The Firm’s Perception of Demand Shocks and the Expected Profitability of Capital under Demand Uncertainty

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ABSTRACT

This paper revisits the results of the pioneering models of the firm under demand uncertainty and analyses the apparent disparity with respect to the signal of the investment-uncertainty relationship predicted by them. In the 1970’s-1980’s the modelling of demand uncertainty at the firm level, taking into account the firm’s optimal choice of factor inputs, constituted a cutting-edge research topic. But while setting the standards in the literature of the firm’s optimal behaviour under uncertainty, those models did not clarify the rationale behind the disparity of the results concerning the impact of increased uncertainty on the firm’s desired investment. In the context of an isoelastic stochastic demand function, where the shock variable may enter either linearly or non-linearly, we show it is the way the firm perceives the demand shocks that, by determining the shape of the profit function, establishes the signal of the investment-uncertainty relationship predicted by the model.

Keywords: Demand Uncertainty; Expected Profitability of Capital; Shock Perception; Jensen’s Inequality.

Classification-JEL: D21; D24
1. INTRODUCTION

Studies of the firm facing random demand follow a long tradition that stretches back to the 1950’s. A strand of the literature, of which Mills (1959) and Leland (1972) are notable examples, investigated the effect of demand uncertainty on a firm’s optimal decisions by analysing models of the firm with different modes of decision-making: price setting, quantity setting or price-quantity setting. In these models, a central consideration is whether control decisions are made before or after the demand shock (in general, a random shock that moves a negatively sloped demand function) is observed, as this determines the results concerning the effect of uncertainty on price and output levels. However, these models assume output and price are direct control variables in the firm’s optimisation problem, and ignore the firm’s optimal choice of factor inputs under uncertainty and the implications of that choice for output.

In contrast, Smith (1969, 1970), Rothschild and Stiglitz (1971), Hartman (1972) and Abel (1983), among others, explored models where factor inputs are the control variables in the firm’s optimisation problem. These models assume a concave (or quasi-concave) production function with two factor inputs: capital and labour; they also assume the firm must choose its capital stock before knowing the demand shock, whereas labour (the perfectly variable factor) is chosen after the shock is observed. However, the assumptions with respect to the way the demand shock enters the model differ: in Smith (1970) and Rothschild and Stiglitz (1971), where the firm is assumed to be quantity-constrained, the demand shock takes the form of changes in the quantity demanded; in Hartman (1972) and Abel (1983), the firm is a price-taker and the demand shock takes the form of changes in the price of output; in Smith (1969), the demand shock moves horizontally a negatively-sloped demand function. These authors studied the impact of demand uncertainty on the optimal stock of capital and, this way, on the firm’s desired investment, concluding, in general, for an investment-uncertainty relationship with a positive sign. Smith (1969)’s model, however, predicts a relationship with a negative sign.

Specifically for the case of a risk-neutral profit-maximising firm facing a Cobb-Douglas production function, Hartman (1972) and Abel (1983) demonstrated that uncertainty over future output prices increases the firm’s incentive to invest because it increases the expected present value of the marginal unit of capital. By Jensen’s inequality, this only requires that the marginal profitability of capital be a convex function of the stochastic variable. Conversely, in a model where the marginal profitability of capital is concave, Jensen’s inequality would...
imply an investment-uncertainty relationship with a negative sign, as observed in Smith (1969).

Drawing from Hartman and Abel’s analytical work, our paper analysis this apparent disparity concerning the investment-uncertainty relationship and its relation to the *shape* of the marginal profitability of capital with respect to the stochastic variable. Therefore, we retain their analytical framework (also shared by Smith, 1969) and explore a model where the firm faces a Cobb-Douglas production function – this assumption ensures that there is a reasonable degree of flexibility of labour relative to capital\(^1\) – and seeks profit-maximisation.\(^2\) We add an isoelastic stochastic demand function, where the shock variable may enter either linearly or non-linearly. Our paper focus on the different modes of decision-making, in the spirit of Mills (1959) and Leland (1972), to argue that, besides factor substitutability, the relevant assumption for the referred convexity/concavity property to hold is the assumption concerning the *implicit* choice variable (price or quantity) in the firm’s profit maximisation problem. We term it ‘implicit’ because within the analytical framework we adopt factor inputs are formally the control variables; the referred distinction between quantity and price pertains to the way the firm *perceives* demand shocks – as changes in the price associated with a given level of output demanded or as changes in the quantity demanded at any given price – and how the firm *reacts* to demand shocks through the choice of its input factors.

In this sense, our results contrast with Leland’s conclusion that the distinction between price and quantity decisions is “immaterial” when they are both made *after* the shock variable is observed (Leland 1972, p. 280). It is a fact that within the analytical framework we adopt price and quantities are both determined along with the choice of labour input, and thus after the demand shock is observed; however, the distinction between price and quantity decisions, as defined in the last paragraph, emerges as relevant in the context of a model that takes explicit consideration of the firm’s choice of factor inputs, where one input is chosen before the demand shock is observed while the other is chosen afterwards.

Observe, for instance, that the Hartman (1972) and Abel (1983)’s assumption of perfect competition implies that the choice variable of the representative firm is output and that price is exogenous, in which case the marginal profitability of capital is a *convex* function of the

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\(^1\) As it is well known, the Cobb-Douglas function exhibits a constant elasticity of factor substitution equal to one.

