Decomposable Effectivity Functions

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Abstract

Decomposable effectivity functions are introduced as an extension of additive effectivity functions. Whereas additive effectivity functions are determined by pairs of additive TU-games, decomposable effectivity functions are generated by pairs of TU-games that need not be additive. It turns out that the class of decomposable effectivity functions does not only contain the class of additive effectivity functions but it also contains the class of effectivity functions corresponding to simple games and the class of effectivity functions corresponding to veto functions.

We examine relations between properties of decomposable effectivity functions and the TU-games by which they are generated. It turns out that a decomposable effectivity function is stable whenever it can be generated by a pair of balanced TU-utility games. Finally, we provide two characterizations of decomposable effectivity functions.

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1 Introduction

Choice correspondences and choice functions,which describe the collective choice of a number of agents, form a central topic of study in social choice theory. It is assumed that these collective decision rules depend on the preferences of the individual agents. As these preferences are private information, strategic aspects play an important role: By feigning or misrepresenting preferences, individuals or coalitions can influence society's choice to their own benefit but at the expense of others. This defect of many rules is well-known. For instance, Condorcet (1785) already criticized Borda's rule (Borda (1781)) on this point.

Using Arrow's impossibility theorem (Arrow (1951)), Gibbard (1973) and Satterthwaite (1975) independently showed that in the case of three or more alternatives, dictatorial rules are the only choice functions which do not exhibit this strategic behavior. So when studying choice functions and choice correspondences in one way or another, we have to cope with strategic behavior. Therefore it is interesting to know the "power distribution" in society which indicates the opportunities for individual or coalitional manipulation at a given collective decision rule. A way to model this power distribution was introduced in Moulin and Peleg (1982) using the concept of an effectivity function. Formally an effectivity function associates to each coalition a collection of subsets of alternatives for which the coalition is *effective*. If a coalition is effective for a certain subset of alternatives, this means that it is able to force the final outcome of the decision rule at hand to be among the elements of this set, or formulated otherwise, this coalition can *veto* all alternatives outside this set of alternatives.

Possible applications of effectivity functions that are discussed by Moulin and Peleg (1982) are effectivity functions associated with monotonic simple games, additive effectivity functions which are related to voting by veto methods (cf. Moulin (1983)), and neutral and A-monotonic effectivity functions which correspond to veto functions (cf. Moulin (1982)). Effectivity functions corresponding to monotonic simple games form a subclass of effectivity functions corresponding to veto functions. However, there is no inclusion relation between the class of additive effectivity functions and the class of effectivity functions corresponding to veto functions. In this paper we introduce another class of effectivity functions, called decomposable effectivity functions, which comprises the classes mentioned above. In both additive effectivity functions and effectivity functions corresponding to veto functions, a coalition S is effective for a set B, if the veto power of S exceeds the veto resistance of $A \setminus B$. Here the veto resistance is an additive measure and the veto power is either an additive measure (in case of an additive effectivity function) or a TU-game (in the other case). For a decomposable effectivity function a coalition S is effective for a set of alternatives B if the veto power of S exceeds the veto resistance of $A \setminus B$. But now the veto power as well as the veto resistance are described by TU-games being not necessarily additive.

The organization of the paper is as follows. Section 2 defines the concept of an effectivity function and recalls some basic properties of effectivity functions. Furthermore, we reconsider the before-mentioned classes of effectivity functions, namely effectivity functions associated with monotonic simple games, additive effectivity functions, and effectivity functions corresponding to veto functions. In Section 3 we introduce decomposable effectivity functions. Section 4 examines relations between the properties of decomposable effectivity functions and the properties of TU-games that generate these effectivity functions. Among others, it is shown that a decomposable effectivity function is monotonic if and only if it can be generated by monotonic TU-games; further a decomposable effectivity function is stable whenever it can be generated by balanced TU-games. Sections 5 and 6 provide two characterizations of decomposable effectivity functions. First, it is shown that an effectivity function is decomposable if and only if it satisfies the revealed power property. This property can be seen as a modification of the more familiar WARP (=weak axiom of revealed preference) condition in revealed preference theory. Secondly, we show that an effectivity function is decomposable if and only if it is possible to represent the effectivity function by a $\{0,1\}$-matrix in echelon form.

