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CHARACTERIZING PROPERTIES OF APPROXIMATE SOLUTIONS FOR OPTIMIZATION PROBLEMS

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Characterizing Properties of Approximate Solutions for Optimization Problems[1]

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Abstract: Approximate solutions for optimization problems become of interest if the "true" optimum can not be found: this may happen for the simple reason that an optimum does not exist or because of the "bounded rationality" (or accuracy) of the optimizer. In this paper we characterize several approximate solutions by means of consistency and invariance properties. In particular, we prove that, besides the trivial ones, there are no consistent solutions satisfying non-emptiness, translation and multiplication invariance.
Key-words: Approximate optimization, consistency, invariance properties.

1 Introduction

In this paper we shall try to give an answer to an apparently silly question: is the concept of "approximate solution" in optimization meaningful? We will show that, moving from exact to approximate optimization, some serious problems may arise.
One reason to focus on approximate optimization can be derived from the increasing interest for the issues related with "bounded rationality" in game

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theory: it turns out that more and more the emphasis is shifted from maximization to approximate maximization. On this point, we only refer to Fudenberg and Tirole (1991), Myerson (1991) and Radner (1980). The latter paper is interesting both, to understand the kind of results that can be achieved along this path and for the remarks about the problems that arise when approximate maximization enters the scene.
The interest for approximate optimization arising from game theory is only one particular case of a general issue. In many cases, given an optimization problem, one does not look for the maximum: this can happen for the obvious reason that a maximum does not exist or for the difficulty of finding it. In both cases, one should have some "rule" that says when the search for a maximum could stop.
Clearly, many different kinds of rules can be devised, from some "rule of thumb" to a sophisticated analysis that compares the computational costs for improving the degree of approximation and the benefits that result.
The approach that will be used in this paper is usually referred to as "axiomatic". That is, we shall state some desirable properties of an "approximate solution concept" and will analyse their consequences and mutual compatibility.
To be more specific, we shall investigate a special issue related with these rules: how they should be if one wants to behave in a consistent way across different approximation problems and, at the same time, one has to take into account some invariance properties.
The invariance requirement is due to the fact that, in many cases, the function to be maximized is only a representative of a class of equivalent functions (let us recall at least utility theory, and the fact that in hard sciences the origin or the scale of measurement quite often can be freely chosen). The remarkable result that we get is a kind of "impossibility theorem", which asserts that there are no consistent rules for choosing truly approximate solutions if one wants to take into account translation and multiplication invariance (as one should do, e.g., when dealing with expected utilities). We also investigate the cases in which one takes into account separately these invariance requirements.
Special emphasis is given to rules that take into account a reasonable monotonicity condition, that can be considered as an instance of the IIA (independence of irrelevant alternatives) principle: the main reason to consider this point of view is that we try to consider classes of optimization problems which contain both bounded and unbounded functions. This interest is an outgrowth of previous research done by the authors in the context of

semi-infinite bimatrix games (see Jurg and Tijs (1993), Lucchetti, Patrone and Tijs (1986) and Norde and Potters (1995)). We get, under appropriate assumptions, a class of approximate solutions to which belong the (ε, k)-solutions investigated in the papers quoted above. We also like to emphasize that some examples show that careless specification of the domain of the rule can give quite strange results.
The two previous paragraphs are an approximate description of the contents of sections 3 and 4 respectively, while section 2 is devoted to the setting of the problem. In section 6 we investigate rules, which make use of sequences. As it could be expected, reasonable conditions restrict the attention to maximizing sequences. So that, in some sense, the "impossibility result" is somehow circumvented: results that guarantee the existence of ε-Nash equilibria for every $\varepsilon > 0$ do have a meaning in this setting (Tijs (1981)); the same can be said, in optimization, about Tykhonov well-posedness (Dontchev-Zolezzi (1993)); see Patrone (1987) for remarks about the invariance of this property). We want to add, however, some warning about sequences. First, for practical reasons one has eventually to give one solution: so, even if sequences can be considered interesting for theoretical reasons, they do not solve the problem of finding an approximate solution for an optimization problem. Secondly, sometimes one is interested in solving (approximately) problems which are an approximate version of the "true" one: in this case, tricks like those of considering asymptotically minimizing sequences (in the terminology of Dontchev-Zolezzi), fall outside the scope of our investigation. Actually, we have completely skipped any reference to "continuity" properties of our solution maps. In our opinion, this is a topic that deserves to be thoroughly studied.

