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Working Paper

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Kiel Working Papers, No. 846

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Kiel Working Papers

Kiel Working Paper No. 846

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January 1998

Institut für Weltwirtschaft an der Universität Kiel
The Kiel Institute of World Economics

ISSN 0342 - 0787

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by Frank Stähler* and Friedrich Wagner+

Abstract

An exhaustible stock of resources may be exploited by N players. An arbitrarily long duration of the game is only possible, if the utility function satisfies certain restrictions at small values R of extraction. We find that stability against unilateral defection occurs if the elasticity of the marginal utility turns out to be larger than $(N-1)/N$, however *independent* of the value of the discount factor. Hence we find that cooperation does not depend on the discount factor for a certain range of elasticities. Analogy to phase transitions in statistical physics is discussed.

JEL classification: C72, C73, Q30.

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1 Introduction

Since its beginnings in economics, a basic question of game theory has been whether cooperative behavior can be expected in an environment in which every player cannot credibly commit himself to a certain strategy. The folk theorem has demonstrated that cooperation may emerge when all players face a sufficiently long time horizon, are sufficiently patient and actions of every player are not unique. If games are repeated, repetition may play a disciplinary role for cooperation because future behavior may be made dependent on past actions. The folk theorem has demonstrated that all rational outcomes including perfect cooperation and no cooperation may be sustained by repetition. Furthermore, it has shown that the chances for cooperation increase with the discount factor. The reason is that unilateral defection from a cooperative behavior is not profitable when future outcomes are sufficiently taken into account and present defection is punished by the opponents in future periods. The strategies taken by players responding to defection of a single player have been discussed extensively in (Abreu, 1988) and (Farrell, 1989).

The proof of the folk theorem for general games in strategic form has been presented by Fudenberg and Maskin (Fudenberg, 1986), Abreu, Dutta and Smith (Abreu, 1994) and Wen (Wen, 1994). Fudenberg, Levine and Maskin (Fudenberg, 1994) have shown that the folk theorem even holds when past actions are not observable but influence a stochastic variable. Benoit and Krishna (Benoit, 1985) have demonstrated that cooperation can be sustained in finitely repeated games if at least two one-shot Nash equilibria exist one of which Pareto-dominates the other. In this case, the agents' strategy is supposed to revert to the "worse" equilibrium if one agent deviates from the cooperative outcome. If no agent has defected before the last period is reached, every agent's strategy is to take the "better" equilibrium in the last period which may prevent defection in the next to last period. Dropping the assumption of complete information, reputation can play a role for sustaining cooperation (for reputation and repetition in prisoners' dilemma games, see (Kreps, 1982). In (Bernheim, 1995) it has been demonstrated that the folk theorem holds also in infinitely repeated games when the discount factor declines. Kandori (Kandori, 1992) and Smith (Smith, 1992) have derived a folk theorem for models with overlapping generations. In all these models, the whole set of attainable outcomes may result from a long-run equilibrium.

The folk theorem holds for repeated games. Repetition is a very specific assumption with respect to the time structure of a game. In general, games involving a time structure can be dynamic so that the action space and the utilities are not stationary but change endogenously as a result of past actions. From this perspective, repeated games are a real subset of the set of dynamic games because they make the specific assumption that neither the action space nor the utility functions change. Dutta (Dutta, 1995) has demonstrated that this distinction is significant because the folk theorem does not hold for dynamic games in general. Repeated games imply a cooperative and a non-cooperative equilibrium which do not depend on the discount factor but the equilibria of dynamic games do in general. The reason is that the time preference does also determine the optimal behavior of the stock variables. The interdependence between discount factor and stock variables implies that the equilibria itself vary with the discount factor, and it implies that the chances for cooperation do not necessarily increase with the discount factor. Consider for example an exhaustible resource which can be exploited simultaneously by N players. Suppose that non-cooperative behavior (i.e. defection) implies that a single player takes the whole rest of the resource, given cooperative behavior of his opponents. When the discount factor increases, two effects can be observed: first, the future non-availability of the resource after defection is given a stronger weight. This effect makes defection less profitable. Second, the cooperative solution is changed in favor of more resource conservation. More resource conservation, however, implies that defection is made more profitable because the remaining stock in every period which may be seized unilaterally is increased.

It is this game which is the starting point of this paper. We are interested in exploring not only the role of the discount factor but the role of the utility function defining the benefits of each player as well. We consider utility functions which can be normalized and we restrict our attention to those utility functions which are able to support long-run cooperation. We discuss the chances of cooperative resource extractions in a fairly general setting of N identical players. Cooperation is immune against defection if the short-run gains of defection fall short of the long-run gains of cooperation for each player. We find that cooperation is immune against defection if the elasticity of the marginal utility does not fall short of $(N-1)/N$, irrespective of the discount factor. This result demonstrates that cooperation is never at a risk in our model for a certain class of utility functions. The behavior as function of the elasticitiy has much in common with phase transitions in physics where a quantity called order parameter is zero below a critical value of a parameter and is different from zero above. In our case, the order parameter is the discount factor which distinguishes a cooperation and a defection phase, and the parameter the elasticitiy. For the range of elasticities between $(N-1)/N$ and 1, the critical value of the discount factor is zero, and we find a phase transition at $(N-1)/N$ where the discount factor becomes relevant for cooperation.

In order to derive our results, we employ the Legendre transformation which transforms the utility function and its first and second derivatives. This method is standard in physics for determining phase transitions. The paper is organized as follows. Section 2 presents the model. Section 3 derives the cooperative solution. Section 4 discusses whether the cooperative solution is immune against defection in two different games, a game under partial control and a game which not subject to any external control. This section derives also the essential results of this paper. Section 5 presents an example and discusses the relationship of our results to the phenomena of phase transitions in physics. Section 6 contains some concluding remarks.