\(^2\) Smith (1970) and Rothschild and Stiglitz (1971) studied the related problem of a firm that minimises costs constrained by a given quantity of demand – our paper does not follow this line of work. Also, those authors explored the possibility of an elasticity of factor substitution different from one.
stochastic variable. However, in the case of a firm facing a downward-sloping demand curve, as in Smith (1969), both output and output price emerge as the possible (implicit) choice variable. We show that, when price is the choice variable, marginal profitability of capital is a concave function of the stochastic variable; hence, by Jensen’s inequality, an increase in uncertainty decreases the expected marginal profitability of capital.

We also show that, within the context of a Cobb-Douglas production function and an isoelastic demand curve, both the share of labour in the production function and the price elasticity of demand have an important impact on the degree of concavity/convexity of the profit function.

This paper is organised as follows. Section 2 introduces the representative firm’s maximisation problem. Section 3 explores a closed-form approach to the firm’s problem, where, however, the demand function is sufficiently general to accommodate either a linear or a non-linear stochastic shock. This section analyses two alternative interpretations of the demand shock and how they relate to the concavity/convexity of the profit function in the stochastic variable. Section 4 explains how the interpretation of the demand shock and the assumption concerning the implicit decision variable in the firm’s maximisation problem is determinant for the behaviour of the expected marginal profitability of capital when the degree of uncertainty varies. Section 5 analyses the impact of changes in the price elasticity of demand and in the share of labour in the production function on the degree of the concavity/convexity of the profit function. Section 6 concludes.

2. THE FIRM’S OPTIMISATION PROBLEM

Let us consider a risk-neutral firm and its investment decisions in a context of variable capacity of production and where uncertainty affects demand conditions faced by the firm. Our exposition is embedded in a dynamic programming approach.

We assume that the firm produces output, \( Q_t \), at time \( t \) using its capital stock, \( K_t \), and perfectly variable factors of production, denoted by \( L_t \), and that the firm sells all of its output. Let \( F \) represent a production function that allows for some substitution between the two inputs, so that:

\[
Q_t = F(L_t, K_t),
\]  

(1)
where $F_K, F_L > 0; \quad F_{KK}, F_{LL} < 0; \quad F_{KL} > 0$ and $F_{KK} F_{LL} - F_{KL}^2 > 0$. The firm retains some pricing power, in the sense that the output price, $P_t$, is determined by a downward-sloping demand curve. The position of the demand curve depends on the value of the stochastic variable $X_t$. Thus, the demand function can be defined by:

$$Q_t^d = D(X_t, P_t),$$

(2)

where $Q_t^d$ is the quantity of output demanded. The shock $X_t$ evolves exogenously accordingly to the following geometric Brownian motion (since the model is developed in a continuous-time and infinite-time horizon framework, henceforth we omit the time subscripts$^3$):

$$dX = \alpha X dt + \sigma X dz, \quad \sigma > 0$$

(3)

where $\alpha X$ is the expected instantaneous drift rate and $(\sigma X)^2$ is the instantaneous variance rate of the stochastic process; $dz$ is the increment of a Wiener process. A similar stochastic process was used, for instance, by Abel and Eberly (1995). The current value of the demand shock is known (the firm observes $X$ changing), but its future values are always uncertain – the firm only knows its distribution of probability. We assume the firm has rational expectations about the underlying stochastic process, so that the firm’s decisions are optimal given (3). The operating profit of the firm, i.e., revenues minus the cost of the perfectly variable factors of production, is:

$$\pi = H(K, X)$$

(4)

where $\pi$ is assumed to account for whatever optimisation the firm can do at every instant on dimensions other than its choice of $K$, given the level of $X$. Thus, we can regard $\pi$ as the outcome of an instantaneous optimisation problem.

For simplicity, we assume linear costs to capital adjustment (e.g., the price of a unit of capital) and no depreciation.$^4$ Given the initial capital stock $K$ and the initial level of the stochastic demand shock $X$, the firm wants to choose the path of its stock of capital in order to maximise the expected present value of its cash flows, that is, its operating profit less the cost of investing (the cost of purchasing capital), over an infinite horizon. The firm is assumed to be

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$^3$ If there is no fixed time horizon for the decision problem, dynamic programming obtains a recursive structure and the calendar date $t$ no longer matters (see, for details, Dixit and Pindyck, 1994, p. 101).

$^4$ Abel (1983) shows that the assumption with respect to the behaviour of the marginal adjustment cost of capital (e.g., whether it is convex or linear in the stock of capital) is not a relevant consideration for the Jensen’s inequality effect.
risk-neutral, as usual in the literature,\(^5\) and to discount future cash flows at the constant positive rate \(r\), with \(r > \alpha\) – otherwise, since \(X\) grows exponentially at a deterministic rate \(\alpha\), waiting longer would always be a better policy and the optimum would not exist. Therefore, the value of the firm is:

\[
V(K, X) = \max_{dK} \left\{ \mathbb{E} \left[ \int_0^\infty e^{-\alpha t} \left[ H(K(t), X(t)) dt \right] - \kappa dK(t) \right] \right\}
\]

where \(\kappa\) is the price of a unit of capital. Since \(K(t)\) is not differentiable with respect to time, the last term in the right-hand side is to be interpreted as a Riemman-Stieltjes integral.\(^6\)

3. A CLOSED-FORM APPROACH TO THE FIRM’S PROBLEM

Drawing from Hartman and Abel’s work, we are interested in studying the behaviour of expected profitability of capital under demand uncertainty. For that end, we use the same specification for the production function as Hartman (1972) and Abel (1983) (one which is also common to many other papers), while the demand function is a generalisation of the specification found, for instance, in Smith (1969) and Abel and Eberly (1995).

