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PREDICTIVE REGRESSIONS:
A PRESENT-VALUE APPROACH

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ABSTRACT

We propose a latent variables approach within a present-value model to estimate the expected returns and expected dividend growth rates of the aggregate stock market. This approach aggregates information contained in the history of price-dividend ratios and dividend growth rates to predict future returns and dividend growth rates. We find that returns and dividend growth rates are predictable with R-squared values ranging from 8.2% to 8.9% for returns and 13.9% to 31.6% for dividend growth rates. Both expected returns and expected dividend growth rates have a persistent component, but expected returns are more persistent than expected dividend growth rates.

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We propose a latent variables approach within a present-value model to estimate the time series of expected returns and expected dividend growth rates of the aggregate stock market. Specifically, we treat conditional expected returns and expected dividend growth rates as latent variables that follow an exogenously specified time-series model, and we combine this model with a Campbell and Shiller (1988) present-value model to derive the implied dynamics of the price-dividend ratio. Then, using a Kalman filter to construct the likelihood of our model, we estimate the parameters of the model by means of maximum likelihood. We find that both expected returns and expected dividend growth rates are time-varying and persistent, but expected returns are more persistent than expected dividend growth rates. The filtered series for expected returns and expected dividend growth rates are good predictors of realized returns and realized dividend growth rates, with R^2 values ranging from 8.2% to 8.9% for returns and 13.9% to 31.6% for dividend growth rates.

We consider an annual model to ensure that the dividend growth predictability we find is not simply driven by the seasonality in dividend payments.[1] However, using an annual dividend growth series implies that we need to take a stance on how dividends received within a particular year are reinvested. Analogous to the way in which different investment strategies lead to different risk-return properties of portfolio returns, different reinvestment strategies for dividends within a year result in different dynamics of dividend growth rates.[2] We study two reinvestment strategies in detail. First, we reinvest dividends in a 30-day T-bill, which we call cash-reinvested dividends. Second, we reinvest dividends in the aggregate stock market, which we refer to as market-reinvested dividends. Market-reinvested dividends have been studied widely in the dividend-growth and return-forecasting literature.[3] We find the reinvestment strategy to matter for the time-series properties of dividend growth. For instance, the volatility of market-reinvested dividend growth is twice as high as the volatility of cash-reinvested dividend growth. Within our model, we derive the link between the time-series models of dividend growth rates for different reinvestment strategies. This analysis demonstrates that if expected cash-reinvested dividend growth follows a first-order autoregressive process, then expected market-reinvested dividend growth has both a first-order autoregressive and a moving-average component. This result is true even if market returns are white noise. Basically, if last year's market-reinvested dividend was inflated by a high market return, this does not carry through to next year's market-reinvested dividend because market returns are white noise, unlike the underlying cash-reinvested dividend. Reinvesting dividends in the market adds noise to the level of dividends, and an AR(1) plus noise is an ARMA(1,1).

The main assumptions we make in this paper concern the time-series properties for expected returns and expected dividend growth rates, which are the primitives of our model. We first consider first-order autoregressive processes for expected *cash-reinvested* dividend growth and returns; we then derive the implied dynamics for expected *market-reinvested* dividend growth rates. Using this specification, we find that both returns and dividend growth rates are predictable, regardless of the reinvestment strategy. We can reject the null hypothesis that either expected returns or expected growth rates are constant at conventional significance levels. Further, for both reinvestment strategies, we find that expected returns are more persistent than expected growth rates using a likelihood ratio test. Also, innovations to both processes are positively correlated. Finally, while we find that future growth rates are predictable, most of the unconditional variance in the price-dividend ratio stems from variation in discount rates, consistent with, for instance, Campbell (1991). If we decompose the conditional variance of stock returns, we find that cash flow news can account for 34.6% to 49.4% of this variance, discount rate news accounts for 118.4% to 215.3%, and the remainder is attributable to the covariance between cash flow and discount rate news.

Our model, in which we consider low-order autoregressive processes for expected returns and expected dividend growth rates, admits an infinite-order VAR representation in terms of dividend growth rates and price-dividend ratios.[4] Cochrane (2008) rigorously derives the link between our model in Section I and the VAR representation. This insightful analysis also demonstrates why our approach can improve upon predictive regressions that include only the current price-dividend ratio to predict future returns and dividend growth rates. Our latent variables approach aggregates the whole history of price-dividend ratios and dividend growth rates to estimate expected returns and expected growth rates. This implies that we expand the information set that we use to predict returns and dividend growth rates. However, instead of adding lags to a VAR model, which would increase the number of parameters to be estimated, the latent variables approach incorporates the information contained in the history of price-dividend ratios and dividend growth rates while keeping the number of parameters low. As Cochrane (2008) shows, our model introduces moving average terms of price-dividend ratios and dividend growth rates, in addition to the current price-dividend ratios, and we find these moving average terms to be relevant in predicting future returns and in particular dividend growth rates.

The insight that return predictability and dividend growth rate predictability are best studied jointly has been pointed out previously by Cochrane (2007), Fama and French (1988), and Campbell and Shiller (1988). The main contribution of our paper is to model expected returns and expected dividend growth rates as latent processes and use filtering techniques to uncover them. Fama and French (1988) note that the price-dividend ratio is a noisy proxy for expected returns when the price-dividend ratio also moves due to expected dividend growth rate variation. This point is also made by Menzly, Santos, and Veronesi (2004) and Goetzmann and Jorion (1995). However, the reverse argument also holds: the price-dividend ratio is a noisy proxy for expected dividend growth when the price-dividend ratio also moves due to expected return variation. Our framework explicitly takes into account the fact that the price-dividend ratio can move due to expected return variation or expected dividend growth rate variation, with the filtering procedure assigning price-dividend ratio shocks to expected return shocks and/or expected dividend growth rate shocks.

One may wonder why we choose an AR(1) process to model expected *cash-reinvested* dividend growth as opposed to expected *market-reinvested* dividend growth. Each reinvestment strategy for dividends corresponds to a different time-series model for expected returns and expected dividend growth rates. It could be the case that, in fact, expected market-reinvested dividend growth is well described by an AR(1) process. Given that most of the literature on return and dividend growth rate predictability focuses on market-reinvested dividend growth rates, this might seem like a more sensible first pass. In the Internet Appendix, we explore this alternative specification and find that the persistence coefficient of expected market-reinvested dividend growth is negative.[5] By fixing the parameter controlling the persistence of expected dividend growth in the estimation of this specification, and maximizing over all other parameters, we show that the model's likelihood is bimodal. This suggests that a simple first-order autoregressive process for expected market-reinvested dividend growth is too restrictive. We next perform a formal specification test and find that the model in Section I, in which expected cash-reinvested dividend growth is an AR(1) process and expected market-reinvested dividend growth is an ARMA(1,1) process, is preferred over a model in which expected market-reinvested dividend growth is an AR(1) process.

Our paper is closely related to the recent literature on present-value models. In particular, our paper is related to Cochrane (2007), Lettau and Van Nieuwerburgh (2008), Pástor and Veronesi (2003, 2006), Pástor, Sinha, and Swaminathan (2008), Bekaert, Engstrom, and Grenadier (2001, 2005), Burnside (1998), Ang and Liu (2004), and Brennan and Xia (2005). All of these papers provide expressions for the price-dividend or market-to-book ratio. However, in the case of Bekaert, Engstrom, and Grenadier (2001), P/'astor and Veronesi (2003, 2006), Ang and Liu (2004), and Brennan and Xia (2005), the price-dividend ratio is an infinite sum or indefinite integral of exponentially quadratic terms, which makes likelihood-based estimation and filtering computationally much more involved. Bekaert, Engstrom, and Grenadier (2001) and Ang and Liu (2004) estimate the model by means of GMM and model expected returns and expected growth rates as an affine function of a set of additional variables. Brennan and Xia (2005) use a two-step procedure to estimate their model and use long-term forecasts for expected returns to recover an estimate of the time series of (instantaneous) expected returns. Alternatively, Lettau and Van Nieuwerburgh (2008) set up a linearized present-value model and recover structural parameters from reduced-form estimators. They then test whether the present-value constraints are violated. The impose the condition, however, that the persistence of expected returns and the persistence of expected growth rates are equal.[6]

Our paper also relates to Brandt and Kang (2004), Pástor and Stambaugh (2006), and Rytchkov (2007), who focus on return predictability using filtering techniques. We contribute to this literature by focusing on the interaction between return and dividend growth predictability, and by showing that the reinvestment strategy of dividends has an impact on the specification of the present-value model.

Further, our paper extends Chen (2009), who also discusses reinvestment strategies. Chen (2009) points out that because a higher price-dividend ratio predicts either higher future returns or lower future growth rates, the combined signal is blurred once stock returns enter the calculation of dividends. We show two additional implications of reinvesting dividends in the aggregate stock market. First, this reinvestment strategy adds an MA component to dividend growth if cash-reinvested dividend growth is an AR(1) process. Second, the market return at time t forecasts market-reinvested dividend growth from time t to $t+1$. Thus, using our framework, we can explicitly account for the reinvestment strategy of dividends in estimating the time series of expected returns and expected dividend growth rates.

The paper proceeds as follows. In Section I, we present the linearized present-value model. In Section II we discuss the data, our estimation procedure, and the link between the two reinvestment strategies. In Section III we present our estimation results and compare our empirical results to predictive regressions. Section IV discusses hypothesis testing, including the tests for (the lack of) return and dividend growth rate predictability. Section V discusses several additional implications and some robustness checks, and Section VI concludes.

I. Present-value Model

In this section we present a log-linearized present-value model in the spirit of Campbell and Shiller (1988).[7] We assume that both expected returns and expected dividend growth rates are latent variables. We first consider a specification in which both latent variables are an AR(1) process.[8] However, we can allow for higher-order VARMA representations for these variables, some of which we explore below when we study different reinvestment strategies.

Let r_{t+1} denote the total log return on the aggregate stock market,

$$r_{t+1} \equiv \log\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right), \tag{1}$$

let PD_t denote the price-dividend ratio of the aggregate stock market,

$$PD_t \equiv \frac{P_t}{D_t},$$

and let Δd_{t+1} denote the aggregate log dividend growth rate,

$$\Delta d_{t+1} \equiv \log\left(\frac{D_{t+1}}{D_t}\right).$$

We model both expected returns (μ_t) and expected dividend growth rates (g_t) as an AR(1) process,

$$\mu_{t+1} = \delta_0 + \delta_1 (\mu_t - \delta_0) + \varepsilon^{\mu}_{t+1}, \quad (2)$$

$$g_{t+1} = \gamma_0 + \gamma_1 (g_t - \gamma_0) + \varepsilon^{g}_{t+1}, \quad (3)$$

where

$$\mu_t \equiv E_t [r_{t+1}],$$

$$g_t \equiv E_t [\Delta d_{t+1}].$$

The distribution of the shocks ε^{μ}_{t+1} and ε^{g}_{t+1} will be specified shortly. The realized dividend growth rate is equal to the expected dividend growth rate plus an orthogonal shock:

$$\Delta d_{t+1} = g_t + \varepsilon^{d}_{t+1}.$$

Defining $pd_t \equiv \log(PD_t)$, we can write the log-linearized return as

$$r_{t+1} \simeq \kappa + \rho pd_{t+1} + \Delta d_{t+1} - pd_t,$$

where $\overline{pd} = E[pd_t]$, $\kappa = \log\left(1 + \exp\left(\overline{pd}\right)\right) - \rho\overline{pd}$, and $\rho = \frac{\exp\left(\overline{pd}\right)}{1+\exp\left(\overline{pd}\right)}$, as in Campbell and Shiller (1988). If we iterate this equation and use the AR(1) assumptions (2) and (3), it follows that

$$pd_t = A - B_1 (\mu_t - \delta_0) + B_2 (g_t - \gamma_0),$$

where $A = \frac{\kappa}{1-\rho} + \frac{\gamma_0 - \delta_0}{1-\rho}$, $B_1 = \frac{1}{1-\rho\delta_1}$, and $B_2 = \frac{1}{1-\rho\gamma_1}$ (see Section IA.A in the Internet Appendix). The log price-dividend ratio is linear in the expected return μ_t and the expected dividend growth rate g_t. The loading of the price-dividend ratio on expected returns and expected dividend growth rates depends on the relative persistence of these variables (δ_1 versus γ_1). The three shocks in the model, namely, shocks to expected dividend growth rates (ε^g_{t+1}), shocks to expected returns (ε^μ_{t+1}), and realized dividend growth shocks (ε^d_{t+1}), have mean zero and covariance matrix

$$\Sigma \equiv var\left(\begin{bmatrix} \varepsilon^g_{t+1} \\ \varepsilon^\mu_{t+1} \\ \varepsilon^d_{t+1} \end{bmatrix}\right) = \begin{bmatrix} \sigma^2_g & \sigma_{g\mu} & \sigma_{gd} \\ \sigma_{g\mu} & \sigma^2_\mu & \sigma_{\mu d} \\ \sigma_{gd} & \sigma_{\mu d} & \sigma^2_d \end{bmatrix},$$

and are independent and identically distributed (i.i.d.) over time. Further, in the maximum likelihood estimation procedure, we assume that the shocks are jointly normally distributed.