Notation Throughout this paper we denote the set $\mathbb{R} \cup \{+\infty\}$ by $\mathbb{R}^*$ and the set $\mathbb{R} \cup \{-\infty, +\infty\}$ by $\overline{\mathbb{R}}$.

2 Optimization problems

An *optimization problem* is a pair (A, u) where A is a non-empty set of alternatives and u is a real-valued function with domain A. Let $\mathcal{P}$ be a non-empty collection of optimization problems. A *solution* β on $\mathcal{P}$ is a map which assigns to every optimization problem $(A, u) \in \mathcal{P}$ a subset of A.

Example 2.1 For the following examples no special restriction is imposed upon $\mathcal{P}$.

a) The solution β_{tot} is defined by

$$\beta_{\text{tot}}(A, u) := A.$$

b) The solution $\beta_{\max}$ is defined by

$$\beta_{\max}(A, u) := \{a \in A : u(a) \geq u(a') \text{ for every } a' \in A\}.$$

c) For $\varepsilon > 0$ the solution β_ε is defined by

$$\beta_\varepsilon(A, u) := \{a \in A : u(a) \geq u(a') - \varepsilon \text{ for every } a' \in A\}.$$

d) For $k \in I\!R$ the solution β^k is defined by

$$\beta^k(A, u) := \{a \in A : u(a) \geq k\}.$$

e) For $\varepsilon > 0, k \in I\!R$ the solution $\beta_{\varepsilon,k}$ is defined by

$$\beta_{\varepsilon,k}(A, u) := \begin{cases} \beta_{\max}(A, u) & \text{if } \beta_{\max}(A, u) \neq \emptyset \\ \beta_\varepsilon(A, u) & \text{if } \beta_{\max}(A, u) = \emptyset \text{ and } \beta_\varepsilon(A, u) \neq \emptyset \\ \beta^k(A, u) & \text{otherwise} \end{cases}.$$

Notice that $\beta_{\max}(A, u)$, $\beta_\varepsilon(A, u)$ and $\beta^k(A, u)$ can be empty; on the contrary, $\beta_{\text{tot}}(A, u) \neq \emptyset$ and $\beta_{\varepsilon,k}(A, u) \neq \emptyset$ for every $(A, u) \in \mathcal{P}$.

Two optimization problems (A, u) and (B, v) are *sup-equivalent* if

$$\sup_{x \in A} u(x) = \sup_{x \in B} v(x).$$

A solution β on $\mathcal{P}$ is *approximation consistent* if for every pair of sup-equivalent problems $(A, u), (B, v) \in \mathcal{P}$ the following statement is true:

if $b \in \beta(B, v)$ and $a \in A$ is such that $u(a) \geq v(b)$ then $a \in \beta(A, u)$.

So, if a solution β is approximation consistent, selection by β of an alternative $b \in B$ in some problem $(B, v) \in \mathcal{P}$, induces selection by β of all 'non-worse' alternatives in sup-equivalent problems. Clearly, the solutions a)-d) in example 2.1 are approximation consistent. For approximation consistent solutions we have the following proposition.

Proposition 2.1 *Let β be an approximation consistent solution on $\mathcal{P}$ and let (A,u), $(B,v) \in \mathcal{P}$ be such that $u(A) = v(B)$. Then there is a subset T of $u(A)(= v(B))$ such that $\beta(A,u) = u^{-1}(T)$ and $\beta(B,v) = v^{-1}(T)$.*

