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VOTERS' POWER IN INDIRECT VOTING SYSTEMS WITH POLITICAL PARTIES: THE SQUARE ROOT EFFECT

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Voters' Power in Indirect Voting Systems with Political Parties: the Square Root Effect*

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Abstract

A modification of the Banzhaf index is proposed which is applicable to indirect voting systems with multiple political parties. By the use of the modified index, it is shown that the square root effect holds even when more than two parties exist. That is, the decreasing influence of voters varies inversely with the square root of the increasing population.

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1 Introduction

The following claim is widely prevailed concerning the fairness of indirect voting systems: "An indirect voting system is fair, or voters have the same influence, if seats are allotted to electoral districts in proportion to their population". The claim implicitly assumes that each voter's influence varies inversely with population of his districts. It has been shown, however, that the assumption is not true from the viewpoint of the Banzhaf power index when each district has one seat. Instead, the so-called "square-root effect" arises; that is, the decreasing influence of voters measured in terms of the Banzhaf index varies inversely with the square root of the increasing population. See Banzhaf [1966] and also Lucas [1983]. [1]

The model assumed in these studies is basically the following. In each district, voters vote "yes" or "no" on a certain bill. Then the representatives vote "yes" or "no" on the bill based on the majority opinion of the voters of their own districts. "Approval" or "disapproval" of the bill is decided through the simple majority rule among the representatives.

In our actual voting systems, however, voters usually vote a political party or vote a person that a party puts forward as a candidate; not directly yes or no.

When there exist only two parties as in the United States, the model above can be straightforwardly applied. The basic idea is the following. In each district voters vote one of the two parties, say Party A or Party B; and then the seat is given to the party that received the majority votes. In the House, representatives belonging to the same party behave as a block; and vote yes or no on each bill proposed. To measure voters' power, we list up all "Party A-Party B" combinations of voters' votes and all "yes-no" combinations of parties' votes. Then for each voter, count the number of combinations in which he can change final decisions in the House from approval to disapproval or vice versa by changing his vote from one party to the other.[2] It is easily seen that this procedure reaches the Banzhaf index of the original "yes-no" setting.

[1] Refer to Dubey and Shapley [1979] for mathematical properties of the Banzhaf index.

[2] Voters can indirectly alter final decisions through the change of the party that represents their district.

When more than two parties exist, as observed in Japan, many European contries, etc., however, the Banzhaf index is not directly applicable since voters have more than two alternatives. The aim of the paper is to study voters' power in such voting systems.

First we need to modify the Banzhaf index preserving at least its spirit. The idea of modification is based on the discussion above for systems with two parties. We first list up all combinations of voters' choices, and also all yes-no combinations of parties' choices. Then for each voter, count the number of combinations in which he can change final decisions in the House from approval to disapproval or vice versa, by changing his vote, say from one party to one of the others. By the use of this modified Banzhaf index, we will show that the square-root effect holds even when more than two parties exist: this is the pricipal finding of the paper.

The rest of the paper will be organized as follows. In Section 2, the model is described on which we work throughout the paper. The modification of the Banzhaf index is given in Section 3. Section 4 presents the main theorem: the proof is given in Sections 5 and 6. Since the proof is complicated, we first present the detailed proof for the simplest three party case in Section 5; and then in Section 6 we briefly explain how the proof can be extended to the case with more than three parties. Some of the mathematical details of the proof are given in the appendix. The paper ends in Section 7 with short remarks.

2 A Model

Let $D = \{d^1, \ldots, d^n\}$ be the set of electoral districts. For each district d^i, let Q^i be the set of its voters, and let $q^i = |Q^i|$, i.e., the number of voters in Q^i. The sets Q^i's are mutually disjoint. Let $P = \{p_1, \ldots, p_m\}$ be the set of political parties. Exactly one seat is allotted to each district.

In each district, there are m candidates: the jth candidate stands for the jth party p_j, $j = 1, \ldots, m$. Each voter has exactly one vote and casts it to one of the candidates; and a candidate who obtains the largest number of votes wins the seat. Ties are resolved by a random choice.

For notational convenience, denote a voter k's vote, $k \in Q^i$, by an m-dimensional vector $v^{ik} = (v_1^{ik}, \ldots, v_m^{ik})$ with one 1 and $(m-1)$ 0's where $v_j^{ik} = 1$ if the voter k votes the jth candidate. Let $v_j^i = \sum_{k \in Q^i} v_j^{ik}$, $j = 1, \ldots, m$, and let $v^i = (v_1^i, \ldots, v_m^i) : v_j^i$ is the total number of votes given to the jth candidate in the district d^i. Let $j^* = \text{argmax}\{v_j^i : j = 1, \ldots, m\}$. Then the j^*th candidate wins the seat. If there are more than one such j^*, each of them may win with equal probability. Let $r^i = (r_1^i, \ldots, r_m^i)$ be an m-dimensional vector with one 1 and $(m-1)$ 0's where $r_j^i = 1$ if the jth candidate, or the jth party, wins the seat in the district d^i.

