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Farsighted Stable Sets [1]

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Abstract

A coalition is usually called stable if nobody has an immediate incentive to leave or to enter the coalition since he does not improve his payoff. This myopic behaviour does not consider further deviations which can take place after the first move. Chwe (1994) incorporated the idea of a farsighted behaviour in his definition of large consistent set (LCS). In some respects, we propose a different idea of dominance relation based on indirect dominance and on a different concept of belief on moving coalitions' behavior. A notion of stability for a coalitional game is introduced by taking into account the different degree of risk/safety of any player participating in a move. Some results about uncovered sets, internal stability are investigated. By exploiting our dominance and stability concepts, the prisoner's dilemma in coalitional form and its Nash equilibrium are studied. Some examples illustrating the differences between the largest consistent set, our stable set and stable set due to von Neumann and Morgenstern (1947) are presented.

1 Preliminaries

Cooperative behaviour often emerges at a group rather than social level. In many instances we observe the formation of independent and sometime competing groups, teams, clubs, cooperatives (coalitions for short) each of them persecuting the same goal (in turn provision of commodities, raising of public funds, standards of behaviour and so on). Examples of this behaviour are numerous both at micro and macro level: scientific research groups, university departments, consumers associations, firms as organizations, consumption and production cooperatives, industrial districts, international commercial treatises among countries are all instances of volunteer agreements among independent parties that coalesce to obtain a same goal. Once coalitions form, society is partitioned in a coalitional structure. Mathematically speaking, a coalitional game can be represented in an effectiveness form in such a way

$$\Gamma = (N, Z, \{\prec_i\}_{i\in N}, \{\rightarrow_S\}_{S\subseteq N, S\neq\emptyset})$$

[1]We want to thank Marta Biancardi for some helpful criticisms and scientific discussions.

where

- N is the set of players; a nonempty subset $S \subseteq N$ is called coalition.
- Z is the set of all the partitions of N whose elements are disjoint coalitions. Any element of Z is called outcome or coalition structure;
- $\prec_i$ is a strong preference relation defined on Z associated to player i;
- $\rightarrow_S$ is a reflexive relation defined on Z associated to $S \subseteq N$.

The family of preference relations $\{\prec_i\}_{i\in N} :=\prec$ can be replaced by a value function $V : Z \rightarrow \mathbb{R}^N$ whose component $V_i(a)$ denotes the payoff obtained by player i if the coalition structure a is formed. In addition, the family $\{\rightarrow_S\}_{S\subseteq N}$ is called effectiveness relation of the game Γ. The game is played in the following manner: when the game begins, there is a status quo outcome, say a. If the members of a coalition S decide to change the status quo to an outcome b, then the status quo becomes b. This means $a \rightarrow_S b$. From this new status quo b, other coalitions might move to c (i.e. $b \rightarrow_T c$) through T and so forth. If the game reaches an outcome from which no coalition moves, the game ends and this outcome has to be necessarily considered stable. Step by step, any player i could prefer an outcome to another one by using his preference relation $\prec_i$ or his value function V_i. All actions are public. If $a \prec_i b$ for all $i \in S$, we write $a \prec_S b$. We say that a is directly dominated by b, or $a < b$, if and only if there exists a coalition S such that $a \rightarrow_S b$ and $a \prec_S b$.

We recall some definitions. In general, let $<$ be a dominance relation defined on Z such that for any $a, b \in Z,\ a < b$ means that a is dominated by b. Then, the pair $(Z, <)$ is called an abstract system [17]. A subset $V \subseteq Z$ is V-M internally stable set if it is free of inner contradiction (i.e. there do not exist $x, y \in V$ such that $x < y$). A subset $V \subseteq Z$ is V-M externally stable set if it accounts for every alternative that excludes, (i.e. if $x \notin V$ it must be the case that there exists $y \in V$ such that $x < y$). A subset V is a stable set for $(Z, <)$ if it is externally and internally stable. So, any coalitional game in effectiveness form with his direct dominance relation $<$ is an abstract system. A good starting point for predicting possible final outcomes is computing V-M internal stable sets, V-M external stable sets and stable sets with respect to $<$. But, these solutions do not incorporate any idea of farsightedness.

Chwe introduces a new dominance relation in a seminal paper.

Definition 1.1 (Chwe, [8]) *Let a, b two outcomes in Z. We say that a is indirectly dominated by b, or $a \ll b$, if there exist a sequence of outcomes $a = a_0, a_1, ..., a_m = b$ and a sequence of coalitions $S_0, S_1, ..., S_{m-1}$ such that $a_i \rightarrow_{S_i} a_{i+1}$ and $a_i \prec_{S_i} b$ for $i = 0, 1, 2, ..., m-1$.*

In such way, an outcome b is said to dominate indirectly another alternative outcome a if b can replace a in a sequence of moves such that at each move the active coalition prefers b to the alternative it faces at that stage. The indirect dominance captures the idea that coalitions can anticipate other coalitions' actions.

A set $Y \subseteq Z$ is consistent if for any $d \in Z$, $S \subseteq N$ such that $a \rightarrow_S d$, there exists $e \in Y$ where $d = e$ or $d \ll e$ such that $a \not\prec_S e$. Roughly speaking, an outcome a belongs to a consistent set Y if whatever deviation some coalition S could make from a, S would reach another outcome $e \in Y$ after some further right deviations; at least, one deviating player $i \in S$ is worse off than he is on a. Conversely, an outcome $a \notin Y$ if there exists at least one coalition S deviating from a such that all the players in S are better off but not equal at the first deviation in y and at every outcomes in Y indirectly dominating the first deviation. If Y is a consistent set and $a \in Y$, the interpretation is not that a is a final outcome but that it is possible for a to be a final outcome. If an outcome $b \notin Y$, the interpretation is that b cannot be a final outcome since there is no consistent story in which b could be so. One of the drawbacks of this idea consists in having been given recursively. In fact, the subset Y is consistent if for any $a \in Y$, one can find out another blocking outcome $e \in Y$ on which some of the initial deviators from a are worse off than they are on a. But, in its turn, $e \in Y$ if there exists another blocking outcome $f \in Y$ on which some of the initial deviators from f are worse off than on e and so forth. Many consistent sets could be in a game in effectiveness form. But, fortunately, the following uniqueness result holds. If N and Z are finite sets and $\prec_i$ is irreflexive, then $LCS(\Gamma) \neq \emptyset$. In addition, it has the external stability property. Moreover, it can also be computed in a finite number of steps by the following iterative mapping algorithm

- $Z_1 = f(Z)$
- $Z_2 = f(Z_1) = f(f(Z)) = f^2(Z)$
- $\cdots$
- $Z_i = f(Z_{i-1}) = \overbrace{(f \circ f \circ \cdots \circ f)}^{i\,\text{times}}(Z)$

then there exists an integer j such that

$$Z_{j+1} = Z_j \neq \emptyset$$

(and this happens in a finite number of steps) then

$$LCS(\Gamma) = Z_j$$

An extension of Chwe's theorem for an uncountable set of outcomes can be found in [21]. This idea of consistent stability has some shortcomings.

Stability of some outcomes could be based on some incredible final outcome. Bhattacharya gets rid of this shortcoming in [6]. The central idea is that if a coalition blocks an outcome by another, then the blocking outcome itself must not be a dominated one. From any status a, if a coalition conceives of blocking it by moving to another outcome b, then it must take into account not all possible paths starting from b and ending with a similar outcome but only ones starting from b and ending with credible outcomes. He introduces a similar notion of dominance with the additional desirable property of credibility. Mathematically speaking, we say that an outcome a is credibly dominated by another one b with respect to a subset Y means that $a \ll b$ and there exists no $c \in Y$ such that $b \ll c$. He constructs a credible consistent set and the largest credible consistent set.

Mauleon and Vannetelbosch, in [16], criticize the solution concept of the largest consistent set that may be too inclusive in some cases. They refine it by assuming that coalitions should contemplate the possibility to end with a positive probability at any coalition structure $a \in Z$ which dominates the first deviation d. They called the set obtained the largest cautious consistent set (LCCS) under this criteria. An outcome a is stable if whatever some coalition S can deviate from it, it is possible to find just an itself stable outcome after some further right deviations (including the first deviation) on which one player $i \in S$ deviating is worse off but not equal at least. Conversely, an outcome is unstable if there exists at least one coalition S deviating from a and for every stable outcomes coming from any possible right deviation (including the first deviation), all the players in the coalition S are better off or equal.