Proof Take $T := u(\beta(A,u))$.
If $a \in u^{-1}(T)$ then $u(a) = u(a')$ for some $a' \in \beta(A,u)$ and hence, by approximation consistency, $a \in \beta(A,u)$. So, $u^{-1}(T) \subseteq \beta(A,u)$. The inclusion $\beta(A,u) \subseteq u^{-1}(T)$ is obvious.
If $b \in v^{-1}(T)$ then $v(b) = u(a')$ for some $a' \in \beta(A,u)$. Since (A,u) and (B,v) are sup-equivalent we get, by approximation consistency, $b \in \beta(B,v)$. So, $v^{-1}(T) \subseteq \beta(B,v)$. If $b \in \beta(B,v)$ then, since $u(A) = v(B)$, there is an $a \in A$ such that $u(a) = v(b)$ and hence, by approximation consistency, $a \in \beta(A,u)$. Therefore, $v(b) = u(a) \in T$ and hence $b \in v^{-1}(T)$. So, $\beta(B,v) \subseteq v^{-1}(T)$. ■

The above proposition shows that, if β is approximation consistent, the set $\beta(A,u)$ only depends on the range $u(A)$ of u. So, if we are interested in approximation consistent solutions only, we may identify an optimization problem (A,u) with $u(A)$, the range of u, which is a subset of $\mathbb{R}$. In the next sections we focus on this approach.

3 Axioms and examples

Let $\mathcal{S}$ be a non-empty collection of non-empty subsets of $\mathbb{R}$. A *solution* σ on $\mathcal{S}$ is a map which assigns to every $S \in \mathcal{S}$ a subset $\sigma(S)$ of S.

A solution σ on $\mathcal{S}$ satisfies (AC) (*approximation consistency*) if for every $S_1, S_2 \in \mathcal{S}$ with $\sup S_1 = \sup S_2$ the following statement is true:

$$\text{if } s_2 \in \sigma(S_2) \text{ and } s_1 \in S_1 \text{ is such that } s_1 \geq s_2 \text{ then } s_1 \in \sigma(S_1).$$

The reason for introducing (AC) is given by proposition 2.1: it is immediate to see that, if a solution β on $\mathcal{P}$ is approximation consistent, as defined in the previous section, then the induced solution σ on the family $\mathcal{S}$ of ranges $u(A)$ ($(A,u) \in \mathcal{P}$), satisfies (AC). Conversely, a solution σ on $\mathcal{S}$, satisfying (AC), induces, for every $\mathcal{P}$ with ranges in $\mathcal{S}$, an approximation consistent solution β.
The (AC) condition is not entirely satisfactory: one expects that $\sigma(S)$ can be described as $\{s \in S : s \geq \gamma\}$ or $\{s \in S : s > \gamma\}$ for some γ (depending on S). However, we can have strict or weak inequality, depending on the value of $\sup S$, as can be seen in the next example.

Example 3.1 Let $\mathcal{S}$ be the collection of all non-empty subsets of $I\!R$. The solution σ_{mix} on $\mathcal{S}$, defined by

$$\sigma_{\mathrm{mix}}(S) := \begin{cases} \{s \in S : s \geq \sup S - 1\} & \text{if } \sup S \leq 0 \\ \{s \in S : s > \sup S - 1\} & \text{if } \sup S \in (0, +\infty) \\ \{s \in S : s \geq 22\} & \text{if } \sup S = +\infty \end{cases},$$

satisfies (AC).

In order to get rid of these kind of approximate solutions one would like to add the requirement that $\sigma(S)$ is a closed subset of S. However, the next example shows us that this addition does not "force" the parentheses to be closed.

Example 3.2 Consider $\mathcal{S} = \{\{0,1\}\} \cup \{[\alpha, 1] : \alpha \in (0,1)\}$ and let σ be the solution on $\mathcal{S}$, defined by $\sigma(\{0,1\}) := \{1\}$ and $\sigma([\alpha,1]) := [\alpha, 1]$ for every $\alpha \in (0,1)$. The solution σ satisfies (AC) and $\sigma(S)$ is a closed subset of S for every $S \in \mathcal{S}$, but there is no $\gamma \in \overline{I\!R}$ such that $\sigma(S) = \{s \in S : s \geq \gamma\}$ for every $S \in \mathcal{S}$.

So, we shall introduce the following axiom, an appropriate strenghtening of (AC).
A solution σ on $\mathcal{S}$ satisfies (SAC) (*strong approximation consistency*) if for every $S, S_1, S_2, \ldots \in \mathcal{S}$ with $\sup S = \sup S_i$ for every $i \in I\!N$ the following statement is true:

if $s_i \in \sigma(S_i)$ for every $i \in I\!N$ and $s \in S$ is such that $s \geq \liminf_{i \to \infty} s_i$ then $s \in \sigma(S)$.