2 The Model

We assume N players which may extract from an exhaustible resource. This exhaustible resource is assumed to be a common pool resource so that it cannot be divided into individual claims of each player. The exhaustible resource stock will be denoted by $S(t)$ with a discrete integer time t. The resource is storable only as a common pool so that any extraction of a player is identical to instantaneous consumption. The equation of motion for the resource stock is

$$S(t+1) = S(t) - \sum_{i=1}^{N} R_i(t) \tag{1}$$

with $R_i(t) \geq 0$, $S(t) \geq 0$ and $S(0) = \bar{S}$. $R_i(t)$ denotes actual resource extraction of player i at time t. We assume that only the initial stock size $\bar{S}$ can be observed, but that future stock sizes cannot. The game ends in t_0, if all players realize an empty stock in time step $t_0 + 1$. An individual resource extraction $R_i(t)$ gives a certain instantaneous utility denoted by $u(R)$ with the following properties:

$$u'(R) > 0 \quad \text{and} \quad u''(R) < 0 \quad \forall\, R \,\epsilon\, (0, \bar{S}) \tag{2}$$

Equ.(2) specifies that the marginal utility is positive and decreasing. The utility function identical for all players depends only on the resource extraction R. Their intertemporal preference is given by a constant discount factor δ . All games which we consider are started in period $t = 0$. The utility function F for all players is given by

$$F(R) = \sum_{i=1}^{N} \sum_{t=0}^{\infty} u(R_i(t))\ \delta^t \tag{3}$$

The strategy space of each player comprises his planned resource extractions $\tilde{R}_i(t)$ at each time t with $0 \leq \tilde{R}_i(t) \leq \bar{S}$. It contains the cooperative solution $R_i^*(t)$, which is given by the maximum of F in (3) under the condition

$$\sum_{i,t} R_i(t) = \bar{S} \tag{4}$$

This solution Pareto-dominates all other non-cooperative equilibria. These depend on the distribution rule to be applied, if the demand $\sum_i \tilde{R}_i(t)$ exceeds the available stock size $S(t)$. We will not discuss different distribution rules because these rules will be relevant only if they affect the planned extractions of the cooperative players. Instead, we assume that the demands of the cooperative players will always be served in any period in which the resource is not yet completely exploited. This assumption implies that defection will not affect extractions of the cooperative players from a nonempty stock. Hence, in all three scenarios we are going to discuss below the following rule must be implied by the distribution mechanism. The defecting player will be denoted with $i = 0$.

Rule 1 (G) *If* $S(t) > 0 \implies R_i(t) = R_i^*(t)$ *for* $i \neq 0$

Rule **(G)** ensures that for a non empty stock the demands of the cooperating players are satisfied. Therefore the optimal strategy for a defecting player is to seize the remaining stock at defection time t_D. Our discussion of defection is valid for any non-cooperative strategies as long as rule **(G)** is implied by the distribution mechanism.
We consider three games which differ by the amount of external control.

- In a game under complete control (hereafter referred by CC) the external control checks ex ante, whether the plan of each player exceeds the cooperative extraction. If it does, all his present and future extractions are set equal to 0. This enforces the cooperative plan, since no player can gain by defection.

- In a game under partial control (PC) the external control performs the same test ex post and excludes the defecting player from future extractions. Hence, it is rational for him to seize the whole remaining stock and thereby the defecting player finishes the game.

- In a game with no control (hereafter referred by NC) the defecting player may optimize his sum of discounted utilities up to time t_D at which the stock is exhausted. Obviously, in this case only the initial stock size is known to the players, but not the time evolution (1).

The cooperative resource extraction policy is sustainable in game PC or NC, respectively, if the benefits of the defection option are less than the benefits of cooperation. Table 1 summarizes the games we consider.

The game will depend on the properties of u. In the remainder of this section we discuss some aspects of u which will be important in later sections. We assume, that $u(R)$ is twice differentiable with positive marginal utility

Game	controlled quantity	punishment	optimal defection
CC	$\tilde{R}_i(t_D) > R_i^*(t_D)$ ex ante	$R_i(t) = 0 \;\; t \geq t_D$	not possible
PC	$\tilde{R}_i(t_D) > R_i^*(t_D)$ ex post	$R_i(t) = 0 \;\; t > t_D$	seize the rest of the resource stock in period t_D
NC	none	none	seize the rest of the resource stock during t_D periods

Table 1: *Controlled quantity, punishment and optimal defection in different games. In all cases rule (G) is implied by the distribution mechanism.*

$u'(R)$ and concave ($u''(R) < 0$) in the intervall $(0, R_{max}]$. Utility functions are defined only up to an additive constant and an arbitrary scale, i.e. they can be changed according

$$u \rightarrow \tilde{u}(R) = \lambda u(R) + u_0 \tag{5}$$

The common utility function (3) changes according

$$F \rightarrow \tilde{F}(R) = F(\lambda R) + u_0/(1 - \delta) \tag{6}$$

only by an irrelevant additive constant. $u(R)$ must be be finite at $R = 0$, otherwise the infinite sum in equ.(3) becomes meaningless for a finite stock size $\bar{S}$. The freedom in the choice of u_0 we can use to require without loss of generality $u(0) = 0$. Therefore our utility functions are non negative functions of R. This seems to exclude the frequently used logarithmic function. Since

$$\ln R = \lim_{\gamma \rightarrow 0} (R^\gamma - 1)/\gamma \tag{7}$$

holds, this function can be approximated by a power law. It is impossible to distinguish $\ln R$ empirically from a power law with small γ within finite measuring accuracy. The normalization $u(0) = 0$ requires in the use of equ. (7) an arbitrary large constant $u_0 = 1/\gamma$. In table 2 we list some examples for commonly used utility functions.

utility function	Legendre Transf.	
$u(R) = Q_0/(1-\epsilon)\ R^{1-\epsilon}$ $u'(R) = Q_0 R^{-\epsilon}$ $\epsilon < 1$	$g(Q) = -Q_0[\epsilon/(1-\epsilon)]$ $\cdot (Q_0/Q)^{(1-\epsilon)/\epsilon}$ $g'(Q) = (Q_0/Q)^{1/\epsilon}$	$t_0 = \infty$ $Q_{max} = \infty$ $\epsilon_{as} = \epsilon$
$u(R) = Q_0 R[1 + \ln(R_0/R)]$ $u'(R) = Q_0 \ln R_0/R,$ $R \leq R_0$	$g(Q) = -Q_0 R_0 \exp(-Q/Q_0)$ $g'(Q) = R_0 \exp(-Q/Q_0)$	$t_0 = \infty$ $Q_{max} = \infty$ $\epsilon_{as} = 0$
$u(R) = Q_0 R_0[1 - \exp(-R/R_0)]$ $u'(R) = Q_0 \exp(-R/R_0)$	$g(Q) = R_0[Q - Q_0$ $+Q \ln(Q_0/Q)]\ ,\ Q \leq Q_0$ $g'(Q) = R_0 \ln(Q_0/Q)$	$t_0 < \infty$ $Q_{max} = Q_0$ $\epsilon_{as} = 0$
$u(R) = Q_0 R(1 - R/(2R_0))$ $u'(R) = Q_0(1 - R/R_0)$ $R \leq R_0$	$g(Q) = R_0 Q(1 - Q/(2Q_0))$ $g'(Q) = R_0(1 - Q/Q_0)$ $Q \leq Q_0$	$t_0 < \infty$ $Q_{max} = Q_0$ $\epsilon_{as} = 0$