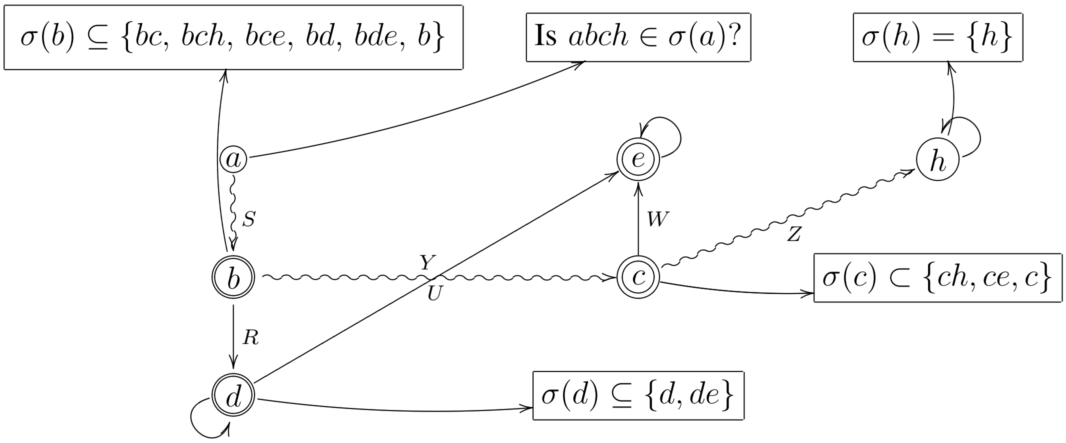
Another criticism to the largest consistent set is done by Xue in [22]. In fact, an outcome a is indirectly dominated by b if there are some coalitions encouraged to move towards the final outcome without predicting about the behaviour of the intermediate coalitions along this path. In fact, indirect dominance does not incorporate any chance that the intermediate coalitions could decide to deviate toward other outcomes more profitable than b for them.

Consider a coalitional game in effectiveness form as an oriented graph. The element ab will be an arc if and only if there exists $S \subseteq N$ such that $a \rightarrow_S b$. We denote by Π_a all the paths reachable starting from a. A path $\alpha \in \Pi_a$ will be a subset of outcomes linked by various arcs starting from a with a terminal node $h = t(\alpha)$. We say that a path α is preferred by player i to β if and only if $t(\alpha) \prec_i t(\beta)$. A standard of behaviour (SB) σ for the situation with perfect foresight is mapping that assigns to every $a \in Z$ a subset $\sigma(a) \subseteq \Pi_a$. In this context, we present some definitions.

Definition 1.2 (Definition of OSSB, [22]) *A standard behaviour (SB) σ for the situation with perfect foresight is optimistic stable if* $\forall a \in Z$, $\alpha \in \Pi_a \setminus \sigma(a)$ $\exists b \in \alpha$, $S \subset N$, $c \in Z$ *such that* $b \rightarrow_S c$ *and* $\exists \beta \in \sigma(c) \, : \, \alpha \prec_S \beta$.

Definition 1.3 (Definition of CSSB, [22]) *A standard behaviour (SB) σ for the situation with perfect foresight is conservative stable if* $\forall a \in Z$, $\alpha \in \Pi_a \setminus \sigma(a)$ $\exists b \in \alpha$, $S \subset$

N, $c \in Z$ *such that* $b \to_S c$ *and* $\forall \beta \in \sigma(c) \,:\, \alpha \prec_S \beta$.



In the scheme above, our question is $abch \in \sigma(a)$? Let a be a node on $abch$, b its unique deviation and suppose $\sigma(b)$ known. If $h = t(abch) \succ_S t(l)\ \forall l \in \sigma(b)$, then a passes the test in conservative sense. If $\exists l \in \sigma(b)\ h = t(abch) \succ_S t(l)$, then a passes the test in optimistic sense. Repeat the same test for any node b, c, h on the path $abch$ for any deviation starting from these nodes by assuming to know $\sigma(c)$, $\sigma(d)$, $\sigma(e)$ and $\sigma(h)$. If b, c, h pass the previous tests, we can say that the path $abch \in \sigma(a)$. By repeating this test for the remaining paths in Π_a like a, ab, abc, $abce$, abd and $abde$, we can compute $\sigma(a) \subseteq \Pi_a$. At the same time, we can extend this algorithm on all the nodes of the graph in diagram above. So, we can compute the values of σ on all the nodes as a mapping satisfying the stability property. But, how can we compute $\sigma(a)$ in the previous graph if we do not know $\sigma(b)$? The technique described above provides a computable result if we start the computation of σ on the terminal nodes of the graph (i.e. e, h) by coming up along the tree for computing the standard behaviour on the remaining nodes. In addition, stables outcomes related to an OSSB/CSSB σ for the situation with perfect foresight are given by $E_\sigma = \{a \in Z \,|a \in \sigma(a)\}$.

Other kind of static solutions can be found in [4]. An interesting paper [5] imposes some axiomatic constraints on any idea of stability solution for coalitional games in effectiveness form. Other kinds of solutions based on an ongoing dynamic process with payoffs generated as coalition form disintegrate or regroup (PCF equilibrium) [14]. In this paper, equilibrium PCFs are connected with multiple absorbing states to the largest consistent set. Another attempt to remedy the shortcomings of existing solution concept and to identify the conseguences of common knowledge of rationality and farsightedness is in [13]. Herings and others propose to apply exstensive-form rationalizability to the framework of social enviroments. The set of socially rationalizable outcomes coincides with the set of outcomes which are rationalizable in a finite well posed multistage game approaching to an infinite multistage game. They prove that the set of rationalizable outcomes is nonempty. Equibrium based on binding agreements are studied in a seminal paper [18] in a context where the payoff to each player depends on the actions of all other players. Farsighted behaviour is studied in the setting of hedonic games where individual's preferences depend solely on the composition of the coalition they belong to [9]. Regarding stability set introduced in a seminal paper [19] as a solution of the voting paradox, Chakravorti, in [7], underlines that the voters are myopic in the sense that they ignore far-sightedness on the part of others and voters look only one step ahead and do not consider events arbitrarily far ahead. In [20], relationship between largest consistent set and stable set in prisoner's dilemma is studied intensively. They underline that the two notions produce completely different outcomes in the mixed extension of prisoner's dilemma.

In section 2, an overview and some examples on the problem are given. In Section 3, we introduce a new idea of dominance relation called believable-path farsightedness which includes a backwards induction proceeding whatever any related idea of stability could predict in terms of final outcomes. Then, we introduce some other dominance relations based on technique. In Section 4, we define a N-vector α for measuring the risk/safety degree of an individual player moving in a coalition. According to this α, we present α-stable sets and main α-stable sets. In Section 5, we study relationship between our concept of stability and some V-M stable sets, uncovered sets. In section 6, we introduce the prisoner's dilemma as a game in effectiveness form in two main setting. Then, we compute the main α-stable sets, the largest consistent set and V-M stable sets. In section 6, some criticisms regarding our definitions and new future research trends are presented.

2 An overview on the problem

In some respects, the idea which is far away any doubts is that modelling stability and farsightedness independently could lead to results which are not comparable one to each other. According to the aforesaid item, the key point is to stress differences between processes of rationalizability between players and coalitions and processes trough which any coalitions can not consider a move from an outcome a convenient move. A coalition moving from an outcome $a \in Z$ to its immediate deviation thinks that some other coalitions can move from its first deviation to another outcome and so forth; till reaching the final outcome b according to Chwe's definition. So, S has a farsighted behavior since it thinks that its final outcome is b. But, it is not so farsighted since he thinks that the intermediate coalitions $S_0,\ S_1, \cdots, S_{m-1}$ are so myopic to predict b as the unique possible outcome for them. The deviating coalition S does not think that the intermediate coalitions could move according to the same idea of farsightedness used by itself. Roughly speaking, it thinks that intermediate coalitions are in a such way more myopic than itself. We can conclude that all the coalitions are farsighted but all the coalitions thinks that other coalitions are not so myopic but not so farsighted as itself. We give an

Table 1: Indirect and Direct Dominances

$a \not\ll a$	$b \not\ll b$	$c \not\ll c$	$d \not\ll d$	$e \not\ll e$	$f \not\ll f$	$g \not\ll g$
$a < b$	$b < c$	$c < d$	$d < e$	$e \not< f$	$f < g$	$g \not\ll a$
$a \not\ll c$	$b \not\ll d$	$c \ll e$	$d \not\ll f$	$e \ll g$	$f \not\ll a$	$g \not\ll b$
$a \not\ll d$	$b \ll e$	$c \not\ll f$	$d \not\ll g$	$e \not\ll a$	$f \not\ll b$	$g \not\ll c$
$a \not\ll e$	$b \not\ll f$	$c \not\ll g$	$d \not\ll a$	$e \not\ll b$	$f \not\ll c$	$g \not\ll d$
$a \not\ll f$	$b \not\ll g$	$c \not\ll a$	$d \not\ll b$	$e \not\ll c$	$f \not\ll d$	$g \not\ll e$
$a \not\ll g$	$b \not\ll a$	$c \not\ll b$	$d \not\ll c$	$e \not\ll d$	$f \not\ll e$	$g \not\ll f$

example which put in evidence a strange solution prescribed by the largest consistent set.