2 Effectivity functions

We start with some basic notations and definitions.
Let X be a finite set. The power set of X is denoted by 2^X, i.e., $2^X := \{Y \mid Y \subset X\}$, and $\mathcal{P}_0(X) := 2^X \setminus \{\emptyset\}$. The cardinality of X is denoted by $|X|$.

Let A be a finite set of *alternatives* and let N be the set $\{1, \ldots, n\}$ ($n \in \mathbb{N}$). N is called a *society*, members of N are called *agents* or *voters*, and non-empty subsets of

N are called *coalitions*. We assume that each agent $i \in N$ has preferences over the set of alternatives which can be represented by a complete and transitive preference relation R_i. Let $a, b \in A$ and $i \in N$. We adopt the usual notation $a\ R_i\ b$ if $(a, b) \in R_i$, and $a\ P_i\ b$ if $a\ R_i\ b$ and not $b\ R_i\ a$. Also as usual $a\ R_i\ b$ is to be interpreted as 'alternative a is at least as good as alternative b according to R_i'. Furthermore, for $S \in \mathcal{P}_0(N)$, $R_S := (R_i)_{i \in S}$. R_N is called a (preference) *profile* on A. The class of all such preference profiles is denoted by $\mathcal{R}_N$.

An *effectivity function* (cf. Moulin and Peleg (1982)) is a map $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ such that

(i) $E(N) = \mathcal{P}_0(A)$

(ii) $A \in E(S)$ for all $S \in \mathcal{P}_0(N)$.

The interpretation of E is as follows: If $B \in E(S)$, then S can force the final decision within the subset B of alternatives. By definition the society N can force the outcome to belong to every (non-empty) subset of alternatives.

An effectivity function E can be represented by means of a $\{0,1\}$-matrix I^E of size $2^n - 1$ by $2^{|A|} - 1$, where for $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$,

$$I^E(S, B) = 1 \text{ if and only if } B \in E(S).$$

We will now consider several properties that effectivity functions might satisfy. We will use these properties later on, but it should be mentioned that this list of properties is certainly not exhaustive. For more properties of effectivity functions we refer the reader to Abdou and Keiding (1992).

Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function.

(i) E is *A-monotonic* if for all $S \in \mathcal{P}_0(N)$ and all $B, B' \in \mathcal{P}_0(A)$ with $B \subset B'$ and $B \in E(S)$, we have $B' \in E(S)$.

(ii) E is *N-monotonic* if for all $S, S' \in \mathcal{P}_0(N)$ with $S \subset S'$, and all $B \in \mathcal{P}_0(A)$, with $B \in E(S)$, we have $B \in E(S')$.

(iii) E is *neutral* if for all $S \in \mathcal{P}_0(N)$, all $B \in E(S)$ and all $B' \in \mathcal{P}_0(A)$ with $|B'| = |B|$, we have $B' \in E(S)$.

(iv) E is *superadditive* if for all $S_1, S_2 \in \mathcal{P}_0(N)$, with $S_1 \cap S_2 = \emptyset$, and all $B_1 \in E(S_1)$, $B_2 \in E(S_2)$, we have $B_1 \cap B_2 \in E(S_1 \cup S_2)$.

(v) E is *convex* if for all $S_1, S_2 \in \mathcal{P}_0(N)$, and all $B_1 \in E(S_1)$, $B_2 \in E(S_2)$, we have $B_1 \cap B_2 \in E(S_1 \cup S_2)$ or $B_1 \cup B_2 \in E(S_1 \cap S_2)$.

It is easy to check that if E is convex, then E is also superadditive, and if E is superadditive, then E is N-monotonic.

Given an effectivity function that describes coalitional power in society and a profile reflecting the individual preferences of all agents, the problem of interest is how to find an alternative, or a set of alternatives, which every agent can agree upon. Since we study situations in which agents behave cooperatively, a rather natural solution concept is the core of an effectivity function (Moulin and Peleg (1982)). The core describes whether the outcome is stable with respect to coalitional deviations.

Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function and $R_N \in \mathcal{R}_N$ a profile. An alternative $a \in A$ is *dominated* by a subset $B \in \mathcal{P}_0(A)$ of alternatives via a coalition $S \in \mathcal{P}_0(N)$ if $B \in E(S)$ and $b\ P_S\ a$ for all $b \in B$. The *core of E at R_N*, $Core(E, R_N)$, consists of all alternatives $a \in A$ which are not dominated by any subset of alternatives via any coalition. An effectivity function E is called *stable* if $Core(E, R_N) \neq \emptyset$ for all profiles $R_N \in \mathcal{R}_N$.
Stability of effectivity functions has been studied by several authors. The first general result on stability of effectivity is due to Peleg (1982), who showed that convex effectivity functions are stable. A complete characterization of stable effectivity functions is due to Keiding (1985).

In the last part of this section we discuss three subclasses of effectivity functions, all introduced in Moulin and Peleg (1982), which play an important role in the literature. Successively, we discuss effectivity functions corresponding to monotonic simple games, additive effectivity functions, and effectivity functions corresponding to veto functions.

Example 2.1 *Simple games*
A *TU-game* on N is a pair (N, v) (often denoted simply by v), where $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. A TU-game v is called a *simple game* if $v(S) \in \{0, 1\}$ for all $S \in 2^N$ and $v(N) = 1$. A simple game v is *monotonic* if for all $S, T \in \mathcal{P}_0(N)$ with

$S \subset T$ and $v(S) = 1$ it holds that $v(T) = 1$. Let S be a coalition. If $v(S) = 1$, then S is a *winning* coalition, and if $v(S) = 0$, then S is called *losing*.

A way of associating an effectivity function $E^v : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ to a monotonic simple game (N, v) is the following. For $S \in \mathcal{P}_0(N)$

$$E^v(S) := \begin{cases} \mathcal{P}_0(A) & \text{if } S \text{ is winning} \\ \{A\} & \text{if } S \text{ is losing.} \end{cases}$$

Winning coalitions have the power to enforce every subset of alternatives, whereas a losing coalition has no power at all. In Peleg (1984a) this effectivity function is called the *standard effectivity function associated with* v. It is clear that E^v is A-monotonic and N-monotonic. Furthermore, if v is *proper*, i.e., $v(S) = 1$ implies $v(N \setminus S) = 0$, then E^v is superadditive, and if v is *strong*, i.e., $v(S) = 0$ implies $v(N \setminus S) = 1$, then E^v is maximal. Finally, if v is *balanced*, i.e., if the *core* $C(v) := \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N\}$ is non-empty, then E^v is stable. A complete characterization of stable effectivity functions associated with a monotonic simple games is provided by Nakamura (1979).

Example 2.2 *Additive effectivity functions*

Let $\lambda \in \mathbb{R}^N$ and $\mu \in \mathbb{R}^A$ be two positive probability measures on N and A, respectively. So, $\lambda_i > 0$ for all $i \in N$ and $\sum_{i \in N} \lambda_i = 1$, and $\mu_a > 0$ for all $a \in A$ and $\sum_{a \in A} \mu_a = 1$. The vectors λ and μ give rise to an effectivity function $E_{\lambda,\mu}$ in the following way. For $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$

$$B \in E_{\lambda,\mu}(S) \text{ if and only if } \sum_{i \in S} \lambda_i > \sum_{a \in A \setminus B} \mu_a.$$

The interpretation is that S is effective for B if the total veto power of S (measured by λ) exceeds the total veto resistance of $A \setminus B$ (measured by μ). Using the fact that $\sum_{a \in A} \mu_a = 1$ we see that

$$B \in E_{\lambda,\mu}(S) \text{ if and only if } \sum_{i \in S} \lambda_i + \sum_{a \in B} \mu_a > 1.$$

It is left to the reader to check that $E_{\lambda,\mu}$ is indeed an effectivity function. An effectivity function $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ is called *additive* if there exist positive probability measures $\lambda \in \mathbb{R}^N$ and $\mu \in \mathbb{R}^A$ such that $E = E_{\lambda,\mu}$. It is clear that these probability measures need not be uniquely determined.

Additive effectivity functions play a prominent role in the literature on effectivity functions. One of the reasons is that additive effectivity functions are convex and hence, stable.

An important application of additive effectivity functions is the class of effectivity functions corresponding to a voting by veto method (cf. Moulin (1983)). Storcken (1994) characterizes the class of additive effectivity functions by associating a simple game with each effectivity function and using Elgot's (1961) characterization of the class of all weighted simple games. For details on this result the reader is referred to Storcken (1994).