Let $r_j = \sum_{i=1}^n r_j^i$, $j = 1, \ldots, m$, and let $r = (r_1, \ldots, r_m)$. The number r_j is the total number of seats that the jth party holds in the House. Note that $\sum_{j=1}^m r_j = n$ since exactly one seat is allotted to each district.

In the House, "Approval (A)" or "Disapproval (D)" of each bill is decided by a simple majority rule among the elected representatives. We assume that representatives belonging to the same party behave as a block; hence all of them vote "Yes (Y)" or all of them vote "No (N)". Thus for each bill, if the set of parties voting "Y" is S, then the bill is approved when and only when S has the majority. To avoid ties in the House, we assume n to be odd; thus the condition is $\sum_{j \in S} r_j \geq (n+1)/2$.

Our main concern is to measure each voter's power, i.e., to evaluate to what extent he has an influence on decisions on bills in the House. The power index used below is similar to the Banzhaf index at least in its spirit. In our model, however, each voter has only an indirect influence on final decisions in the House. What he may directly do is to alter the party that wins the seat in his district. It then may alter the number of representatives belonging to each party in the House; and thus, it may change final decisions on bills.

3 A Modified Banzhaf Index

The Banzhaf index, in usual direct voting systems, measures each voter's power by counting the number of "Y" – "N" combinations of votes in which he may change final outcomes by changing his vote. More precisely, consider all "Y" – "N" combinations of all voters. In each combination, a voter is called a swing if he may change a final outcome from "A" to "D" (or from "D" to "A") by changing his vote from "Y" to "N" (or from "N" to "Y"). The Banzhaf index, more precisely the absolute Banzhaf index, of a voter is given by the fraction of his being a swing over all "Y" - "N" combinations of votes, or in other words, by the probability that he is a swing provided that each combination is equiprobable.

In our model, however, each voter may only indirectly alter a final outcome; and further he may have multiple ways to change his vote. Recall there are m candidates in his district, and thus he has $(m-1)$ possible ways to change his vote. On the basis of these characteristics of our model, we define a power index for our model in the following manner.

Consider first all possible combinations of voters' votes in all districs; in addition, consider all combinations of parties' votes in the House. Let Π^i be the set of all combinations of votes in the ith district, and Π^H be the set of all combinations of parties' votes in the House. Since there are m parties, each voter has m alternatives to choose. Each of the m parties has two alternatives to choose, i.e., "Y" or "N" in the House. Thus $|\Pi^i| = m^{q^i}$ and $|\Pi^H| = 2^m$: recall q^i is the number of voters in the ith district. Let $\Pi = \Pi^1 \times \ldots \times \Pi^n \times \Pi^H$; and thus, $|\Pi| = m^{q1} \times \ldots \times m^{qn} \times 2^m$.

Take a combination $\pi = (\pi^1, \ldots, \pi^n, \pi^H) \in \Pi$. Suppose the final decision induced by the combination π is "A". Pick up a voter k in the ith district, and suppose he votes for the jth candidate in the combination π. Then he may change his vote to one of the other $(m-1)$ candidates; thus he has $(m-1)$ different ways of changing his vote. Let $s^k(\pi)$ denote the number that k can be a swing in π, i.e., the number of ways that k may change the final outcome from "A" to "D". Thus $0 \leq s^k(\pi) \leq m-1$. If $s^k(\pi) = 0$, then k has no way to change the final outcome; and if $s^k(\pi) = m-1$, then k may change the final outcome by changing his vote from the jth candidate to any of the others. If $0 < s^k(\pi) < m-1$, then we have an inbetween of these two extremes. Assuming that each of $m^{q1} \times \ldots \times m^{qn} \times 2^m$ combinations is equiprobable and that ties of the largest number of votes are resolved by a random choice, we define a modified Banzaf index β^k of the voter k by the expected number of his being a swing. It is to be noted that since voters in the same district are symmetric their modified Banzhaf indices are identical.

Thus in what follows we denote by β^i the modified Banzhaf index of a voter in the ith district.

4 Passage to the Limit and the Main Theorem

Our concern is to evaluate each voter's relative power measured by the modified Banzhaf index when every district's population is quite large. For this purpose, we augment every district's population keeping their proportions fixed, and study asymptotic behavior of ratios of voters' indices in different districts. Let $\alpha^1, \dots, \alpha^n$ be rational numbers representing proportions of districts' populations, and K be their least common multiple. Let M be a positive integer, $M = 1, 2, \dots$, and $q^i(M) = \alpha^i KM$, $i = 1, \dots, n$. Let $\beta^i(M)$ be the modified Banzhaf index of a voter in the ith district when the district has $q^i(M)$ voters, $i = 1, \dots, n$. We want to examine the asymptotic behavior of $\beta^i(M)/\beta^{i'}(M)$ as $M \to \infty$ for each pair of $i, i' = 1, \dots, n$, $i \neq i'$. As a main theorem, we obtain the following. The theorem shows that the relative power per capita of each district is inversely proportional to the square root of its population. Hence the "square root effect" shown in Banzhaf [1966] and Lucas [1983] also holds even if there are more than two parties.