One easily verifies that (SAC) induces (AC). Moreover, (SAC) implies that $\sigma(S)$ is a closed subset of S for every $S \in \mathcal{S}$. In fact, if $\mathcal{S}$ is a collection of intervals, then σ satisfies (SAC) if and only if σ satisfies (AC) and $\sigma(S)$ is a closed subset of S for every $S \in \mathcal{S}$.
In the sequel we also make use of the following axioms.
A solution σ on $\mathcal{S}$ satisfies (NEM) (*non-emptiness*) if for every $S \in \mathcal{S}$ we have

$$\sigma(S) \neq \emptyset.$$

The collection $\mathcal{S}$ is *closed under translation* (CL+) if for every $S \in \mathcal{S}$ and $t \in I\!R$ we have $t + S := \{t + s : s \in S\} \in \mathcal{S}$. A solution σ on $\mathcal{S}$, obeying (CL+), satisfies (TI) (*translation invariance*) if for every $S \in \mathcal{S}$ and $t \in I\!R$ we have

$$\sigma(t + S) = t + \sigma(S).$$

The collection $\mathcal{S}$ is *closed under multiplication* (CL*) if for every $S \in \mathcal{S}$ and $\lambda > 0$ we have $\lambda S := \{\lambda s : s \in S\} \in \mathcal{S}$. A solution σ on $\mathcal{S}$, obeying (CL*), satisfies (MI) (*multiplication invariance*) if for every $S \in \mathcal{S}$ and $\lambda > 0$ we have

$$\sigma(\lambda S) = \lambda \sigma(S).$$

A solution σ on $\mathcal{S}$ satisfies (IIA) (*independence of irrelevant alternatives*) if for every $S, T \in \mathcal{S}$ with $S \subseteq T$ one has

$$\sigma(T) \cap S \subseteq \sigma(S).$$

So, if σ satisfies (IIA), selection by σ of an element $s \in T$, implies selection by σ of s in any subset S of T with $s \in S$. This notion of (IIA) is weaker than the one used in Kaneko (1980) and Peters (1992).

Example 3.3 For the following examples suppose that $\mathcal{S}$ is the collection of all non-empty subsets of $\mathbb{R}$.

a) The solution σ_{mix}, defined in example 3.1, satisfies (AC) and (NEM).

b) The solution σ_{rat}, defined by

$$\sigma_{\text{rat}}(S) := \{s \in S : s \text{ is rational }\},$$

only satisfies (IIA).

c) The solution σ_{tot}, defined by

$$\sigma_{\text{tot}}(S) := S,$$

satisfies (SAC), (NEM), (TI), (MI) and (IIA).

d) The solution σ_{max}, defined by

$$\sigma_{\text{max}}(S) := \{s \in S : s \geq s' \text{ for every } s' \in S\},$$

satisfies (SAC), (TI), (MI) and (IIA).

e) The solution σ_ε (where $\varepsilon > 0$), defined by

$$\sigma_\varepsilon(S) := \{s \in S : s \geq \sup S - \varepsilon\},$$

satisfies (SAC), (TI) and (IIA).

f) The solution σ^k (where $k \in I\!R$), defined by

$$\sigma^k(S) := \{s \in S \,:\, s \geq k\},$$

satisfies (SAC) and (IIA).

g) The solution $\sigma_{\varepsilon,k}$ (where $\varepsilon > 0, k \in I\!R$), defined by

$$\sigma_{\varepsilon,k}(S) := \begin{cases} \sigma_{\max}(S) & \text{if } \sigma_{\max}(S) \neq \emptyset \\ \sigma_{\varepsilon}(S) & \text{if } \sigma_{\max}(S) = \emptyset \text{ and } \sigma_{\varepsilon}(S) \neq \emptyset \\ \sigma^k(S) & \text{otherwise} \end{cases},$$

only satisfies (NEM).