3.1. The Expected Present Value of the Marginal Operating Profit

Consider a firm that faces an isoelastic demand curve:

\[
Q^d = X^\varepsilon P^{-\varepsilon}, \quad \varepsilon > 1; \quad \phi > 0; \quad X > 0, \quad (5)
\]

where \(\varepsilon\) is the price elasticity of demand and \(\phi\) is a parameter that allows \(X\) to enter the demand function both linearly and non-linearly. The firm produces non-storable output \(Q\) according to the Cobb-Douglas production function:

\[
Q = L^w K^v, \quad w > 0, v > 0, \quad w + v \leq 1, \quad (6)
\]

where \(L\) is labour, \(K\) is the capital stock, \(w\) is the labour share and \(v\) is the capital share (note that Hartman, 1972, Abel, 1983, and others, restrict themselves to the constant-returns-to-

\(^5\) However, e.g., Leland (1972) considers the case of a risk-averse firm that maximises the expected utility of profit.

\(^6\) Since \(X\) follows a continuous non-differentiable time path (because it is governed by a Brownian motion) and there are only linear costs to capital adjustment (\(\kappa\)), then \(K\) will also follow a continuous non-differentiable time path. However, as we explain below, the model meets the conditions that guarantee the firm has a determinate size.
scale case \( w = 1 - \nu \)). At every instant the firm chooses \( L \) to maximise its operating profits \( PQ - WL \), given the levels of \( K \) and \( X \); the wage rate \( W \) is exogenous and assumed to be constant over time. Notice that \( K \) has to be chosen knowing only the probability distribution of \( X \), while \( L \) can be chosen after the realisation of its actual value. Assuming that, in equilibrium, output must be equal to demand, i.e. \( Q = Q^d \), the instantaneously maximised value of operating profit and marginal operating profit are given, respectively, by:

\[
H(K, X) = C X^\gamma K^\theta \\
H_k(K, X) = C X^\gamma K^{\theta - 1},
\]

where \( C = \theta \left( \frac{1}{\gamma^\theta} \right)^{\nu \varepsilon - 1} \nu^{\nu - 1} W^{1 - \nu} \) is a positive constant and where, it is easy to show, the elasticity parameters \( \theta \) and \( \gamma \) depend on \( \varepsilon, \phi, v \) and \( w \) as follows:

\[
\theta = \frac{v(\varepsilon - 1)}{\varepsilon (1 - w) + w} \quad (9) \\
\gamma = \frac{\phi}{\varepsilon - w(\varepsilon - 1)} \quad (10)
\]

Since we are assuming linear costs to capital adjustment, we must impose the condition \( \theta < 1 \), so that \( H_k(K, X) \) is not independent of \( K \) and, thus, there is a finite and determinate optimal level of \( K \) that maximises the value of the firm, \( V(K, X) \). For that condition to be met, notice that we must have (i) \( w + v < 1 \), if \( \varepsilon \to \infty \), and (ii) \( \varepsilon \) finite, if \( w + v = 1 \). The parameter \( \gamma \) takes strictly positive values, given the assumption of \( \varepsilon > 1 \).

We will now calculate the expected present value of \( H_k(K, X) \) holding \( K \) fixed, which we will represent by \( \Pi_k \) hereafter. Suppose first that \( F(X) = X^\gamma \) and recall that \( dX = \alpha X \, dt + \sigma X \, dz \), from Eq. (3), above.

Then, applying Ito’s Lemma, we get:

\[
dF = \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} (dX)^2 = \\
= \frac{\partial F}{\partial X} (\alpha X \, dt + \sigma X \, dz) + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 F}{\partial X^2} dt =
\]
\[ F(t) = \left( \gamma \alpha + \frac{1}{2} \gamma(\gamma - 1)\sigma^2 \right) t + \sigma \gamma F \, dz. \]

Notice that since higher-order terms go to zero faster than \( dt \) as it becomes infinitesimally small, we ignore them and write \( (dX)^2 = \sigma^2 X^2 \, dt \). Thus, we find that \( F \) follows a geometric Brownian motion with variance rate \( (\sigma \gamma)^2 \) and expected drift rate \( \frac{1}{dt} E \left[ \frac{dF}{F} \right] = \gamma \alpha + \frac{1}{2} \gamma(\gamma - 1)\sigma^2 \). The latter is an ordinary differential equation that, as it is easily shown, has the solution:

\[ E[F(X,t)] = F(X_0) e^{\left[ \gamma \alpha + \frac{1}{2} \gamma(\gamma - 1)\sigma^2 \right] t} = X^r e^{\left[ \gamma \alpha + \frac{1}{2} \gamma(\gamma - 1)\sigma^2 \right] t}. \]