Main Theorem: *For each two districts d^i and $d^{i'}$,*

$$\lim_{M \to \infty} \frac{\beta^i(M)}{\beta^{i'}(M)} = \frac{\sqrt{\alpha^{i'}}}{\sqrt{\alpha^i}}.$$

5 A Proof of the Main Theorem: the Case with Three Parties

For simplicity of presentation, we first present a proof for the simplest case with three parties. A similar proof holds even if there are more than three paries, which will be given in the next section. Only a major difference is that the normal distribution which will appear in the following discussion is replaced by a multi-variate normal.

5.1 Decomposition of $\beta^i(M)$

Let p_1, p_2 and p_3 be the three parties, and suppose each district i has $\alpha^i KM$ voters, $i = 1, \ldots, n$. Pick up a district, say the ith district, and take one of its voters, say the kth voter. We first rewrite the modified Banzhaf index $\beta^i(M)$ of this voter. Recall every voter in this district has the same index, and thus the notation $\beta^i(M)$ is used. In what follows, we use the following notations to represent several events:

$V(j), j = 1, 2, 3$: the voter votes the jth candidate: recall that the candidate belongs to the jth party;

$V(j \to j'), j, j' = 1, 2, 3, j \neq j'$: the voter changes his vote from the jth to the j'th candidate;

$R^i(j), j = 1, 2, 3$: the jth party wins the seat of the ith disctrict;

$R^i(j \to j'), j, j' = 1, 2, 3, j \neq j'$: the party winning the seat is changed from the jth to the j'th.

$A \leftrightarrow D$: a final outcome in the House is changed from A to D or from D to A.

Then since the modified Banzhaf index $\beta^i(M)$ is defined as the expected number of the kth voter's being a swing, $\beta^i(M)$ is rewritten as

$$\beta^i(M) = \sum_{j=1}^{3} \left(\sum_{\substack{j' = 1 \\ j' \neq j}}^{3} \text{Prob } (A \leftrightarrow D | V(j \to j')) \text{ Prob } (V(j)) \right). \tag{5.1}$$

Here Prob$(\cdot)$ denotes the probability that an event $\cdot$ occurs. Since the kth voter may change a final outcome through the change of a winning party of the district, the inner sum of (5.1) is further rewritten as

$$\sum_{\substack{j'=1 \\ j' \neq j}}^{3} \left(\sum_{\substack{f,g=1 \\ f \neq g}}^{3} \text{Prob } (A \leftrightarrow D | R^i(f \to g)) \times \text{Prob } (R^i(f \to g) | V(j \to j')). \right. \quad (5.2)$$

Prob$(\cdot | \cdot\cdot)$ denotes the conditional probability of $\cdot$ given $\cdot\cdot$.

5.2 Evaluation of Prob$(A \leftrightarrow D | R^i(f \to g)), f, g = 1, 2, 3, f \neq g$

Let h be the party other than f, g. Denote a combination of parties' voters by $(*, *, *) : *$ stands for Y or N, and the first (second, third, resp.) element corresponds to a vote by the party f (g, h, resp.). For example, (Y,N,Y) implies that f and h vote Y and g votes N.

Recall first there are $2^3 = 8$ combinations of parties' votes in the House. Thus the desired probability, henceforth denoted $a(f, g)$ for saving spaces, is rewritten as

$$\begin{aligned} a(f,g) &= \text{Prob}(A \leftrightarrow D \ \& \ (Y,Y,Y) | R^i(f \to g)) \\ &+ \ \ldots + \text{Prob } (A \leftrightarrow D \ \& \ (N,N,N) | R^i(f \to g)) \end{aligned} \quad (5.3)$$

Let $a(f, g; (*, *, *)) = \text{Prob}(A \leftrightarrow D \ \& \ (*, *, *) | R^i(f \to g))$ where $*$ stands for Y or N.

Suppose both of f and g vote Y or both vote N. Then a final outcome never changes even if a winning party of the district changes from f to g. Thus we obtain

$$\begin{aligned} a(f,g;(Y,Y,Y)) &= a(f,g;(Y,Y,N)) \\ &= a(f,g;(N,N,Y)) = a(f,g;(N,N,N)) = 0 \end{aligned} \quad (5.4)$$

Take $a(f, g; (Y, N, Y))$. It is easily seen that under the combination (Y, N, Y) the change of a winning party from f to g alters a final outcome only when $r_f + r_h = (n+1)/2$: in this case the final outcome changes from A to D. Recall r_j is the total number of seats that the party j holds in the House, $j = 1, 2, 3$. Therefore

$$\begin{aligned} a(f,g;(Y,N,Y)) &= \text{Prob } (r_f + r_h = (n+1)/2 \ \ \& \ \ (Y,N,Y)|R^i(f \to g)) \\ &= \text{Prob } (r_g^{-i} = (n-1)/2 \ \ \& \ \ (Y,N,Y)|R^i(f \to g)) \\ &\quad \text{where } r_g^{-i} = \sum_{\substack{i'=1 \\ i' \neq i}}^{n} r_g^{i'}. \end{aligned} \tag{5.5}$$

r_g^{-i} is the number of seats that the party g holds in districts other than the ith.