II. Data and Estimation

A. Data

We obtain with-dividend and without-dividend monthly returns on the value-weighted portfolio of all NYSE, Amex, and NASDAQ stocks for the period 1946 to 2007 from the Center for Research in Security Prices (CRSP). We obtain data for the S&P500 index over the same sample period from S&P Index Services. We use these data to construct our annual data for aggregate dividends and prices. We consider two reinvestment strategies. First, we consider dividends reinvested in 30-day T-bills and compute the corresponding series for dividend growth, the price-dividend ratio, and returns. Data on the 30-day T-bill rate also come from CRSP. Second, we consider dividends reinvested in the aggregate stock market (or the S&P500 index) and compute the corresponding series for dividend growth, the price-dividend ratio, and returns. The latter reinvestment strategy, which is commonly used in the return predictability literature, causes annual dividend growth to be highly volatile with an annual unconditional volatility of 12.3% versus a volatility of 6.2% for cash-reinvested dividend growth, as summarized in Table I and plotted in

Figure 1. In the next three subsections we present our estimation procedure. In Section II.B we discuss our estimation procedure for the case in which dividends are reinvested cash. Section II.C discusses the link between the two models. In Section II.D we discuss our estimation procedure for the case in which dividends are reinvested in the market.

[TABLE I AND FIGURE I ABOUT HERE]

B. State-space Representation: Cash-reinvested Dividends

Our model features two latent state variables, μ_t and g_t. We assume that each of these is an AR(1) process. The de-meaned state variables are

$$\begin{aligned} \hat{\mu}_t &= \mu_t - \delta_0, \\ \hat{g}_t &= g_t - \gamma_0. \end{aligned}$$

The model has two transition equations,

$$\begin{aligned} \hat{g}_{t+1} &= \gamma_1 \hat{g}_t + \varepsilon^g_{t+1}, \\ \hat{\mu}_{t+1} &= \delta_1 \hat{\mu}_t + \varepsilon^\mu_{t+1}, \end{aligned}$$

and two measurement equations,

$$\begin{aligned} \Delta d_{t+1} &= \gamma_0 + \hat{g}_t + \varepsilon^d_{t+1}, \\ pd_t &= A - B_1 \hat{\mu}_t + B_2 \hat{g}_t. \end{aligned}$$

Because the second measurement equation contains no error term, we can substitute the equation for pd_t into the transition equation for de-meaned expected returns to arrive at our final system that has just one transition and two measurement equations:

$$\begin{aligned} \hat{g}_{t+1} &= \gamma_1 \hat{g}_t + \varepsilon^g_{t+1}, & (4) \\ \Delta d_{t+1} &= \gamma_0 + \hat{g}_t + \varepsilon^d_{t+1}, & (5) \\ pd_{t+1} &= (1-\delta_1) A + B_2 (\gamma_1 - \delta_1) \hat{g}_t + \delta_1 pd_t - B_1 \varepsilon^\mu_{t+1} + B_2 \varepsilon^g_{t+1}. & (6) \end{aligned}$$

It may be surprising that there is no measurement equation for returns. However, the measurement equation for dividend growth rates and the price-dividend ratio together imply the measurement equation for returns. As all equations are linear, we can compute the likelihood of the model using a Kalman filter (Hamilton (1994)). We then use conditional maximum-likelihood estimation (MLE) to estimate the vector of parameters:

$$\Theta \equiv (\gamma_0, \delta_0, \gamma_1, \delta_1, \sigma_g, \sigma_\mu, \sigma_d, \rho_{g\mu}, \rho_{gd}, \rho_{\mu d}).$$

The details of this estimation procedure are described in Appendix A at the end of this text. We use conditional maximum likelihood to facilitate the comparison with the standard predictive regressions approach. We maximize the likelihood using simulated annealing. This maximization algorithm is designed to search for the global maximum (Goffe, Ferrier, and Rogers (1994)).

C. Reinvesting Dividends and Modeling Growth Rates

We assume that cash-reinvested expected growth rates are an AR(1) process. In this section, we derive the observable implications for market-reinvested dividends. To illustrate why the reinvestment strategy is potentially important for the time-series model of dividend growth rates, we present the following extreme example. Consider the case in which year t prices are recorded on December 31st and year $t+1$ dividends are all paid out one day later, on January 1st. Denote by D_{t+1} the dividends paid out on January 1st. Assuming (for ease of exposition) that the one-year interest rate is zero, the end-of-year cash-reinvested dividends are simply given by D_{t+1}. However, the end-of-year market-reinvested dividends are given by

$$D_{t+1}^{M} = D_{t+1} \exp\left(r_{t+1}\right),$$

where r_{t+1} denotes the aggregate stock market return defined in (1). Even though realized dividend growth rates are strongly dependent on the reinvestment strategy, the aggregate stock market return is not. The correlation between cum-dividend returns where dividends are reinvested in the market, denoted by r_{t+1}^{M}, and cum-dividend returns where dividends are reinvested at the risk-free rate, denoted by r_{t+1}, is 0.9999. As such, from an empirical perspective, these two series can be used interchangeably. The observed market-reinvested dividend growth rates are then given by

$$\Delta d_{t+1}^{M} = \log\left(\frac{D_{t+1}^{M}}{D_{t}^{M}}\right) = \log\left(\frac{D_{t+1}}{D_{t}}\right) + r_{t+1} - r_{t}.$$

This expression suggests that the lagged return on the market is a candidate predictor of market-reinvested dividend growth rates where a high past return predicts low future dividend growth. If the return on the market in period t is high, this increases the dividend growth rate at time t, but it implies a lower dividend growth rate at time $t+1$ relative to cash-reinvested dividends. The expression above further suggests that reinvesting dividends in the market can add substantial volatility to dividend growth rates.

In reality, dividends are paid out throughout the year, for example, at the end of each quarter. To capture the impact of reinvesting dividends in the aggregate stock market, we consider the following reduced-form representation:

$$D^M_{t+1} = D_{t+1}\exp(\varepsilon^M_{t+1}),$$

where D_{t+1} denotes the cash-reinvested dividend. We assume that ε^M_{t+1} is i.i.d. over time with mean zero and standard deviation σ_M. Further, we allow for correlation between ε^M_{t+1} and aggregate market returns:

$$\rho_M \quad = \quad corr\left(\varepsilon^M_{t+1}, \varepsilon^r_{t+1}\right),$$

where $\varepsilon^r_{t+1} \equiv r_{t+1} - \mu_t \approx -B_1\rho\varepsilon^\mu_{t+1} + B_2\rho\varepsilon^g_{t+1} + \varepsilon^d_{t+1}$. In our previous example, in which all dividends are paid out at the beginning of the year, this correlation is close to one and $r_{t+1} \approx \varepsilon^M_{t+1}$. If dividend payments are made throughout the year, we expect a positive value for ρ_M, but not necessarily close to one. Using this model, we can decompose ε^M_{t+1} into a part that is correlated with ε^r_{t+1} and a part that is orthogonal to ε^r_{t+1},

$$\varepsilon^M_{t+1} = \beta_M\varepsilon^r_{t+1} + \varepsilon^{M\perp}_{t+1},$$

where $\beta_M = \rho_M\sigma_M/\sigma_r$, $\sigma_r = \sqrt{var(\varepsilon^r_{t+1})}$ and $\varepsilon^{M\perp}_{t+1}$ is orthogonal to ε^r_{t+1}. To keep the model parsimonious, we assume that all correlation between ε^M_{t+1} and the structural shocks in our model, that is, ε^g_{t+1}, ε^μ_{t+1}, and ε^d_{t+1}, comes via the aggregate market return. The latter assumption implies that $cov(\varepsilon^{M\perp}_{t+1}, \varepsilon^g_{t+1}) = cov(\varepsilon^{M\perp}_{t+1}, \varepsilon^d_{t+1}) = cov(\varepsilon^{M\perp}_{t+1}, \varepsilon^\mu_{t+1}) = 0$.

Given that expected growth rates for cash-reinvested dividends follow an AR(1) process, we can derive the expected growth rate of market-reinvested dividends as follows:

$$\begin{aligned}\Delta d^M_{t+1} \quad &= \quad \Delta d_{t+1} + \varepsilon^M_{t+1} - \varepsilon^M_t \\ &= \quad g_t + \varepsilon^d_{t+1} + \varepsilon^M_{t+1} - \varepsilon^M_t,\end{aligned}$$

where $g_t \equiv E_t[\Delta d_{t+1}]$. This implies that $g_t^M \equiv E_t[\Delta d_{t+1}^M] = g_t - \varepsilon_t^M$. The dynamics of expected market-reinvested dividend growth are therefore given by

$$\begin{aligned} g_{t+1}^M &= g_{t+1} - \varepsilon_{t+1}^M \\ &= \gamma_0 + \gamma_1(g_t - \gamma_0) + \varepsilon_{t+1}^g - \varepsilon_{t+1}^M \\ &= \gamma_0 + \gamma_1(g_t^M - \gamma_0) + \gamma_1\varepsilon_t^M + \varepsilon_{t+1}^g - \varepsilon_{t+1}^M. \end{aligned}$$

This shows that expected market-reinvested dividend growth is not a first-order autoregressive process, but instead an ARMA(1,1) process.

Summary of the model: We can now summarize the model for market-reinvested dividend growth as follows:

$$\begin{aligned} \Delta d_{t+1}^M &= g_t^M + \varepsilon_{t+1}^{dM}, \\ g_{t+1}^M &= \gamma_0 + \gamma_1(g_t^M - \gamma_0) + \gamma_1\varepsilon_t^M + \varepsilon_{t+1}^{gM}, \\ \mu_{t+1} &= \delta_0 + \delta_1(\mu_t - \delta_0) + \varepsilon_{t+1}^\mu, \\ pd_t^M &= A - B_1(\mu_t - \delta_0) + B_2(g_t^M - \gamma_0) + (B_2 - 1)\,\varepsilon_t^M, \end{aligned}$$

where we define

$$\begin{aligned} \varepsilon_{t+1}^{dM} &\equiv \varepsilon_{t+1}^d + \varepsilon_{t+1}^M, \\ \varepsilon_{t+1}^{gM} &\equiv \varepsilon_{t+1}^g - \varepsilon_{t+1}^M, \end{aligned}$$

and recall that $g_t^M = g_t - \varepsilon_t^M$. Market-reinvested dividend growth rates Δd_{t+1}^M are equal to expected market-reinvested dividend growth rates g_t^M plus an orthogonal shock ε_{t+1}^{dM}. This orthogonal shock consists of two parts: one part is due to the unexpected change in the payout of firms, ε_{t+1}^d, and the second part is related to the performance of the stock market during the current year, ε_{t+1}^M. The expected market-reinvested dividend growth rate g_t^M also consists of two parts. The first part, g_t, is driven by the expected change in the payout of firms and the second part, ε_t^M, relates to the performance of the stock market in the previous year. As g_t is an AR(1) process, market-reinvested dividend growth is an ARMA(1,1) process. Finally, the expected return continues to be an AR(1)-process as in the case of cash-reinvested dividends.

The parameters of the covariance matrix are given by

$$
\begin{aligned}
\sigma_d^{M2} &\equiv var(\varepsilon_{t+1}^{dM}) = \sigma_d^2 + \sigma_M^2 + 2\sigma_{dM}, \\
\sigma_g^{M2} &\equiv var(\varepsilon_{t+1}^{gM}) = \sigma_g^2 + \sigma_M^2 - 2\sigma_{gM}, \\
\sigma_{\mu d}^{M2} &\equiv cov(\varepsilon_{t+1}^{\mu}, \varepsilon_{t+1}^{dM}) = \sigma_{\mu d} + \sigma_{\mu M}, \\
\sigma_{\mu g}^{M2} &\equiv cov(\varepsilon_{t+1}^{\mu}, \varepsilon_{t+1}^{gM}) = \sigma_{\mu g} - \sigma_{\mu M}, \\
\sigma_{gd}^{M2} &\equiv cov(\varepsilon_{t+1}^{gM}, \varepsilon_{t+1}^{dM}) = -\sigma_M^2,
\end{aligned}
$$

and σ_{dM}, σ_{gM}, and $\sigma_{\mu M}$ are derived in Appendix A. The correlations are subsequently defined as $\rho_{gd}^M \equiv \sigma_{gd}^M/(\sigma_g^M \sigma_d^M)$, $\rho_{\mu d}^M \equiv \sigma_{\mu d}^M/(\sigma_\mu \sigma_d^M)$, and $\rho_{\mu g}^M \equiv \sigma_{\mu g}^M/(\sigma_\mu \sigma_g^M)$.

D. State-space Representation: Market-reinvested Dividends

We define the two de-meaned state variables as

$$
\begin{aligned}
\mu_t &= \delta_0 + \hat{\mu}_t, \\
g_t &= \gamma_0 + \hat{g}_t.
\end{aligned}
$$

Again, the model has two transition equations,

$$
\begin{aligned}
\hat{g}_{t+1} &= \gamma_1 \hat{g}_t + \varepsilon_{t+1}^g, \\
\hat{\mu}_{t+1} &= \delta_1 \hat{\mu}_t + \varepsilon_{t+1}^{\mu},
\end{aligned}
$$

and two measurement equations,

$$\begin{aligned}\Delta d_{t+1}^M &= \gamma_0 + \hat{g}_t + \varepsilon_{t+1}^d + \varepsilon_{t+1}^M - \varepsilon_t^M,\\ pd_t^M &= A - B_1\hat{\mu}_t + B_2\hat{g}_t - \varepsilon_t^M.\end{aligned}$$

We are now using dividends reinvested in the market to compute the log dividend growth rate and the log price-dividend ratio. As before, we can substitute the price-dividend ratio for one latent variable to arrive at a final system consisting of two measurement equations and one transition equation:

$$\Delta d_{t+1}^M = \gamma_0 + \hat{g}_t + \varepsilon_{t+1}^d + \varepsilon_{t+1}^M - \varepsilon_t^M, \tag{7}$$

$$pd_{t+1}^M = (1-\delta_1)\,A + B_2\,(\gamma_1 - \delta_1)\,\hat{g}_t + \delta_1 pd_t^M - B_1\varepsilon_{t+1}^\mu + B_2\varepsilon_{t+1}^g - \varepsilon_{t+1}^M + \delta_1\varepsilon_t^M, \tag{8}$$

$$\hat{g}_{t+1} = \gamma_1\hat{g}_t + \varepsilon_{t+1}^g. \tag{9}$$

As all equations are still linear, we can compute the likelihood of the model using the Kalman filter, and use conditional MLE to estimate the parameters.

