0$.

Lemma 4 *If the functions $x(t)$ and $s(t)$ are solutions to the problem just described then, provided $x \neq \alpha$, there exists a function $\theta(t)$ such that x, s and θ satisfy the following system of differential equations*

$$\dot{x} = \frac{k}{\alpha - x} \tag{8}$$

$$\dot{\theta} = -1 + \frac{s}{x - \alpha} \tag{9}$$

$$\dot{s} = \frac{k}{x} + \frac{k}{\alpha - x}s + \frac{k^2}{x(\alpha - x)^2}\theta. \tag{10}$$

The proof follows that of Theorem 1 using $\mathcal{L} = \pi(x, s) + \theta(k - dW(x)/dt)$.

Again, we can integrate the Equation (8) to obtain

$$x^2 - 2\alpha x + 2kt + \beta = 0, \tag{11}$$

where again β is a positive constant arising in the integration procedure.

Theorem 3 *For values of time such that $t > (\alpha^2 - \beta)/(2k)$, Equation (11) has no solution. Otherwise stated, in the long run, there will be no output x and no subsidy s that will maximize the profit and guarantee the growth of the welfare and therefore, it will not pay for the government to subsidize Firm 1.*

The proof is, similarly to that of Theorem 2, based on the study of the discriminant of Equation (11).

Lemma 5 *For each choice of α and β such that $\alpha^2 \geq \beta$ there are two choices for the initial value of the output. These are $x_0 = \alpha - \sqrt{\alpha^2 - \beta}$ and $x_0 = \alpha + \sqrt{\alpha^2 - \beta}$.*

The proof follows that of Lemma 1.

For $t = (\alpha^2 - \beta)/(2k)$ we have $x = \alpha$, that is, a constant output. As depicted in Figure 2, we can see that for initial values of the output $x_0 < \alpha$ the output grows with time and the opposite happens for initial values greater than α.

Figure 2 shows how the solution $x(t)$ varies with different values of α, β and k. It also illustrates the results of the following lemma.

PLEASE, INSERT FIGURE 2 HERE.

Lemma 6 *The interval of time over which a solution can be found*

- *decreases with k if $\alpha^2 > \beta$;*
- *increases with α;*
- *decreases with β.*

The case $\alpha^2 < \beta$ does not correspond to a solution.

The proof follows that of Lemma 2. Note that the statements of both lemmas are identical showing that the qualitative behavior persists without the discount factor.

We can now use Equation (10) to study the variation of the subsidy. This is achieved by deciding when $\dot{s}$ changes sign. A positive value of $\dot{s}$ means that the subsidy is an increasing function of time t. The subsidy is decreasing otherwise. If $\dot{s} = 0$, we may have attained an extreme value for the subsidy, the study of which requires the computation of the second order derivative of s. This situation occurs for values of the subsidy equal to $s_0 = (k\theta + (\alpha - x)^2)/(x(x - \alpha))$. We have the following possibilities for the sign of $\dot{s}$.

a. If $x < \alpha$ then

(a) If $k\theta + (\alpha - x)^2 > 0$ then the subsidy is always increasing, provided it is positive, that is, not a tariff.

(b) If $k\theta + (\alpha - x)^2 < 0$ then $\dot{s}$ is positive if the value of the subsidy is greater than s_0. Note that this case corresponds to a negative shadow-price.

b. If $x > \alpha$ then

(a) If $k\theta + (\alpha - x)^2 > 0$ then $\dot{s}$ is positive if the value of the subsidy is less than s_0.

(b) If $k\theta + (\alpha - x)^2 < 0$ then a growing subsidy corresponds to a negative one, that is, the subsidy will grow only when it is a tariff.

The scenario above shows that the government will vary the subsidy according to values of the output as expected and also according to the shadow-price, the growth rate of the welfare and the parameters defining the profit functions of both the domestic and the foreign firms.

6 Conclusion

We have shown that even when we are dealing with public and regulated firms an export subsidy may be justified in terms of the domestic welfare in a dynamic version of the third market model. This will be true only for a bounded interval of time with a variable duration.

As seen in Theorems 2 and 3, the problem treated in this work can only be solved using calculus of variations if the time interval is bounded. These theorems provide an upper bound for the time interval for which a solution exists. Lemmas 2 and 6 provide strategies by which the length of the time interval can be changed. This may be done by varying the initial production of the firm (associated to the parameter β), the marginal costs c and c^*, the demand parameter a and the interest rate r, or the growth rate k of the welfare in the non-discounted setting. These results are essentially connected to decisions made by the government and allow for an informed decision on the attribution of a subsidy.

Lemmas 1 and 5 establish initial values for the production for which a subsidy will be a good strategy used by the government. Note that, similar results may be obtained for final values of the output if the upper limit T of the time interval is specified. These results provide the domestic firm with enough information to induce a subsidy, provided it can control its initial output. The analogous results concerning the final values of the output (not stated in this work) may be interpreted as values of the output to aim for at final time.

We stress again that the use of calculus of variations is very much suited to models involving public or regulated firms. It has the advantage, over more standard methods, of allowing the public or regulated firm to worry about its profits as well as the welfare of the country.

Our conclusions are generic. Solutions can be computed numerically from our model by anyone having econometric values for the parameters of the model.

7 Appendix

Here we derive the Euler equations for a problem of calculus of variations where one of the integrating functions depends on the other, that is, where the integrating function is of the form $f(t, x(s(t)), s(t), \dot{x}, \dot{s})$. The construction of the Euler equations is a standard procedure and can be found in virtually all books on Calculus of Variations. However, the authors have not found this particular case treated anywhere.

Suppose that x^* and s^* are extremals for the problem and let $p(t)$ and $q(t)$ be perturbations with value zero at initial and final time. We write

$$x(s(t)) = x^*(s^*(t) + \epsilon p(t)) + \epsilon q(t) \text{ and } s(t) = s^*(t) + \epsilon p(t).$$

Thus we obtain

$$V(\epsilon) = \int_0^T f(t, x^*(s^* + \epsilon p) + \epsilon q, s^* + \epsilon p, \dot{x}^* + \epsilon \dot{q}, \dot{s}^* + \epsilon \dot{p}) dt,$$

which has a maximum for $\epsilon = 0$. We proceed as in the standard derivation of the Euler equations, noting that $\dot{x} = (dx/ds)\dot{s}$, to obtain

$$\begin{aligned} \frac{dV}{d\epsilon}(0) \quad &= \quad \int_0^T (f_{s^*} - \frac{d}{dt} f_{\dot{s}^*}) p dt + \\ &\int_0^T (f_{x^*} - \frac{d}{dt} f_{\dot{x}^*}) q dt + \\ &\int_0^T (f_{x^*} - \frac{d}{dt} f_{\dot{x}^*}) \frac{dx^*}{ds} p dt = 0. \end{aligned}$$

In the above equations, f_z indicates partial derivative with respect to the variable z. Since the perturbations p and q are arbitrary, extremals must satisfy the usual Euler equations.

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