Example 2.1 *Let* $\{1, 2, 3\}$ *and* $\{a, b, c, d, e, f, g\}$ *be three players and the set composed by seven outcomes. Suppose that the game in effectiveness form is described as in the scheme. Every outcome is identified with a triple which describes payoff for every player whenever the associated outcome is formed.*

$$\begin{tikzcd}
\overset{a}{(1, \frac{3}{4}, 0)} \arrow[r, "\{1\}"] & \overset{b}{(\frac{3}{2}, 1, 1)} \arrow[r, "\{2\}"] & \overset{c}{(\frac{1}{2}, 2, \frac{1}{2})} \arrow[r, "\{1{,}3\}"] & \overset{d}{(\frac{9}{8}, \frac{1}{2}, 1)} \arrow[d, "\{3\}"] \\
& \overset{g}{(\frac{4}{5}, 3, \frac{3}{4})} & \overset{f}{(\frac{1}{2}, 2, \frac{1}{4})} \arrow[l, "\{2\}"'] & \overset{e}{(\frac{3}{4}, \frac{5}{2}, 2)} \arrow[l, "\{1\}"']
\end{tikzcd}$$

We show direct and direct dominances in Table 2.

the largest consistent set is $\{b, d, g\}$. If player 1 is not farsighted, he should think that a move from a to b is good since $1 = V_1(a) < V_1(b) = \frac{3}{2}$. So, for a not farsighted player 1, a is considered unstable. But, according to Chwe's indirect dominance, a is not a possible final outcome notwithstanding players are farsighted. In fact, suppose that player 1 moves from a to b. So c, e are the only outcomes such that $b \ll c$, $b \ll e$. But, $1 = V_1(a) > V_1(c) = \frac{1}{2}$ and $1 = V_1(a) > V_1(e) = \frac{3}{4}$. We focus our attention on e: e is not a possible final outcome. In fact, suppose that player 1 deviates from e to f. So, $\frac{3}{4} = V_1(e) > V_1(f) = \frac{1}{2}$. But, in its turn, f is not a possible final outcome since player 2 prefers to move from f to g since $2 = V_2(f) < V_2(g) = 3$. In addition, it is true that $f \ll g$ but $\frac{3}{4} = V_1(e) < V_1(g) = \frac{4}{5}$. So, e is not be considered a final outcome. We focus our attention on c: e is not a possible final outcome. In fact, taking d as the first deviation from c, note that $\frac{1}{2} = V_3(c) < V_3(d) = 1$; e is the only one such that $d \ll e$. But, e is not a possible final outcome as shown above. Finally, a is not a possible final outcome. Roughly speaking, a is unstable. According to this scheme, being so much farsighted (in Chwe's meaning) is analogous to be not farsighted at all !!

Suppose that player 1 is farsighted in a different way. Suppose that he is able to see one shot-deviation in its turn and predict successive move by the next moving coalition and so on, by taking into account that he could move again along the game. If player 1 thinks in this simple and common way: "If I deviate from a to b, player 2 would deviate from b to c since $1 = V_2(b) < V_2(c) = 2$. Then, coalition $\{1, 3\}$ would deviate from c to d since $\frac{1}{2} = V_1(c) < V_1(d) = \frac{9}{8}$ and $\frac{1}{2} = V_3(c) < V_3(d) = 1$. Then, player 3 would deviate from d to e since $1 = V_3(d) < V_3(e) = 2$. Then, I should enforce f notwithstanding I get a loose since player 2 leads me to g since $2 = V_2(f) < V_2(g) = 3$ where I should be better than I am on e. But, g is worst off for me than a, so I prefer to remain at a." So, a can be considered stable. According to his belief, player 1 ends in g which is in the largest consistent set.

Another problem is related to the fact that largest consistent set is hardly related to a complete pessimistic belief of a moving coalition according to the spirit of social situations [10]. Let be the following example.

Example 2.2 *Let $\{1, 2\}$ and $\{a,\, b,\, c,\, d,\}$ be two players and the set composed by four outcomes. Suppose that the game in effectiveness form is described as in the scheme.*

$$
\begin{tikzcd}
\overset{a}{(1,2)} \arrow[d, "\{1\}"'] & \\
\overset{b}{(2,0)} \arrow[r, "\{2\}"] \arrow[d, "\{2\}"'] & \overset{c}{(0,1)} \\
\overset{d}{(10^{10},1)} &
\end{tikzcd}
$$

Note that LCS=$\{a, c, d\}$. In this case, a is a bit strange solution prescribed by largest consistent set. In fact, after the first deviation from a to b, player 2 can deviate to c or d. But, both deviations are credible deviations according to Bhattacharia' s spirit. But, $a \notin \sigma(a)$ with an OSSB σ. Then, a is not stable according to an optimistic standard behavior with perfect foresight. But, player 1 has an earn in d enormously greater than the loose he receives in c. But, an OSSB can fail in putting this optimistic behavior in evidence as shown in this example.

Example 2.3 *Let $\{1, 2\}$ and $\{a,\, b,\, c,\, d,\}$ be two players and the set composed by four outcomes. Suppose that the game in effectiveness form is described as in the scheme.*

$$\begin{tikzcd}
\overset{a}{(1,0)} \arrow[d, "\{1\}"'] & \\
\overset{b}{(2,0)} \arrow[r, "\{2\}"] \arrow[d, "\{1\}"'] & \overset{c}{(0,2)} \\
\overset{d}{(10^{10},1)} &
\end{tikzcd}$$

Note that LCS=$\{a, c, d\}$. In this case, a is a bit strange solution prescribed by largest consistent set. In fact, after the first deviation from a to b, player 2 can deviate to c or d. But, both deviations are credible deviations according to Bhattacharia' s spirit. But, $a \in \sigma(a)$ with any σ OSSB/CSSB since $\sigma(b) = \emptyset$. Then, a is stable according to any standard behavior with perfect foresight. But, player 1 continues to have an earn in d enormously greater than the loose he receives in c.

3 New dominance relations

In this section, we define our concept of dominance between outcomes on Z by taking into account shortcomings shown in Example 2.1. The idea is based on backwards induction along the path joining two outcomes indirectly.

Definition 3.1 (path-believable farsighted dominance) *Given two outcomes a, b, we say that a is believable-path dominated by b if and only if there exists a chain of coalitions $S_0, S_1 \dots S_{m-1}$ and a chain of outcomes $a_0, a_1 \dots a_m$ such that*

$$a = a_0 \rightarrow_{S_0} a_1 \rightarrow_{S_1} a_2 \dots\dots a_{m-1} \rightarrow_{S_{m-1}} a_m = b$$

$$\emptyset \neq B_h \subseteq P_{h+1} \qquad h = 0 \dots m-1$$

or

$$\exists\, h \in \{0, \dots, m-1\} \;:\; B_h = \emptyset \Rightarrow B_{h-1} \subseteq P_k \qquad k = h \dots m \tag{3.1}$$

where

$$B_0 = \{c \in Z \mid b \ll c\} \tag{3.2}$$

$$B_h = \{c \in Z \mid k \ll c \quad \forall\, k \in B_{h-1}\} \qquad h = 1 \dots m-1 \tag{3.3}$$

$$P_h = \{a \in Z \mid V_i(a) > V_i(a_{m-h}) \quad \forall i \in S_{m-h}\} \qquad h = 1 \dots m. \tag{3.4}$$

We denote it by $a \ll_{pbf} b$.

Definition 3.2 (weak path-believable farsighted dominance) *In the same settings of Definition 3.1, we say that a is weakly believable-path dominated by b if* (3.1) *is replaced by*

$$B_h \cap P_{h+1} \neq \emptyset \qquad h = 0 \ldots m-1$$

or

$$\exists\, h \in \{0, \ldots, m-1\} \;:\; B_h = \emptyset \Rightarrow B_{h-1} \cap P_k \neq \emptyset \qquad k = h \ldots m.$$

We denote it by $a \ll_{wpbf} b$.

Definition 3.3 (weak* path-believable farsighted dominance) *In the same settings of Definition 3.1, we say that a is weakly* believable-path dominated by b if* (3.1), (3.3) *are replaced by*

$$B_h \cap P_{h+1} \neq \emptyset \qquad h = 0 \ldots m-1$$

or

$$\exists\, h \in \{0, \ldots, m-1\} \;:\; B_h = \emptyset \Rightarrow B_{h-1} \cap P_k \neq \emptyset \qquad k = h \ldots m.$$

and

$$B_h = \{c \in Z \mid k \ll c \quad \forall\, k \in B_{h-1} \cap P_h\} \qquad h = 1 \ldots m-1.$$

We denote it by $a \ll_{w^*pbf} b$.