h) The solution $\hat{\sigma}_{\varepsilon,k}$ (where $\varepsilon > 0, k \in I\!R$), defined by

$$\hat{\sigma}_{\varepsilon,k}(S) := \begin{cases} \sigma_{\varepsilon}(S) & \text{if } \sup S \leq k + \varepsilon \\ \sigma^k(S) & \text{if } \sup S > k + \varepsilon \end{cases},$$

satisfies (SAC), (NEM) and (IIA). Notice that $\hat{\sigma}_{\varepsilon,k}(S) = \sigma_{\varepsilon}(S) \cup \sigma^k(S)$.

i) The solution $\sigma_{\text{pro}(\alpha,\beta)}(S)$ (where $\alpha > 1, \beta < 1$), defined by

$$\sigma_{\text{pro}(\alpha,\beta)}(S) := \begin{cases} \sigma^{\alpha s}(S) & \text{if } \sup(S) = s \in (-\infty, 0) \\ S & \text{if } \sup(S) = 0 \\ \sigma^{\beta s}(S) & \text{if } \sup(S) = s \in (0, +\infty) \\ S & \text{if } \sup(S) = +\infty \end{cases},$$

satisfies (SAC), (NEM) and (MI).

The following table summarizes the statements above.

	(AC)	(SAC)	(NEM)	(TI)	(MI)	(IIA)
σ_{mix}	+	−	+	−	−	−
σ_{rat}	−	−	−	−	−	+
σ_{tot}	+	+	+	+	+	+
σ_{max}	+	+	−	+	+	+
σ_{ε}	+	+	−	+	−	+
σ^{k}	+	+	−	−	−	+
$\sigma_{\varepsilon,k}$	−	−	+	−	−	−
$\hat{\sigma}_{\varepsilon,k}$	+	+	+	−	−	+
$\sigma_{\text{pro}(\alpha,\beta)}$	+	+	+	−	+	−

4 Characterizations for translation and multiplication invariant solutions

Let $\mathcal{S}$ be a collection of non-empty subsets of $\mathbb{R}$. We write $\mathcal{S} = \cup_{k \in \mathbb{R}^*} \mathcal{S}_k$ where $\mathcal{S}_k := \{S \in \mathcal{S} \,:\, \sup S = k\}$ for every $k \in \mathbb{R}^*$. For a function $a : \mathbb{R}^* \to \overline{\mathbb{R}}$ we define the solution σ_a on $\mathcal{S}$ by

$$\sigma_a(S) := \{s \in S \,:\, s \geq a(\sup(S))\}.$$

So, σ_a selects, for every $S \in \mathcal{S}_k$, the elements $s \in S$ with $s \geq a(k)$. Clearly, σ_a satisfies (SAC). The following proposition shows that the solutions σ_a are completely characterized by (SAC). Note that the solutions, defined in example 3.3 c)-f),h),i) , satisfy (SAC) and are, in fact, σ_a for some suitably chosen a.

Proposition 4.1 *Let S be a collection of non-empty subsets of $\mathbb{R}$ and let σ be a solution on S. The solution σ satisfies (SAC) if and only if $\sigma = \sigma_a$ for some function a.*

Proof We only prove the only-if-part. So, assume that σ satisfies (SAC). First we define the function $a : I\!R^* \to \overline{I\!R}$. Let $k \in I\!R^*$. Define

$$a(k) := \begin{cases} \text{arbitrarily} & \text{if } \mathcal{S}_k = \emptyset \\ \inf(\cup_{S \in \mathcal{S}_k} \sigma(S)) & \text{otherwise} \end{cases} \tag{1}$$

(with the convention that $\inf \emptyset = +\infty$). Now we have to prove that $\sigma = \sigma_a$, i.e. we have to prove that $\sigma(S) = \sigma_a(S)$ for every $S \in \mathcal{S}$. So, let $S \in \mathcal{S}$ and let $k := \sup S$ (which trivially induces that $\mathcal{S}_k \neq \emptyset$). For every $s \in \sigma(S)$ we have $s \geq a(k)$ by definition of $a(k)$. Therefore $\sigma(S) \subseteq \sigma_a(S)$. Note that the converse inclusion $\sigma_a(S) \subseteq \sigma(S)$ is trivial when $\sigma_a(S) = \emptyset$. So, assume $\sigma_a(S) \neq \emptyset$ and let $s \in \sigma_a(S)$. Then $s \geq a(k)$ which implies $a(k) \neq +\infty$. Therefore, $\cup_{S \in \mathcal{S}_k} \sigma(S) \neq \emptyset$, and hence, by definition of $a(k)$, there is a sequence $S_1, S_2, \ldots \in \mathcal{S}_k$ and, for every $i \in I\!N$, an $s_i \in \sigma(S_i)$ such that $a(k) = \lim_{i \to \infty} s_i$. By (SAC) we get $s \in \sigma(S)$. Therefore $\sigma_a(S) \subseteq \sigma(S)$, which finishes the proof. ■