E. Identification

In our model, all but one of the parameters in the covariance matrix are identified.[9] We choose to normalize the correlation between realized dividend growth shocks (ε_{t+1}^d) and expected dividend growth shocks (ε_{t+1}^g) to zero.

III. Results

A. Estimation Results

Table II shows the maximum likelihood estimates of the parameters for the model with cash-reinvested dividends (equations (4) to (6)) as well as the model with market-reinvested dividends (equations (7) to (9)).

[TABLE II ABOUT HERE]

For cash-reinvested dividends, we estimate the unconditional expected log return to be $\delta_0 = 9.0\%$ and the unconditional expected log growth rate of dividends to be $\gamma_0 = 6.2\%$. Further, we find expected returns to be highly persistent, consistent with Fama and French (1988), Campbell and Cochrane (1999), Ferson, Sarkissian, and Simin (2003), and Pástor and Stambaugh (2006), with an annual persistence coefficient (δ_1) of 0.932. The estimated persistence of expected dividend growth rates equals 0.354, which is less than the estimated persistence of expected returns. We test whether this difference is significant with a likelihood ratio test in Section IV. Further, shocks to expected returns and expected dividend growth rates are positively correlated.[10] In our state-space model, we compute the R^2 values for returns and dividend growth rates as (see also Harvey (1989))

$$
\begin{aligned}
R^2_{Ret} &= 1 - \frac{v\hat{a}r\left(r_{t+1} - \mu_t^F\right)}{v\hat{a}r\left(r_t\right)}, \\
R^2_{Div} &= 1 - \frac{v\hat{a}r\left(\Delta d_{t+1} - g_t^F\right)}{v\hat{a}r\left(\Delta d_{t+1}\right)},
\end{aligned}
$$

where $v\hat{a}r$ is the sample variance, μ_t^F is the filtered series for expected returns (μ_t), and g_t^F is the filtered series for expected dividend growth rates (g_t). Alternatively, we can compute the R^2 values within our model as if g_t and μ_t are observed and do not need to be filtered. However, to compare our results to OLS, we use the filtered series because g_t and μ_t are latent processes. The R^2 value for returns is equal to 8.2% and that for dividend growth rates it equal to 13.9%.

For market-reinvested dividends, we find $\delta_0 = 8.6\%$ and $\gamma_0 = 6.0\%$, which are slightly lower than for the cash-reinvested case. Further, we find that the persistence coefficient of expected returns (δ_1) equals 0.957 and the persistence of expected dividend growth rates equals 0.638, indicating, as before, that expected returns are more persistent than expected dividend growth rates. Compared to the case of cash-reinvested dividends, we find a higher correlation $\rho_{\mu g}$ and a higher persistence coefficient γ_1. When we reinvest dividends in the market, a high expected return will lead to a high expected growth rate. This increases the correlation between expected returns and expected dividend growth rates. Further, the part of the expected dividend growth rate that relates to the higher expected return will be persistent, due the high persistence of expected returns. This increases the estimated persistence coefficient of expected dividend growth.

The R^2 values in the case of market-reinvested dividends are:

$$
\begin{aligned}
R^2_{Ret_M} &= 1 - \frac{v\hat{a}r\left(r^M_{t+1} - \mu^F_t\right)}{v\hat{a}r\left(r^M_t\right)}, \\
R^2_{Div_M} &= 1 - \frac{v\hat{a}r\left(\Delta d^M_{t+1} - g^{M,F}_t\right)}{v\hat{a}r\left(\Delta d^M_{t+1}\right)},
\end{aligned}
$$

where $v\hat{a}r$ is the sample variance, μ^F_t is the filtered series for expected returns (μ_t), and $g^{M,F}_t$ is the filtered series for expected dividend growth rates (g^M_t). For returns, we find an R^2 value of 8.9% and for dividend growth rates, we find an R^2 value of 31.6%. Further, the standard deviation of the shock ε^M_t equals 5.4% and the correlation between ε^M_t and the unexpected return on the aggregate market is $\rho_M = 0.59$. If all dividends were paid out at the beginning of the year, ε^M_t would closely resemble the market return and we would expect a standard deviation σ_M equal to that of the aggregate market and a correlation close to one. If all dividend payments were instead paid out at the end of the year, we would expect a value of σ_M close to zero and a correlation close to zero. For intermediate cases, that is, if dividends are paid out throughout the year, as they are in our data set, we find that σ_M and ρ_M above take values in between these two extreme cases. This suggests that this reduced-form representation indeed captures, at least in part, the reinvestment of dividends in the aggregate market.

B. Comparison with OLS Regressions

As a benchmark for our latent variables approach, we also report results from the following predictive OLS regressions:[11]

$$
\begin{aligned}
r_{t+1} &= \alpha_r + \beta_r pd_t + \varepsilon_{t+1}^{r,OLS}, \\
\Delta d_{t+1} &= \alpha_d + \beta_d pd_t + \varepsilon_{t+1}^{d,OLS}.
\end{aligned}
$$

The results are summarized in Table III. For market-reinvested dividends, the return regression has a predictive coefficient of $\beta_r = -0.10$ with an R^2 value of 7.96% and a t-statistic of -2.19, where we use OLS standard errors to compute the t-statistic. The dividend growth rate regression results in a predictive coefficient of $\beta_d = -0.04$, with an R^2 value of 1.56% and a t-statistic of -0.91. The dividend growth rate regression has an insignificant coefficient, which seems to have the wrong sign in the sense that a *high* price-dividend ratio predicts a *low* expected dividend growth rate as opposed to a high expected dividend growth rate.[12] Appendix B shows why our filtering approach can improve upon predictive regressions. In addition to the lagged price-dividend ratio, we use the entire history of dividend growth rates and price-dividend ratios to predict future growth rates and returns:

$$
\begin{aligned}
\Delta d_t &= a_0^d + \sum_{i=0}^{\infty} a_{1i}^d pd_{t-i-1} + \sum_{i=0}^{\infty} a_{2i}^d \Delta d_{t-i-1} + \varepsilon_t^{d*}, \\
r_t &= a_0^r + \sum_{i=0}^{\infty} a_{1i}^r pd_{t-i-1} + \sum_{i=0}^{\infty} a_{2i}^r \Delta d_{t-i-1} + \varepsilon_t^{r*},
\end{aligned}
$$

where the coefficients and innovations (ε_t^{d*} and ε_t^{r*}) are defined in Appendix B. Our filtering approach thus uses more information and aggregates this information in a parsimonious way (see also Cochrane (2008)). The present-value approach we propose, in combination with the Kalman filter, allows us therefore to expand the information set without increasing the number of parameters.

TABLE III ABOUT HERE

For cash-reinvested dividends, the return regression has a predictive coefficient of $\beta_r = -0.10$ with an R^2 value of 8.20% and a t-statistic of -2.32. The dividend growth rate regression results in a predictive coefficient of $\beta_d = -0.01$, with an R^2 value of 0.01% and a t-statistic of -0.91.

We have argued that the reinvestment strategy matters for realized dividend growth and that market-reinvested dividend growth is more volatile than cash-reinvested dividend growth due to the volatility of stock returns. Further, we have argued that apart from this added volatility as a result of the reinvestment return, realized market-reinvested dividend growth can be well described by an ARMA(1,1) process. To further explore this argument we present results for several OLS regressions of market-reinvested dividends. The results are summarized in Table IV. When we include in the regression a constant term and an AR(1) term, we find a negative coefficient that is significant at the 10% level. The R^2 is low and equal to 5.0%. When we estimate an ARMA(1,1) process while controlling for the lagged return (r_{t-1}), we find an AR(1) coefficient of 0.559 and an MA(1) coefficient of -0.569, both statistically significant. Further, the lagged return enters significantly, as expected, with a negative coefficient of -0.378. The R^2 of the latter regression equals 27.8%. In fact, including the lagged return as the sole regressor already leads to an R^2 of 22.3%.[13] The R^2 of 27.8% is still lower than the 31.6% that we achieve by filtering, even though the OLS regressions allow for an additional degree of freedom compared to the specification for dividend growth in the filtering procedure. To increase the R^2 further we need to include the information contained in the price-dividend ratio, which in the OLS regressions above we have not yet explored. However, including the lagged price-dividend ratio in the regression does not lead to a higher R^2, the coefficient is not significant, and the coefficient has the wrong sign, consistent with the OLS regression above where the price-dividend ratio is the only regressor. This is not surprising. When the price-dividend ratio moves both due to expected returns and expected dividend growth rates, the price-dividend ratio is a noisy proxy for expected dividend growth rates. Our filtering approach explicitly takes into account the possibility that the price-dividend ratio also moves due to expected return variation, which allows us to filter out the relevant expected dividend growth rate information and achieve an R^2 for dividend growth equal to 31.6%.

TABLE IV ABOUT HERE

IV. Hypothesis Testing

Our estimates reveal several important properties of expected returns and expected dividend growth rates. In particular, both expected returns and expected growth rates appear to vary over time, expected returns appear to be more persistent than expected growth rates, and both appear to contain a persistent component. In this section, we perform a series of hypothesis tests to establish the statistical significance of these results.

Our likelihood-based estimation approach provides a straightforward way to address these questions using the likelihood ratio (LR) test. Denote the log-likelihood that corresponds to the unconstrained model by $\mathcal{L}^1$. The log-likelihood that follows from estimating the model under the null hypothesis is denoted by $\mathcal{L}^0$. The likelihood ratio test statistic is then given by

$$LR = 2(\mathcal{L}^1 - \mathcal{L}^0),$$

which is asymptotically chi-squared distributed with degrees of freedom equal to the number of constrained parameters. We perform our test for both market-reinvested dividend growth rates and cash-reinvested dividend growth rates.

First, we test for a lack of return predictability. The associated null hypothesis is

$$H_0 : \delta_1 = \sigma_\mu = \rho_{\mu g} = \rho_{\mu d} = 0,$$

where the LR statistic has a χ^2_4-distribution. Under this null hypothesis, all variation in the log price-dividend ratio comes from variation in expected growth rates. In this case, we can uncover expected dividend growth rates through an OLS regression of dividend growth rates on the lagged price-dividend ratio.

Second, we test for the lack of dividend growth rate predictability. The null hypothesis that corresponds to this test for cash-reinvested dividends reads

$$H_0 : \gamma_1 = \sigma_g = \rho_{\mu g} = 0,$$

where the LR statistic follows a χ^2_3-distribution. If dividend growth is unpredictable, we can uncover expected returns through an OLS regression of returns on the lagged price-dividend ratio. For market-reinvested dividends, the null hypothesis is

$$H_0 : \gamma_1 = \sigma_g = \rho_{\mu g} = \sigma_M = \rho_M = 0,$$

where the LR statistic has a χ^2_5-distribution. The absence of dividend growth predictability also requires that σ_M and ρ_M be zero. If not, ε^M_{t+1} correlates with returns and forecasts subsequent dividend growth rates. Under this null hypothesis, all variation in the log price-dividend ratio comes from variation in expected returns.

Third, we test whether the persistence coefficient of expected dividend growth rates equals zero. The null hypothesis that corresponds to this test is

$$H_0 : \gamma_1 = 0,$$

where the LR statistic has a χ^2_1-distribution. The question of whether expected dividend growth rates are time-varying, and what their persistence is, plays an important role in general equilibrium models with long-run risk (Bansal and Yaron (2004) and Hansen, Heaton, and Li (2008)).

Fourth, we test whether the persistence coefficients of expected dividend growth rates and expected returns are equal, which has been assumed by Cochrane (2007) and Lettau and Van Nieuwerburgh (2008) for analytical convenience. The null hypothesis for this test is

$$H_0 : \gamma_1 = \delta_1,$$

where the LR statistic has a χ^2_1-distribution. Under the null hypothesis of equal persistence coefficients, the price-dividend ratio is an AR(1) process, which has been used as a reduced-form model by many authors.[14] Under the alternative hypothesis, the price-dividend ratio is not an AR(1) process, as the sum of two AR(1) processes is an ARMA(2,1) process.

Finally, we test whether the inclusion of ε_t^M adds significantly to the fit of the model. The null hypotheses we test are

$$H_0 : \sigma_M = 0,$$

and

$$H_0 : \rho_M = 0.$$

In both cases, the LR statistic has a χ^2_1-distribution.

We summarize the LR statistics of all these tests in a table in the Internet Appendix. The table also contains the critical values at the 5% and 1% significance levels for the χ^2 with N degrees of freedom. The table shows that all the null hypotheses stated above can be rejected at the 5% level. This suggests, in the context of our model, that both returns and dividend growth rates are predictable. Furthermore, it seems that expected returns are more persistent than expected dividend growth rates, given that (i) we find a lower value of γ_1 than for δ_1 in our unconstrained estimates and (ii) the hypothesis that these two coefficients are equal can be rejected at the 1% level. Finally, the inclusion of the term ε_t^M in our specification for market-reinvested dividend growth seems to add significantly to the fit of the model. The correlation between returns and ε_t^M is significantly different from 0 and positive, lending further support to our interpretation of ε_M as a reduced-form representation for reinvesting dividends in the market throughout the year.