Now since in the i'th district, $i' \neq i$, combinations of voters' votes are equiprobable, and further ties of the largest votes are resolved by a random choice, we have

$$\text{Prob } (R^{i'}(f)) = \text{Prob } (R^{i'}(h)) = \text{Prob } (R^{i'}(h)) = 1/3.$$

Recall $R^{i'}(j)$ denotes the event that the party j wins the seat of the i'th district, $j = 1,2,3$. Therefore (5.5) is rewritten as

$$a(f,g;(Y,N,Y)) = \frac{1}{2^3} \times \sum_{\substack{x,y \geq 0 \\ x+y=(n-1)/2 \\ x,y \text{ integers}}} \frac{(n-1)!}{((n-1)/2)!x!y!} \left(\frac{1}{3}\right)^{n-1} \tag{5.6}$$

From the same reason, we have

$$a(f,g;(Y,N,Y)) = a(f,g;(N,Y,N)) = \text{the r.h.s. of (5.6)}. \tag{5.7}$$

Hereafter we denote the sum of the r.h.s. of (5.6) by X. It should be noted that X depends neither on f nor on g. Thus (5.6) and (5.7) hold for all $f, g = 1,2,3, \ f \neq g$. Thus from (5.3), (5.4), (5.6) and (5.7), we obtain

$$a(f,g) = X/2 \text{ for all } f,g = 1,2,3, \ f \neq g. \tag{5.8}$$

It is to be noted further that X is independent of districts' population.

5.3 Evaluation of Prob $(R^i(f \to g)|(V(j \to j'))$, $f, g = 1, 2, 3$, $f \neq g$

In what follows denote this probability by $b(f, g; j, j')$. Let j'' be the party other than j and j'. First let $f = j'$. Then $v^i(j') \geq \max(v^i(j), v^i(j''))$ since the party j' wins the seat. Recall $v^i(j)(v^i(j'), v^i(j''),$ resp.) is the total number of votes that the candidate (or the party) $j(j', j''$, resp.) obtains. Thus under $V(j \to j')$, the party j' still wins the seat. Hence

$$b(j', j; j, j') = b(j', j''; j, j') = 0 \tag{5.9}$$

In a similar manner, we obtain

$$b(j'', j; j, j') = 0. \tag{5.10}$$

We now examine the remaining three probabilities $b(j, j'; j, j')$, $b(j, j''; j, j')$, and $b(j'', j; j, j')$. Recall the ith district has $\alpha^i KM$ voters. We first assume $\alpha^i KM$ is a multiple of three, and let $\alpha^i KM = 3\ell$ where ℓ is a positive integer. Then we have the following five cases in which $V(j \to j')$ induces $R^i(j \to j')$.

(1) $v^i(j) = v^i(j') = v^i(j'') = \ell$;

(2) $v^i(j) = v^i(j') \geq v^i(j'') + 3$;

(3) $v^i(j) = v^i(j') + 1 \geq v^i(j'') + 2$;

(4) $v^i(j) = v^i(j') + 2 \geq v^i(j'') + 4$;

(5) $v^i(j) = v^i(j') + 1 = v^i(j'') + 2$.

Hence

$$
\begin{aligned}
b(j,j';j,j') &= \text{Prob}\ (R^i(j \to j')\ \ \&\ \ (1)|V(j \to j')) \\
&+\ \ldots + \text{Prob}\ (R^i(j \to j')\ \ \&\ \ (5)|V(j \to j')).
\end{aligned}
\tag{5.11}
$$

In the following we explain in detail the two cases: Cases (1) and (3).

Case (1): Since ties are resolved by a random choice, each of the three parties may win the seat with equal probability 1/3. When the voter k changes his vote from j to j', the party j' wins the seat for sure. Thus

$$
\text{the first term of the r.h.s. of (5.11)} = \frac{1}{3} \times \frac{3\ell - 1)!}{(\ell - 1)!\ell!\ell!} \times \left(\frac{1}{3}\right)^{3\ell-1}. \tag{5.12}
$$

Applying Stirling's approximation to the r.h.s. of (5.12) we obtain the following proposition.

Proposition 5.1 :

$$
\textit{the first term of (5.11)} \underset{\ell}{\sim} \frac{1}{2\sqrt{3}\pi} \times \frac{1}{\ell} \tag{5.13}
$$

where $\underset{\ell}{\sim}$ *implies that the ratio of both sides converges to 1 as* $\ell \to \infty$.