Notice that, given the recursive structure of the infinite horizon problem, we can assume that \( X_0 = X \). The expected present value of \( F(X) \) is thus:

\[ E \left[ \int_0^\infty F(X,t) \, e^{-rt} \, dt \right] = \frac{X^r}{r - \gamma \alpha - \frac{1}{2} \gamma(\gamma - 1)\sigma^2} \]

provided the denominator is positive. Then, substituting in (8), we see that \( \Pi_K(K,X) \), the expected present value of the flow of marginal profit \( H_K(K,X) \), is:

\[ \Pi_K(K,X) = \frac{C}{r - \gamma \alpha - \frac{1}{2} \gamma(\gamma - 1)\sigma^2} X^r K^{\theta-1}. \]

### 3.2. Convexity versus Concavity of the Profit Function in the Stochastic Shock

Let us explore two alternative interpretations of the demand shock: as changes in quantity of output demanded at any given price or as changes in the price associated with any given quantity demanded. Analytically, the first case corresponds to Eq. (5), above:

\[ Q^d = X^\phi P^{-\epsilon}, \]

whereas the second case corresponds to the inverse function of (5):

\[ \phi = \epsilon; \alpha = 0; \theta = 1; \text{ and } \gamma = 1/(1-\nu) \]

as \( \epsilon \to \infty \).
\[ P = X^{\theta/\varepsilon} (Q^{\varepsilon})^{-1/\varepsilon} \]  \hspace{1cm} (12)

Given these two alternative formulations of the same demand curve, the relevant shock variable also assumes two alternative formulations: \( X^{\theta} \) in the former, with an elasticity of \( \gamma / \phi \) in (7) and (8); and \( X^{\theta/\varepsilon} \) in the latter, with an elasticity of \( \gamma / (\phi / \varepsilon) \) in (7) and (8).

Given the assumptions of \( \varepsilon > 1 \) and \( \omega < 1 \), and having in mind the expression for \( \gamma \) given by (10), we see that both \( H(K,X) \) and \( H_{k}(K,Y) \) are concave in the stochastic variable \( X^{\theta} \), since:

\[ \gamma / \phi \equiv \frac{1}{\varepsilon - w(\varepsilon - 1)} < 1 \]  \hspace{1cm} (13)

However, \( H(K,X) \) and \( H_{k}(K,Y) \) are convex in the stochastic variable \( X^{\theta/\varepsilon} \), since:

\[ \gamma / (\phi / \varepsilon) \equiv \frac{\varepsilon}{\varepsilon - w(\varepsilon - 1)} > 1 \]  \hspace{1cm} (14)

Notice that if we set \( \phi = 1 \), then the relevant demand shock in (5) becomes \( X \), with a corresponding elasticity of \( \gamma \equiv 1/[(\varepsilon - w(\varepsilon - 1)] \) in (7) and (8) (see (13) above). This result is similar to that obtained by Smith (1969) and Abel and Eberly (1995), although they assumed a constant-return-to-scale Cobb-Douglas production function. In contrast, if we set \( \phi = \varepsilon \), then the relevant demand shock in (12) becomes \( X \), with an elasticity of \( \gamma \equiv \varepsilon /[(\varepsilon - w(\varepsilon - 1)] \) in (7) and (8) (see (14) above). If we also let \( \varepsilon \to \infty \) (i.e., the perfect competition case), then we get \( \gamma \equiv 1/(1-w) \). This corresponds to the specification in Hartman (1972) and Abel (1983).

### 3.3. The Economic Reason Behind the Shape of the Profit Function

We now try to elucidate the economic reason for the shape of the profit function with respect to the stochastic variable.

The rationale for the convexity property is already well established in the literature of microeconomic analysis. If the demand shock is represented as in (12), meaning that it takes the form of changes in the price associated with any given level of output demanded, the firm sees price as an exogenous stochastic variable and thus focuses on its variance; therefore, it will produce more output when the price is high and less when the price is low. As a result,
profit will exhibit increasing marginal returns in prices, which is to say the profit function is convex in the stochastic variable. See, e.g., Varian (1992, pp. 42-3) for a formal proof.

However, if the demand shock is represented as in (5), meaning that it comes about as changes in the quantity demanded at any given price, the firm sees quantity demanded as an exogenous stochastic variable and thus focuses on its variance. As the firm optimally increases $L$ (the instantaneously variable factor), for a given $K$, to take advantage of the demand shock (higher $Q^d$ for a given $P$), the firm runs into the decreasing marginal productivity of factor $L$. As a result, profit will exhibit decreasing marginal returns in quantity demanded, which is to say the profit function is concave in the stochastic variable.

A simple verification consists of analysing the role of the elasticity of $L$ in the production function (6). If we set $w = 1$ in each specification of the demand function, meaning that $L$ is characterised by constant marginal productivity, we see that the elasticity of $X^\phi$ becomes $\gamma / \phi = 1$, i.e., the profit function becomes linear in the stochastic variable. In contrast, the elasticity of $X^{\phi/\varepsilon}$ becomes $\gamma / (\phi / \varepsilon) = \varepsilon$, which means that it continues to be greater than one, i.e., the profit function is convex in the stochastic variable as before (Section 5.2, below, further elaborates on this point).