If we impose some feasibility condition upon the function a we get solutions which are characterized by (SAC) and (NEM).

Proposition 4.2 *Let $\mathcal{S}$ be a collection of non-empty subsets of $I\!R$ and let σ be a solution on $\mathcal{S}$. The solution σ satisfies (SAC) and (NEM) if and only if $\sigma = \sigma_a$ for some function $a : I\!R^* \to \overline{I\!R}$ satisfying $a(k) \leq k$ for every $k \in I\!R^*$, with strict inequality for every $k \in I\!R^*$ for which there is an $S \in \mathcal{S}_k$ with $k \notin S$.*

Proof Let $a : I\!R^* \to \overline{I\!R}$ be such that $a(k) \leq k$ for every $k \in I\!R^*$, with strict inequality for every $k \in I\!R^*$ for which there is an $S \in \mathcal{S}_k$ with $k \notin S$ and let $\sigma = \sigma_a$. Clearly, σ_a satisfies (SAC). Let $S \in \mathcal{S}$ and let $k := \sup S$. If $k \in S$ then $k \in \sigma_a(S)$. If $k \notin S$ (which is, e.g., the case if $k = +\infty$) then $a(k) < k$ and we may choose an $s \in S$ with $s \geq a(k)$ and, as a consequence, $s \in \sigma_a(S)$. So, σ_a also satisfies (NEM).
In order to prove the only-if-part assume that σ satisfies (SAC) and (NEM). By (SAC) and the proof of proposition (4.1) we know that $\sigma = \sigma_a$, where a is defined by (1). Now we have to prove that a has the desired properties. So let $k \in I\!R^*$. If $\mathcal{S}_k = \emptyset$ we choose $a(k) \leq k$. If $\mathcal{S}_k \neq \emptyset$ there is an $S \in \mathcal{S}_k$ and we may, by (NEM), take some element $s \in \sigma(S) = \sigma_a(S)$. We get $a(k) \leq s \leq k$, where the last inequality is strict if $k \notin S$. ■

The next theorem describes the class of solutions, which are characterized by

(SAC), (NEM) and (TI). It turns out that these solutions coincide with the collection of 'ε-optimal' solutions for some $\varepsilon \in [0, +\infty]$, if $\mathcal{S}$ only contains upper bounded sets.

Proposition 4.3 *Let $\mathcal{S}$ be a collection of non-empty subsets of $\mathbb{R}$ which satisfies (CL+) and let σ be a solution on $\mathcal{S}$. The solution σ satisfies (SAC), (NEM) and (TI) if and only if there is an $\varepsilon \geq 0$ ($\varepsilon > 0$ in case there is an upper bounded $S \in \mathcal{S}$ which has no maximum) such that $\sigma = \sigma_a$, where $a : \mathbb{R}^* \to \overline{\mathbb{R}}$ is defined by*

$$\begin{cases} a(k) := k - \varepsilon & \text{for every } k \in \mathbb{R} \\ a(+\infty) = -\infty \end{cases} . \tag{2}$$