Proof. See the appendix.

Case (3): The party j wins the seat for sure; whereas if the voter k changes his vote from j to j', the party j' wins the seat. Thus

$$
\text{the third term of (5.11)} = \sum_{x=\ell}^{[(3\ell-1)/2]} \frac{(3\ell - 1)!}{x!x!(3\ell - 2x - 1)!} \times \left(\frac{1}{3}\right)^{3\ell-1} \tag{5.14}
$$

where $[\cdot]$ is the largest integer in $\cdot$. Using a similar device to the proof of DeMoivre-Laplace's convergence theorem (the convergence of a binomial to a normal distribution, e.g. refer to Feller [1957]), we obtain the following proposition: the detailed proof is given in the appendix.

Proposition 5.2 :

$$\textit{the third term of (5.11)} \underset{\ell}{\sim} \frac{1}{2\sqrt{\pi\ell}} \times (1 - \Phi(0)) = \frac{1}{4\sqrt{\pi\ell}} \tag{5.15}$$

where $\Phi(\cdot)$ denotes the normal distribution function with mean 0 and variance 1.

Proof. See the appendix.

As for the remaining three cases, we have similar asymptotic values. In Case (2), the parties j, j' may win the seat with equal probability $1/2$, and when the voter k changes his vote from j to j', the party j' wins for sure. In Case (4), the party j wins the seat, and the parties j, j' may win the seat with equal probability $1/2$ after the change of voter k's vote. In Case (5), the party j wins the seat, and after the change of voter k's vote, three parties may win the seat with equal probability $1/3$. Thus in a similar manner as above, we obtain the following proposition.

Proposition 5.3 :

$$\begin{aligned} \textit{the second and fourth terms} &\underset{\ell}{\sim} \frac{1}{4\sqrt{\pi\ell}} \times \frac{1}{2} \\ \textit{and the fifth term} &\underset{\ell}{\sim} \frac{1}{2\sqrt{3}\pi\ell}. \end{aligned} \tag{5.16}$$

Hence, from (5.11), (5.13), (5.15) and (5.16) we obtain

$$b(j, j'; j, j') \underset{\ell}{\sim} \frac{1}{2\sqrt{\pi\ell}}. \tag{5.17}$$

We next examine $b(j, j''; j, j')$. We have the following three cases in which $V(j \to j')$ induces $R^i(j \to j'')$.

1. $v^i(j) = v^i(j'') \geq v^i(j') + 3$;

2. $v^i(j) = v^i(j'') + 1 = v^i(j') + 2$;

3. $v^i(j) = v^i(j'') + 1, v^i(j'') \geq v^i(j') + 4$:

recall that $\alpha^i KM$ is a multiple of three. Thus in a similar manner as above, we obtain

$$b(j, j''; j, j') \underset{\ell}{\sim} \frac{1}{4\sqrt{\pi\ell}}. \tag{5.18}$$

As to $b(j'', j'; j, j')$, we similarly obtain

$$b(j'', j'; j, j') \underset{\ell}{\sim} \frac{1}{4\sqrt{\pi\ell}}. \tag{5.19}$$

Since $\alpha^i KM = \ell$, we have from (5.17), (5.18) and (5.19)

$$b(j, j'; j, j') + b(j, j''; j, j') + b(j'', j'; j, j') \underset{m}{\sim} \frac{1}{\sqrt{\pi(\alpha^i KM/3)}}. \tag{5.20}$$

It is easily shown that (5.20) holds even if $\alpha^i KM$ is not a multiple of three.

5.4 Evaluation of $\beta^i(M)$ and of the ratio $\beta^i(M)/\beta^{i'}(M)$

We first obtain from (5.2), (5.8), (5.9), (5.10) and (5.20) that the asymptotic value of (5.2) (the inner sum of (5.1)) is given by

$$\sum_{\substack{j'=1 \\ j' \neq j}}^{3} \text{Prob } (A \leftrightarrow D | V(j \to j')) \underset{M}{\sim} \sum_{\substack{j'=1 \\ j' \neq j}}^{3} \frac{X}{2} \times \frac{1}{\sqrt{\pi(\alpha^i KM/3)}} \tag{5.21}$$

where X is the sum of the r.h.s. of (5.6). Recall that X does not depend on districts' populations. Since every combination of voters' votes is equiprobable, we have Prob $(V(j)) = 1/3$. Thus it follows from (5.1) and (5.21) that

$$\beta^i(M) \underset{M}{\sim} \frac{X}{2\sqrt{\pi\alpha^i KM/3}}.$$

Therefore for each i and i' we obtain the desired relation

$$\lim_{M\to\infty} \frac{\beta^i(M)}{\beta^{i'}(M)} = \frac{1/\sqrt{\alpha^i}}{1/\sqrt{\alpha^{i'}}} = \frac{\sqrt{\alpha^{i'}}}{\sqrt{\alpha^i}}.$$