Example 2.3 *Veto functions*

A *veto function* (cf. Moulin (1982)) is a function $\nu : 2^N \to \{0, 1, \ldots, |A| - 1\}$ with $\nu(\emptyset) := 0$ and $\nu(N) := |A| - 1$. (Notice that a veto function can be regarded as a TU-game on N.) Given a veto function ν, the *effectivity function E^ν corresponding to ν* is defined by

$$E^\nu(S) := \{B \in \mathcal{P}_0(A) \mid \nu(S) \geq |A \setminus B|\}$$

for all $S \in \mathcal{P}_0(N)$.

We leave it to the reader to verify that E^ν is an effectivity function. Again, the interpretation is that a coalition S is effective for a subset of alternatives if S can veto all alternatives outside B, where the veto power of coalitions is described by the veto function ν (which is a TU-game). Since in this case the veto power of coalitions need not be additive, it is clear that effectivity functions corresponding to veto functions need not be additive effectivity functions.

Several properties of E^ν can be formulated in terms of the veto function ν (cf. Abdou and Keiding (1992)). For example, E^ν is superadditive if and only if ν is *superadditive*, i.e., $\nu(S_1 \cup S_2) \geq \nu(S_1) + \nu(S_2)$ for all $S_1, S_2 \in \mathcal{P}_0(N)$ with $S_1 \cap S_2 = \emptyset$. Otten (1995) shows that, analogous to effectivity functions corresponding to monotonic simple games, balancedness of ν is a sufficient condition for stability of E^ν.

Contrary to the class of additive effectivity functions, it is rather easy to characterize the class of effectivity functions corresponding to veto functions. The effectivity function E^ν corresponding to veto function ν is neutral and A-monotonic. Conversely, every neutral and A-monotonic effectivity function E generates a veto function ν^E defined by

$$\nu^E(S) := \max\{|A \setminus B| \mid B \in E(S)\}$$

for all $S \in \mathcal{P}_0(N)$, such that $E^{\nu^E} = E$.

Since effectivity functions associated with monotonic simple games are both neutral and A-monotonic, it follows that this class is a subclass of the effectivity functions corresponding to veto functions. Additive effectivity functions however, need not be neutral, so this class is not a subclass of the class of effectivity functions corresponding to veto functions.

In the next section we introduce another class of effectivity functions, called decomposable effectivity functions, which incorporates all three classes of effectivity functions that we discussed in this section.

3 Decomposable effectivity functions

Based on the observation that additive effectivity functions can be generated by positive probability measures on N and A, which can be regarded as *additive* TU-games on N and A, we introduce the following generalization of additive effectivity functions.

Let $v : 2^N \to [0,1]$ and $w : 2^A \to [0,1]$ be TU-games on N and A, which satisfy $v(N) = 1$ and $v(S) > 0$ for all $S \in \mathcal{P}_0(N)$, $w(A) = 1$ and $w(B) > 0$ for all $B \in \mathcal{P}_0(A)$. The games v and w generate an effectivity function $E(v,w) : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ as follows. For $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$

$$B \in E(v,w)(S) \text{ if and only if } v(S) + w(B) > 1.$$

An effectivity function $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ is called *decomposable* if there exist TU-games v and w as above such that $E = E(v,w)$. For such TU-games v and w, $E(v,w)$ is called *the effectivity function generated by v and w.*

Here the TU-game v represents the veto power of coalitions and w represents the veto resistance of subsets of alternatives.

It readily follows from this definition that additive effectivity functions are decomposable. The following proposition illustrates that also effectivity functions corresponding to veto functions are decomposable.