Table 2: *Examples for utility functions $u(R)$ (column 1). In column 2 its Legendre tranformed function is given and in column 3 its properties. The first example can be immune against defection, the second cannot.*

The scalefactor in (5) defines the units the utility function is measured with. A nontrivial case is obtained, if this scale can be tight on scale changes of the resource R by

$$u(\lambda R) = \lambda^{1-\epsilon} u(R) \qquad (8)$$

A scaling law like (8) occurs in physics, especially in chaotic systems, in a number of different cases (e.g. fractal dimensions, phase transitions, quantum field theory of renormalizable fields a.s.o.). The exponent $-\epsilon$ is called the anomalous dimension of u. Requiring equ.(8) at all R would restrict $u(R)$ to a powerlike behavior as example 1 in table 2. For some of our applications it is sufficient that the scaling law is only valid for small R, which is expressed by the following definition:

Definition 2.1 *$u(R)$ is asymptic scaling with anomalous dimension $-\epsilon_{as}$ if for sufficient small R the inequality*

$$|u(R) - a_1 \cdot R^{1-\epsilon_{as}}| < a_2 R^{\omega}$$

holds with positive constants a_1, a_2 and ω.

Small R means, that the corrections to scaling proportional to R^{ω} can be neglected. u has to exist at $R \to 0$, therefore ϵ_{as} has to be restricted to $\epsilon_{as} < 1$. Only the first example in table 2 has a nonvanishing anomalous dimension $-\epsilon_{as}$. ϵ_{as} coincides with the value of the elasticity of $u'(R)$ at $R \to 0$:

$$\epsilon_{as} = - \lim_{R \to 0} \frac{u''(R)R}{u'(R)} \tag{9}$$

It is easy to prove that asymptic scaling utility functions satisfy equ.(8) at small R.
Frequently the utility function u appears in an optimization problem. A common method applied in statistical physics (Landau, 1969) is the use of the so called Legendre transformed function $g(Q)$ (Rockafellar, 1970), which is defined by the transformation

$$Q = u'(R) \tag{10}$$

$$g(Q) = Q \cdot g'(Q) - u(R(Q)) \tag{11}$$

In (11) $R(Q)$ is obtained from (10) by inverting the function $u'(R)$, which is given in terms of g as

$$R = g'(Q) \tag{12}$$

It is easy to show that $g(Q)$ has the same convexity properties as $u(R)$, i.e. $g'(Q) > 0$ and $g''(Q) < 0$. A problem can be characterized equally well by $g(Q)$ as by $u(R)$. For our examples the Legendre transformed can be computed (column 2 of table 2). Note that linear combinations of positive utility functions with positive coefficients are again positive utility function respecting (2). An important property is the support of g given by equ.(10). The intervall $R \,\epsilon\, [\, 0,\ R_{max}\,]$ is mapped on the intervall $Q \,\epsilon\, [\, Q_{max}\ ,\ Q_{min}\,]$. For our game utility functions with the property

Definition 2.2 *$u(R)$ has infinite range, if $Q_{max} = \infty$ and $u''(R)R/u'(R)$ is integrable near $R = 0$.*

can support infinite long duration of the game, as will be shown in the next section.

3 Cooperative game

In the cooperative game the extraction $R_i(t)$ of player i at time t follows from the maximum of the common utility function

$$F(R) = \sum_{i=1}^{N} \sum_{t=0}^{t_0} u\left(R_i(t)\right) \delta^t \tag{13}$$

with respect to R with the subsidiary condition

$$\sum_{i} \sum_{t=0}^{t_0} R_i(t) = \bar{S} \tag{14}$$

In equ. (13) it is assumed, that the game ends after step t_0. Since $u(0) = 0$, all sums extent only up to t_0. The actual duration can be found by maximizing F with respect to t_0. If this optimal duration t_0 vanishes, the outcome of the game is the same as in a non-cooperative solution where each player plans to extract the whole stock and receives the share $\bar{S}/N$. Long term cooperation is characterized by $t_0 = \infty$. The solution R^* of the maximum of (13) can be expressed in terms of the Legendre transformed $g(Q)$ (see for example Rockafellar(1970)) in the following way

$$R_i^*(t) = g'(\mu\delta^{-t}) \tag{15}$$

The shadow price $\mu(\bar{S}, \delta)$ has to be determined from condition (14)

$$N \sum_{t=0}^{t_0} g'(\mu\delta^{-t}) = \bar{S} \tag{16}$$

Equ. (16) yields a one to one relation between μ and $\bar{S}$. The stock can be equally well characterized by shadow price μ instead of $\bar{S}$. Therefore the nonlinear equation (16) needs not to be solved explicitly, which is one of main advantages of using Legendre transformations. The optimal value of the utility function (13)

$$F^*(\mu, \delta, t_0) = F(R^*) \tag{17}$$

depends also on t_0. Since $\mu\delta^{-t}$ becomes large at $t \to \infty$, a necessary condition for infinite duration is that $u(R)$ leads to $Q_{max} = \infty$. In addition the now infinite sum in (16) has to be convergent, which can be expressed as an integrability condition of $u(R)$. Thereby we arrive at our necessary condition for infinite duration (the proof is given in the appendix A.1)

Criterion 3.1 *An infinite duration is only possible if u has infinite range according definition 2.2*

Criterion 3.1 is met by the first two examples in table 2. Criterion 3.1 alone does not guarantee infinite duration. If $u'(R)$ diverges too fast at $R \to 0$, then contributions of high t in equns.(16) and (13) become very small, and finite t_0 will be favoured. This is formulated more precisely by the following theorem which is is proven in appendix A.2.

