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A GENERALIZATION OF RADER'S UTILITY REPRESENTATION THEOREM

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Abstract

Rader's utility representation theorem guarantees the existence of an upper semicontinuous utility function u for any upper semicontinuous total preorder \lesssim on a second countable topological space (X, τ) . In this paper we present a generalization of Rader's theorem to not necessarily total preorders that are *weakly upper semicontinuous*.

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1 Introduction

Rader's utility representation theorem (see e.g. Rader [6], Mehta [5], Richter [7], Bosi and Mehta [3] and Isler [4]), according to which every upper semicontinuous total preorder on a second countable topological space has an upper semicontinuous utility representation, is one of the most famous results in mathematical utility theory.

We recall that Rader's theorem is particularly important in economic applications concerning the existence of maximal elements because every upper semicontinuous real-valued function has a maximum on a compact set (see e.g. Bosi and Herden [2] and Bosi and Mehta [3]).

In a slightly different context, the existence of an upper semicontinuous *weak utility* for an acyclic binary relation on a topological space was characterized by Alcantud [1].

In this paper we introduce the concept of a *weakly upper semicontinuous* preorder and then we generalize Rader's theorem to the case of not necessarily total preorders that are weakly upper semicontinuous on a second countable topological space.

In order to illustrate the situations that we are going to consider in this paper, we present the following simple example.

Example 1.1 Let X be the real interval $[0, 1]$ and consider the nontotal preorder \preceq on X defined as follows:

$$x \preceq y \Leftrightarrow \begin{cases} x \leq y \text{ and } x, y \in \mathbb{Q} \cap [0, 1] \\ \text{or} \\ x \leq y \text{ and } x, y \in [0, 1] \setminus \mathbb{Q} \end{cases}.$$

Then denote by τ the upper order topology on X associated to the natural total preorder \leq on X (i.e., $\tau = \tau_u^{\leq}$ is the topology generated by the order intervals $L_{<}(x) = \{z \in X : z < x\}$). The topology τ is second countable since it is a subtopology of a second countable topology (the *order topology* on X associated to the natural (pre)order \leq) and in addition it is totally ordered by set inclusion. It is immediate to check that the identity function $u = i_X$ is an upper semicontinuous utility function $u : (X, \preceq, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$. On the other hand, we have that the preorder \preceq is not *upper semicontinuous* (actually, $L_{<}(x) = \{z \in X : z \prec x\}$ is not open for all $x \in X$). Nevertheless, it should be noted that $L_{<}(x)$ is a τ -open \preceq -decreasing subset of X excluding x and containing $L_{<}(x)$ for all $x \in X$.

2 Notation and definitions

A binary relation \preceq on a set X is said to be a *preorder* if \preceq is *reflexive* and *transitive*. If \preceq is a preorder on X then denote by \prec the *strict part* of \preceq (i.e., $x \prec y$ is equivalent to $x \preceq y$ and not $y \preceq x$ for all $x, y \in X$).

If (X, \preceq) is a preordered set, then a subset D of X is said to be *decreasing* if, for all $x, y \in X$, $x \preceq y$ and $y \in D$ imply $x \in D$.

A preorder \preceq on X is said to be *total* if, for all $x, y \in X$, either $x \preceq y$ or $y \preceq x$.

If (X, \preceq) is a preordered set, then a function $u : (X, \preceq) \longrightarrow (\mathbb{R}, \leq)$ is said to be a *utility function* on (X, \preceq) if it is *order preserving* (i.e., $x \preceq y$ implies $u(x) \leq u(y)$ and $x \prec y$ implies $u(x) < u(y)$ for all $x, y \in X$).

We recall that a *preorder* \preceq on a topological space (X, τ) is said to be *upper semicontinuous* if $L_{<}(x) = \{z \in X : z \prec x\}$ is an open subset of X for all $x \in X$.

Let us now present the definition of a *weakly upper semicontinuous* preorder on a topological space.

Definition 2.1 We say that a preorder \preceq on a topological space (X, τ) is *weakly upper semicontinuous* if it is possible to associate to every $x \in X$ a uniquely determined open decreasing subset O_x of X in such a way that the following conditions are verified:

- (i) $x \notin O_x$ for every $x \in X$;
- (ii) $x \prec y$ implies that $x \in O_y$ for all $x, y \in X$;
- (iii) the family $\{O_x\}_{x \in X}$ is totally ordered by set inclusion.

It is immediate to check that if \preceq is an upper semicontinuous total preorder on a topological space (X, τ) then \preceq is weakly upper semicontinuous (just define $O_x = L_{\prec}(x)$ for all $x \in X$ and consider the fact that the family $\{L_{\prec}(x)\}_{x \in X}$ is totally ordered by set inclusion). Conversely, a weakly upper semicontinuous total preorder \preceq on a topological space is upper semicontinuous. Indeed in this case it must be $O_x = L_{\prec}(x)$ for all $x \in X$ (if there exists $w \in O_x \setminus L_{\prec}(x)$, then we must have that $x \preceq w$ since \preceq is total and therefore we arrive at the contradiction $x \in O_x$ using the fact that O_x is decreasing).