Proposition 3.1 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. The following statements are equivalent

(i) E is decomposable

(ii) there exist $v : \mathcal{P}_0(N) \to [0,1]$ and $w : \mathcal{P}_0(A) \to [0,1]$ such that for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$ it holds that

$$B \in E(S) \text{ if and only if } v(S) + w(B) > 1$$

(iii) there exist $v_1 : \mathcal{P}_0(N) \to [0,1]$ and $w_1 : \mathcal{P}_0(A) \to [0,1]$ such that for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$ it holds that

$$B \in E(S) \text{ if and only if } v_1(S) + w_1(B) \geq 1$$

(iv) there exist $v_2 : \mathcal{P}_0(N) \to [0,1]$ and $w_2 : 2^A \to [0,1]$ with $w_2(\emptyset) := 0$ such that for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$ it holds that

$$B \in E(S) \text{ if and only if } v_2(S) \geq w_2(A \setminus B)$$

(v) there exist $v_3 : \mathcal{P}_0(N) \to [0,1]$ and $w_3 : 2^A \to [0,1]$ with $w_3(\emptyset) := 0$ such that for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$ it holds that

$$B \in E(S) \text{ if and only if } v_3(S) > w_3(A \setminus B).$$

As the proof of this proposition is straightforward, it is omitted.[1]
From Proposition 3.1 (iv) we can derive the following corollary.

Corollary 3.2 Effectivity functions associated with monotonic simple games and effectivity functions corresponding to veto functions are decomposable.

4 Properties of TU-games and decomposable effectivity functions

In this section we examine relations between properties of the TU-games v and w and the effectivity function $E(v, w)$.
The following proposition shows that if v and w are monotonic, then $E(v, w)$ is N- and A-monotonic. The proof is straightforward.

Proposition 4.1 Let $E = E(v, w) : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be the decomposable effectivity function generated by the TU-games v and w. Then

[1] The TU-games v and w constructed in the proof of Theorem 5.4 can be used to define the functions v_i and w_i $(i = 1, 2, 3)$.

(i) if v is monotonic, then E is N-monotonic

(ii) if w is monotonic, then E is A-monotonic.

With respect to the converse of this proposition it can be seen that if E is N-monotonic (A-monotonic) and decomposable, then there exist TU-games v and w with v (w) monotonic such that $E = E(v, w)$. (The TU-games v and w constructed in the proof of Theorem 5.4 are monotonic if E is monotonic.)
Proposition 4.2 shows that a decomposable effectivity function is convex if it can be generated by convex TU-games.

Proposition 4.2 Let $E = E(v, w) : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be the decomposable effectivity function generated by the TU-games v and w. If v and w are convex, then E is convex.

Proof. Let v be convex, i.e., for all $S, T \in \mathcal{P}_0(N)$: $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, and let w be convex. Let $S, T \in \mathcal{P}_0(N)$, $B \in E(S)$ and $D \in E(T)$. We have to show that E is convex, i.e., $B \cap D \in E(S \cup T)$ or $B \cup D \in E(S \cap T)$.
Since $v(S) + w(B) > 1$ and $v(T) + w(D) > 1$, we have

$$v(S) + v(T) + w(B) + w(D) > 2.$$

Using convexity of v and w now yields

$$v(S \cup T) + w(B \cap D) + v(S \cap T) + w(B \cup D) > 2.$$

Hence, $v(S \cup T) + w(B \cap D) > 1$ or $v(S \cap T) + w(B \cup D) > 1$. So, $B \cap D \in E(S \cup T)$ or $B \cup D \in E(S \cap T)$. $\square$

The next example shows that $E(v, w)$ is not necessarily superadditive, if both v and w are superadditive.

Example 4.3 Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Define $v : 2^N \to [0, 1]$ by $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 1/3$, $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 2/3$, and $v(N) = 1$, and define $w : 2^A \to [0, 1]$ by $w(\emptyset) = 0$, $w(\{a\}) = w(\{b\}) = w(\{c\}) = 1/4$, $w(\{a, b\}) = w(\{a, c\}) = w(\{b, c\}) = 3/4$, and $w(A) = 1$. Then for all $S, T \in \mathcal{P}_0(N)$ with $S \cap T = \emptyset$ we have $v(S) + v(T) \leq v(S \cup T)$, and for all $B, D \in \mathcal{P}_0(A)$ with $B \cap D = \emptyset$ we have $w(B) + w(D) \leq w(B \cup D)$. So v and w are superadditive. Furthermore, $\{a, b\} \in E(v, w)(\{1\})$ and $\{a, c\} \in E(v, w)(\{2\})$, but $\{a\} \notin E(v, w)(\{1, 2\})$. Hence, $E(v, w)$ is not superadditive.

It can be shown that $E(v,w)$ is superadditive whenever v is superadditive and w convex.