6 A Proof of the Main Theorem: the Case with More Than Three Parties

Suppose there are m parties ($m > 3$), and let the set of parties $P = \{p_1, \ldots, p_m\}$. Then similarly to (5.1) and (5.2), we have

$$\beta^i(M) = \sum_{j=1}^{m} \left(\sum_{\substack{j'=1 \\ j' \neq j}}^{m} \text{Prob } (A \leftrightarrow D | V(j \to j')) \right) \text{Prob } (V(j)) \tag{6.1}$$

and the inner sum of (6.1) is rewritten as

$$\sum_{\substack{j'=1 \\ j' \neq j}}^{m} \left(\sum_{\substack{f,g=1 \\ f \neq g}}^{m} \text{Prob } (A \leftrightarrow D | R^i(f \to g) \text{Prob } (R^i(f \to g) | V(j \to j')) \right) \tag{6.2}$$

6.1 Prob $(A \leftrightarrow D|R^i(f \to g))$

As in Section 5.2, $R^i(f \to g)$ may induce $A \leftrightarrow D$ only when the parties f, g vote differently, i.e., f votes Y and g votes N, or f votes N and g votes Y. Thus letting S be a set of parties which vote Y, we obtain

$$\text{Prob } (A \leftrightarrow D|R^i(f \to g)) = \sum_{|S|=0}^{m} {}_mC_{|S|} \sum_{\substack{\sum_{i \in S} x_i = (n-1)/2 \\ \sum_{i=1}^{m} x_i = n-1}} \frac{(n-1)!}{x_1! \dots x_m!} \left(\frac{1}{m}\right)^{n-1} \times \frac{1}{2}$$

where ${}_mC_{|S|}$ denotes the number of combinations of taking $|S|$ out of m elements. This probability does not depend on districts' populations similarly to the three party case.

6.2 Prob $(R^i(f \to g)|V(j \to j'))$:

We first notice that, as in Section 5.3, $V(j \to j')$ may induce $R^i(f \to g)$ only in the following three cases: (1) $f = j, g = j'$; (2) $f = j, g = j''$; and (3) $f = j'', g = j'$ where $j'' \neq j, j'$. One of the typical cases which induce the case (1) is

$$v^i(j) = v^i(j') + 1, v^i(j') \geq v^i(j'') \text{ for all } j'' \neq j, j' : \tag{6.3}$$

this case corresponds to Case (3) of the evaluation of $b(j, j'; j, j')$ in Section 5.3. We obtain

$$\begin{aligned} &\text{Prob } (R^i(j \to j') \ \& \ (6.3)|V(j \to j')) \\ &= \sum_{\substack{x = [(\alpha^i KM - 1)/m] \\ y_1 + \dots + y_{m-2} = \alpha^i KM - 1 - 2x \\ y_1, \dots, y_{m-2} \leq x}}^{[(\alpha^i KM - 1)/2]} \frac{(\alpha^i KM)!}{x! x! y_1! \dots y_{m-2}!} \left(\frac{1}{m}\right)^{\alpha^i KM - 1}. \end{aligned}$$

It is shown in a similar manner to the proof of Proposition 5.2 that this probably is of the order $\sqrt{1/\alpha^i KM}$; but the normal distribution appeared in the proposition must be replaced by a multi-variate normal. By the use of this fact, the desired result follows in a similar manner to that in the three party case.

7 Concluding remarks

We have studied voters' power in indirect voting systems with multiple, in particular more than two, parties. Since voters have more than two alternatives, we have first propose a modification of the Bamzhaf index. Then using the modified index, we have shown the square root effect holds even when more than two parties exist. We conclude the paper with short remarks concerning possible future research directions.

The first is to extend the model so that it may cover the case where districts may have unequal numbers of seats. In the case of two parties, studies were done by Owen [1975] and Muto [1989].[3] The former assumed the party received majority of votes takes all seats, and analyzed voters' power in the U.S. Presidential Election; while the latter assumed the seats are given to each party in proportion to the votes they received, and demonstrated the following: (1) voters in districts with even number of seats are completely powerless, and (2) voters' power in districts with odd seats depends only on districts' population and never depends on the number of seats allotted to districts. It must be interesting to study their counterparts in the case with more than two parties.

The second and more conceptually difficult question is on the application of the Shapley-Shubik index. Since the Shapley-Shubik index is defined on the basis of coalition formation, it seems much more difficult to extend this notion so that it may be applied to voting systems with more than two parties. The difficulty arises since coalition formation among political parties must be taken into consideration together with coalition formation among voters.

These problems will be studied in future papers.

[3] Banzhaf [1966] also analyzed the case with unequal numbers of seats; but his analysis was limited to voters' influences within districts. Voters' influences on final decisions made by representatives were not considered.