Further, a not necessarily total preorder \preceq on a topological space (X, τ) is weakly upper semicontinuous as soon as there exists an upper semicontinuous utility function $u : (X, \preceq, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ (just define $O_x = u^{-1}(]-\infty, u(x)[)$ for all $x \in X$ in order to immediately verify that properties (i), (ii) and (iii) in Definition 2.1 hold).

3 The main result

The following theorem provides the aforementioned generalization of Rader's theorem to the case of a weakly upper semicontinuous and not necessarily total preorder on a topological space.

Theorem 3.1 *Let \preceq be a weakly upper semicontinuous preorder on a second countable topological space (X, τ) . Then there exists an upper semicontinuous utility function $u : (X, \preceq, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$.*

Proof. Let \preceq be a weakly upper semicontinuous preorder on a second countable topological space (X, τ) . Then denote by $\tau_{\mathcal{O}}$ the topology generated by the family $\mathcal{O} = \{O_x\}_{x \in X}$. Since $\tau_{\mathcal{O}}$ is a linearly ordered subtopology of τ and τ is second countable, we have that also $\tau_{\mathcal{O}}$ is second countable (see Bosi and Herden [2]). Let $\{O_n\}_{n \in \mathbb{N}}$ be a countable base for the topology $\tau_{\mathcal{O}}$ on X consisting of (open) decreasing subsets of X . Since the preorder \preceq on (X, τ)

is weakly upper semicontinuous, we have that for all $x, y \in X$ such that $x \prec y$ there exists $n \in \mathbb{N}$ such that $x \in O_n \subset O_y$, $y \notin O_y$.

Now consider, for every $n \in \mathbb{N}$, the upper semicontinuous increasing function with values in $[0, 1]$ defined as follows:

$$u_n(x) = \begin{cases} 0 & \text{if } x \in O_n \\ 1 & \text{if } x \notin O_n \end{cases}.$$

It is now almost immediate to check that the function

$$u = \sum_{n \in \mathbb{N}} 2^{-n} u_n$$

is an upper semicontinuous utility function for \succsim on (X, τ) . Indeed, it is clear that u is upper semicontinuous since u_n is upper semicontinuous for all $n \in \mathbb{N}$. Further, u is a utility function for the preorder \succsim on X since u is increasing and for all $x, y \in X$ such that $x \prec y$ there exists some $n \in \mathbb{N}$ with $u_n(x) = 0$ and $u_n(y) = 1$. So the proof is complete. \square

From the above considerations we immediately recover Rader's theorem as a corollary of the previous result.

Corollary 3.2 (Rader's theorem) *Let \succsim be an upper semicontinuous total preorder on a second countable topological space (X, τ) . Then there exists an upper semicontinuous utility function $u : (X, \succsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$.*

Finally, Theorem 3.1 can be used in order to provide a new characterization of the existence of an upper semicontinuous utility function for a not necessarily total preorder on a topological space. We recall that a characterization of this kind was presented, for example, by Bosi and Mehta [3, Corollary 3].

Corollary 3.3 *Let \succsim be a preorder on a topological space (X, τ) . There exists an upper semicontinuous utility function $u : (X, \succsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ if and only if there exists a second countable subtopology τ' of τ such that \succsim is weakly upper semicontinuous on (X, \succsim, τ') .*

Proof. The “if” part is a direct consequence of Theorem 3.1 since it is clear that an upper semicontinuous utility function $u : (X, \succsim, \tau') \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ is also an upper semicontinuous utility function on (X, \succsim, τ) due to the fact that τ' is a subtopology of τ .

In order to prove the “only if” part, assume that there exists an upper semicontinuous utility function $u : (X, \succsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ and consider the second countable subtopology τ' of τ that is generated by the family

$$\mathcal{O} = \{O_q = u^{-1}(] - \infty, q])\}_{q \in \mathbb{Q}}.$$

Then define, for all $x \in X$,

$$O_x = u^{-1}(]-\infty, u(x)[).$$

It is simple to check that the preorder \preceq is weakly upper semicontinuous on (X, \preceq, τ') . Indeed, it is clear that O_x is a decreasing subset of X and that O_x excludes x and contains $L_{\preceq}(x)$ for all $x \in X$. Further, we have already observed that the family $\{O_x\}_{x \in X}$ is totally ordered by set inclusion. We just need to show that O_x is τ' -open for all $x \in X$. If $z \in X$ is such that $u(z) < u(x)$ then $z \in O_q \subset O_x$ for all $q \in \mathbb{Q}$ such that $u(z) < q < u(x)$. This consideration completes the proof. \square

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