Theorem 4.4 states that if both v and w have a non-empty core, then also the core of $E(v,w)$ is non-empty for every preference profile.

Theorem 4.4 Let $E = E(v,w) : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be the decomposable effectivity function generated by the TU-games v and w. If v and w are balanced, then E is stable.

Proof. Let $x \in C(v)$ and $y \in C(w)$. Then $\sum_{i \in N} x_i = 1$ and $\sum_{a \in A} y_a = 1$. Furthermore, $x_i \geq v(\{i\}) > 0$ for all $i \in N$ and $y_a \geq w(\{a\}) > 0$ for all $a \in A$. So, the vectors x and y determine an additive effectivity function $E_{x,y}$. Moreover, $E(S) \subset E_{x,y}(S)$ for all $S \in \mathcal{P}_0(N)$, since $v(S) + w(B) > 1$ implies $\sum_{i \in S} x_i + \sum_{a \in B} y_a > 1$. Now stability of E follows directly from the fact that $E_{x,y}$ is stable. $\square$

It is an open problem whether each stable decomposable effectivity function can be generated by TU-games v and w both having an non-empty core.

5 A characterization of decomposable effectivity functions

Moulin and Peleg (1982) showed that each neutral and A-monotonic effectivity function corresponds to a veto function and conversely. A characterization of additive effectivity functions is provided by Storcken (1994) using a property that strengthens convexity. In this section we will provide a characterization of the class of decomposable effectivity functions using a modification of the 'weak axiom of revealed preference' in the theory of preference revelation. This property is called the revealed power property.

Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. Suppose that for all coalitions $S, T \in \mathcal{P}_0(N)$ and all subsets $B \in \mathcal{P}_0(A)$ of alternatives with $B \in E(S)$ and $B \notin E(T)$ we have, if $D \in \mathcal{P}_0(A)$ and $D \in E(T)$, then $D \in E(S)$. In this case we say that E satisfies the *revealed power property.*

The interpretation of this property is the following: If an effectivity function satisfies the revealed power property and a coalition S is effective for a certain subset of

alternatives for which coalition T is not effective, then this 'reveals' that S has more power than T, i.e., S is effective for every subset that T is effective for.
It is clear that an effectivity function E satisfies the revealed power property if and only if for all $S, T \in \mathcal{P}_0(N)$ we have

$$E(S) \subset E(T) \text{ or } E(T) \subset E(S).$$

The following proposition shows that the revealed power property is a necessary condition to characterize decomposable effectivity functions.

Proposition 5.1 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. If E is decomposable, then E satisfies the revealed power property.

Proof. Let E be decomposable. Then there exist TU-games v and w such that $E = E(v, w)$. Let $S, T \in \mathcal{P}_0(N)$ with $E(S) \not\subset E(T)$. We show that $E(T) \subset E(S)$. Since there is a $B \in \mathcal{P}_0(A)$ with $v(S)+w(B) > 1$ and $v(T)+w(B) \leq 1$, it follows hat $v(S) > v(T)$. Now let $D \in E(T)$. Then $v(T)+w(D) > 1$ and hence $v(S)+w(D) > 1$, which implies that $D \in E(S)$. So we may conclude that $E(T) \subset E(S)$. $\square$

It turns out that the revealed power property is also a sufficient condition to characterize decomposability. In order to prove this, we first introduce some additional notation.
Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. The *dual of* E (Peleg (1984b)), $E^d : \mathcal{P}_0(A) \to 2^{\mathcal{P}_0(N)}$ is defined as follows. For $B \in \mathcal{P}_0(A)$

$$E^d(B) = \{S \in \mathcal{P}_0(N) \mid B \in E(S)\}.$$

We can restate the revealed power property in terms of the dual of an effectivity function.

Lemma 5.2 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. Then E satisfies the revealed power property if and only if for all $B, D \in \mathcal{P}_0(A)$ we have $E^d(B) \subset E^d(D)$ or $E^d(D) \subset E^d(B)$.