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Appendix

Proof of Proposition 5.1: It follows from (5.12) that

$$\begin{aligned}\text{the first term of (5.11)} &= \frac{1}{3}\times\frac{(3\ell-1)!}{(\ell-1)!\ell!\ell!}\times\left(\frac{1}{3}\right)^{3\ell-1}\\ &= \frac{(3\ell)!}{\ell!\ell!\ell!}\times\frac{\ell}{3\ell-1}\times\left(\frac{1}{3}\right)^{3\ell}.\end{aligned}$$

By Stirling's approximation, we obtain

$$\begin{aligned}\frac{(3\ell)!}{\ell!\ell!\ell!}\times\left(\frac{1}{3}\right)^{3\ell} &\underset{\ell}{\sim} \frac{\sqrt{2\pi(3\ell)}(3\ell)^{3\ell}\exp(-3\ell)}{(\sqrt{2\pi\ell}(\ell)^{\ell}\exp(-\ell))^3}\times\frac{1}{3^{3\ell}}\\ &= \frac{\sqrt{3}}{2\pi\ell}.\end{aligned}$$

Further $\lim_{\ell\to\infty} \ell/(3\ell-1) = 1/3$. Hence we obtain

$$\text{the first term of (5.11)} \underset{\ell}{\sim} \frac{1}{2\sqrt{3}\pi} \times \frac{1}{\ell}. \qquad Q.E.D.$$

Proof of Proposition 5.2: We first recall (5.14), i.e.,

$$\text{the third term of (5.11)} = \sum_{x=\ell}^{[(3\ell-1)/2]} \frac{(3\ell-1)!}{x!x!(3\ell-2x-1)!} \times \left(\frac{1}{3}\right)^{3\ell-1}. \tag{A.1}$$

Denote the r.h.s. of (A.1) by A. Just for simplicity, suppose $3\ell-1$ is even in what follows. Let $y = x-\ell$, and thus $3\ell-2x-1 = \ell-1-2y$. Then

$$A = \sum_{y=0}^{(3\ell-1)/2} \frac{(3\ell-1)!}{(\ell+y)!(\ell+y)!(\ell-1-2y)!} \times \left(\frac{1}{3}\right)^{3\ell-1}. \tag{A.2}$$

Denote each component of the sum of (A.2) by $t(y)$, and for each s, s' with $0 \le s < s' \le (\ell-1)/2$, denote $\sum_{y=s}^{s'} t(y)$ by $t(s,s')$. Further let $\ell^* = [\ell^{7/12}]$. Then we have

$$\begin{aligned} A = t\left(0, \frac{\ell-1}{2}\right) &= t(0,\ell^*) + t(\ell^*+1, \frac{\ell-1}{2}) \\ &= t(0,\ell^*)\left(1 + \frac{t(\ell^*+1,(\ell-1)/2)}{t(0,\ell^*)}\right) \end{aligned} \tag{A.3}$$

Claim A.1:

$$\lim_{\ell\to\infty} \frac{t(\ell^*+1,(\ell-1)/2)}{t(0,\ell^*)} = 0.$$

Proof of Claim A.1: We first obtain

$$\frac{t(y)}{t(y-1)} = \frac{(\ell+1-2y)(\ell-2y)}{(\ell+y)(\ell+y)} \text{ for all } y = 1,\ldots,(\ell-1)/2.$$

As for the r.s.h., we easily obtain

$$0 < \frac{(\ell+1-2y)(\ell-2y)}{(\ell+y)(\ell+y)} < 1 \text{ for all } y = 1,\ldots,(\ell-1)/2; \tag{A.4}$$

and further this term is decreasing in y. Thus we have

$$\frac{t(y)}{t(y-1)} \leq \frac{(\ell-2\ell^*-1)(\ell-2\ell^*-2)}{(\ell+\ell^*+1)(\ell+\ell^*+1)} \text{ for all } y = \ell^*+1,\ldots,(\ell-1)/2. \tag{A.5}$$

Denote the r.h.s. of (A.5) by $B : 0 < B < 1$ from (A.4). Then we obtain from (A.5)

$$\begin{aligned} \frac{t(\ell^*+u)}{t(\ell^*)} &= \frac{t(\ell^*+1)}{t(\ell^*)} \times \frac{t(\ell^*+2)}{t(\ell^*+1)} \times \ldots \times \frac{t(\ell^*+u)}{t(\ell^*+u-1)} \\ &\leq B^u \quad \text{for all } u = 1,\ldots,(\ell-1)/2-\ell^*. \end{aligned}$$

Thus

$$\begin{aligned} t(\ell^*+1,(\ell-1)/2) &\leq t(\ell^*)(B^1+B^2+\ldots+B^{(\ell-1)/2-\ell^*}) \\ &\leq t(\ell^*)(B^1+B^2+\ldots\ldots\ldots) \\ &= t(\ell^*)(B/(1-B)) : \text{ recall } 0 < B < 1. \end{aligned} \tag{A.6}$$

In the meanwhile, it follows from (A.4) that $t(0) > t(1) > \ldots > t((\ell-1)/2)$. Hence

$$t(0,\ell^*) > (\ell^*+1)t(\ell^*) \tag{A.7}$$

Thus from (A.6) and (A.7),

$$\frac{t(\ell^* + 1, (\ell - 1)/2)}{t(0, \ell^*)} < \frac{1}{\ell^* + 1} \times \frac{B}{1 - B}. \tag{A.8}$$

Since B is the r.h.s. of (A.5) and $\ell^* = [\ell^{7/12}]$, it follows from a straightforward calculation that the r.h.s. of (A.8) is of the order $1/\ell^{1/6}$. Noting that the l.h.s. of (A.8) is positive by (A.4), we thus obtain that the l.h.s. of (A.8) converges to 0 as $\ell \to \infty$. Q.E.D.