It is trivial to show that if $a \ll_{pbf} b$ it implies $a \ll_{wpbf} b$. These dominance relations can be considered in a more *large* meaning [16]. But these slight variations does not involve the backwards-induction spirit of our Definition 3.1.

Definition 3.4 (Large path-believable farsighted dominance) *In the same settings of Definition 3.1, we say that a is strictly believable-path dominated by b if* (3.4) *is replaced by*

$$P_h = \{a \in Z \mid V_i(a) \geq V_i(a_{m-h})\ \forall i \in S_{m-h},\ \exists j \in S_{m-h}\ V_j(a) > V_j(a_{m-h})\} \qquad h = 1 \ldots m.$$

We denote it by $a \ll_{lpbf} b$.

Definition 3.5 (Large weak path-believable farsighted dominance) *In the same settings of Definition 3.1, we say that a is strictly weakly believable-path dominated by b if* (3.1), (3.4) *are replaced by*

$$B_h \cap P_{h+1} \neq \emptyset \qquad h = 0 \ldots m-1$$

or

$$\exists\, h \in \{0, \ldots, m-1\} \;:\; B_h = \emptyset \Rightarrow B_{h-1} \cap P_k \neq \emptyset \qquad k = h \ldots m$$

$$P_h = \{a \in Z \mid V_i(a) \geq V_i(a_{m-h})\ \forall i \in S_{m-h},\ \exists j \in S_{m-h}\ V_j(a) > V_j(a_{m-h})\} \qquad h = 1 \ldots m.$$

We denote it by $a \ll_{lwpbf} b$.

Definition 3.6 (Large weak* path-believable farsighted dominance) *In the same settings of Definition 3.1, we say that a is strictly weakly* believable-path dominated by b if* (3.1), (3.3)(3.4) *are replaced by*

$$B_h \cap P_{h+1} \neq \emptyset \qquad h = 0 \ldots m-1$$

or

$$\exists\, h \in \{0, \ldots, m-1\} \ : \ B_h = \emptyset \Rightarrow B_{h-1} \cap P_k \neq \emptyset \qquad k = h \ldots m$$

$$B_h = \{c \in Z \mid k \ll c \quad \forall\, k \in B_{h-1} \cap P_h\} \qquad h = 1 \ldots m-1$$

$$P_h = \{a \in Z \mid V_i(a) \geq V_i(a_{m-h}) \ \forall i \in S_{m-h}, \ \exists j \in S_{m-h} \ V_j(a) > V_j(a_{m-h})\} \qquad h = 1 \ldots m.$$

We denote it by $a \ll_{lw^*pbf} b$.

For any $a, e \in Z$ such that $a \ll e$ according to the previous definitions, the set of outcomes in B_{m-1} is denoted by $B_{a,e}$. This subset $B_{a,e}$ is named the set of final outcomes.

We can compute PBF, WPBF dominances in Example 2.1. These results are described in Tables 2, 3.

Table 2: PBF dominances

$a \not\ll_{pbf} a$	$b \not\ll_{pbf} b$	$c \not\ll_{pbf} c$	$d \not\ll_{pbf} d$	$e \not\ll_{pbf} e$	$f \not\ll_{pbf} f$	$g \not\ll_{pbf} g$
$a \not\ll_{pbf} b$	$b \not\ll_{pbf} c$	$c \ll_{pbf} d$	$d \not\ll_{pbf} e$	$e \ll_{pbf} f$	$f \ll_{pbf} g$	$g \not\ll_{pbf} a$
$a \not\ll_{pbf} c$	$b \ll_{pbf} d$	$c \not\ll_{pbf} e$	$d \not\ll_{pbf} f$	$e \ll_{pbf} g$	$f \not\ll_{pbf} a$	$g \not\ll_{pbf} b$
$a \not\ll_{pbf} d$	$b \not\ll_{pbf} e$	$c \not\ll_{pbf} f$	$d \not\ll_{pbf} g$	$e \not\ll_{pbf} a$	$f \not\ll_{pbf} b$	$g \not\ll_{pbf} c$
$a \not\ll_{pbf} e$	$b \not\ll_{pbf} f$	$c \not\ll_{pbf} g$	$d \not\ll_{pbf} a$	$e \not\ll_{pbf} b$	$f \not\ll_{pbf} c$	$g \not\ll_{pbf} d$
$a \not\ll_{pbf} f$	$b \not\ll_{pbf} g$	$c \not\ll_{pbf} a$	$d \not\ll_{pbf} b$	$e \not\ll_{pbf} c$	$f \not\ll_{pbf} d$	$g \not\ll_{pbf} e$
$a \not\ll_{pbf} g$	$b \not\ll_{pbf} a$	$c \not\ll_{pbf} b$	$d \not\ll_{pbf} c$	$e \not\ll_{pbf} d$	$f \not\ll_{pbf} e$	$g \not\ll_{pbf} f$

Note that $b \not\ll d$ since player 2 prefers b to d. But $B_0 = \{e\}$, so coalition $\{1, 3\}$ prefers e to c. But $B_1 = \{g\}$, so player 2 prefers g to b. Therefore, $b \ll_{pbf} d$. At the same way, note that $b \ll e$ but $b \not\ll_{pbf} e$, $a < b$ but $a \not\ll_{pbf} b$ and $e \not< f$ but $e \ll_{pbf} f$. This implies that our concept is really unrelated to direct dominance and Chwe's indirect dominance. It is acceptable since our definition is based on believability between coalitions. It is easy to prove

Proposition 3.7 *Let $\Gamma = (N, Z, \{\prec_i\}_{i \in N}, \{\rightarrow_S\}_{S \subseteq N, S \neq \emptyset})$ be a coalitional game in effectiveness form. Let $a, b \in Z$ such that no $y \in Z$ $a \ll y$. Then, $b \ll a \iff b \ll_{pbf} a$.*

Table 3: WPBF dominances

$a \not\ll_{wpbf} a$	$b \not\ll_{wpbf} b$	$c \not\ll_{wpbf} c$	$d \not\ll_{wpbf} d$	$e \not\ll_{wpbf} e$	$f \not\ll_{wpbf} f$	$g \not\ll_{wpbf} g$
$a \not\ll_{wpbf} b$	$b \ll_{wpbf} c$	$c \ll_{wpbf} d$	$d \not\ll_{wpbf} e$	$e \ll_{wpbf} f$	$f \ll_{wpbf} g$	$g \not\ll_{wpbf} a$
$a \not\ll_{wpbf} c$	$b \ll_{wpbf} d$	$c \not\ll_{wpbf} e$	$d \not\ll_{wpbf} f$	$e \ll_{wpbf} g$	$f \not\ll_{wpbf} a$	$g \not\ll_{wpbf} b$
$a \not\ll_{wpbf} d$	$b \not\ll_{wpbf} e$	$c \not\ll_{wpbf} f$	$d \not\ll_{wpbf} g$	$e \not\ll_{wpbf} a$	$f \not\ll_{wpbf} b$	$g \not\ll_{wpbf} c$
$a \not\ll_{wpbf} e$	$b \not\ll_{wpbf} f$	$c \not\ll_{wpbf} g$	$d \not\ll_{wpbf} a$	$e \not\ll_{wpbf} b$	$f \not\ll_{wpbf} c$	$g \not\ll_{wpbf} d$
$a \not\ll_{wpbf} f$	$b \not\ll_{wpbf} g$	$c \not\ll_{wpbf} a$	$d \not\ll_{wpbf} b$	$e \not\ll_{wpbf} c$	$f \not\ll_{wpbf} d$	$g \not\ll_{wpbf} e$
$a \not\ll_{wpbf} g$	$b \not\ll_{wpbf} a$	$c \not\ll_{wpbf} b$	$d \not\ll_{wpbf} c$	$e \not\ll_{wpbf} d$	$f \not\ll_{wpbf} e$	$g \not\ll_{wpbf} f$

4 α-Stable Sets

In this section, we define new concepts of stability in coalitional games in effectiveness form based on dominance relations in Section 5.2, by taking into account drawbacks shown in Examples 2.2, 2.3. In this framework, we set a convex coefficient $\alpha_i \in [0, 1]$ through which any player i expects his value in front of other two possible credible outcomes $a, b \in Z$ as a convex combination $(1 - \alpha_i)\, V_i(a) + \alpha_i\, V_i(b)$, whatever the outcome he is at. If player i lies on c such that $V_i(c) < V_i(a) < V_i(b)$, his expected payoff is higher than the value in c; if player i lies on c such that $V_i(a) < V_i(b) < V_i(c)$, his expected payoff is lesser than the value in c; if player i lies on c such that $V_i(a) < V_i(c) < V_i(b)$, his expected payoff could be higher or lesser than the value in c. The last issue depends on his evaluation coefficient α_i. In this construction, we assume that this coefficient does not depend on Z but only on player i. Associating such a coefficient to any player does not involve any idea of agreement between players or coalitions; but, it makes more realistic any strategy by players and, successively, by coalitions which can enforce some outcomes. So, any coalitional game can be considered added by a vector $\alpha = (\alpha_i)_{i=1,\ldots N}$ which takes into account the safety/risk level of any player participating in a move of a coalition. According to this idea, we introduce the following definition.