Proof. Let E satisfy the revealed power property. Let $B, D \in \mathcal{P}_0(A)$ with $E^d(B) \not\subset E^d(D)$. Then there exists a coalition $S \in \mathcal{P}_0(N)$ with $B \in E(S)$ and $D \notin E(S)$. Now let $T \in E^d(D)$. Then $D \in E(T)$, and since E satisfies the revealed power property, we have $E(S) \subset E(T)$. Since $B \in E(S)$, it follows that $B \in E(T)$, which implies $T \in E^d(B)$. Hence, $E^d(D) \subset E^d(B)$.

Since $(E^d)^d = E$ the other implication follows.

□

In the following we use the equivalence relations $\sim_N$ on $\mathcal{P}_0(N)$ and $\sim_A$ on $\mathcal{P}_0(A)$, corresponding to an arbitrary effectivity function E, defined by

$$S \sim_N T \quad \Leftrightarrow \quad E(S) = E(T) \qquad \text{for all } S, T \in \mathcal{P}_0(N), \tag{1}$$

$$B \sim_A D \quad \Leftrightarrow \quad E^d(B) = E^d(D) \qquad \text{for all } B, D \in \mathcal{P}_0(A). \tag{2}$$

If E satisfies the revealed power property, it is possible to order the equivalence classes $[S_1], [S_2], \ldots, [S_k]$ induced by $\sim_N$ in a decreasing way, i.e.,

$$S \in [S_i],\ T \in [S_j],\ i < j \Rightarrow E(S) \supsetneq E(T). \tag{3}$$

(Note that $N \in [S_1]$).

By Lemma 5.2 it follows that if E satisfies the revealed power property, it is possible to order the equivalence classes $[B_1], [B_2], \ldots, [B_l]$ induced by $\sim_A$ such that

$$B \in [B_r],\ D \in [B_s],\ r < s \Rightarrow E^d(B) \supsetneq E^d(D). \tag{4}$$

(Note that $A \in [B_1]$).

Lemma 5.3 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function which satisfies the revealed power property. Let $\sim_N$ and $\sim_A$ be the equivalence relations as defined in (1) and (2). Let the corresponding equivalence classes $[S_1], [S_2], \ldots, [S_k]$ and $[B_1], [B_2], \ldots, [B_l]$ be ordered as in (3) and (4), respectively. Then we have

(i) for all $i \in \{1, \ldots, k\}$ there exists an $s \in \{1, \ldots, l\}$ such that for all $S \in [S_i]$

$$E(S) = \bigcup_{r=1}^{s} [B_r]$$

(ii) for all $r \in \{1, \ldots, l\}$ there exists an $j \in \{1, \ldots, k\}$ such that for all $B \in [B_r]$

$$E^d(B) = \bigcup_{i=1}^{j} [S_i]$$

(iii) $k = l$

(iv) for all $i \in \{1, \ldots, k\}$ and $S \in [S_i]$

$$E(S) = \bigcup_{r=1}^{k+1-i} [B_r].$$

Proof. (i) Let $i \in \{1, \ldots, k\}$ and $S \in [S_i]$. It suffices to show that for $t \in \{1, \ldots, l\}$, for $B \in [B_t]$ with $B \in E(S)$, and for $D \in [B_r]$ with $1 \leq r \leq t$, we have $D \in E(S)$. This follows immediately from the fact that $E^d(B) \subset E^d(D)$.
(ii) Similar to (i).
(iii) From (i) we derive that $l \geq k$ and from (ii) it follows that $k \geq l$. Hence, $k = l$.
(iv) Follows immediately from (i) and (iii). □

Now we are able to prove

Theorem 5.4 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. Then E is decomposable if and only if it satisfies the revealed power property.

Proof. The only if part follows from Proposition 5.1. To prove the if part, let E satisfy the revealed power property. Let $[S_1], [S_2], \ldots, [S_k]$ be the equivalence classes corresponding to $\sim_N$ ordered as in (3), and let $[B_1], [B_2], \ldots, [B_l]$ be the equivalence classes corresponding to $\sim_A$ ordered as in (4). By Lemma 5.3 we have $k = l$ and for all $S \in [S_i]$ it holds that

$$E(S) = \bigcup_{r=1}^{k+1-i} [B_r].$$

Now define TU-games $v : 2^N \to [0, 1]$ and $w : 2^A \to [0, 1]$ as follows.
$v(\emptyset) := 0$, $w(\emptyset) := 0$, and

$$\begin{aligned} v(S) &:= (k + 1 - i)/k \qquad \text{for all } S \in [S_i] \text{ and } i \in \{1, \ldots, k\}, \\ w(B) &:= (k + 1 - r)/k \qquad \text{for all } B \in [B_r] \text{ and } r \in \{1, \ldots, k\}. \end{aligned}$$