Theorem 3.1 *If u has infinite range according definition 2.2 and satisfies* $\lim_{R \to 0} R \cdot u'(R) = 0$, *the optimal duration* t_0 *becomes arbitrary large* ($t_0 = \infty$).

The additional restriction of vanishing $R \cdot u'(R)$ at $R = 0$ is not very severe, since it needs a rather singular function $u(R)$ to have simultaneously $u(0) = 0$ and $R \cdot u'(R) \neq 0$ at $R = 0$. As a consequence all examples in table 2 do not violate this restriction. The first two examples satisfy also criterion 3.1 and support therefore infinitely long cooperation. They differ, however, with respect to immunity against defection we will discuss in the next section.

4 Stability against defection

The CC game discussed in the previous section has a Pareto optimal solution given by (15). However, the game is in general not stable against unilateral defection, if the PC or NC rules are imposed. For the rest of this section we assume an utility function having infinite long duration under CC-rules. We want to investigate conditions, under which a player may deviate from the optimal extractions R^* given by equ. (15). In PC one player may cooperate $t_D - 1$ time steps and seize the rest of the stock after the extraction of the other players. He will be inclined to do this, if his total gain

$$f^{(PC)} = \sum_{t=0}^{t_D-1} u\left(g'(\mu\delta^{-t})\right)\delta^t + \left[u(S(t_D)) - (N-1)g'(\mu\delta^{-t_D}))\right]\delta^{t_D} \qquad (18)$$

exceeds his gain by cooperation

$$f^{(CC)} = \sum_{t=0}^{\infty} u\left(g'(\mu\delta^{-t})\right)\delta^t \qquad (19)$$

In equ.(18) $S(t_D)$ denotes the stock size in time step t_D given by

$$S(t_D) = \bar{S} - N\sum_{t=0}^{t_D-1} g'(\mu\delta^{-t}) \qquad (20)$$

Since the other players know that this player defects if the stability condition

$$\Delta f^{(PC)} = f^{(CC)} - f^{(PC)} \geq 0 \qquad (21)$$

is violated, we may say that the outcome of the game will be Pareto-inferior in this case. Only if (21) is satisfied, we know that cooperation is immune against unilateral defection. To study the effect of condition (21) we eliminate as before $\bar{S}$ in favour of μ by equ.(16) with $t_0 = \infty$ and $S(t_D)$ by introducing $Q^{(PC)}$ defined by

$$g'(Q^{(PC)}\delta^{-t_D}) - g'(\mu\delta^{-t_D}) = N\sum_{t>t_D} g'(\mu\delta^{-t}) \qquad (22)$$

With the help of $Q^{(PC)}$ the function $f^{(PC)}$ can be written

$$f^{(PC)} = \sum_{t=0}^{t_D-1} \delta^t u(g'(\mu\delta^{-t})) + \delta^{t_D} u(g'(Q^{(PC)}\delta^{-t_D})) \qquad (23)$$

Now we replace u by its Laplace transformed g. After using equ.(22) we can arrange $\Delta f^{(PC)}$ as the sum of two terms

$$\Delta f^{(PC)} = \Delta f_1^{(PC)} + \Delta f_2 \tag{24}$$

with

$$\Delta f_1^{(PC)} = (\mu - Q^{(PC)})g'(Q^{(PC)}\delta^{-t_D}) + \delta^{t_D}(g(Q^{(PC)}\delta^{-t_D}) - g(\mu\delta^{-t_D})) \tag{25}$$

and

$$\Delta f_2 = -\sum_{t>t_D}[\delta^t g(\mu\delta^{-t}) + (N-1)\mu g'(\mu\delta^{-t})] \tag{26}$$

The first term is positive and depends also on $Q^{(PC)}(\mu,\delta)$ given by equ.(22) as function of μ and δ. Using the convexity property of g one proves the inequality (for a proof see appendix B)

$$0 \leq \Delta f_1^{(PC)} \leq (\mu - Q^{(PC)})(g'(Q^{(PC)}\delta^{-t_D}) - g'(\mu\delta^{-t_D})) \tag{27}$$

Δf_2 is independent of $Q^{(PC)}$, but not necessarily positive. If $\Delta f_2 \geq 0$ under certain conditions, than $\Delta f^{(PC)} \geq 0$ holds and cooperation is immune against unilateral defection. Before doing this we discuss the analogous expressions in the NC game. The difference to the PC game is, that one player can take more than his share in a cooperative game so that the stock is exhausted after time step t_D and the game will be finished. The other $N-1$ players cannot observe this defection and will continue to take their share given by equ.(15). The actual resources taken by the defecting player involve an optimization problem similar to the one discussed in section 3 and also the solution procedes in a similar way as in the PC-game discussed in the beginning of this section. These calculations are given in appendix B. The result for the difference $\Delta f^{(NC)} = f^{(NC)} - f^{(CC)}$ is the following

$$\Delta f^{(NC)} = \Delta f_1^{(NC)} + \Delta f_2 \tag{28}$$

with

$$\Delta f_1^{(NC)} = \sum_{t\leq t_D}(\mu - Q^{(NC)})g'(Q^{(NC)}\delta^{-t}) + \delta^t[g(Q^{(NC)}\delta^{-t}) - g(\mu\delta^{-t})] \tag{29}$$

and Δf_2 the same as before. $Q^{(NC)}$ is given by

$$\sum_{t\leq t_D} g'(Q^{(NC)}\delta^{-t}) - g'(\mu\delta^{-t}) = N\sum_{t>t_D} g'(\mu\delta^{-t}) \tag{30}$$

$\Delta f_1^{(NC)}$ is positive and obeys the inquality

$$0 \leq \Delta f_1^{(NC)} \leq \sum_{t \leq t_D} (\mu - Q^{(NC)})(g'(Q^{(NC)}\delta^{-t}) - g'(\mu\delta^{-t})) \tag{31}$$