We now examine the asymptotic behavior of $t(0, \ell^*)$. We first note that $t(y)$ is rewritten as

$$\begin{aligned} t(y) &= \frac{(3\ell - 1)!}{(\ell + y)!(\ell + y)!(\ell - 1 - 2y)!} \times \left(\frac{1}{3}\right)^{3\ell - 1} \\ &= \frac{(3\ell)!}{(\ell + y)!(\ell + y)!(\ell - 2y)!} \times \left(\frac{1}{3}\right)^{3\ell} \times \left(\frac{\ell - 2y}{\ell}\right). \end{aligned} \tag{A.9}$$

Let

$$C = \frac{(3\ell)!}{(\ell + y)!(\ell + y)!(\ell - 2y)!} \times \left(\frac{1}{3}\right)^{3\ell}.$$

Then by Stirling's approximation,

$$\begin{aligned} C &\underset{\ell}{\sim} \frac{\sqrt{3\ell}}{2\pi(\ell + y)\sqrt{\ell - 2y}} \times \frac{\ell^{3\ell}}{(\ell + y)^{2(\ell + y)}(\ell - 2y)^{\ell - 2y}} \\ &= \frac{\sqrt{3\ell}}{2\pi(\ell + y)\sqrt{\ell - 2y}} \times \frac{1}{(1 + y/\ell)^{2\ell + 2y}(\ell - 2y/\ell)^{\ell - 2y}}. \end{aligned} \tag{A.10}$$

We are examining $t(0, \ell^*)$, and thus $y \leq \ell^* = [\ell^{1/12}]$. Thus we obtain

$$\text{the first term of (A.10)} \underset{\ell}{\sim} \frac{\sqrt{3}}{2\pi\ell}. \tag{A.11}$$

We now examine the second term of (A.10). Let its denominator be D. Then

$$\log_e D = (2\ell + 2y)\log_e(1 + y/\ell) + (\ell - 2y)\log_e(1 - 2y/\ell).$$

By Taylor's expansion, $\log_e D = 3y^2/\ell + y^3/\ell^2 + 6y^4/\ell^3 + \ldots$. Since $y \le \ell^* = [\ell^{7/12}]$, $y^3/\ell^2, y^4/\ell^3, \ldots \to 0$ as $\ell \to \infty$. Hence

$$\log_e D \underset{\ell}{\sim} 3y^2/\ell, \text{ or } D \underset{\ell}{\sim} \exp(3y^2/\ell). \tag{A.12}$$

Substituting (A.11) and (A.12) to (A.10), we obtain

$$\begin{aligned} C &\underset{\ell}{\sim} \frac{\sqrt{3}}{2\pi\ell} \times \exp\left(\frac{-3y^2}{\ell}\right) \\ &= \frac{1}{2\sqrt{\pi\ell}} \times \frac{1}{\sqrt{2\pi(\ell/6)}} \times \exp\left(\frac{-1}{2} \times \frac{6y^2}{\ell}\right). \end{aligned}$$

Since $(\ell - 2y)/\ell \to 1$ as $\ell \to \infty$, we have

$$t(y) \underset{\ell}{\sim} \frac{1}{2\sqrt{\pi\ell}} \times \frac{1}{\sqrt{2\pi}\sqrt{\ell/6}} \times \exp\left(\frac{-1}{2} \times \frac{6y^2}{\ell}\right).$$

Let $h = 1/(\sqrt{\ell/6})$ and $z_y = y/(\sqrt{\ell/6})$. Then

$$t(0, \ell^*) = \sum_{y=o}^{\ell^*} t(y) \underset{\ell}{\sim} \frac{1}{2\sqrt{\pi\ell}} \sum_{y=o}^{\ell^*} h\emptyset(z_y)$$

$$\text{where } \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right).$$

Recalling $\ell^* = [\ell^{7/12}]$, we have

$$z_{\ell^*} = \frac{\ell^*}{\sqrt{\ell/6}} \to \infty \quad \text{as} \quad \ell \to \infty.$$

Further $z_0 = 0$. Thus we obtain

$$t(0, \ell^*) \underset{\ell}{\sim} \frac{1}{2\sqrt{\pi\ell}}(1 - \Phi(0)).$$

Thus together with (A.3) and Claim A.1 the proof of Proposition 5.2 is completed.

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