Definition 4.1 (α-PBF Stability) *A subset $Y \subseteq Z$ is path-believable farsightedly stable if for any $a \in Y$ $\forall S, d$ such that $a \to_S d$, we have that $\forall e$ such that $d \ll_{pbf} e \in Y$*

$$\exists\ i \in S \quad V_i(a) \geq (1 - \alpha_i) \min_{b \in B_{d,e}} V_i(b) + \alpha_i \max_{b \in B_{d,e}} V_i(b); \tag{4.1}$$

$\exists\ i \in S$ $V_i(a) \geq V_i(d)$ or if no outcome $e \in Y$ is such that $d \ll_{pbf} e$ and $d \in Y$; or no outcome $e \in Y$ is such that $d \ll_{pbf} e$ and $d \notin Y$. The subset $B_{d,e}$ is the collection of any $B_{\overline{m}-1}$ with $\overline{m}$ the minimum length of any chain through which $d \ll_{pbf} e$, $B_{\overline{m}-1}$ the last subset of any minimal length chains starting from e as described in Definition 3.1.

The last subset $B_{d,e}$ is named the set of final outcomes for any element (d, e) belonging to $\ll_{pbf}$.

Suppose that a coalition S moves from a to d. Let Y be a path-believable farsightedly stable set. Let e be any outcome such that $d \ll_{pbf} e \in Y$. If $\alpha_i = 0$ for some player $i \in S$, player i evaluates only the minimum value that he receives in any outcome in $B_{d,e}$. If the value received in a is greater than the minimum of the values received for any outcome in $B_{d,e}$, he prefers to remain at a. So, $a \in Y$. If $\alpha_i = 1$ for some player $i \in S$, he evaluates only the maximum value that he receives in any outcome in $B_{d,e}$. If the value received in a is greater or equal than the maximum value received for any outcome in $B_{d,e}$, he prefers to remain at a. So, $a \in Y$. Note that such a player i can receive a minimum value for some outcome in B_{m-1} lesser than he receives in a and a maximum value for some outcome in $B_{d,e}$ greater than he receives in a. But, if his evaluation coefficient α_i is such that his expected return is lesser or equal than his payoff in a, then player i does not move from a. So, $a \in Y$. In addition, if the subset $B_{d,e}$ is reduced to a singleton, then the maximum value coincides to minimum one. Therefore, player i is not involved in setting his evaluation coefficient α_i. According to the different ideas of dominance relations, we construct associated notions of stability.

Definition 4.2 (α-WPBF-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF Stability) *A subset $Y \subseteq Z$ is weak* path-believable farsightedly stable (weak* path-believable farsightedly stable, large path-believable farsightedly stable, large weak path-believable farsightedly stable, large weak* path-believable farsightedly stable) if for any $a \in Y$ $\forall S, d$ we have that $\forall e$ such that $d \ll_{wpbf} e \in Y$ ($d \ll_{w^*pbf} e \in Y$, $d \ll_{lpbf} e \in Y$, $d \ll_{lwpbf} e \in Y$, $d \ll_{lw^*pbf} e \in Y$)*

$$\exists\ i \in S \quad V_i(a) \geq (1 - \alpha_i) \min_{b \in B_{d,e}} V_i(b) + \alpha_i \max_{b \in B_{d,e}} V_i(b); \tag{4.2}$$

$\exists\ i \in S \quad V_i(a) \geq V_i(d)$ if no outcome $e \in Y$ is such that $d \ll_{wpbf} e$ ($d \ll_{w^*pbf} e \in Y$, $d \ll_{lpbf} e \in Y$, $d \ll_{lwpbf} e \in Y$, $d \ll_{lw^*pbf} e \in Y$) and $d \in Y$; or no outcome $e \in Y$ is such that $d \ll_{wpbf} e$ ($d \ll_{w^*pbf} e \in Y$, $d \ll_{lpbf} e \in Y$, $d \ll_{lwpbf} e \in Y$, $d \ll_{lw^*pbf} e \in Y$) and $d \notin Y$. The subset $B_{d,e}$ is the collection of any $B_{\overline{m}-1}$ with $\overline{m}$ the minimum length of any chain through which $d \ll_{wpbf} e$ ($d \ll_{w^*pbf} e \in Y$, $d \ll_{lpbf} e \in Y$, $d \ll_{lwpbf} e \in Y$, $d \ll_{lw^*pbf} e \in Y$), $B_{\overline{m}-1}$ the last subset of any minimal length chains starting from e as described in Definition 3.1. The last subset $B_{d,e}$ is named the set of final outcomes for any element (d, e).

We give a particular class of stable sets.

Definition 4.3 (Main α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF Stable Sets) *We say that any α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF stability set is a main*

*α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF stable set* [2] *if and only if V is so constructed in one of these alternative situations:*

- *Step 1*

  Take any outcome in Z and check (4.1), (4.2). *Then,*

  $$V = Z \qquad \textit{if} \quad Z \quad \textit{is stable}.$$

- *Step 2*

  Suppose this is not true. Collect all the outcomes not stable in X in a subset E. We can say that $X \setminus E$ is a stable set. Take $z \in E$. Check if $(Z \setminus E) \cup \{z\}$ is stable. If it is not stable, cut z; otherwise, throw z in $\bar{E} \subseteq E$. Repeat this procedure for all elements in E. Finally, we have a subset $\bar{E} \subseteq E$. So,

  $$V = Z \setminus E \qquad \textit{if}\ \bar{E} = \emptyset.$$

- *Step 3*

  If $\bar{E} \neq \emptyset$, let $z \in \bar{E}$ be. Take any element $A_z \in P(\bar{E}) \setminus \{\bar{E}\}$ such that $z \in A_z$. Consider all the subsets A_z such that $(X \setminus E) \cup A_z$ is stable. This family A_z is totally ordered with respect to inclusion relation; so, take the maximal element $\bar{A}_z$. Repeat this argument for all $z \in \bar{E}$. Therefore, we have a subset $\bar{E}$ such that to any element in $\bar{E}$ it is associated a subset $\bar{A}_z \subset \bar{E}$ such that $z \in \bar{A}_z$. Select all the maximal $H \in P(\bar{E})$ such that

  $$x \in \cap_{z \in H} \bar{A}_z \qquad \forall x \in H.$$

  Finally,

  $$V = (Z \setminus E) \cup H.$$

Proposition 4.4 *A main α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF stable set is a α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF stable set.*

Remark 4.5 *In general, main α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF stable sets does not admit the largest main stable set. In fact, let be the simple following coalitional game in effectiveness form.*

$$\overset{a}{(1,0)} \underset{\{1\}}{\overset{\{2\}}{\rightleftarrows}} \overset{b}{(0,1)}$$

In this example, $\{a\}, \{b\}$ are two main PBF stable sets. But $\{a, b\}$ is not a main stable set but it is the largest consistent set.

[2] In the following three steps, we use the term stable as α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF stable.

In Example 2.1, the main 0-PBF stable set is unique and coincides with $\{a, d, g\}$. In addition, it is externally stable since $b \ll_{pbf} d$, $c \ll_{pbf} d$, $e \ll_{pbf} g$ and $f \ll_{pbf} g$. Moreover, it is internally stable with respect to PBF dominance since any element in the main 0-PBF stable set is not dominated with respect to $\ll_{bpf}$. Note that the main 0-PBF stable set is not externally stable with respect to $\ll$ since $b \notin \{a, d, g\}$ is only dominated by $c, e \notin \{a, d, g\}$. Note that $b \notin \{a, d, g\}$. If player 2, starting from b, is endowed by the same common-sense behaviour described in Example 2.1, player 2 predicts e as a final outcome which prescribes him to move from b. Therefore, it would seem that our definition gets this common-sense behaviour.

5 Properties on α-Stable Sets

In this section, we study some relationships between our concept of stability and internally stable sets with respect to $<$, uncovered set with respect to $\ll_{pbf}$, exernally stable sets with respect to $\ll_{pbf}$. First, our stable sets do not imply internal stability with respect to $<$ as shown by Proposition 5.1. In fact, the main 0-PBF stable set is unique and coincides with $\{a, d, g\}$ which is not internally stable.