As we see the NC game differs form the PC game only that in the definition of Δf, the condition for $Q^{(NC)}$ and the inequality a sum $\sum_{t \leq t_D}$ appears instead of a single term.
For both games cooperation will be supported if $\Delta f_2 \geq 0$. Let us first consider the case of small, but positive δ. The utility function satisfies the necessary criterion (3.1), therefore the right hand sides of equ. (22) and (30) are small, which implies a small difference $\mu - Q^{(NC)}$. Since $\Delta f_1^{(PC)}$ and $\Delta f_1^{(NC)}$ are of second order in this small difference, they can be neglected. If in addition u satisfies asymptotic scaling (see definition 2.1), the sum for Δf_2 in (26) can be carried out for small δ

$$\Delta f_2 = \frac{a_1^{1/\epsilon_{as}}}{1 - \epsilon_{as}} \mu^{1-1/\epsilon_{as}} \cdot N \left(\epsilon_{as} - \frac{N-1}{N} \right) \cdot \delta^{(1+t_D)/\epsilon_{as}} \tag{32}$$

The neglected terms in (32) are of order $\delta^{(2+t_D)/\epsilon_{as}}$ and $\delta^{(1+\omega+t_D)/\epsilon_{as}}$. Δf_2 is only positive if the asymptotic elasticity is larger than $\epsilon_c = (N-1)/N$. This completes the proof of the following theorem

Theorem 4.1 *If the utility function has infinite range according definition 2.2 and satisfies asymptotic scaling with the anomalous dimension ϵ_{as}, the game is for small δ immune against NC or PC defection if and only if $\epsilon_{as} \geq \epsilon_c = (N-1)/N$.*

At finite δ the situation is more complicated, since a negative Δf_2 can be balanced by the positive contribution of $\Delta f_1^{(NC)}$ or $\Delta f_1^{(PC)}$, which depends on the details of the utility function. Therefore the requirement of positive Δf_2 is only sufficient. Δf_2 is a sum of terms

$$r(Q) = -(g(Q) + (N-1)Qg'(Q)) \tag{33}$$

For the derivative of $r(Q)$ we obtain

$$r'(Q) = -Ng'(Q) - (N-1)Qg''(Q) \tag{34}$$

By replacing $g''(Q)$ by the elasticity $\epsilon(R) = -g'(Q)/(Qg''(Q))$

$$r'(Q) = -NR\left(1 - \frac{\epsilon_c}{\epsilon(R)}\right) \tag{35}$$

we find, that $r'(Q)$ is negative for all $\epsilon(R) \geq \epsilon_c$ with $R = g'(Q)$. If we know, that for values $Q \to \infty$ $r(Q)$ is positive, equ. (35) implies that $r(Q)$ is positive for all Q with $\epsilon(R) \geq \epsilon_c$. $Q \to \infty$ corresponds to the case $\delta \to 0$, for which we can apply theorem 4.1. This finding we express in the following theorem which gives the most salient result of our paper.

Theorem 4.2 *If the utility function has infinite range according definition 2.2 and satisfies for the elasticity $\epsilon(R) \geq \epsilon_c$ for all $R \leq \bar{S}$, the game is immune against NC or PC defection for all $\delta > 0$.*

Theorem 4.2 states that the discount factor plays no role for sustaining cooperation for a certain range of elasticities. Instead, cooperation can emerge irrespective of the discount factor. This result is in deep contrast with the findings of the theory of repeated games. Theorem 4.2 says that the discount factor is irrelevant under certain conditions. In this case, time preferences play no role for the sustainability of cooperation. If N is small, there is always a chance that all players cooperate irrespective of the discount factor. If N is equal to 2, the range of possible elasticities supporting long-run cooperation irrespective of the discount factor is divided equally into a subrange in which the discount factor is irrelevant and into a subrange in which the discount factor matters. As we learn from the example in the next section *sufficiently large δ may lead to cooperation even if $\epsilon(R) \geq \epsilon_c$ is violated*, and there are cases, where $\epsilon(R) \geq \epsilon_c$ is sufficient for cooperation. Therefore no general statement is possible, unless more restrictions are imposed on the utility function.

5 Example and Phase Transition

As an example we discuss the utility function of example 1 in table 2 given by

$$u(R) = \frac{R^{1-\epsilon}}{1-\epsilon} \tag{36}$$

and its Laplace transformed function

$$g(Q) = \frac{\epsilon}{\epsilon - 1} Q^{(\epsilon-1)/\epsilon} \tag{37}$$

$u(R)$ has infinite range (definition 2.2). The elasticity ϵ is independent of R and therefore it is also asymptotic scaling with $\epsilon_{as} = \epsilon$. Acording criterion 3.1 it supports infinite duration of the cooperative game. From theorem 4.2 we learn, that this cooperation will be sustained for

$$\epsilon \geq \epsilon_c = \frac{N-1}{N} \tag{38}$$

With the equations (36) and (37) the optimal extractions $R_i^*(t)$ for the controlled game CC and from there the differences $\Delta f^{(i)}$ for defection with type $i = PC, NC$ can be computed

$$R_i^*(t) = \frac{\bar{S}}{N}(1-x)x^t \tag{39}$$

$$\Delta f^{(PC)} = \left(\frac{\bar{S}}{N}\right)^{1-\epsilon} \frac{x^{t_D}}{1-\epsilon}[(1-x)^{-\epsilon} - (1+(N-1)x)^{1-\epsilon}] \tag{40}$$

$$\begin{aligned} \Delta f^{(NC)} &= \left(\frac{\bar{S}}{N}\right)^{1-\epsilon} \frac{(1-x)^{-\epsilon}}{1-\epsilon} \\ &\quad \left[\, 1 - (1+(N-1)x^{t_D+1})^{1-\epsilon}(1-x^{t_D+1})^{\epsilon} \,\right] \end{aligned} \tag{41}$$

(42)

where we used the abbreviation $x = \delta^{1/\epsilon}$. The $\Delta f^{(i)}$ change sign at $\delta = \delta_c^{(i)}(\epsilon, t_D)$. For the partial controlled game $\delta_c^{(PC)}(\epsilon)$ is independent of the time t_D where defection happens. Since the gain $-\Delta f^{(PC)}$ decreases with t_D, the optimal time of defection is $t_D = 0$. The critical curve $\delta_c^{(PC)}(\epsilon)$ is shown for $N = 2, 4, 10$ in figure 1. For $\epsilon > \epsilon_c = (N-1)/N$ the function