To conclude, in both cases we assist to the endogenous response of perfectly variable production factors to exogenous demand shocks. Whether that generates a profit function that is convex (with increasing marginal returns) or concave (decreasing marginal returns) in the stochastic variable depends on the type of demand shock we are assuming: shock on prices or on quantities demanded.

4. THE IMPLICIT CHOICE VARIABLE AND THE EFFECT OF INCREASED UNCERTAINTY

The choice between (5) and (12) is not just a normalisation, as Abel and Eberly (1995, p. 15) recognised. It has both qualitative and quantitative importance, namely by determining the sign of the ‘Jensen’s inequality’ effect of uncertainty on the expected marginal profitability of capital and, thus, on the firm’s incentive to invest. Therefore, we are interested in examining the effects of an increase in uncertainty for each formulation of the demand function. In order

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8 Recall that in the model presented here, the firm continuously chooses $L$ (the perfectly variable production factor) to maximise its operating profit taking into account, at each point of time, the level of the demand shock variable. This illustrates the importance of factor substitutability in the Cobb-Douglas production function.
to study these effects we would like to focus on mean-preserving increases in the variance of the stochastic process.

Let us first consider the demand function as in (12), according to which \( P \) depends linearly on \( \varepsilon/\phi X \). This formulation of the demand curve implies that firms see (exogenous) demand shocks as changes in the price associated with any given level of output demanded, and thus \( X^{\phi/\varepsilon} \) is the relevant shock variable. This is to say that the implicit choice variable of the firm facing uncertain demand is output (\( Q \)) and that price is exogenous. We must thus study the effects of an increase in uncertainty (variance) that leaves the expected value of \( X^{\phi/\varepsilon} \) unchanged, so as to obtain a mean-preserving increase in the variance, given the stochastic process represented by (3), above. Therefore, we establish:

**Proposition 1:** If the implicit choice variable of the firm is \( Q \), meaning that the firm perceives the demand shock as represented in (12), then the discounted present value of the marginal profit, \( \Pi_K \), increases with a mean-preserving increase in the variance of the demand shock.

**Proof:** First, repeating the calculations for \( X^\gamma \) presented in Section 3.1 above, we see that the expected drift rate of \( X^{\phi/\varepsilon} \) is:

\[
\frac{\phi}{\varepsilon} \alpha + \frac{\phi}{2 \varepsilon} \left( \frac{\phi}{\varepsilon} - 1 \right) \sigma^2 \equiv M(\phi/\varepsilon),
\]

whereas the variance rate is \( \left[ \sigma(\phi/\varepsilon) \right]^2 \). Thus, in order to leave the expected value of \( X^{\phi/\varepsilon} \) unchanged as its variance increases, the increase in \( \sigma^2 \) must be accompanied by a change in \( \alpha \). Setting \( M(\phi/\varepsilon) = M_0 \), where \( M_0 \) is a constant, and applying the implicit function theorem, we get:

\[
\frac{\partial M}{\partial \alpha} \frac{d\alpha}{d\sigma^2} + \frac{\partial M}{\partial \sigma^2} = 0 \iff \frac{d\alpha}{d\sigma^2} = \frac{1 - \phi}{\varepsilon},
\]

which is positive if \( \phi < \varepsilon \) and negative if \( \phi > \varepsilon \). The impact of a mean-preserving increase in \( \sigma^2 \) on \( \Pi_K \) can be found by analysing the way the denominator in (11) changes. Total differentiation yields:

\[
-\gamma \frac{1}{2} \left( 1 - \frac{\phi}{\varepsilon} \right) d\sigma^2 - \frac{1}{2} \gamma(\gamma - 1) d\sigma^2 = \]
\[
\frac{\gamma}{2} d\sigma^2 \left( \frac{\phi}{\varepsilon} - \gamma \right),
\]

which, it is easily shown, is always negative with \( \phi > 0 \) and \( \varepsilon > 1 \). Thus, a mean-preserving increase in the variance of the stochastic variable \( X^{\phi/\varepsilon} \) increases \( \Pi_K \).

This is the ‘Jensen’s inequality’ effect of uncertainty on investment with a positive sign, which results from the convexity of \( H_K(K,X) \) in \( X^{\phi/\varepsilon} \) (i.e., \( \gamma / (\phi / \varepsilon) > 1 \)). This effect means that greater uncertainty increases the incentive to investment, since the marginal unit of capital generates a higher expected flow of future (operating) profits, in present value.\(^9\)

Proposition 1 also applies to the case of a competitive firm, as in Hartman (1972) and Abel (1983). Indeed, in this case, the price facing the firm is the natural demand shock variable to focus on since firms are price-takers and output is the only possible choice variable of the representative firm. Of course, in this context, Eq. (12) must be seen as the industry-wide demand curve, while the individual competitive firm focuses on mean-preserving increases in the variance of price (note that \( P \) is proportional to \( X \) in the inverse demand function in (12) when \( \phi = \varepsilon \), which means the stochastic processes that governs \( P \) when \( Q \) is fixed is

\[dP = \alpha P \, dt + \sigma P \, dz.\]

Nevertheless, in the case of a firm facing a downward-sloping demand curve, both output \( (Q) \) and output price \( (P) \) emerge as the possible implicit choice variable of the firm facing uncertain demand. If that variable is \( P \), then expression (5) is appropriate to represent the firm’s demand curve, because this formulation means the firm perceives (exogenous) demand shocks as changes in the quantity demanded at any given price. Since, in (5), \( Q \) depends linearly on \( X^\phi \), this emerges as the relevant shock variable. Therefore, we establish:

**Proposition 2:** If the implicit choice variable of the firm is \( P \), meaning that the firm perceives the demand shock as represented in (5), then the discounted present value of the marginal profit, \( \Pi_K \), decreases with a mean-preserving increase in the variance of the demand shock.