Proposition 5.1 *In Example 2.1, a stable set with respect to $\ll$ does not exist. In addition, $\{a, c, e, g\}$ is a stable set with respect to $<$.*

Proof. Let V be a stable set with respect to $\ll$. But, we prove that V has a cardinality of $[\frac{7}{2}] + 1 = 4$ at least. Let $D(z)$ be the set of all outcomes dominating z with respect to $\ll$. $g \in V$ since $D(z) = \emptyset$ If $e \in V$, then $\{e, g\} \subseteq V$. But V is is not internally stable since $g \in D(e)$. But, $D(d) = \{e\}$ then $d \in V$. So, $e \notin V$ but $d, g \in V$. In addition, $D(a) = \{b\}$, $D(b) = \{c, e\}$, $D(c) = \{d, e\}$, $D(f) = \{g\}$. Since $e \notin V$, there is no case some different outcomes in $C_Z(V) \setminus \{e\}$ are dominated by the same outcome in V. Then $|C_Z(V) \setminus \{e\}| \leq [\frac{7-3}{2}] = 2$. So, $|C_Z(V)| \leq 3 < 4$. Then, $|V| \geq 4$.

V is internally stable with respect to $\ll$; then, it is internally stable set with respect to $<$ [3]. But, we claim that any internally stable set with respect to $<$ has a cardinality of $[\frac{7}{2}] + 1 = 4$ at the most as shown in the second row of Table 2. Then, $|V| \leq 4$. Finally, $|V| = 4$.

The only internally stable sets with respect to $<$ whose cardinality is 4 are $\{a, c, e, g\}$ and $\{a, c, e, f\}$. But, $\{a, c, e, g\}$ $\{a, c, e, f\}$ are not internally stable with respect to $\ll$ since $D(e) = \{g\}$ and $e \in D(c)$, respectively. This means that any stable set with respect to $\ll$ does not exist. By simple computations, the unique stable set with respect to $<$ is $\{a, c, e, g\}$. $\square$

[3] Any internally stable set with respect to $\ll$ is internally stable set with respect to $<$. Any externally stable set with respect to $<$ is externally stable set with respect to $\ll$.

We say that $\rightarrow$ is a function as functional relation (i.e. there exists some coalition T such that $a \rightarrow_T b$). We denote by $\rightarrow^{-1}$ the inverse multifunction of $\rightarrow$. Then, the following proposition holds.

Theorem 5.2 *Let $\Gamma = (N, Z, \{\prec_i\}_{i\in N}, \{\rightarrow_S\}_{S\subseteq N, S\neq\emptyset})$ be a coalitional game in effectiveness form. Let V be α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF stable set. Suppose*

1. *$\forall a, b \in V$, $a \ll_{pbf} b$: $\cup_i S_i \subseteq S_0$ where S_i are such that $\rightarrow_{S_i}^{-1} (a) \cap V \neq \emptyset$ and S_0 is the first coalition in the chain between a, b as prescribed in Definition 3.1.*

Then, V is internally stable with respect to $<$.

Proof. Suppose V an α-PBF stable set without loosing generality. Let $a, a^{'} \in V$ be. There exists S a coalition such that $a \rightarrow_S a^{'}$. Suppose that $D(a^{'}) = \{e \in V \,|a^{'} \ll_{pbf} e\} \neq \emptyset$. Let $e \in D(a^{'})$,$B_{a^{'},e}$ be the set of final outcomes along the path between $a^{'}, e$ of minimal lenght. Then,

$$\exists\, i \in S \qquad V_i(a) \geq (1-\alpha_i) \min_{b\in B_{a^{'},e}} V_i(b) + \alpha_i \max_{b\in B_{a^{'},e}} V_i(b) \tag{5.1}$$

since $a \in V$. But, $a^{'} \ll_{pbf} e$. This implies that there exist a chain starting from $a^{'}$ to e. This implies a coalition S_0 , $a^{''} \in Z$ such that $a^{'} \rightarrow_{S_0} a^{''}$ and

$$V_i(a^{'}) < \min_{b\in B_{a^{'},e}} V_i(b) \qquad \forall i \in S_0. \tag{5.2}$$

By condition 1, $i \in S \subseteq S_0$. From (5.1),(5.2), $a \prec_S a^{'}$ is not true. By condition 1, by arbitrariety of a, S, $a < a^{'}$ is not true. Suppose that $D(a^{'}) = \emptyset$. Then, $V_i(a^{'}) \leq V_i(a)$ for some $i \in S$ since $a, a^{'} \in V$. Then $a <_S a^{'}$ is not true. By arbitrariety of $a, a^{'}$ $a^{'} <_S a$ is not true. □

We say that effectiveness relation is acyclic if there exists $a, b \in Z$ such that $\rightarrow (a) = \emptyset$, $\rightarrow^{-1} (b) = \emptyset$. We say that effectiveness relation is monotonic if $a \rightarrow_S b$, $b \rightarrow_T c$ for some $a, b, c \in Z$, for some coalitions S, T implies $S \subseteq T$. We say that that effectiveness relation is antisymmetric if $a \rightarrow_S b$ implies $b \rightarrow_T a$.

Corollary 5.3 *Let $\Gamma = (N, Z, \{\prec_i\}_{i\in N}, \{\rightarrow_S\}_{S\subseteq N, S\neq\emptyset})$ be a coalitional game in effectiveness form. Let V be α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF stable set. Suppose that $\rightarrow$ is a monotonic acyclic antisymmetric function. Let K be V-M core. Then, V is internally stable with respect to $<$. In addition,*

$$|V| \leq \left[\frac{|Z| + |K|}{2}\right].$$

These results are useful necessary conditions of α-PBF,-WPBF,-W*PBF, -LPBF,-LWPBF,-LW*PBF stable set. The uncovered set is the set of maximal elements with respect to our dominances.

Theorem 5.4 *Let be a coalitional game in effectiveness form* $\Gamma = (N, Z, \{\prec_i\}_{i\in N}, \{\rightarrow_S\}_{S\subseteq N, S\neq\emptyset})$ *be a coalitional game in effectiveness form. Suppose* $UC \neq \emptyset$. *Suppose*

1. $\rightarrow \subseteq \ll_{pbf} \; (\ll_{wpbf}, \ll_{w^*pbf})$
2. *Suppose* $(a, d) \in \rightarrow$. *Let* e *be such that* $a, d \ll_{pbf} e \quad (a, d \ll_{wpbf} e \; a, d \ll_{w^*pbf} e)$. *If* $B_{a,e} \neq B_{d,e}$ *then* $\{a \in Z \;:\; a \ll b \; b \in B_{a,e}\} \subseteq B_{d,e}$.
3. *Suppose* $(a, d) \in \rightarrow_S$ *for some coalition* S. *Let* e *be such that* $d \ll_{pbf} e$. *Then,* $S \subseteq \bar{S}$ *such that* $\rightarrow_{\bar{S}} (z) \neq \emptyset \;\; \forall z \in B_{d,e}$.

Then, $\exists\, \alpha \geq 0$, $\exists\, V$ α*-PBF,(-WPBF,-W*PBF) stable set such that* $UC \subseteq V'$ *with* V' α' *-PBF,(-WPBF,-W*PBF) stable set for any* $\alpha' \geq \alpha$.

Proof. Take an element $a \in UC$. Suppose that $\rightarrow (a) \neq \emptyset$. Consider a deviation d from a thorugh a coalition S. So, there are two alternatives: $D(d) = \{e \in V \;|d \ll_{pbf} e\} = \emptyset$ or not. Let the first case be true. By 1, $a \ll_{pbf} d$. But, no outcome dominates d with respect to $\ll_{pbf}$. It means that $a < d$. By cover relation's definition, it means that $a <_c d$. Then $a \notin UC$. This is an absurd. Suppose that $D(d) = \{e \in V \;|d \ll_{pbf} e\} \neq \emptyset$. Take any element $e \in D(d)$. So, $a \ll_{pbf} d$ and $d \ll_{pbf} e$. This implies that $a \not\ll_{pbf} e$ since $a \in UC$. Then, compute $B_{a,e}$ along the paths of minimal length between a, e. So,

$$\exists\, b \in B_{a,e},\; i \in S \qquad V_i(a) \geq V_i(b).$$

By 2, there exists $\bar{b} \in B_{d,e}$ such that $\bar{b} \ll b$, $\rightarrow_{\bar{S}} (\bar{b}) \neq \emptyset$ for some coalition $\overline{S}$. This implies

$$V_i(\bar{b}) < V_i(b) \qquad i \in \overline{S}.$$

Let

$$\alpha^*(i, b, d, e) = \frac{V_i(b) - \min\limits_{z\in B_{d,e}} V_i(z)}{\max\limits_{z\in B_{d,e}} V_i(z) - \min\limits_{z\in B_{d,e}} V_i(z)} \geq 0, \quad b \in B_{d,e} \text{ is not a singleton}$$
$$\alpha^*(i, b) = 1 \quad b \in B_{d,e} \text{ is a singleton}$$

be for $d, e \in Z, d \ll_{pbf} e$. Then, by 3, $i \in S \subseteq \overline{S}$ (depending on $\bar{b}$) such that

$$V_{\bar{i}}(a) > V_{\bar{i}}(\bar{b}) = (1 - \alpha^*(\bar{i}, \bar{b}, d, e)) \min_{b\in B_{d,e}} V_i(b) + \alpha^*(\bar{i}, \bar{b}, d, e) \max_{b\in B_{d,e}} V_i(b).$$