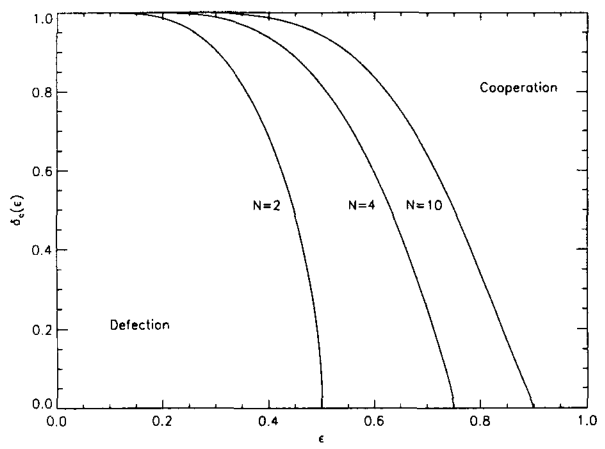

Figure 1: *Critical curves δ_c within PC-defection are shown in the $(\delta, \epsilon)-$plane for various number of players N. To the right of each curve are regions where cooperation id sustained, to the left regions where one player will defect.*

$\delta_c(\epsilon) = 0$ vanishes in agreement with theorem 4.2. For $\epsilon < \epsilon_c$ the game (PC) still supports cooperation for sufficient large $\delta > \delta_c(\epsilon)$.
In contrast to the (PC) case the critical curve $\delta_c^{(NC)}(\epsilon, t_D)$ depends on t_D. Again the relation $\delta_c^{(NC)}(\epsilon, t_D) = 0$ holds for $\epsilon > \epsilon_c$. For $\epsilon < \epsilon_c$ the function $-\Delta f^{(NC)}$ can be maximized with respect to t_D

$$t_D^{max} + 1 = \epsilon \cdot \frac{\ln(1 - \epsilon/\epsilon_c)}{\ln \delta} \tag{43}$$

with $\epsilon_c = (N-1)/N$. Since t_D has to be a non negative integer, we get regions in the ϵ, δ plane where $-\Delta f^{(NC)}(\epsilon, t_D)$ is maximal. These regions are shown in figure 2 in the (ϵ, δ)-plane. For $\epsilon < \epsilon_c$ and sufficient small δ the time step $t_D = 0$ is optimal. In this case there is no difference between PC and NC strategy. If ϵ approaches ϵ_c, t_D gets larger and larger and approaches the value $t_D = \infty$, where NC becomes identical to the CC strategy. Therefore for all $\epsilon < \epsilon_c$ there will be a timestep t_D where defection becomes profitable for one player, within NC rules. In contrast to NC rules PC rules allow also cooperation for $\epsilon < \epsilon_c$ if the discount factor δ is sufficiently large.

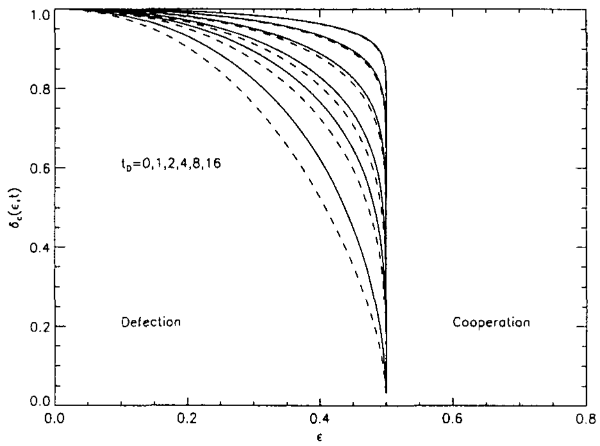

Figure 2: *Critical curves δ_c within NC-defection at various defection times t_D are shown in the (δ, ϵ)−plane for $N = 2$ players. Dashed lines give the loci of maximal $\Delta f^{(NC)}(t_D)$ with respect to t_D, solid lines $\Delta f^{(NC)}(t_D) = \Delta f^{(NC)}(t_D + 1)$ holds. As in figure 1 regions of cooperation are to the left of the curves.*

The critical curve $\delta_c^{(PC)}(\epsilon)$ shown in figure 1 allows an interpretation in physical terms. The behavior of δ_c as function of ϵ

$$\delta_c^{(PC)}(\epsilon) = \begin{cases} 0 & : \quad \epsilon > \epsilon_c \\ 2^{\epsilon_c}(1 - \epsilon/\epsilon_c)^{\epsilon_c} & : \quad \epsilon \lesssim \epsilon_c \\ 1 - \epsilon(1/2)^{\epsilon} & : \quad \epsilon \to 0 \end{cases} \tag{44}$$

is typical for a second order phase transition (Binney, 1992). In that case δ_c would be called an order parameter (magnetization, difference of density of liquid and gas, ratio of components in alloys or mixtures of liquids, asymmetry of a cristal, orientation of polar molecules in a LCD device a.s.o.). It vanishes as function of ϵ above a critical value ϵ_c of an parameter or field ϵ (which may be the temperature, pressure, chemical potential, voltage a.s.o.), exhibits a power law near ϵ_c as in equ. (44) and saturates for small ϵ as in equ.(44). It divides in the (δ, ϵ)-plane regions of two phases, which are distinguished by $\delta_c = 0$ (disordered phase) and $\delta_c \neq 0$ (ordered phase). The importance of those phase transition in physical applications rests on two properties:

- Near the critical point all systems exhibit long range correlation in the ordered phase and physical quantities have anomalous dimension (sometimes called soft chaotic behavior), which means observables like δ_c obey power laws as (44).
- The power laws are universal in the sense, that the exponents depend only on the type of order parameter (scalar, vector, phase factor a.s.o.) and the dimension of the system, but are independent of the details of the dynamics.