**Proof:** First, note that the expected drift rate of \( X^\phi \) is:

\[\frac{\gamma}{2} d\sigma^2 \left( \frac{\phi}{\varepsilon} - \gamma \right),\]

\(^9\) According to Eq. (11), \( \Pi_K \) depends on both \( K \) and \( X \), in contrast to the models by Hartman (1972) and Abel (1983). However, Abel and Eberly (1993, p. 23) showed under rather general assumptions that the level of the existing capital stock does not qualitatively affect the impact of uncertainty on investment through the expected marginal profitability of capital.
\[ \phi \alpha + \frac{\phi}{2} (\phi - 1) \sigma^2 \equiv M(\phi), \]

whereas the variance rate is \((\sigma^2 \phi)^2\). Thus, in order to leave the expected value of \(X^\phi\) unchanged as its variance increases, the increase in \(\sigma^2\) must be accompanied by a change in \(\alpha\). Setting \(M(\phi) = M_0\), where \(M_0\) is a constant, and applying the implicit function theorem, we get:

\[ \frac{d\alpha}{d\sigma^2} = \frac{1}{2} (1 - \phi), \]

which is positive if \(\phi < 1\) and negative if \(\phi > 1\). Again, the impact of a mean-preserving increase in \(\sigma^2\) on \(\Pi_k\) can be found by analysing the way the denominator in (11) changes. Total differentiation yields:

\[ \frac{\gamma}{2} \frac{d\sigma^2}{d\sigma^2} (\phi - \gamma), \]

which can be easily shown to be always positive with \(\phi > 0\) and \(\epsilon > 1\). Thus, a mean-preserving increase in the variance of the stochastic variable \(X^\phi\) decreases \(\Pi_k\).

This is the ‘Jensen’s inequality’ effect of uncertainty on investment with a negative sign, which results from the concavity of \(H_K(K, X)\) in \(X^\phi\) (i.e., \(\gamma / \phi < 1\)). This effect means that greater uncertainty implies less willingness to invest, since the marginal unit of capital generates a lower expected flow of future (operating) profits, in present value.

Therefore, within the analytical framework we adopt, the assumption concerning the implicit decision variable (price or quantity) in the firm’s maximisation problem determines the shape of the marginal profitability of capital and, this way, the sign of the investment-uncertainty relationship. The distinction between price and quantity pertains to the way the firm perceives demand shocks and how the firm reacts to them through the choice of labour input, which is formally the control variable. Notice that, in our model, it is equivalent to specify the firm’s optimisation problem as a profit maximisation with respect to labour and output, so that the output price adjusts to clear the market, or with respect to labour and output price, with quantities adjusting to clear the market. In fact, price and quantity are determined simultaneously, along with the labour input and, thus, after the shock variable is observed. According to Leland (1972, p. 280), this simultaneity should imply the irrelevance of the distinction between a quantity-setting and a price-setting firm. However, the significance of
that distinction in our model results from the consideration of the firm’s choice of factor inputs, where one input is chosen before the demand shock is observed while the other is chosen afterwards.

5. Sensitivity Analysis of the Elasticity of the Stochastic Shock

5.1. Sensitivity to $\varepsilon$

Assuming that (5) and (12) are alternative expressions for the firm’s demand curve, we analyse now the impact of changes in the price elasticity of demand $\varepsilon$ (i.e., the inverse of the pricing power of the firm) on the concavity/convexity of the profit function with respect to the shock variable, which, as we have seen, is measured by $\gamma / \phi < 1$ (Eq. 13) in case the demand function is formulated as in (5) and by $\gamma / (\phi / \varepsilon) > 1$ (Eq. 14) in case (12) is adopted.

**Proposition 3**: If $\gamma / \phi$ is the elasticity of the stochastic shock, then the profit function becomes more concave in the demand shock, $X^\phi$, as $\varepsilon$ increases.\(^\text{10}\)

**Proof**: From (13) we see that $\frac{d(\gamma / \phi)}{d\varepsilon} < 0$.

**Proposition 4**: If $\gamma / (\phi / \varepsilon)$ is the elasticity of the stochastic shock, then the profit function becomes more convex in the demand shock, $X^{\phi / \varepsilon}$, as $\varepsilon$ increases (eventually approaching the perfect competition case, i.e., $\varepsilon \to \infty$).