V. Additional Results

A. Comparing the Filtered Series

In this section, we compare the filtered series for both expected returns and expected dividend growth rates for both reinvestment strategies. In Figure 2, we plot the filtered series for μ_t as well as the realized log return when dividends are reinvested in the risk-free rate. We compare these series to the fitted return series from an OLS regression of realized log returns on the lagged price-dividend ratio. The figure shows that the two expected return series are almost identical, consistent with the comparable R^2 values that we find for both approaches. In Figure 3, we plot the same series when dividends are reinvested in the market. In this case, the expected return series of our filtering procedure is different from the OLS series. The filtered series is lower in the 1980s and higher by the end of the 1990s. Consequently, the OLS regression predicts a negative return in the 1990s, whereas the filtered series remains positive.[15]

FIGURE II AND III ABOUT HERE

In Figure 4 we plot the filtered series for g_t when dividends are reinvested in the risk-free rate as well as the fitted value from an OLS regression of realized log dividend growth rates (again reinvested in the risk-free rate) on the lagged price-dividend ratio. The difference between the two series is large. The filtered series picks up much more of the variation in realized dividend growth than the fitted values from the OLS regression do. Further, it appears that expected dividend growth has a positive autocorrelation, but its persistence is not as high as that of the price-dividend ratio. The price-dividend ratio is mainly driven by expected returns, which are more persistent than expected dividend growth rates, as we formally tested in Section IV. In Figure 5 we plot the same series, but now for the reinvestment strategy that reinvests dividends in the market. The filtered series picks up a large fraction of the variation in market-reinvested dividend growth rates. This implies that a substantial fraction of market-reinvested dividend growth is predictable.

FIGURE IV AND V HERE

B. Variance Decompositions

We now derive variance decompositions of both the price-dividend ratio and unexpected returns in both models. The variance decomposition of the price-dividend ratio for cash-reinvested dividends is given by

$$\begin{aligned} var(pd_t) &= B_1^2 var(\mu_t) + B_2^2 var(g_t) - 2B_1 B_2 cov(\mu_t, g_t) \\ &= \frac{(B_1 \sigma_\mu)^2}{1-\delta_1^2} + \frac{(B_2 \sigma_g)^2}{1-\gamma_1^2} - \frac{2B_1 B_2 \sigma_{g\mu}}{1-\delta_1 \gamma_1}. \end{aligned} \quad (10)$$

The first term, $B_1^2 var(\mu_t)$, represents the variation in the price-dividend ratio due to discount rate variation. The second term, $B_2^2 var(g_t)$, measures the variation in the price-dividend ratio due to expected dividend growth rate variation. The last term measures the covariation between these two components. For market-reinvested dividends, the variance decomposition is given by

$$var\left(pd_t^M\right) = B_1^2 var\left(\mu_t\right) + var\left(B_2 g_t^M + (B_2-1)\varepsilon_t^M\right) + 2cov\left(B_1 \mu_t, B_2 g_t^M + \left(B_2 - 1\right)\varepsilon_t^M\right). \quad (11)$$

We include the variance due to ε_t^M as part of expected dividend growth variation. This enhances the comparison with cash-reinvested dividends because we can now also summarize the decomposition using three terms: variation due to discount rates, $B_1^2 var(\mu_t)$, variation due to expected dividend growth variation, $var\left(B_2 g_t^M + (B_2-1)\varepsilon_t^M\right)$, and the covariance between these two components. Table V summarizes the results, where we use sample covariances and we standardize all terms on the right-hand side of (10) and (11) by the left-hand side, so that the sum of the terms is 100%. We find that for both reinvestment strategies, most of the variation in the price-dividend ratio is related to expected return variation.

TABLE V ABOUT HERE

We decompose the variance of unexpected stock returns as in Campbell (1991). In the case of cash-reinvested dividend growth rates, the unexpected return can be written as

$$r_{t+1} - \mu_t = -\rho B_1 \varepsilon^{\mu}_{t+1} + \rho B_2 \varepsilon^{g}_{t+1} + \varepsilon^{d}_{t+1}. \tag{12}$$

We group the last two terms together to decompose the unexpected return into the influence of discount rates, dividend growth variation, and the covariance between the two. In the case of market-reinvested dividend growth rates, the unexpected return can be written as[16]

$$\begin{aligned} r_{t+1} - \mu_t &= -\rho B_1 \varepsilon^{\mu}_{t+1} + \rho B_2 \varepsilon^{g}_{t+1} + \varepsilon^{d}_{t+1} + (1-\rho)\varepsilon^{M}_{t+1} \\ &= -\rho B_1 \varepsilon^{\mu}_{t+1} + \rho B_2 \varepsilon^{gM}_{t+1} + \varepsilon^{dM}_{t+1} + (B_2 - 1)\,\varepsilon^{M}_{t}. \end{aligned} \tag{13}$$

As before, we group all the terms after $-\rho B_1 \varepsilon^{\mu}_{t+1}$ together and compute the influence of discount rates, dividend growth rates, and the covariance between these two components. In the results we report below, we use sample covariances and standardize all terms on the right-hand side of equations (12) and (13) by the left-hand side, so that the sum of the terms is 100%.

The variance decomposition of unexpected returns is quite different across reinvestment strategies. This difference is caused by the difference in the correlation between ε^{μ} and ε^{g}, which is higher in the case of market-reinvested dividends, and the difference in the persistence of expected dividend growth rates, γ_1, which is higher in the case of market-reinvested dividends. Finally, the decomposition of unexpected returns suggests that dividend growth variation plays a significant role in explaining unexpected returns.

C. S&P500 Index

In this section, we repeat our estimation for cash-reinvested dividends using data for the S&P500 index.[17] As before, we use monthly returns with and without dividends to construct an annual dividend and price series. We reinvest dividends paid out throughout the year in the risk-free rate. The estimation results are summarized in Table VI. While the results are generally similar to the results in Table II, a first difference is that the persistence of expected dividend growth rates is slightly higher using the S&P500 data, with a value of 0.485. Second, the correlation between innovations to expected returns and expected growth rates, $\rho_{\mu g}$, is a estimated to be higher with a value of 0.494, and the correlation between innovations to expected returns and unexpected dividend shocks is also estimated to be higher with a value of 0.853. Third, the R^2 values for returns as well as for dividend growth rates are higher using S&P500 data than the corresponding numbers reported in Table II. The R^2 for returns now equals 9.8% and the R^2 for dividend growth is 24.2%.

TABLE VI ABOUT HERE

D. Out-of-Sample Predictability

The R^2 values we have reported so far are in-sample measures of fit. To assess the out-of-sample predictability of our model, we follow Campbell and Thompson (2008) and

Goyal and Welch (2006) and compute the mean squared error,

$$R^2_{OS-Ret} = 1 - \frac{\sum_{t=0}^{T-1} \left(r_{t+1} - \widetilde{\mu}_t\right)^2}{\sum_{t=0}^{T-1} \left(r_{t+1} - \overline{r}_t\right)^2},$$

where $\widetilde{\mu}_t$ is the filtered value of the expected return using only data up until time t to filter and to estimate the parameters of the model. The denominator $\overline{r}_t$ is the historical mean of returns up until time t.

Similarly, we compute the out-of-sample mean squared error for dividend growth,

$$R^2_{OS-Div} = 1 - \frac{\sum_{t=0}^{T-1} \left(\Delta d_{t+1} - \widetilde{g}_t\right)^2}{\sum_{t=0}^{T-1} \left(\Delta d_{t+1} - \overline{\Delta d}_t\right)^2},$$

where $\widetilde{g}_t$ is the filtered value of the expected dividend growth rate using data up until time t to filter and to estimate the parameters of the model. The denominator $\overline{\Delta d}_t$ is the historical mean of dividend growth rates up until time t.

We start our out-of-sample computations in 1972. Using the data between 1946 and 1972 to compute the parameters of the model, we compute the expected return (expected dividend growth rate) for 1973. We compare this prediction with the realized return (dividend growth rate). We then use the data between 1946 and 1973 to compute the parameters of the model and compute predictions for 1974. We proceed in this way up until 2007.

The results are summarized in Table VII. The table shows that our model performs somewhat better than standard predictive regressions in terms of out-of-sample predictability. Over this sample period, our model generates an out-of-sample mean squared error of 1.1% for returns and 5.7% for dividend growth rates. For standard predictive regressions, these numbers are -1.8% for returns and -5.6% for dividend growth rates.

TABLE VII ABOUT HERE

E. Robustness to Log-linearizations

In deriving the expression for the log price-dividend ratio in Section I, we use the approximation to the log total stock return in equation (4). In Binsbergen and Koijen (2010) we study a nonlinear present-value model within the class of linearity-inducing models developed by Menzly, Santos, and Veronesi (2004) and generalized by Gabaix (2009). Because the transition equation is nonlinear in this model, we use nonlinear filtering techniques to estimate the time series of expected returns. More specifically, we use an unscented Kalman filter (Julier and Uhlmann (1997)) and a particle filter. We find that the main results that we report in this paper are not sensitive to the linearization of log total stock returns. Both expected returns and expected growth rates are persistent processes, but expected returns are more persistent than expected growth rates. Innovations to expected returns and expected growth rates are positively correlated, and we find that the filtered series are good predictors of future returns and dividend growth rates.

VI. Conclusion

We propose a new approach to predictive regressions by assuming that conditional expected returns and conditional expected dividend growth rates are latent, following an exogenously specified ARMA model. We combine this model with a Campbell and Shiller (1988) present-value model to derive the implied dynamics of the price-dividend ratio, and use filtering techniques to uncover estimated series of expected returns and expected dividend growth rates. The filtered series turn out to be good predictors for future returns and for future dividend growth rates.

We find that the reinvestment strategy of dividends received within a particular year can have a nonnegligible effect on dividend growth rates. For instance, if dividends are reinvested in the aggregate stock market instead of the T-bill rate, the annual volatility of dividend growth is twice as high. We provide a parsimonious model to relate the two reinvestment strategies. The model shows, for instance, that if cash-reinvested expected growth rates are an AR(1) process, market-invested expected growth rates are an ARMA(1,1) process.

Our likelihood setup allows for straightforward hypothesis testing using the likelihood ratio test. We can statistically reject the hypotheses that returns and dividend growth rates are unpredictable or that they are not persistent. Further, we can reject the hypothesis that expected returns and expected dividend growth rates are equally persistent. Rather, we find that expected dividend growth rates are less persistent than expected returns.

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Appendix A. Kalman Filter

In this section we provide details on the Kalman filtering procedure of our model. The discussion pertains to the general case in which dividends are reinvested in the market; the other models considered in the paper are special cases of this general setup.

We first reformulate the model in standard state-space form. Define an expanded state vector

$$X_t = \begin{bmatrix} \hat{g}_{t-1} \\ \epsilon_t^d \\ \epsilon_t^g \\ \epsilon_t^\mu \\ \epsilon_t^M \\ \epsilon_{t-1}^M, \end{bmatrix}$$

that satisfies

$$X_{t+1} = FX_t + \Gamma \epsilon_{t+1}^X,$$

where

$$F = \begin{bmatrix} \gamma_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \varepsilon_{t+1}^X \begin{bmatrix} \varepsilon_{t+1}^d \\ \varepsilon_{t+1}^g \\ \varepsilon_{t+1}^\mu \\ \varepsilon_{t+1}^M \end{bmatrix},$$

which we assume to be jointly normally distributed.

The measurement equation, which has the observables $Y_t = (\Delta d_t^M, pd_t^M)$, is given by:

$$Y_t = M_0 + M_1 Y_{t-1} + M_2 X_t,$$

where

$$\begin{aligned} M_0 &= \begin{bmatrix} \gamma_0 \\ (1-\delta_1)A \end{bmatrix}, \\ M_1 &= \begin{bmatrix} 0 & 0 \\ 0 & \delta_1 \end{bmatrix}, \\ M_2 &= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & -1 \\ B_2(\gamma_1 - \delta_1) & 0 & B_2 & -B_1 & -1 & \delta_1 \end{bmatrix}. \end{aligned}$$

The Kalman procedure is then given by

$$\begin{aligned}
X_{0|0} &= E[X_0] = 0_{6\times 1}, \\
P_{0|0} &= E[X_t X_t'], \\
X_{t|t-1} &= F X_{t-1|t-1}, \\
P_{t|t-1} &= F P_{t-1|t-1} F' + \Gamma\Sigma\Gamma', \\
\eta_t &= Y_t - M_0 - M_1 Y_{t-1} - M_2 X_{t|t-1}, \\
S_t &= M_2 P_{t|t-1} M_2', \\
K_t &= P_{t|t-1} M_2' S_t^{-1}, \\
X_{t|t} &= X_{t|t-1} + K_t \eta_t, \\
P_{t|t} &= (I - K_t M_2) P_{t|t-1}.
\end{aligned}$$

The likelihood is based on prediction errors (η_t) and their covariance matrix, which is subject to change in every iteration:

$$L = -\sum_{t=1}^{T} \log(\det(S_t)) - \sum_{t=1}^{T} \eta_t' S_t^{-1} \eta_t.$$

Finally, the covariance matrix of the shocks is

$$\Sigma \equiv var\left(\left[\begin{array}{c} \varepsilon_{t+1}^{g} \\ \varepsilon_{t+1}^{\mu} \\ \varepsilon_{t+1}^{d} \\ \varepsilon_{t+1}^{M} \end{array}\right]\right) = \left[\begin{array}{cccc} \sigma_g^2 & \sigma_{g\mu} & \sigma_{gd} & \sigma_{gM} \\ \sigma_{g\mu} & \sigma_\mu^2 & \sigma_{\mu d} & \sigma_{\mu M} \\ \sigma_{gd} & \sigma_{\mu d} & \sigma_d^2 & \sigma_{dM} \\ \sigma_{gM} & \sigma_{\mu M} & \sigma_{dM} & \sigma_M^2 \end{array}\right].$$