In physical applications the underlying dynamic is known at least qualitatively. In our case we have in the cooperation phase only an equilibrium condition, that the common utility function (3) ought to have a maximum. In the defection phase even this condition is missing. This phase is only characterized by the inequality $\Delta f^{(i)} \leq 0$. As a consequence one cannot claim that $\delta_c(\epsilon)$ is the true orderparameter, but any function of δ_c may serve the purpose equally well. Interpreting the time steps as a one dimensioned lattice space, the cooperation phase corresponds to the ordered phase, since cooperation establishes a long range correlation in time, whereas the defection may be called the disordered phase. Our phase transition is of second order since at the critical line $\Delta f^{(i)} = 0$ and $\partial \Delta f^{(i)} / \partial \bar{S} = 0$ hold simultaneously. This follows from the fact that $\bar{S}$ dependence of $\Delta f^{(i)}$ is merely a factor $\bar{S}^{1-\epsilon}$. An important support for this interpretation is the universality. Near the critical point only values of δ near zero are important. As we have shown in section 4, for small δ only the behavior of $u(R)$ near $R \to 0$ is important. Any utility function obeying asymptotic scaling will lead to a phase diagram near $\delta = 0, \epsilon_{as} = \epsilon_c$ similar to the one shown in figures 1 and 2. The exponent ϵ_c appearing in (44) is also universal in the sense it does not depend on the utility function, but only on the number of players N. This size dependence is somewhat disturbing from a physical point of view. As excuse we can argue, that the order parameter may not be δ_c, but rather $x_c = \delta_c^{1/\epsilon}$. Then a universal exponent 1 would follow. Another argument could follow from the fact, that our model is one dimensional (t is a one dimensional discrete index). According to a well known theorem (Mermin, 1966) those systems should not lead to a phase transition unless in the dynamic forces of infinite range exist. Since in our case past and future are known to all players, this exception will happen. In this case a N dependent exponent may be reasonable.

6 Concluding Remarks

This paper has explored the scope for cooperation in an exhaustible resource extraction game. In order to face a relevant problem, we have assumed an utility function which reduces to a power law at small values of resource extractions. In view of theorem 3.1 this is a rather mild additional assumption beyond the necessary infinite duration. For those utility functions we have been able to show that cooperation can emerge irrespective of the discount factor for a certain range of elasticities. If the elasticity of the marginal utility does not fall short of $(N-1)/N$ with N the number of players, cooperation is never at a risk. We have observed that the minimum value for cooperation of the discount factor behaves similar as an order parameter in second order phase transitions in statistical physics. For elasticities above the critical value, the discount factor plays no role, for elasticities below the critical level, cooperation depends on the discount factor.

Compared to the folk theorem in repeated games, our results stress the relevance of the elasticity of the marginal utilities. Increasing f.e. the elasticity has two effects. Firstly, the resource extraction decreases in the beginning of the game making defection more attractive. Secondly, defection itself is made less profitable since the total gain for each player increases with the elasticity. Our result shows that the second effect overcompensates the first effect for all elasticities above the critical level. The example has demonstrated that below the critical value the game can exhibit a behavior expected from the folk theorem, that cooperation depends on the discount factor.

The logarithmic function is of special relevance for our results. If the number of players becomes large, cooperation independent of the discount factor is possible only if the elasticity ϵ is close to 1, which is equivalent to a logarithmic utility function. For such functions cooperation is sustained irrespectively of the discount factor and also of the number of players. Although logarithmic functions are often thought of to explain observable intertemporal behavior, for example saving behavior, one cannot claim this as evidence in real-life resource extraction games with long-term cooperation. The prediction of our model that long-term cooperation with many participants independent of the discount factor should imply logarithmic behavior at small values of resource extractions, could be tested empirically.

Acknowledgements. The authors would like to thank Horst Herberg, Grenot Klepper and Oliver Lorz for valuable discussion.

A General Proofs

A.1 Proof of Criterion 3.1

Suppose the duration t_0 of the game becomes $t_0 = \infty$. The normalization condition (16) requires $g'(Q)$ at arguments $Q = \mu\delta^{-t}$ becoming arbitrarity large for $t \to \infty$. Therefore the support of the function $g'(Q)$ must include $Q_{max} = \infty$. The sum over t in (16) involves decreasing positive terms. It exists only if the integral over t exists:

$$\sum_t g'(\mu\delta^{-t}) \leq \int_{t_0}^{\infty} dt g'(\mu\delta^{-t}) \tag{45}$$

In the integral on the r.h.s. of (45) we replace the variable t by $R = g'(\mu\delta^{-t})$ and obtain

$$\sum_t g'(\mu\delta^{-t}) \leq (-1/\ln\delta) \int_0^{R_0} dR\, R\, (-u''(R))/u'(R) \tag{46}$$

According the assumption in criterion 3.1 the integral exists. In a similar way one can estimate, that the discrete sum is larger than an integral like in equ. (46) with a different upper limit R_0' (or t_0'). The finding $Q_{max} = \infty$ and the necessary existence of the integral completes the proof of criterion 3.1.

A.2 Proof of Theorem 3.1

The optimal duration t_0 of the game will tend to $t_0 \to \infty$ if we can prove the inequality

$$\Delta F = F^*(\mu(t_0+1), t_0+1) - F^*(\mu(t_0), t_0) \geq 0 \tag{47}$$

First we prove two lemmata.

Lemma A.1 *If the utility function $u(R)$ satisfies $u(0) = 0$ and $lim_{R\to 0} Ru'(R) = 0$, its Legendre transformed is negative $g(Q) \leq 0$.*

The Legendre transformation maps the point $R = 0$ to $Q_{max} = u'(0)$. Setting in the definition (11) of $g(Q)$

$$g(Q) = Ru'(R) - u(R)$$

$Q = Q_{max}$ and $R = 0$ we find $g(Q_{max}) = 0$. Since $g'(Q) \geq 0$ all $Q \leq Q_{max}$, this implies $g(Q) \leq 0$.