Take $\alpha = \min \alpha^*(\bar{i}, \bar{b}, d, e) \geq 0$, then

$$V_{\bar{i}}(a) \geq (1-\alpha) \min_{b \in B_{d,e}} V_i(b) + \alpha \max_{b \in B_{d,e}} V_i(b).$$

Suppose $\rightarrow_m (a) = \emptyset$. [4] Then, a is in any V α-PBF stable set. $\Box$

The hypothesis 2 is a kind of backwards-decreasing property on the set of final outcomes along the paths of a game with respect to inclusion. As corollary of Proposition 3.7, we have

Corollary 5.5 *Let* $\Gamma = (N, Z, \{\prec_i\}_{i \in N}, \{\rightarrow_S\}_{S \subseteq N, S \neq \emptyset})$ *be a coalitional game in effectiveness form. The core with respect to* $\ll$ *is externally stable with respect to* $\ll$ $\iff$ *the core with respect to* $\ll_{pbf}$ *is externally stable with respect to* $\ll_{pbf}$.

6 Applications to Prisoner's Dilemma

A normal form game is a triple $G = \{N, Z_i, u_i\}$ where N is the set of players, Z_i is the nonempty set of strategies and $u_i : Z := \prod_{i=1}^{N} Z_i$ are utility functions for every player $i \in N$. The traditional stability solution concepts are strong Nash equilibria, α core, β core in [1] [2],[3]; other solutions in [12] [11], [18]. We recall how a game in normal form can be seen as a game in effectiveness form. Each individual, facing a proposed strategy profile $c \in Z$, can declare: "If you all other players stick to play $C_{N\setminus\{i\}}$, I will play $d_i \in Z_i$ instead of c_i. " Each player can make such contingent threats in turn. Players can revise their threats: no one is committed to anything. Let $a, b \in Z$, we say that $a \prec_i b$ if and only if $u_i(a) < u_i(b)$; we say that $a \xrightarrow{i} b$ if and only if $a_j = b_j$ for all $j \in N$. If coalitions can form, we say that $a \prec_S b$ if and only if $u_i(a) < u_i(b)$ for all $i \in S$; we say that $a \xrightarrow{S} b$ if and only if $a_j = b_j$ for all $j \in N \setminus S$.

Studies about normal form game played in such a way can be found in [8], [10] and [15]. For instance, according to rules above, the prisoner's dilemma can be represented

[4] If no movement from a is allowed, then a cannot be dominated by any element $x \in Z$ with respect to $\ll_{pbf}$. This implies that a cannot be dominated by any element $x \in Z$ with respect to cover relation on $\ll_{pbf}$. So an element $a \in UC$ could satisfy $\rightarrow_m (a) = \emptyset$.

by this graph

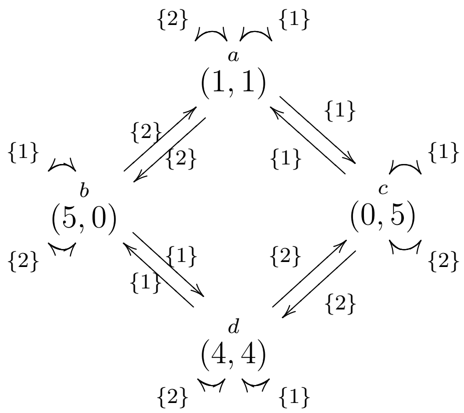


Table 4: Direct and indirect Dominances

$a \not\ll a$	$b < a$	$c < a$	$d \not\ll a$
$a \not\ll b$	$b \not\ll b$	$c \not\ll b$	$d < b$
$a \not\ll c$	$b \not\ll c$	$c \not\ll c$	$d < c$
$a \not\ll d$	$b \not\ll d$	$c \not\ll d$	$d \not\ll d$

Table 5: PBF Dominances

$a \not\ll_{pbf} a$	$b \ll_{pbf} a$	$c \ll_{pbf} a$	$d \not\ll_{pbf} a$
$a \not\ll_{pbf} b$	$b \ll_{pbf} b$	$c \not\ll_{pbf} b$	$d \not\ll_{pbf} b$
$a \not\ll_{pbf} c$	$b \not\ll_{pbf} c$	$c \ll_{pbf} c$	$d \not\ll_{pbf} c$
$a \not\ll_{pbf} d$	$b \not\ll_{pbf} d$	$c \not\ll_{pbf}$d	$d \ll_{pbf} d$

Table 6: Final outcomes for PBF Dominances

$\emptyset$	$\{a\}$	$\{a\}$	$\emptyset$
$\emptyset$	$\{a\}$	$\emptyset$	$\emptyset$
$\emptyset$	$\emptyset$	$\{a\}$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Tables regarding WPBF, W*PBF, LPBF, LWPBF, LW*PBF dominances are equal to Tables 5,6.

Proposition 6.1 *The unique stable set with respect to indirect dominance (or direct one) is $\{a, d\}$ in prisoner's dilemma played by individual contingent threats.*

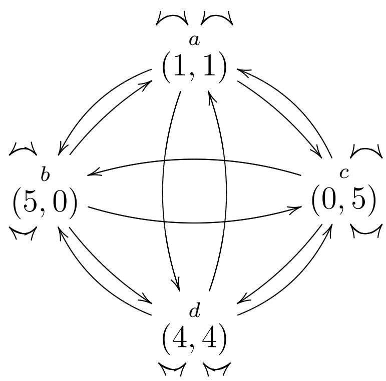
Proposition 6.2 *The largest consistent set is $\{a, d\}$ in prisoner's dilemma played by individual contingent threats.*

Proof. [A sketch of the proof] Suppose that players 1 and 2 deviate from a to b and c, respectively. But b is not consistent with a. In fact, $b \xrightarrow{2} a$ and the set of outcomes dominating a is empty. But $b \prec_2 a$. So, b is unstable. The same argument can be used for c by taking a deviation to a through player 1. So, the outcome a cannot be consistent in terms of stability with b and c. But the only outcome dominating b or c is a. So, a is stable being consistent with itself. $\square$

Proposition 6.3 *The main α-PBF, -WPBF, -W*PBF, -LBF, -LWPB, LW*PBF stable set for any $\alpha \in [0, 1]$ is unique and coincides with $\{a, d\}$ in prisoner's dilemma played by individual contingent threats.*

The interesting note is that our concept of stability incorporates the outcome d which is the "cooperative" solution of the game without letting formation of coalitions possible. The unique stable set generated by all three dominances is $\{a, d\}$. We can say that the so called *cooperative* solution is a *farsighted* solution in a setting in which any player acts by itself.

We suppose that prisoner's dilemma is played by Greenberg's coalitional contingent situation threats in [10]. Not only each individual but each coalition S facing a proposed strategy profile can declare: "If all you other players stick to play $z_{N \setminus S}$, we will play $d_S \in Z_S$ instead of c_S".



All the arrows in the following graph are labelled by coalition $\{1, 2\}$. In this case, we suppose that all the rules prescribed by two graphs above are true. The reader has to superimpose two graphs above for reading all the possible effectiveness relations between outcomes. Every dominance with respect (L)BPF, (L)WBPF, (L)W*BPF relation between two outcomes generates a subset of final outcomes according to previous definitions. We compute direct and indirect dominance, (L)BPF dominances, (L)WBPF dominances (L)W*BPF dominances and their final outcomes. For instance, $b \ll_{pbf} a$ in Table 8; the subset of final outcomes generated by $b \ll_{pbf} a$ is at the same entry in Table 9. The same reading scheme is used in the construction of Tables 10 and 11, Tables 12 and 13.