Recall that we have assumed that

$$\varepsilon_{t+1}^{M} = \beta_M \varepsilon_{t+1}^{r} + \varepsilon_{t+1}^{M\perp},$$

where $\beta_M = \rho_M \sigma_M / \sigma_r$ and $\sigma_r = \sqrt{var(\varepsilon_{t+1}^r)}$, and

$$\varepsilon_{t+1}^{r} \equiv r_{t+1} - \mu_t \approx -B_1 \rho \varepsilon_{t+1}^{\mu} + B_2 \rho \varepsilon_{t+1}^{g} + \varepsilon_{t+1}^{d}.$$

It follows that

$$\begin{aligned}
\sigma_r^2 &= \sigma_d^2 + \rho^2 B_1^2 \sigma_\mu^2 + \rho^2 B_2^2 \sigma_g^2 - 2\rho B_1 \sigma_{\mu d} - 2\rho^2 B_1 B_2 \sigma_{\mu g}, \\
\sigma_{gM} &= -\beta_M \rho B_1 \sigma_{g\mu} + \beta_M \rho B_2 \sigma_g^2, \\
\sigma_{\mu M} &= \beta_M \sigma_{\mu d} - \beta_M \rho B_1 \sigma_\mu^2 + \beta_M B_2 \rho \sigma_{\mu g}, \\
\sigma_{dM} &= \beta_M \sigma_d^2 - \beta_M \rho B_1 \sigma_{\mu d}.
\end{aligned}$$

We subsequently maximize the likelihood over the parameters:

$$\Theta \equiv (\gamma_0, \delta_0, \gamma_1, \delta_1, \sigma_g, \sigma_\mu, \sigma_d, \rho_{g\mu}, \rho_{gd}, \rho_{\mu d}, \sigma_M, \rho_M).$$

Appendix B. Wold Decomposition

Using the Kalman filter in Appendix A in stationary state ($K_t = K$), we can express the filtered state in terms of historical growth rates and price-dividend ratios:

$$\begin{aligned} X_{t|t} &= X_{t|t-1} + K\left(Y_t - M_0 - M_1 Y_{t-1} - M_2 X_{t|t-1}\right) \\ &= (I - KM_2) X_{t|t-1} + K(Y_t - M_0 - M_1 Y_{t-1}) \\ &= (I - KM_2) F X_{t-1|t-1} + K(Y_t - M_0 - M_1 Y_{t-1}) \\ &= \dots \\ &= \sum_{i=0}^{\infty} \left[(I - KM_2) F\right]^i K (Y_{t-i} - M_0 - M_1 Y_{t-1-i}). \end{aligned}$$

Using the return definition, we have:

$$r_{t+1} = \kappa + \rho pd_{t+1} + \Delta d_{t+1} - pd_t,$$

which implies a representation in terms of price-dividend ratios and returns. The first element of X_t is $\hat{g}_{t-1} = g_{t-1} - \gamma_0$. Hence, it is more natural to think about $X_{t|t-1}$:

$$\begin{aligned} X_{t|t-1} &= F X_{t-1|t-1} \\ &= F \sum_{i=0}^{\infty} \left[(I - KM_2) F\right]^i K (Y_{t-1-i} - M_0 - M_1 Y_{t-2-i}), \end{aligned}$$

implying that the first element of $X_{t|t-1}$ equals $\hat{g}_{t-1|t-1}$, the filtered value of expected growth rates up to time $t-1$. Using the expression for the log price-dividend ratio, we obtain a similar representation for the filtered value of expected returns, $\hat{\mu}_{t-1|t-1}$:

$$\hat{\mu}_{t-1|t-1} = B_1^{-1}\left(pd_{t-1} - A - B_2 \hat{g}_{t-1|t-1}\right).$$

This system represents expected returns and expected growth rates as a function of lagged growth rates and price-dividend ratios. Define $\varepsilon_t^{d*} \equiv \Delta d_t - \gamma_0 - g_{t-1|t-1}$. We obtain:

$$\begin{aligned} \Delta d_t &= \gamma_0 + e_1' X_{t|t-1} + \varepsilon_t^{d*} \\ &= \gamma_0 + e_1' F \sum_{i=0}^{\infty} \left[(I - KM_2) F\right]^i K (Y_{t-1-i} - M_0 - M_1 Y_{t-2-i}) + \varepsilon_t^{d*} \\ &= a_0^d + \sum_{i=0}^{\infty} a_{1i}^d pd_{t-i-1} + \sum_{i=0}^{\infty} a_{2i}^d \Delta d_{t-i-1} + \varepsilon_t^{d*}, \end{aligned}$$

where

$$
\begin{aligned}
a_0^d &= \gamma_0 - e_1' F \sum_{i=0}^{\infty} \left[(I - KM_2) F \right]^i KM_0, \\
a_{1i}^d &= e_1' F K e_1, \text{ if } i = 0 \\
&= e_1' F \left[(I - KM_2) F \right]^{i-1} \left((I - KM_2) FK - KM_1 \right) e_1, \text{ if } i \neq 0, \\
a_{2i}^d &= e_1' F K e_2, \text{ if } i = 0 \\
&= e_1' F \left[(I - KM_2) F \right]^{i-1} \left((I - KM_2) FK - KM_1 \right) e_2, \text{ if } i \neq 0.
\end{aligned}
$$

For returns, we have

$$
r_t = a_0^r + \sum_{i=0}^{\infty} a_{1i}^r pd_{t-i-1} + \sum_{i=0}^{\infty} a_{2i}^r \Delta d_{t-i-1} + \varepsilon_t^{r*},
$$

where

$$
\begin{aligned}
a_0^r &= a_0^d - B_1^{-1} A, \\
a_{1i}^r &= -\frac{B_2}{B_1} a_{1i}^d + \frac{1}{B_1}, \text{ if } i = 0, \\
&= -\frac{B_2}{B_1} a_{1i}^d, \text{ if } i \neq 0, \\
a_{2i}^r &= -\frac{B_2}{B_1} a_{2i}^d.
\end{aligned}
$$

Table I

Summary Statistics of Dividend Growth Rates

The table shows summary statistics for both market-reinvested and cash-reinvested dividend growth rates using data between 1946 and 2007.

	Δd_t^M	Δd_t
Mean	0.0586	0.0611
Median	0.0558	0.0540
Standard Deviation	0.1232	0.0622
Maximum	0.3699	0.2616
Minimum	-0.2912	-0.0579

Table II

Estimation Results

In the second and third columns, we present the estimation results of the present-value model in equations (4) to (6) with cash-reinvested dividends. In the fourth and fifth columns, we present the estimation results of the present-value model in equations (7) to (9) with market-reinvested dividends. The models are estimated by conditional maximum likelihood using data between 1946 and 2007 on dividend growth rates and the corresponding price-dividend ratio. Panel A presents the estimates of the coefficients of the underlying processes (bootstrapped standard errors between parentheses). Panel B reports the resulting coefficients of the present-value model ($pd_t = A - B_1 (\mu_t - \delta_0) + B_2 (g_t - \gamma_0)$). The constants A, B_1 and B_2 are nonlinear transformations of the underlying present-value parameters. Therefore, when interpreting the standard errors, it should be taken into account that the distribution of these constants is not symmetric. In Panel C we report the R^2 values for returns and dividend growth rates.

Panel A: Maximum likelihood estimates

	Cash-reinvested dividends		Market-reinvested dividends	
	Estimate	S.e.	Estimate	S.e.
δ_0	0.090	(0.020)	0.086	(0.039)
γ_0	0.062	(0.011)	0.060	(0.014)
δ_1	0.932	(0.128)	0.957	(0.055)
γ_1	0.354	(0.271)	0.638	(0.170)
σ_μ	0.016	(0.013)	0.016	(0.012)
σ_g	0.058	(0.017)	-	-
σ_g^M	-	-	0.077	(0.015)
σ_d	0.002	(0.022)	-	-
σ_d^M	-	-	0.089	(0.011)
$\rho_{d\mu}$	-0.147	(0.579)	-	-
$\rho_{d\mu}^M$	-	-	-0.344	(0.171)
$\rho_{\mu g}$	0.417	(0.375)	-	-
$\rho_{\mu g}^M$	-	-	0.805	(0.078)
σ_M	-	-	0.054	(0.016)
ρ_M	-	-	0.586	(0.191)

Panel B: Implied present-value model parameters

	Cash-reinvested dividends		Market-reinvested dividends	
A	3.571	(0.421)	3.612	(0.953)
B_1	10.334	(4.088)	13.484	(5.626)
B_2	1.523	(2.001)	2.616	(2.723)
ρ	0.969	-	0.968	-

Panel C: R^2 values

	Cash-reinvested dividends		Market-reinvested dividends	
R^2_{Ret}	8.2%	-	8.9%	-
R^2_{Div}	13.9%	-	31.6%	-

Table III

OLS Predictive Regressions

The table reports the OLS regression results of log returns and log dividend growth rates on the lagged log price-dividend ratio using data between 1946 and 2007. The second and fourth columns report the results using dividends that are reinvested in the aggregate stock market, whereas the third and fifth columns report the results using cash-reinvested dividends. Two asterisks (**) indicates significance at the 5% level, and three asterisks indicates significance at the 1% level.

Dependent Variable	r_t^M		r_t		Δd_t^M	Δd_t
constant	0.4539	***	0.4555	***	0.1814	0.1085
	(0.1537)		(0.1524)		(0.1266)	(0.0645)
pd_{t-1}^M	-0.1023	**	-		-0.0361	-
	(0.0449)		-		(0.0370)	-
pd_{t-1}	-		-0.1020	**	-	-0.0138
	-		(0.0441)		-	(0.0186)
R^2	7.96%		8.20%		1.56%	0.90%
Adj. R^2	6.43%		6.67%		-0.07%	-0.75%

Table IV

Predictive Regressions of Market Reinvested Dividend Growth

The table reports the results for several reduced-form specifications of realized log dividend growth estimated using data between 1946 and 2007. Dividends are reinvested in the aggregate stock market. One asterisk (*) denotes significance at the 10% level, two asterisks indicates significance at the 5% level, and three asterisks indicates significance at the 1% level. In the last specification, the reported constant term is the estimated unconditional mean of dividend growth (γ_0).

Dependent Variable: Δd_t^M						
constant	0.0605	***	0.1007	***	0.0594	***
	(0.0127)		(0.0170)		(0.0146)	
AR(1)	-0.2214	*	-		0.5594	***
	(0.1255)		-		(0.1517)	
MA(1)	-		-		-0.5688	**
	-		-		(0.2080)	
r_{t-1}	-		-0.3717	***	-0.3784	***
	-		(0.0904)		(0.1250)	
R^2	5.01%		22.27%		27.79%	
Adj. R^2	3.40%		20.95%		24.06%	

Table V

Variance Decompositions of the Price-dividend Ratio and Unexpected Returns

Panel A: Decomposition of the price-dividend ratio

Reinvestment strategy	Discount rates	Div. Growth	Covariance
Cash	104.6%	4.6%	-9.2%
Market	117.9%	4.9%	-22.8%

Panel B: Decomposition of unexpected returns

Reinvestment strategy	Discount rates	Div. Growth	Covariance
Cash	118.4%	34.6%	-53.0%
Market	215.3%	49.4%	-164.7%

Table VI

S&P500 Estimation Results for Cash-Reinvested Dividends

We present the estimation results of the present-value model in equations (4) to (6) using cash-reinvested dividend and price data from the S&P 500 index. The model is estimated by conditional maximum likelihood using data between 1946 and 2007 on the dividend growth rate and the price-dividend ratio. Panel A presents the estimates of the coefficients of the underlying processes (bootstrapped standard errors in parentheses). Panel B reports the resulting coefficients of the present-value model ($pd_t = A - B_1 (\mu_t - \delta_0) + B_2 (g_t - \gamma_0)$). These parameters are nonlinear transformations of the original present-value parameters. When interpreting the standard errors, it should be taken into account that the distribution of the coefficients is not symmetric. In Panel C we report the R^2 values for returns and dividend growth rates.

Panel A: Maximum likelihood estimates					
	Estimate	S.e.		Estimate	S.e.
δ_0	0.090	(0.018)	γ_0	0.062	(0.012)
δ_1	0.927	(0.084)	γ_1	0.485	(0.148)
σ_μ	0.013	(0.013)	σ_g	0.046	(0.009)
$\rho_{d\mu}$	0.858	(0.511)	σ_d	0.004	(0.011)
$\rho_{\mu g}$	0.494	(0.195)			
Panel B: Implied present-value model parameters					
A	3.541	(0.392)	ρ	0.968	
B_1	9.716	(3.752)	B_2	1.887	(1.408)
Panel C: R^2 values					
$R^2_{Returns}$	9.8%		R^2_{Div}	24.2%	

Table VII

Out-of-Sample Predictability

We compute the mean squared error of our present-value model and of standard predictive regressions and divide them by the mean squared error generated by the historical mean of returns and dividend growth rates. We present results for cash-reinvested dividends and use the model in equations (4) to (6).