Let $S \in [S_i]$ and $B \in [B_r]$. Then

$$\begin{aligned} v(S) + w(B) > 1 \quad &\Leftrightarrow \quad (k + 1 - i)/k + (k + 1 - r)/k > 1 \\ &\Leftrightarrow \quad k + 2 - i > r \\ &\Leftrightarrow \quad r \leq k + 1 - i \\ &\Leftrightarrow \quad B \in E(S). \end{aligned}$$

Hence $E = E(v, w)$, which completes the proof. □

6 Decomposability and echelon matrices

In Section 2 we have seen that an effectivity function E can be represented by a $\{0,1\}$-matrix I^E of size 2^n-1 by $2^{|A|}-1$, where for $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$,

$$I^E(S,B) = 1 \text{ if and only if } B \in E(S).$$

In this section we provide a characterization of decomposable effectivity functions in terms of matrices. We show that an effectivity function is decomposable if and only if it can be represented by a $\{0,1\}$-matrix in echelon form in which the 1's are 'separated' from the 0's. (see Figure 1)

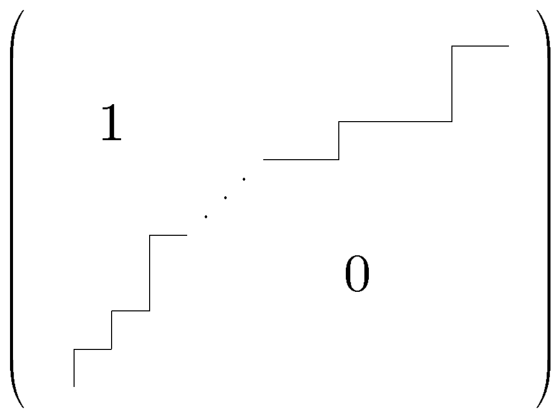

Figure 1.

Theorem 6.1 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function and I^E a matrix that represents E. Then E is decomposable if and only if it is possible to rearrange the rows and columns of I^E in such a way that the rearranged matrix has an echelon form as in Figure 1.

Proof. Let E be decomposable. By Theorem 5.4, E satisfies the revealed power property. Let $[S_1], [S_2], \ldots, [S_k]$ be the equivalence classes corresponding to $\sim_N$ ordered as in (3), and let $[B_1], [B_2], \ldots, [B_k]$ be the equivalence classes corresponding to $\sim_A$ ordered as in (4). Rearrange the rows and columns of I^E according to these equivalence classes. Consider the column corresponding to a coalition $S \in \mathcal{P}_0(N)$. $S \in [S_i]$ for some $i \in \{1,\ldots,k\}$ and so by Lemma 5.3 (iv) we have $E(S) = \bigcup_{r=1}^{k+1-i}[B_r]$. From this observation it immediately follows that every row of the rearranged matrix has the form $(1,\ldots,1,0,\ldots,0)$. Analogously, every column of this matrix has the form $(1,\ldots,1,0,\ldots,0)^T$ (x^T denotes the transposed of a vector x). Hence, the rearranged matrix has the echelon form of Figure 1.

To prove the if part, suppose it is possible to rearrange the rows and columns of I^E in such a way that we obtain a matrix in the form of Figure 1. Suppose the columns of this matrix are arranged in the order $B_1, \ldots, B_{2^{|A|-1}}$. Let $S \in \mathcal{P}_0(N)$. Define $m(S) := \max\{r \in \{1, \ldots, 2^{|A|-1}\} \mid I^E(S, B_r) = 1\}$. Since the row corresponding to S has the form $(1, \ldots, 1, 0, \ldots, 0)$, it follows that $E(S) = \{B_1, \ldots, B_{m(S)}\}$. From this observation it immediately follows that for $S, T \in \mathcal{P}_0(N)$ we have $E(S) \subset E(T)$ if and only if $m(S) \leq m(T)$. Hence, E satisfies the revealed power property and hence, by Theorem 5.4, E is decomposable. □

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