**Proof**: From (14) we see that $\frac{d[\gamma / (\phi / \varepsilon)]}{d\varepsilon} > 0$.

Figure 1, below, illustrates these results. The economic rationale is as follows: in case $P$ is the implicit choice variable (implying a concave profit function), the higher the elasticity of demand (i.e., the lower the pricing power of the firm), the harder it is for the firm to increase $P$ in response to a positive demand shock without forcing a fall in quantity demanded; in case $Q$ is the implicit choice variable (convex profit function), the higher the elasticity of demand, the easier it is for the firm to increase $Q$ in response to a positive demand shock without forcing a fall in price. We conclude that, in our model, the price elasticity of demand only has an impact on the degree of the concavity/convexity of the marginal profit function. The

\(^{10}\) However, since (5) and (13) apply only to the non-competitive case (firms are not price-takers and thus price may emerge as a choice variable), we must have $\varepsilon < \infty$.\[\]
convexity/concavity of the marginal profit function in the stochastic variable does not depend on any given magnitude of the price elasticity of demand (provided $\varepsilon > 1$), and thus – as far as the response of the discounted value of marginal profits to increased uncertainty is concerned – neither does the sign of the investment-uncertainty relationship. As already seen in Section 4, the crucial factor that distinguishes the non-competitive firm case (studied by Smith, 1969, and Abel and Eberly, 1995) from the competitive firm limiting case (studied by Hartman, 1972, and Abel, 1983) is the assumption about the implicit choice variable of the firm.

Figure 1 - Sensitivity Analysis of Eq. (13) and Eq. (14) to $\varepsilon$, with $w = 0.6$

To finalise, recall that our model incorporates the assumption of $\varepsilon > 1$, in line with Smith (1969), Abel and Eberly (1995) and others, meaning that demand is ‘sufficiently’ elastic with respect to output price. If we assumed $\varepsilon < 1$ instead, the results of our model would be different. In the limit, if $\varepsilon = 0$, the demand function would collapse into $P = 0$, in case of Eq. (12) (if $\phi = \varepsilon$), and into $Q = X$, in case of Eq. (5) (if $\phi = 1$). If the former can be dismissed as uninteresting in economic terms, the latter may pertain to the case of a firm facing demand constraints, where the demand shocks take the form of random exogenous changes in the quantity demanded, an approach followed by Smith (1970) and Rothschild and Stiglitz.
In this case, as figure 1 shows, we would have $\gamma/\phi > 1$ in (13) and, thus, an investment-uncertainty relationship with a positive sign, just as predicted in those two papers.

5.2. Sensitivity to $w$

Notice that the particular values chosen for $w$ and $v$, the elasticities of labour and capital in the Cobb-Douglas production function, do not alter the results described in Sections 4 and 5.1, as long as $w$ and $v$ take positive values not greater than unity. Nevertheless, together with the price elasticity of demand, the specific value taken by $w$ (which may be seen as the contribution of labour to production) is found to be relevant for the degree of concavity/convexity of the profit function in the stochastic variable and thus for its sensitivity to changes in uncertainty.

As shown in figure 2, when the firm faces a production technology with $w$ taking a value near zero, $\gamma/\phi$ (Eq. 13) rapidly approaches zero as the elasticity-price of demand $\varepsilon$ grows, whereas $\gamma/(\phi/\varepsilon)$ (Eq. 14) rapidly stabilises just above unity. When the former is true, we fall back into the deterministic case – the position of the profit function does not depend on the stochastic variable. The latter means that, being the profit function almost linear in the
stochastic variable, the expected present value of marginal profitability of capital is rather insensitive to changes in the degree of uncertainty. In both cases, what happens is that the effect of the endogenous response of labour (the perfectly variable input factor) to exogenous demand shocks is dampened by its very low contribution to production (i.e., the very small $w$).

Figure 3, below, depicts the opposite extreme case (already alluded to in Section 3.3, above). If $w \to 1$, then $\gamma / \phi$ approaches unity for every value of $\varepsilon$, whereas $\gamma / (\phi / \varepsilon)$ continues to be greater than one;\footnote{Of course, in this extreme case we would have to impose $v = 0$, to guarantee that $w + v \leq 1$.} moreover, $\gamma / (\phi / \varepsilon)$ now grows linearly with $\varepsilon$. This result contrasts with the fact that, when $L$ exhibits decreasing marginal returns (i.e., $w < 1$), both $\gamma / \phi$ and $\gamma / (\phi / \varepsilon)$ change by a decreasing rate as $\varepsilon$ grows.

Overall, we observe that the influence of (a non-zero) $w$ on $\gamma / (\phi / \varepsilon)$ is confined to the impact on the rate of change of the latter in response to $\varepsilon$, being the convexity of the profit function preserved even in the limiting case of $w = 1$. In contrast, with $\gamma / \phi$, as $w$ approaches unity and thus $L$ exhibits less pronounced decreasing marginal returns, the concavity of the profit

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Sensitivity Analysis of Eq. (13) and Eq. (14) to $\varepsilon$, with $w = 1$}
\end{figure}
function gradually vanishes. The profit function eventually becomes linear in the stochastic variable, implying a neutral Jensen’s inequality effect of uncertainty on investment. The disparity concerning the impact of \( w \) on the concavity/convexity of the profit function in the stochastic variable mirrors the fact that the elasticity of labour in the production function exerts a mere second-order effect when quantity is the implicit choice variable and, thus, the elasticity of the demand shock is \( \gamma/\phi \), in contrast with the dominant effect exerted when price is the implicit choice variable and the elasticity of the demand shock is \( \gamma/\phi \).