Out-of-sample predictability		
	Predictive regression	Present-value model
Returns	-0.0178	0.0106
Dividend growth rates	-0.0559	0.0576

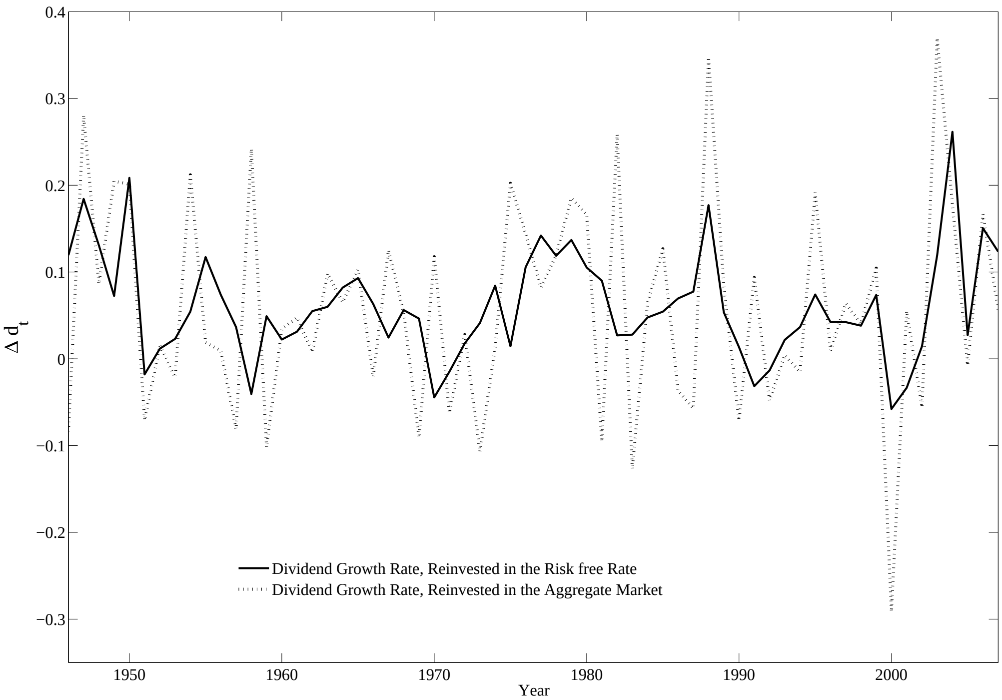

Figure 1. Dividend-growth rates: Reinvesting in either the risk-free rate or in the market. The graph plots the log dividend growth rate for two dividend reinvestment strategies: reinvesting in the risk-free rate and reinvesting in the market.

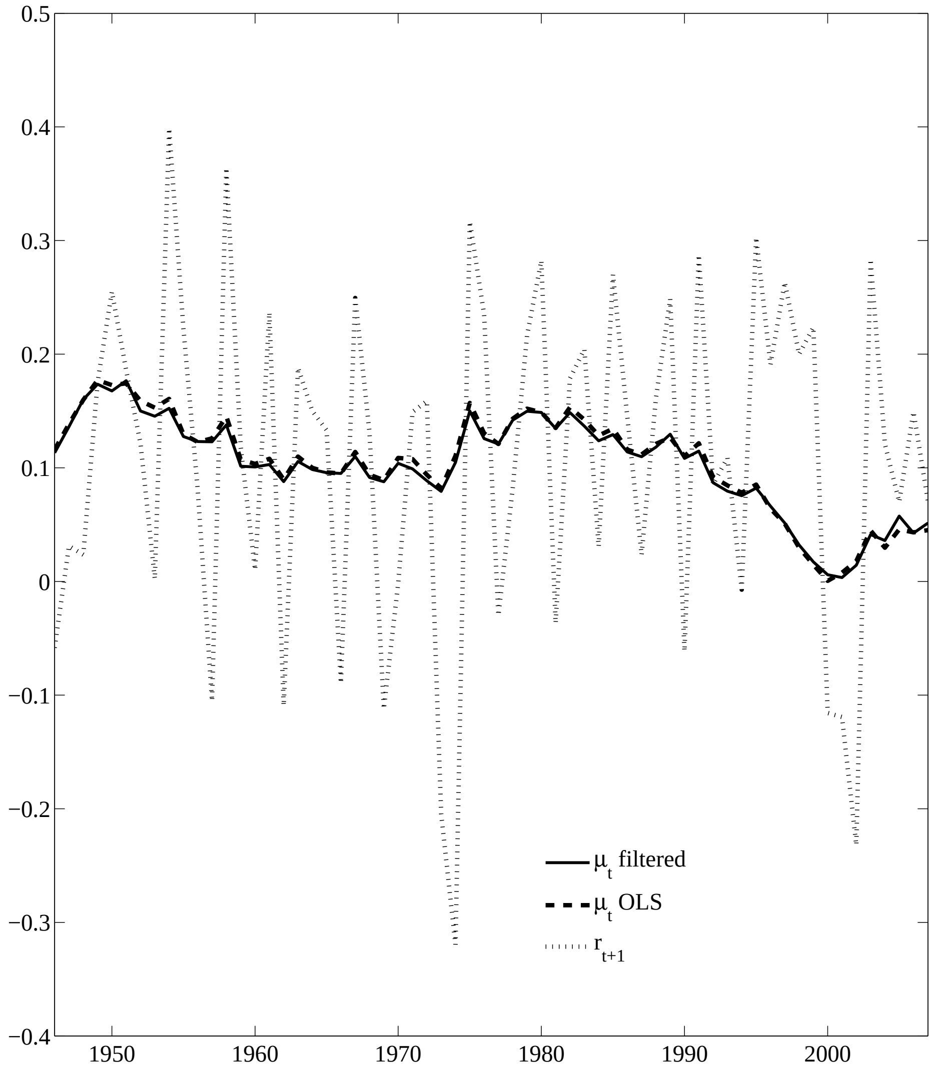

Figure 2. Filtered series for expected returns for reinvesting in the risk-free rate. The graph plots the filtered series of expected returns (μ_t) when dividends are reinvested in the risk-free rate. The graph also plots the realized return r_{t+1} as well as the expected return obtained from an OLS regression of r_{t+1} on the lagged price-dividend ratio.

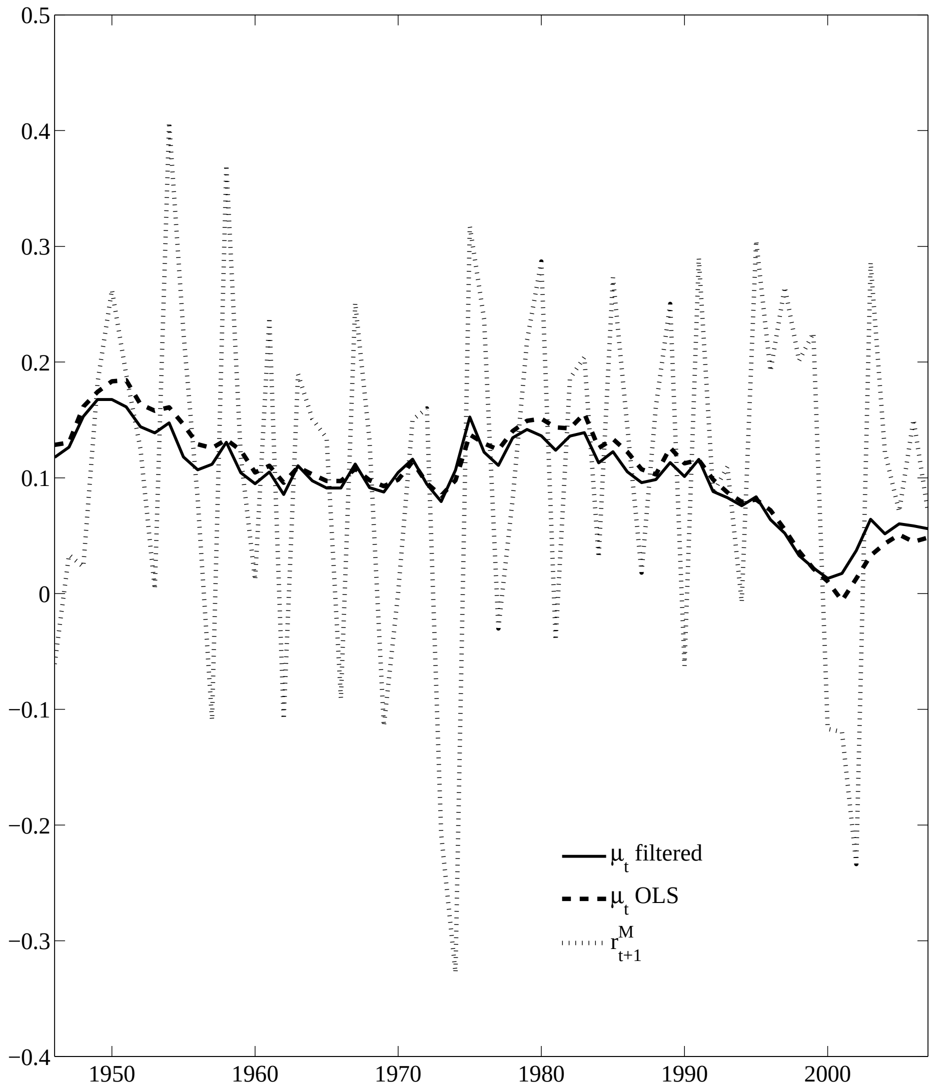

Figure 3. Filtered series for expected returns for reinvesting in the market. The graph plots the filtered series of expected returns (μ_t) when dividends are reinvested in the market. The graph also plots the realized return r^M_{t+1} (again when dividends are reinvested in the market) as well as the expected return obtained from an OLS regression of r^M_{t+1} on the lagged price-dividend ratio.

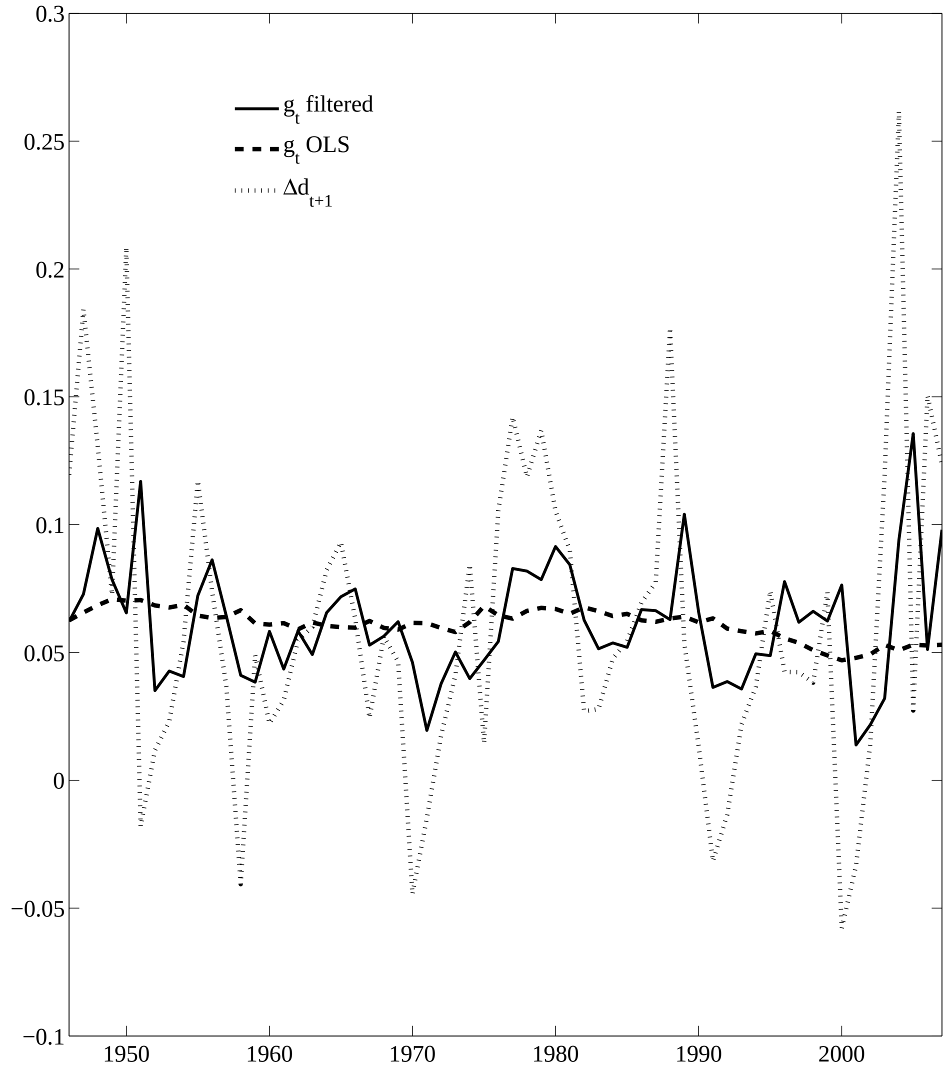

Figure 4. Filtered series for expected dividend growth for reinvesting in the risk-free rate. The graph plots the filtered series of expected dividend growth (g_t) when dividends are reinvested in the risk-free rate. The graph also plots the realized dividend growth Δd_{t+1} (again when dividends are reinvested in the risk-free rate) as well as the expected dividend growth rate obtained from an OLS regression of realized dividend growth Δd_{t+1} on the lagged price-dividend ratio.