Proof Clearly, $\sigma = \sigma_a$ satisfies (SAC), (NEM) and (TI) if a is defined by (2). In order to prove the only-if-part suppose that σ satisfies (SAC), (NEM) and (TI). Again, by (SAC) and the proof of proposition 4.1, we know that $\sigma = \sigma_a$, where a is defined by (1). Notice that, by (CL+), we have for every $k \in \mathbb{R}$

$$\mathcal{S}_k = \{k + S \, : \, S \in \mathcal{S}_0\}.$$

So, if $\mathcal{S}_k = \emptyset$ for some $k \in \mathbb{R}$, then $\mathcal{S}_k = \emptyset$ for every $k \in \mathbb{R}$ and we choose $a(k) = k - 37$ for every $k \in \mathbb{R}$. If $\mathcal{S}_k \neq \emptyset$ for some $k \in \mathbb{R}$, then $\mathcal{S}_k \neq \emptyset$ for every $k \in \mathbb{R}$. Moreover, we get by (TI),

$$\begin{aligned} a(k) &= \inf(\cup_{S \in \mathcal{S}_k} \sigma(S)) &&= \inf(\cup_{S \in \mathcal{S}_0} \sigma(k + S)) &= \\ &= \inf(\cup_{S \in \mathcal{S}_0} (k + \sigma(S))) &&= k + \inf(\cup_{S \in \mathcal{S}_0} \sigma(S)) &= \\ &= k + a(0). \end{aligned}$$

By (NEM) we get $a(0) \leq 0$. Moreover, if there is an upper bounded $S \in \mathcal{S}$ which has no maximum, then, by (CL+), there is an $S \in \mathcal{S}_0$ which has no maximum and we have $a(0) < 0$. So, take $\varepsilon = -a(0)$.
If $\mathcal{S}_{+\infty} = \emptyset$ we choose $a(+\infty) = -\infty$. Otherwise there is an $S \in \mathcal{S}_{+\infty}$ and, by (NEM), an $s \in \sigma(S)$. Since, by (CL+), $t + S \in \mathcal{S}_{+\infty}$ and, by (TI), $t + s \in \sigma(t + S)$ for every $t \in \mathbb{R}$ we get $a(+\infty) \leq t + s$ for every $t \in \mathbb{R}$. Hence, $a(+\infty) = -\infty$. ∎

The solution $\hat{\sigma}_{\varepsilon,k}$ of example 3.3 satisfies (SAC) and (NEM) but not (TI), the solution σ_ε satisfies (SAC) and (TI) but not (NEM) and one easily constructs a solution satisfying (NEM) and (TI) but not (SAC) (simply by defining the solution for S with $\sup S \in \{0, +\infty\}$ in an arbitrary but not approximation consistent way and extending this solution by translation

invariance). Therefore the axioms (SAC), (NEM) and (TI) are logically independent.
In the following proposition we characterize the 'proportional' solutions by (SAC), (NEM) and (MI).

Proposition 4.4 *Let $\mathcal{S}$ be a collection of non-empty subsets of $\mathbb{R}$ which satisfies (CL*) and let σ be a solution on $\mathcal{S}$. The solution σ satisfies (SAC), (NEM) and (MI) if and only if there are $\alpha \geq 1$ and $\beta \leq 1$ such that $\sigma = \sigma_a$, where $a : \mathbb{R}^* \to \overline{\mathbb{R}}$ is defined by*

$$\begin{cases} a(k) := \alpha k & \text{for every } k < 0 \\ a(0) \in \{-\infty, 0\} & \\ a(k) := \beta k & \text{for every } k \in (0, \infty) \\ a(+\infty) \in \{-\infty, 0\} & \end{cases} \tag{3}$$

(where $\alpha > 1$ if there is a $k < 0$ and an $S \in \mathcal{S}_k$ with $k \notin S$, $\beta < 1$ if there is a $k > 0$ and an $S \in \mathcal{S}_k$ with $k \notin S$ and $a(0) = -\infty$ if there is an $S \in \mathcal{S}_0$ with $0 \notin S$).