Lemma A.2 *The shadow price $\mu(t_0)$ is an increasing function of t_0, or alternatively the difference $\Delta\mu = \mu(t_0+1) - \mu(t_0)$ is nonnegative: $\Delta\mu \geq 0$*

Subtracting the equation (16) for t_0 from the corresponding for $t_0 + 1$ we obtain

$$g'\left((\mu+\Delta\mu)\delta^{-t_0-1}\right) + \sum_{t=0}^{t_0} g'\left((\mu+\Delta\mu)\delta^{-t}\right) - g'(\mu\delta^{-t}) = 0 \qquad (48)$$

From the mean value theorem of the analysis we write for $g'\left((\mu+\Delta\mu)\delta^{-t}\right)$

$$g'\left((\mu+\Delta\mu)\delta^{-t}\right) = g'(\mu\delta^{-t}) + \Delta\mu\delta^{-t}g''(\bar{\mu}\delta^{-t}) \qquad (49)$$

with $\mu \leq \bar{\mu} \leq \mu + \Delta\mu$. Inserting (49) into (48) we obtain

$$\Delta\mu = -g'\left((\mu+\Delta\mu)\delta^{-t_0-1}\right) / \sum_{t=0}^{t_0} g''(\bar{\mu}\delta^{-t})\delta^{-t}$$

Since $g' \geq 0$ and $g'' \leq 0$, the inquality $\Delta\mu \geq 0$ follows. The difference ΔF can be expressed in terms of the Laplace transformed g:

$$\begin{aligned}\Delta F &= \bar{S}\Delta\mu - \delta^{t_0+1}g\left(\mu(t_0+1)\delta^{-t_0-1}\right) \\ &\quad - \sum_{t=0}^{t_0}\delta^{t}\left(g((\mu+\Delta\mu)\delta^{-t}) - g(\mu\delta^{-t})\right) \qquad (50)\end{aligned}$$

The mean value theorem applied twice leads to

$$g((\mu+\Delta\mu)\delta^{-t}) = g(\mu\delta^{-t}) + \Delta\mu\delta^{-t}g'(\mu\delta^{-t}) + \Delta\mu(\mu_1-\mu)g''(\mu_2\delta^{-t})\delta^{-2t} \quad (51)$$

with $\mu \leq \mu_2 \leq \mu_1 \leq \Delta\mu + \mu$. Inserting the expression (51) into (50), we get

$$\Delta F = \delta^{t_0+1}\left[-g(\mu(t_0+1)\delta^{-t_0-1})\right] + \Delta\mu(\mu_1-\mu)\sum_{t=0}^{t_0}\left[-g''(\mu_2\delta^{-t})\delta^{-t}\right]$$

According the lemmata $g < 0$ and $\Delta\mu > 0$ hold. Since $\mu_1 \geq \mu$ and $g'' < 0$ ΔF is a sum of only positive terms. Positive ΔF means $F^{*}(t_0,\mu)$ is an increasing function of t_0 which proves the optimal t_0 tends to ∞.

B Optimization for NC defection

The defecting player's utility function is given by

$$f^{(NC)} = \sum_{t=0}^{t_D} u(R(t))\delta^t \tag{52}$$

He maximizes $f^{(NC)}$ with respect to his extractions $R(t)$ and the subsidiary condition

$$\sum_{t=0}^{t_D} R(t) + (N-1)\sum_{t=0}^{t_D} g'(\mu\delta^{-t}) = \bar{S} \tag{53}$$

The condition for the maximum reads

$$R(t) = g'(Q^{(NC)}\delta^{-t}) \tag{54}$$

where the defecting shadow price $Q^{(NC)}$ is determined from equ. (53). Eliminating $\bar{S}$ in favor of μ we obtain

$$\sum_{t=0}^{t_D}[g'(Q^{(NC)}\delta^{-t}) - g'(\mu\delta^{-t})] = N\sum_{t>t_D} g'(\mu\delta^{-t}) \tag{55}$$

Since g' is positive and decreasing, the inequality

$$Q^{(NC)} \leq \mu \tag{56}$$

holds. His optimized gain is

$$f^{(NC)} = \sum_{t=0}^{t_D}[Q^{(NC)}g'(Q^{(NC)}\delta^{-t}) - \delta^t g(Q^{(NC)}\delta^{-t})] \tag{57}$$

if we use the function $g(Q) = Q\cdot R - u(R)$ instead of u. Subtracting the gain $f^{(CC)}$ given by equ. (19), we obtain for the difference

$$\begin{aligned}\Delta f^{(NC)} &= \sum_{t\leq t_D} \mu g'(\mu\delta^{-t}) - Q^{(NC)}g'(Q^{(NC)}\delta^{-t}) + \delta^t(g(Q^{(NC)}\delta^{-t}) - g(\mu\delta^{-t}) \\ &+ \sum_{t=0}^{t_D}[\mu g'(\mu\delta^{-t}) - g(\mu\delta^{-t})\delta^t]\end{aligned} \tag{58}$$

Eliminating $\sum_{t\leq t_D} g'(\mu\delta^{-t})$ by equ. (56) $\Delta f^{(NC)}$ can be cast in the form equ. (28)

$$\Delta f^{(NC)} = \Delta f_1^{(NC)} + \Delta f_2 \tag{59}$$

with Δf_2 given by (26) and

$$\Delta f_1^{(NC)} = \sum_{t \leq t_D} [(\mu - Q^{(NC)})g'(Q^{(NC)}\delta^{-t}) + \delta^t(g(Q^{(NC)}\delta^{-t} - g(\mu\delta^{-t}))] \quad (60)$$

To prove the inequalities (27) and (31) we consider the function $d(x)$

$$d(x) = (x - x_0)g'(x_0) + g(x_0) - g(x) \quad (61)$$

Since the tangent of a concave function is always above the function, we have the following inequality for any t and x

$$g(t) \leq g(x) + (t - x)g'(x) \quad (62)$$

Setting $t = x_0$ we get

$$g(x_0) \leq g(x) + (x_0 - x)g'(x) \quad (63)$$

A second inequality is obtained by interchanging x_0 and x

$$g(x) \leq g(x_0) + (x - x_0)g'(x_0) \quad (64)$$

Both inequalities (63) and (64) inserted into (61) leads to

$$0 \leq d(x) \leq (x - x_0)\,(g'(x_0) - g'(x)) \quad (65)$$

If we set $x = \mu\delta^{-t_D}$ and $x_0 = Q^{(NC)}\delta^{-t_D}$ equ.(65) is equivalent to equ.(27). Since $\Delta f_1^{(NC)}$ is a sum of d functions with $x = \mu\delta^{-t}$ and $x_0 = Q^{(NC)}\delta^{-t}$ (65) immediately proves the inequality (31).

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