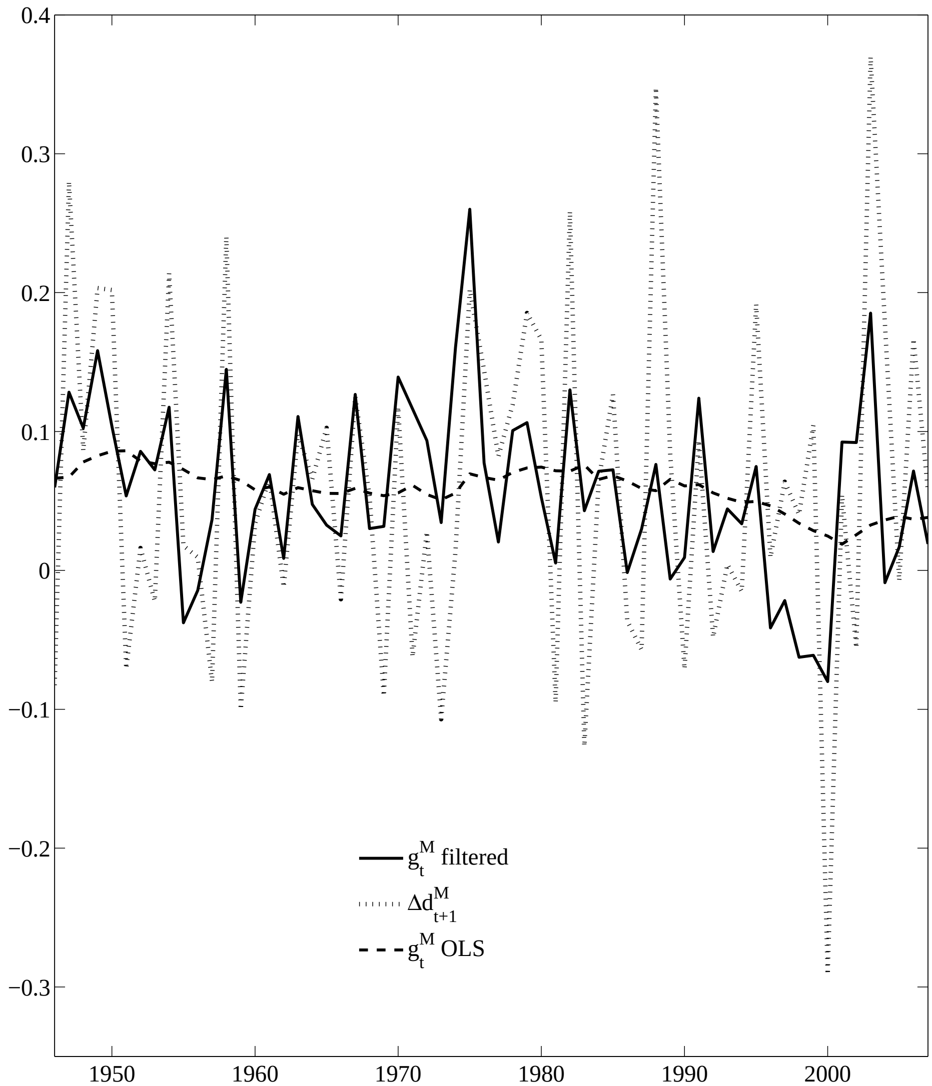

Figure 5. Filtered series for expected dividend growth for reinvestment in the market. The graph plots the filtered series of expected dividend growth (g_t^M) for market-reinvested dividends, the fitted OLS value, where log dividend growth rates are regressed on the lagged price-dividend ratio, and the realized dividend growth rate Δd_{t+1}^M.

Notes

[1]See also Cochrane (1994) and Lettau and Ludvigson (2005).

[2]For instance, the risk-return properties of the end-of-period capital will be different if an investor allocates its capital to stocks instead of Treasury bonds. By the same token, the properties of dividend growth rates depend on the reinvestment strategy chosen for dividends that are received within a particular year.

[3]See for instance Lettau and Ludvigson (2005), Cochrane (2007), and Lettau and Van Nieuwerburgh (2008).

[4]Pástor and Stambaugh (2006) show a similar result for return predictability. They abstract, however, from dividend growth predictability and do not impose the present-value relationship.

[5]The Internet Appendix is available on the Journal of Finance website at http://www.afajof.org/supplements.asp.

[6]Under these assumptions, no filtering is required to uncover expected returns and expected dividend growth rates. We test, using a likelihood ratio test, whether the persistence of expected returns and expected dividend growth rates is equal, and we reject this hypothesis.

[7]Cochrane (1991, 2007), and Lettau and Van Nieuwerburgh (2008) present a version of this model in which it is assumed that expected growth rates are constant, or that expected returns are equally persistent as expected growth rates.

[8]Many authors have argued that expected returns are likely to be persistent, including Fama and French (1988), Campbell and Cochrane (1999), Ferson, Sarkissian, and Simin (2003), and Pástor and Stambaugh (2006). Further, it has been argued that expected dividend growth rates have a persistent component; see, for example, Bansal and Yaron (2004), Menzly, Santos, and Veronesi (2004), and Lettau and Ludvigson (2005).

[9]See also Cochrane (2008) and Rytchkov (2007).

[10]This is consistent with Menzly, Santos, and Veronesi (2004) and Lettau and Ludvigson (2005).

[11]These regressions have been studied widely in the literature. An incomplete list of references includes Cochrane (2007), Lettau and Van Nieuwerburgh (2008), and Stambaugh (1999).

[12]Given that for values of γ_1 smaller than $1/\rho$, the coefficient B_2 is greater than zero, we might expect a positive sign in this regression. However, the price-dividend ratio is a noisy proxy for expected dividend growth rates when the price-dividend ratio also moves due to expected return variation, which can lead to the wrong sign in the regression. Binsbergen and Koijen (2010) show that the price-dividend ratio relates negatively to expected growth rates if $\frac{\sigma_g^2}{1-\gamma_1^2} < \frac{B_1}{B_2}\frac{\sigma_{\mu g}}{1-\gamma_1\delta_1}$.

[13]Fama and French (1988) add up dividends throughout the year, which is close to our cash-reinvested dividend reinvestment strategy. They find that dividend growth rates are positively correlated with past returns. When dividends are reinvested in the market, this induces a negative correlation that more than offsets the positive correlation found by Fama and French (1988).

[14]See for example Stambaugh (1999), Lewellen (2004), and Lettau and Van Nieuwerburgh (2008).

[15]Campbell and Thompson (2008) suggest to impose that the equity risk premium be restricted to always be positive in predictive regressions, which, as they show, enhances the out-of-sample predictability of stock returns.

[16]The first of these two equations illustrates why the quantitative influence of ε_{t+1}^M on unexpected returns is negligible: it is premultiplied by $1-\rho$, which equals 0.032. In the variance decomposition, the contribution of the term is less than 1%.

[17] Ang and Bekaert (2007) also use S&P500 data in their analysis of return predictability.

Internet Appendix for
"Predictive Regressions: A Present-Value Approach"*

IA.A Derivation of the Present-value Model

We consider the model

$$
\begin{aligned}
\Delta d_{t+1} &= g_t + \varepsilon^d_{t+1}, \\
g_{t+1} &= \gamma_0 + \gamma_1 (g_t - \gamma_0) + \varepsilon^g_{t+1}, \\
\mu_{t+1} &= \delta_0 + \delta_1 (\mu_t - \delta_0) + \varepsilon^\mu_{t+1},
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta d_{t+1} &\equiv \log\left(\frac{D_{t+1}}{D_t}\right), \\
\mu_t &\equiv E_t[r_{t+1}], \\
r_{t+1} &\equiv \log\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right).
\end{aligned}
$$

We also define $pd_t = \log(PD_t)$. Now consider the log-linearized return, with $\overline{pd} = E[pd_t]$:

$$
\begin{aligned}
r_{t+1} &= \log(1 + \exp(pd_{t+1})) + \Delta d_{t+1} - pd_t \\
&\simeq \log\left(1 + \exp\left(\overline{pd}\right)\right) + \frac{\exp\left(\overline{pd}\right)}{1 + \exp\left(\overline{pd}\right)}\left(pd_{t+1} - \overline{pd}\right) + \Delta d_{t+1} - pd_t \\
&= \kappa + \rho pd_{t+1} + \Delta d_{t+1} - pd_t.
\end{aligned}
$$

Equivalently, we have

$$
pd_t = \kappa + \rho pd_{t+1} + \Delta d_{t+1} - r_{t+1},
$$

where

$$
\begin{aligned}
\kappa &= \log\left(1 + \exp\left(\overline{pd}\right)\right) - \rho\overline{pd}, \\
\rho &= \frac{\exp\left(\overline{pd}\right)}{1 + \exp\left(\overline{pd}\right)}.
\end{aligned}
$$

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By iterating this equation we find

$$\begin{aligned} pd_t &= \kappa + \rho pd_{t+1} + \Delta d_{t+1} - r_{t+1} \\ &= \kappa + \rho\left(\kappa + \rho pd_{t+2} + \Delta d_{t+2} - r_{t+2}\right) + \Delta d_{t+1} - r_{t+1} \\ &= \kappa + \rho\kappa + \rho^2 pd_{t+2} + \Delta d_{t+1} - r_{t+1} + \rho\left(\Delta d_{t+2} - r_{t+2}\right) \\ &= \sum_{j=0}^{\infty} \rho^j \kappa + \rho^{\infty} pd_{\infty} + \sum_{j=1}^{\infty} \rho^{j-1}\left(\Delta d_{t+j} - r_{t+j}\right) \\ &= \frac{\kappa}{1-\rho} + \sum_{j=1}^{\infty} \rho^{j-1}\left(\Delta d_{t+j} - r_{t+j}\right), \end{aligned}$$

assuming that $\rho^{\infty} pd_{\infty} = \lim_{j\to\infty} \rho^j pd_{t+j} = 0$ (in expectation would suffice for our purpose). Next, we take expectations conditional upon time t:

$$\begin{aligned} pd_t &= \frac{\kappa}{1-\rho} + \sum_{j=1}^{\infty} \rho^{j-1} E_t\left[\Delta d_{t+j} - r_{t+j}\right] \\ &= \frac{\kappa}{1-\rho} + \sum_{j=1}^{\infty} \rho^{j-1} E_t\left[g_{t+j-1} - \mu_{t+j-1}\right] \\ &= \frac{\kappa}{1-\rho} + \sum_{j=0}^{\infty} \rho^{j} E_t\left[g_{t+j} - \mu_{t+j}\right], \\ &= \frac{\kappa}{1-\rho} + \sum_{j=0}^{\infty} \rho^{j} \left(\gamma_0 + \gamma_1^j\left(g_t - \gamma_0\right) - \delta_0 - \delta_1^j\left(\mu_t - \delta_0\right)\right) \\ &= \frac{\kappa}{1-\rho} + \frac{\gamma_0 - \delta_0}{1-\rho} + \sum_{j=0}^{\infty} \rho^{j} \left(\gamma_1^j\left(g_t - \gamma_0\right) - \delta_1^j\left(\mu_t - \delta_0\right)\right) \\ &= \frac{\kappa}{1-\rho} + \frac{\gamma_0 - \delta_0}{1-\rho} + \frac{g_t - \gamma_0}{1-\rho\gamma_1} - \frac{\mu_t - \delta_0}{1-\rho\delta_1}, \end{aligned}$$

which uses

$$E_t\left[x_{t+j}\right] = \alpha_0 + \alpha_1^j\left(x_t - \alpha_0\right),$$

provided that

$$x_{t+1} = \alpha_0 + \alpha_1\left(x_t - \alpha_0\right) + \varepsilon_{t+1}.$$

IA.B Finite-sample Properties

For this section, we analyze the finite-sample properties of our maximum likelihood estimators. We focus on the model for cash-reinvested dividends in Section I. We simulate 1,000 samples with the same number of observations as in the data, starting with a draw from the unconditional distribution of the state variables. We use the point estimates of Table II in the simulation. We subsequently estimate the model for each of the simulated samples. Table IA.I reports the true parameters along with the average, standard deviation, and quantiles of the distribution of 1,000 parameter estimates in Panel A. Panel B reports the correlation between the parameter estimates.

Panel A shows that δ_1 is somewhat downward biased, while γ_1 is upward biased. This corresponds to an upward bias in σ_μ and a downward bias in σ_g. Further, it appears that the correlation between expected returns and unexpected growth rates, $\rho_{\mu d}$, is not estimated precisely. Panel B shows that the estimates for the persistence of expected returns (δ_1) and the persistence of expected growth rates (γ_1) are negatively correlated. Also, we find the persistence parameters and the conditional volatility parameters to be negatively correlated (e.g., δ_1 and σ_μ).

[Table IA.VIII about here]

IA.C Reinvestment Strategy and Model Specification

In the main article, we assume that the conditional expected dividend growth rate is an AR(1) process if dividends are reinvested in the risk-free rate. We subsequently derive the implied dynamics for market-reinvested dividends. We stress again that there is a present-value model for each reinvestment strategy of dividends, reflected in the time-series properties of expected returns and expected dividend growth rates. We now consider the present-value model in equations (4) to (6) for market-reinvested dividends instead of cash-reinvested dividends. That is, we estimate an alternative specification in which expected growth rates of *market-reinvested* dividends are modeled as an AR(1) process. The parameter estimates of this model are presented in Table IA.II. The table shows that the estimated value of γ_1 is not only lower than in the model in which cash-reinvested expected dividend growth is an AR(1) process, but it is in fact estimated to be *negative*. Despite this negative value for γ_1, we still find relatively high R^2 values for both returns and dividend growth rates.

[Table IA.IX about here]

To further explore this evidence of a negative estimated value for γ_1 in this model, we construct a grid of possible levels of γ_1. For each point in the grid, we optimize over the other parameters and record the associated likelihoods and parameter estimates, as shown in Table IA.III. The main results are summarized in Panel A of Figure IA. 1, where we plot the likelihood as a function of γ_1. The picture shows that the likelihood has two peaks, of which one is positive; the other is negative. Panels B and C show plots of the R-squared values for returns and dividend growth rates as a function of γ_1. The R^2 value for dividend growth rates also exhibits a bimodel shape, and perhaps surprisingly, the R^2 value is higher for the positive root than for the negative root of γ_1. Furthermore, the R^2 value for returns is also higher for the positive root of γ_1. The figures therefore illustrate that maximizing R^2 values is not necessarily equivalent to maximizing the likelihood.[18]

[Table IA.X about here]

[Figure IA. 1 about here]

Table IA.I. Finite-sample properties of the maximum-likelihood estimators.