Table 7: Direct and Indirect Dominances

$a \not\ll a$	$b < a$	$c < a$	$d \not\ll a$
$a \not\ll b$	$b \not\ll b$	$c \not\ll b$	$d < b$
$a \not\ll c$	$b \not\ll c$	$c \not\ll c$	$d < c$
$a < d$	$b \ll d$	$c \ll d$	$d \not\ll d$

Table 8: (L)PBF Dominances

$a \ll_{pbf} a$	$b \ll_{pbf} a$	$c \ll_{pbf} a$	$d \not\ll_{pbf} a$
$a \not\ll_{pbf} b$	$b \ll_{pbf} b$	$c \not\ll_{pbf} b$	$d \not\ll_{pbf} b$
$a \not\ll_{pbf} c$	$b \not\ll_{pbf} c$	$c \ll_{pbf} c$	$d \not\ll_{pbf} c$
$a \not\ll_{pbf} d$	$b \not\ll_{pbf} d$	$c \not\ll_{pbf} d$	$d \not\ll_{pbf} d$

Table 9: Final outcomes for (L)BPF Dominances

$\{d\}$	$\{d\}$	$\{d\}$	$\emptyset$
$\emptyset$	$\{a, d\}$	$\emptyset$	$\emptyset$
$\emptyset$	$\emptyset$	$\{a, d\}$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Proposition 6.4 *The unique stable set with respect to $\ll$ is $\{d\}$ in prisoner's dilemma played by coalitional contingent threats. A stable set with respect to $<$ does not exist in prisoner's dilemma played by coalitional contingent threats.*

Proof. Let V be a stable set with respect to $\ll$. Suppose that $d \notin V$, So, by external stability of V, $b \in V$ or $c \in V$. But, $b \in V$ or $c \in V$ implies $a \notin V$ by internal stability. But, if $a \notin V$ there exists no outcome dominating a which is in V since $d \notin V$. But this is a contradiction since V is externally stable. This means that $d \in V$. V can not

Table 10: (L)WPBF Dominances

$a \ll_{wpbf} a$	$b \ll_{wpbf} a$	$c \ll_{wpbf} a$	$d \ll_{wpbf} a$
$a \ll_{wpbf} b$	$b \ll_{wpbf} b$	$c \ll_{wpbf} b$	$d \ll_{wpbf} b$
$a \ll_{wpbf} c$	$b \ll_{wpbf} c$	$c \ll_{wpbf} c$	$d \ll_{wpbf} c$
$a \not\ll_{wpbf} d$	$b \not\ll_{wpbf} d$	$c \not\ll_{wpbf} d$	$d \ll_{wpbf} d$

Table 11: Final outcomes for (L)WPBF Dominances

$\{d\}$	$\{d\}$	$\{d\}$	$\{b, c\}$
$\{a, d\}$	$\{a, d\}$	$\{b, c, d\}$	$\{a, d\}$
$\{a, d\}$	$\{b, c, d\}$	$\{a, d\}$	$\{a, d\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\{b, c\}$

include any other outcome distinct from d since $a \ll d, b \ll d, c \ll d$. Then, $V = \{d\}$ which is stable with respect to $\ll$.

Let V be a stable set with respect to $<$. Suppose that $d \notin V$, So, by external stability of V, $b \in V$ or $c \in V$. But, $b \in V$ or $c \in V$ implies $a \notin V$ by internal stability. But, if $a \notin V$ there exists no outcome dominating a which are in V since $d \notin V$. But this is a contradiction since V is externally stable. This means that $d \in V$. V has to include b, c since no other outcome dominates b or c in a direct way except for $a \notin V$. So, $b \in V$ or $c \in V$. But, it is an absurd because $d \ll b$ or $d \ll c$. Then, $V = \{d\}$. But, it is not externally stable. Then V does not exist. □

Proposition 6.5 *The largest consistent set is $\{d\}$ in prisoner's dilemma played by coalitional contingent threats.*

Proof. [A sketch of proof] Suppose that players 2 or coalition {1,2} deviate from a to b, respectively. But b is not consistent with a. In fact, $b \xrightarrow{2} a$ but $b \prec_2 a$. The set of outcomes dominating a consists in d. But, $b \prec_2 d$. So, b is unstable. So, the outcome a cannot be consistent in terms of stability with b. But the only outcomes dominating b are a, d. So, a is stable being consistent with itself notwithstanding any initial deviator from a (i.e. $\{1\}$ or $\{1, 2\}$) is better in d than a. The same argument can be used for c by taking a deviation to a through player 1 or coalition $\{1, 2\}$.

In this case, only the coalition $\{1, 2\}$ can take a deviation from a to d. But, $d \ll b$ and $d \ll c$. player 1 is better in a than c and player 2 is better in a than b. So, there exists at least one player belonging to the initial deviators who prefers the initial outcome to the ending ones. This is wrong!!! In fact, b and c are not stable outcomes. So, a becomes unstable. □

Table 12: (L)W*PBF Dominances

$a \ll_{wpbf} a$	$b \ll_{wpbf} a$	$c \ll_{wpbf} a$	$d \ll_{wpbf} a$
$a \ll_{wpbf} b$	$b \ll_{wpbf} b$	$c \ll_{wpbf} b$	$d \not\ll_{wpbf} b$
$a \ll_{wpbf} c$	$b \ll_{wpbf} c$	$c \ll_{wpbf} c$	$d \not\ll_{wpbf} c$
$a \not\ll_{wpbf} d$	$b \not\ll_{wpbf} d$	$c \not\ll_{wpbf} d$	$d \ll_{wpbf} d$

Table 13: Final outcomes for (L)W^*PBF Dominances

$\{d\}$	$\{d\}$	$\{d\}$	$\{b,c\}$
$\{d\}$	$\{a,d\}$	$\{b\}$	$\emptyset$
$\{d\}$	$\{c\}$	$\{a,d\}$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\{b,c\}$

Proposition 6.6 *The stable set with respect to $\ll_{pbf}$ does not exist in prisoner's dilemma played by coalitional contingent threats.*

Proof. Suppose that there exist a stable set V with respect to $\ll_{pbf}$. The outcomes a, b, c can not belong to V by internal stability since they are dominated by themselves with respect to $\ll_{pbf}$. So, suppose that $V = \{d\}$ since $d \not\ll_{pbf} d$. But, $c \notin V$ and $c \not\ll_{pbf} d$. Therefore, $\{d\}$ is not externally stable. Finally, this a contradiction. □

Proposition 6.7 *The stable set with respect to $\ll_{wpbf}$ does not exist in prisoner's dilemma played by coalitional contingent threats.*

Proposition 6.8 *The stable set with respect to $\ll_{w^*pbf}$ does not exist in prisoner's dilemma played by coalitional contingent threats.*

Let R,R' and V,V' be two relations on Z and their relative VNM stable sets, respectively. If $R \subset R'$ does not imply that $V \subset V'$. So, neither Proposition 6.7 does not imply Proposition 6.6 according to $\ll_{pbf} \subset \ll_{wpbf}$ nor Proposition 6.8 does not imply Proposition 6.7 according to $\ll_{w^*pbf} \subset \ll_{wpbf}$.

In addition, prisoner's dilemma played by coalitional contingent threats satisfies hypotheses of Corollary **??**. So, a main α-(L)PBF stability set is not $\{a, d\}$ since it is not internally stable with respect to $<$.

Proposition 6.9 *In prisoner's dilemma played by coalitional contingent threats, the main α-(L)PBF stability set is unique and coincides with $\{d\}$ for any $\alpha \in [0, 1]$; the main α-(L)WPBF stability set is unique and coincides with $\{d\}$ for any $\alpha \in [0, \frac{4}{5}]^2$; the main α-(L)WPBF stability set is unique and coincides with an $\emptyset$ for any $\alpha \in]\frac{4}{5}, 1]^2$; the main α-(L)W*PBF stability set is unique and coincides with $\{d\}$ for any $\alpha \in [0, 1]$.*

7 Some final considerations

Some objections to this idea could be found in our definitions. We suppose that the first deviator can consider that the intermediate coalitions are farsighted as him. But this definition suffers from the fact that the intermediate coalitions, in their turn, could consider their intermediate coalitions, leading themselves to ending final outcome, not so farsighted. So, this objection is strong. If the last thing were true, we should be upset in computing our dominances between two outcomes: or, better, it would need to know other dominances between other outcomes for computing dominances between two fixed outcomes and viceversa. So, this could imply an implicit definition. Therefore, we suppose a first degree of knowledge. Thinking of infinite processes on rationalizability for all coalitions, moving by a belief of moves of other coalitions who, in its turn, move by a belief of moves of other moving coalitions and so on, is an hard problem. Our dominance's concept takes into account the length of path joining two outcomes instead of Chwe's definition. We propose some questions for a interested reader. In Section 4, if does the α vector depend on Z what could happen in terms of stability results? In the case Z is a subset of $\mathbb{R}$ or in the case of a countable infinite number of players, does α need to have some good property for making our stability sets not empty? If yes, what kind of properties? In Section 4, how can we construct different concept of stability sets by not choosing paths of minimal length in Definitions 4.1, 4.2? In this case, can we suppose that dominance prescribes paths of same length for reaching final outcomes? Can we interpreter the last assumption as a same length farsightedness degree for all coalitions?

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