The table below makes the synthesis of the results for the extreme values of \( w \):

<table>
<thead>
<tr>
<th></th>
<th>( \gamma/\phi ) (Eq. 13) &lt; 1</th>
<th>( \gamma/\phi ) (Eq. 14) &gt; 1</th>
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<tr>
<td></td>
<td>(demand shock variable: Q)</td>
<td>(demand shock variable: P)</td>
</tr>
<tr>
<td></td>
<td>(implicit choice variable: P)</td>
<td>(implicit choice variable: Q)</td>
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<tr>
<td>( w \to 0 )</td>
<td>( \gamma/\varepsilon \to 0 )</td>
<td>( \gamma/\phi/\varepsilon \to 1 )</td>
</tr>
<tr>
<td></td>
<td>(deterministic case)</td>
<td>(neutral JIE)</td>
</tr>
<tr>
<td>( w \to 1 )</td>
<td>( \gamma/\varepsilon \to 1 )</td>
<td>( \gamma/\phi/\varepsilon &gt; 1 ), grows with ( \varepsilon )</td>
</tr>
<tr>
<td></td>
<td>(neutral JIE)</td>
<td>(positive JIE)</td>
</tr>
</tbody>
</table>

Note: JIE = Jensen’s Inequality Effect

Figures 4 and 5 in the appendix further illustrate these results, by comparing the sensitivity of the expected marginal profitability of capital to changes in the degree of uncertainty for selected values of \( \gamma/\phi \) and \( \gamma/\phi/\varepsilon \).

6. CONCLUSION

This paper revisited the results of the pioneering models of the firm under demand uncertainty, which displayed a disparity concerning the predicted signal of the investment-uncertainty relationship. In the 1970’s-1980’s the modelling of demand uncertainty at the firm level, taking into account the firm’s optimal choice of factor inputs in a context of specific functional forms, constituted a cutting-edge research topic. But while setting the standards in the literature of the firm’s optimal behaviour under uncertainty, those models did not clarify
the rationale behind the disparity of the results concerning the impact of increased uncertainty on the firm’s desired investment.

By focusing on the different modes of decision-making, in the spirit of Mills (1959) and Leland (1972), while drawing from Hartman (1972) and Abel (1983)’s analytical work, this paper has analysed the referred apparent disparity concerning the investment-uncertainty relationship and its relation to the shape of the marginal profitability of capital with respect to the stochastic variable. Within the context of a model of a firm facing a Cobb-Douglas production function and an isoelastic stochastic demand curve, we have shown that:

- The relevant assumption for the convexity/concavity of the profit function, besides factor substitutability, concerns the implicit variable of choice in the firm’s maximisation problem: if that variable is price (quantities), then the marginal profit function is concave (convex); the distinction between price and quantity pertains to the way the firm perceives demand shocks and how the firm reacts to them through the choice of labour input, which is formally the control variable in the firm’s maximisation problem.

- The price elasticity of demand and the share of labour in the production function influence the degree, but not the sign, of the relationship between marginal profitability of capital and uncertainty.

APPENDIX

We illustrate the impact of changes in the degree of uncertainty on the expected present value of marginal profitability of capital, $\Pi_k$, performing a simulation exercise with $\phi = 1.0$, $r = 3.0$, $\varepsilon = 2.5$ and $\alpha = 0.5$ (initial value). The expected marginal profitability of capital is normalised to unity.

Figure 4, below, depicts $\Pi_k$ as a function of $\sigma$ when the shock variable is $X^{\phi/\varepsilon}$ ($Q$ is the implicit choice variable) and its elasticities in the profit function are $\gamma/(\phi/\varepsilon) = 1.06$ and $\gamma/(\phi/\varepsilon) = 1.56$, corresponding respectively to a low value (0.1) and a high value (0.6) of $w$ (see figures 1 and 2, in Section 5.2). As we can see, when $\gamma/(\phi/\varepsilon)$ is above but near to unity, the positive reaction of $\Pi_k$ to changes in uncertainty only becomes evident for rather high values of $\sigma$. This result is in line with Hartman (1972; see Table I, p. 265).
Figure 4 - Sensitivity Analysis of $\Pi_k$ to $\sigma$ (shock variable $X^\phi$)

Figure 5, below, depicts $\Pi_k$ as a function of $\sigma$ when the shock variable is $X^\phi$ ($P$ is the implicit choice variable) and its elasticities in the profit function are $\gamma/\phi = 0.1$ and $\gamma/\phi = 0.9$; these may be seen as the values of $\gamma/\phi$ corresponding, respectively, to extreme low values (near zero) and high values (near one) of $w$, for a given $\varepsilon$. We also include $\gamma/\phi = 0.5$ with the purpose of comparison. Notice that when the profit function is concave in the stochastic variable, the expected marginal profitability of capital displays maximum sensitivity to the degree of uncertainty when $\gamma/\phi = 0.5$. For values of $\gamma/\phi$ above that level, the sensitivity of the expected marginal profitability of capital to changes in uncertainty levels decreases because the profit function becomes ever more linear in the stochastic variable. For values of $\gamma/\phi$ below 0.5, the position of the profit function tends to be independent of the stochastic variable and thus becomes closer to the deterministic case.
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