Panel A: Mean, standard deviation, and quantiles

	True	Average	St.dev.	Q(0.10)	Q(0.25)	Q(0.50)	Q(0.75)	Q(0.90)
δ_0	0.090	0.090	0.020	0.067	0.077	0.089	0.101	0.113
δ_1	0.932	0.864	0.128	0.765	0.837	0.887	0.926	0.952
γ_0	0.062	0.061	0.011	0.047	0.054	0.061	0.069	0.076
γ_1	0.354	0.429	0.271	0.218	0.304	0.417	0.565	0.764
σ_μ	0.016	0.025	0.013	0.012	0.016	0.022	0.030	0.041
σ_g	0.058	0.045	0.017	0.017	0.036	0.052	0.057	0.061
σ_d	0.002	0.022	0.019	0.003	0.006	0.014	0.040	0.051
$\rho_{g\mu}$	0.417	0.318	0.375	-0.009	0.254	0.403	0.516	0.605
$\rho_{\mu d}$	-0.147	0.176	0.579	-0.808	-0.180	0.298	0.640	0.860
A	3.612	3.546	0.421	3.135	3.345	3.551	3.771	3.979
B_1	13.484	8.009	4.088	3.870	5.288	7.116	9.709	12.891
B_2	2.616	2.281	2.001	1.268	1.418	1.678	2.212	3.855

Panel B: Correlation matrix

	δ_0	δ_1	γ_0	γ_1	σ_μ	σ_g	σ_d	$\rho_{g\mu}$	$\rho_{\mu d}$
δ_0	1.000	0.008	0.783	0.063	-0.021	0.011	-0.022	-0.021	-0.015
δ_1	0.008	1.000	0.007	-0.175	-0.686	0.235	-0.189	0.061	-0.149
γ_0	0.783	0.007	1.000	0.067	-0.024	0.012	-0.029	-0.034	-0.034
γ_1	0.063	-0.175	0.067	1.000	0.107	-0.280	0.337	0.152	0.072
σ_μ	-0.021	-0.686	-0.024	0.107	1.000	-0.105	0.104	0.105	0.233
σ_g	0.011	0.235	0.012	-0.280	-0.105	1.000	-0.885	0.496	-0.167
σ_d	-0.022	-0.189	-0.029	0.337	0.104	-0.885	1.000	-0.402	0.194
$\rho_{g\mu}$	-0.021	0.061	-0.034	0.152	0.105	0.496	-0.402	1.000	-0.072
$\rho_{\mu d}$	-0.015	-0.149	-0.034	0.072	0.233	-0.167	0.194	-0.072	1.000

The table contains results about the finite-sample properties of our maximum-likelihood estimators. We focus on the model for cash-reinvested dividends in Section I. We simulate 1,000 samples with the same number of observations as in the data, starting with a draw from the unconditional distribution of the state variables. We use the point estimates of Table II in the simulation. We subsequently estimate the model for each of the simulated samples. Panel A reports the true parameters along with the average, standard deviation, and quantiles of distribution of 1,000 parameter estimates. Panel B depicts the correlation between the parameter estimates.

Table IA.II. Estimation results of the model in (4)-(6) using market-invested dividends.

Panel A: Maximum likelihood estimates					
	Estimate	S.e.		Estimate	S.e.
δ_0	0.085	(0.019)	γ_0	0.059	(0.012)
δ_1	0.933	(0.148)	γ_1	-0.324	(0.282)
σ_μ	0.015	(0.014)	σ_g	0.094	(0.026)
$\rho_{d\mu}$	-0.422	(0.276)	σ_d	0.065	(0.022)
$\rho_{\mu g}$	0.905	(0.076)			
Panel B: Implied present-value model parameters					
A	3.596	(0.349)	ρ	0.968	
B_1	10.263	(3.439)	B_2	0.761	(2.883)
Panel C: R^2 values					
R^2_{Ret}	8.6%		R^2_{Div}	18.7%	

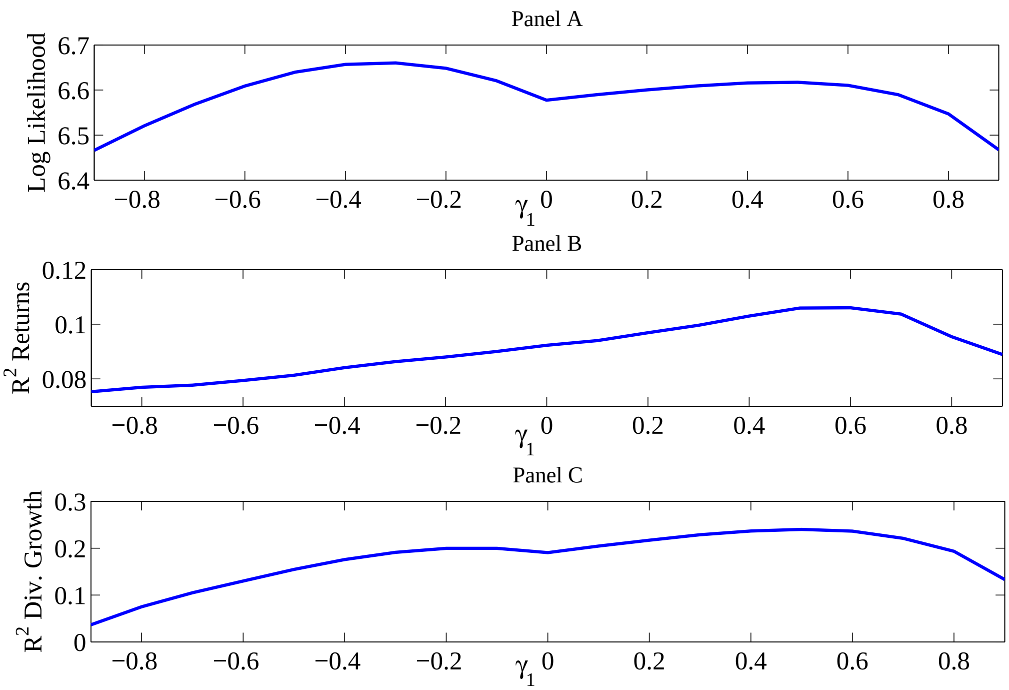
We present the estimation results of the present-value model in equations (4) to (6) using market-reinvested dividend data. The model is estimated by conditional maximum likelihood using data from 1946 to 2007 on the dividend growth rate and the price-dividend ratio. Panel A presents the estimates of the coefficients of the underlying processes (bootstrapped standard errors in parentheses). Panel B reports the resulting coefficients of the present-value model ($pd_t = A - B_1(\mu_t - \delta_0) + B_2(g_t - \gamma_0)$). These parameters are non-linear transformations of the original present-value parameters. When interpreting the standard errors, it should be taken into account that the distribution of the coefficients is not symmetric. In Panel C we report the R^2 values for returns and dividend growth rates.

Table IA.III. Estimating a model with an AR(1)-process for expected growth rates in case of market-invested dividends.

		γ_1																	
	> 0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
δ_0	0.084	0.083	0.083	0.085	0.083	0.085	0.085	0.085	0.086	0.087	0.086	0.087	0.085	0.086	0.085	0.083	0.084	0.083	0.077
δ_1	0.933	0.957	0.954	0.952	0.949	0.9441	0.936	0.931	0.926	0.921	0.918	0.920	0.922	0.924	0.926	0.933	0.944	0.957	0.976
γ_0	0.058	0.059	0.059	0.060	0.059	0.060	0.059	0.059	0.059	0.059	0.058	0.059	0.058	0.058	0.058	0.057	0.058	0.059	0.059
γ_1	0.472	-0.900	-0.800	-0.700	-0.600	-0.500	-0.400	-0.300	-0.200	-0.100	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800
σ_μ	0.022	0.010	0.010	0.011	0.011	0.012	0.014	0.015	0.017	0.019	0.020	0.021	0.021	0.022	0.023	0.022	0.022	0.020	0.017
σ_g	0.053	0.029	0.047	0.062	0.076	0.084	0.089	0.095	0.101	0.105	0.051	0.051	0.053	0.053	0.053	0.052	0.050	0.045	0.040
σ_d	0.107	0.109	0.101	0.092	0.082	0.075	0.069	0.062	0.057	0.053	0.109	0.109	0.108	0.106	0.106	0.106	0.107	0.108	0.109
$\rho_{g\mu}$	0.978	0.857	0.887	0.915	0.944	0.928	0.915	0.902	0.897	0.897	0.953	0.968	0.971	0.974	0.977	0.980	0.982	0.986	0.990
$\rho_{\mu d}$	0.208	0.093	-0.030	-0.158	-0.314	-0.369	-0.403	-0.431	-0.442	-0.442	0.255	0.249	0.237	0.225	0.213	0.200	0.187	0.169	0.142
R^2_{Ret}	0.105	0.075	0.077	0.078	0.079	0.081	0.084	0.086	0.088	0.090	0.092	0.094	0.097	0.100	0.103	0.106	0.106	0.104	0.095
R^2_{Div}	0.242	0.036	0.075	0.105	0.130	0.155	0.176	0.191	0.200	0.200	0.191	0.204	0.217	0.229	0.237	0.240	0.236	0.221	0.193
Log L	6.617	6.466	6.521	6.568	6.609	6.640	6.657	6.660	6.648	6.621	6.578	6.590	6.600	6.609	6.616	6.617	6.611	6.590	6.547

In the column "> 0" we report the maximum likelihood estimates of equations (4) to (6), but using dividends that are reinvested in the market. In the first column, we impose the condition that the persistence coefficient of expected dividend growth rates be positive. We then define a grid for γ_1 between -0.9 and 0.8 with increments of 0.1, and compute for each of these values of γ_1 the likelihood while optimizing over all the other parameters.

Figure IA. 1. Log likelihood and R^2 values as a function of γ_1.



The graph plots the log likelihood and the R^2 values as a function of the persistence of expected dividend growth, γ_1, using the system described in equations (4) to (6) and data where dividends are reinvested in the aggregate market. We define a grid for γ_1 between -0.9 and 0.9 with step size 0.1, and compute for each of these grid points the likelihood and R^2 values of the model while optimizing over all the other parameters.

IA.D Likelihood Ratio Statistics

Table IA.IV. Likelihood-ratio tests.

					Parameters under H_0										
	LR	Sign	Log Lik. H_0	Log Lik. H_a	δ_0	δ_1	γ_0	γ_1	σ_μ	σ_g	σ_d	$\rho_{g\mu}$	$\rho_{\mu d}$	σ_M	ρ_M
Test for lack of return predictability															
Cash-reinv. dividends	**28.67**	***	7.0593	7.5218	0.0936	0	0.0637	0.9900	0	0.0065	0.0659	0	0	-	-
Market-reinv. dividends	**22.37**	***	6.4773	6.8381	0.0926	0	0.0666	0.9936	0	0.0057	0.0780	0	0	0.0607	0.8521
Test for lack of div. growth predictability															
Cash-reinv. dividends	**9.23**	**	7.3730	7.5218	0.0882	0.9261	0.0607	0	0.0164	0	0.0617	0	0.3494	-	-
Market-reinv. dividends	**29.59**	***	6.3609	6.8381	0.0833	0.9514	0.0587	0	0.0104	0	0.1222	0	0.2973	0	0
Test for lack of persistence in expected div. growth															
Cash-reinv. dividends	**8.26**	***	7.3886	7.5218	0.0882	0.9288	0.0610	0	0.0156	0.0605	0.0121	0.2550	0.2636	-	-
Market-reinv. dividends	**5.89**	**	6.7431	6.8381	0.0852	0.9262	0.0584	0	0.0174	0.0619	0.0470	0.7449	-0.2207	0.0501	0.6792
Test whether g_t and μ_t are equally persistent															
Cash-reinv. dividends	**8.60**	***	7.3831	7.5218	0.0867	0.9437	0.0595	0.9437	0.0157	0.0022	0.0617	0.9493	0.3090	-	-
Market-reinv. dividends	**5.10**	**	6.7558	6.8381	0.0782	0.9478	0.0548	0.9478	0.0166	0.0033	0.0764	0.9351	0.3541	0.0631	0.9254
Test for exclusion of ε_M															
Market-reinv. dividends	**11.00**	***	6.6607	6.8381	0.0854	0.9324	0.0591	-0.3253	0.0149	0.0939	0.0635	0.9064	-0.4212	0	0
Test $\rho_M = 0$															
Market-reinv. dividends	**6.93**	***	6.7264	6.8381	0.0853	0.9321	0.0584	0.4419	0.0209	0.0595	0.0633	0.9945	-0.1048	0.0479	0

We report the LR statistics for the tests described in Section IV; in particular, we report the results for the first four tests for the two specifications that we explore in this paper. “Cash” refers to the system in equations (4)to(6) using the data where dividends are reinvested at the risk-free rate. “Market” refers to the system in equations (7)to(9) using the data where dividends are reinvested in the aggregate stock market. Two asterisks (**) denotes that we reject the hypothesis at the 5% level and three asterisks (***) indicates that we reject the hypothesis at the 1% level. The critical values for the χ^2 statistic at the five percent level are given by 3.841, 5.991, 7.815, 9.488, and 11.070 for degrees of freedom equal to 1,2...,5, respectively. These five critical values are equal to 6.635, 9.210, 11.345, 13.277, and 15.086 for the 1% level.