Proof Clearly, $\sigma = \sigma_a$ satisfies (SAC), (NEM) and (MI) if a is defined by (3). In order to prove the only-if-part suppose that σ satisfies (SAC), (NEM) and (MI). Again, by (SAC) and the proof of proposition 4.1, we know that $\sigma = \sigma_a$, where a is defined by (1). Notice that, by (CL*), we have for every $k \in (0, +\infty)$

$$\mathcal{S}_k = \{kS \,:\, S \in \mathcal{S}_1\}.$$

So, if $\mathcal{S}_k = \emptyset$ for some $k \in (0, +\infty)$, then $\mathcal{S}_k = \emptyset$ for every $k \in (0, +\infty)$ and we choose $a(k) = \frac{1}{37}k$ for every $k \in \mathbb{R}$. If $\mathcal{S}_k \neq \emptyset$ for some $k \in (0, +\infty)$, then $\mathcal{S}_k \neq \emptyset$ for every $k \in (0, +\infty)$. Moreover, we get by (MI),

$$\begin{aligned} a(k) &= \inf(\cup_{S \in \mathcal{S}_k} \sigma(S)) &&= \inf(\cup_{S \in \mathcal{S}_1} \sigma(kS)) &&= \\ &= \inf(\cup_{S \in \mathcal{S}_1} k\sigma(S)) &&= k \inf(\cup_{S \in \mathcal{S}_1} \sigma(S)) &&= \\ &= ka(1). \end{aligned}$$

By (NEM) we get $a(1) \leq 1$. Moreover, if there is a $k \in (0, +\infty)$ and an $S \in \mathcal{S}_k$ with $k \notin S$, then, by (CL*), there is an $S \in \mathcal{S}_1$ with $1 \notin S$ and we have $a(1) < 1$. So, take $\beta = a(1)$.
In an analogous way we prove, in case $\mathcal{S}_k \neq \emptyset$ for every $k \in (-\infty, 0)$, that $a(k) = k(-a(-1))$, for every $k \in (-\infty, 0)$, where $a(-1) \leq -1$ and $a(-1) < -1$ if there is an $S \in S_{-1}$ with $-1 \notin S$ (or equivalently, there is a

$k < 0$ and an $S \in \mathcal{S}_k$ with $k \notin S$).
Suppose $\mathcal{S}_0 \neq \emptyset$ and $a(0) < 0$. Then there is an $S \in \mathcal{S}$ and an $s \in \sigma(S)$ with $s < 0$. Since, by (CL*), $\lambda S \in \mathcal{S}_0$ and, by (MI), $\lambda s \in \sigma(\lambda S)$ for every $\lambda > 0$ we get $a(0) \leq \lambda s$ for every $\lambda > 0$. Hence $a(0) = -\infty$. In an analogous way one proves that $a(+\infty) \in \{-\infty, 0\}$. ■

The solution $\hat{\sigma}_{\varepsilon,k}$ of example 3.3 satisfies (SAC) and (NEM) but not (MI), the solution $\sigma_{\max}$ satisfies (SAC) and (MI) but not (NEM) and one easily constructs a solution satisfying (NEM) and (MI) but not (SAC) (simply by defining the solution for S with $\sup S \in \{-1, 0, 1, +\infty\}$ in an arbitrary but not approximation consistent way and extending this solution by multiplication invariance). Therefore the axioms (SAC), (NEM) and (MI) are logically independent.
Clearly, the trivial solution σ_{tot} satisfies (SAC), (NEM), (TI) and (MI). Moreover, if the collection $\mathcal{S}$ is such that every upper bounded $S \in \mathcal{S}$ has a maximum, then the solution, selecting the maximum for every upper bounded $S \in \mathcal{S}$ and selecting S for every $S \in \mathcal{S}$ which is not upper bounded, does either. In the following proposition we show the impossibility of finding another solution, satisfying these four properties.

Proposition 4.5 *Let $\mathcal{S}$ be a collection of non-empty subsets of $\mathbb{R}$, which satisfies (CL+) and (CL*). Suppose, moreover, that there is at least one upper bounded $S \in \mathcal{S}$ which has no maximum. Let σ be a solution on $\mathcal{S}$. The solution σ satisfies (SAC), (NEM), (TI) and (MI) if and only if $\sigma = \sigma_a$, where $a : \mathbb{R}^* \to \overline{\mathbb{R}}$ is defined by $a(k) := -\infty$ for every $k \in \mathbb{R}^*$ (i.e. $\sigma_a = \sigma_{\text{tot}}$).*

Proof Again we only prove the only-if-part. Suppose σ satisfies (SAC), (NEM), (TI) and (MI). By (SAC) we may conclude that $\sigma = \sigma_a$ where $a : \mathbb{R}^* \to \overline{\mathbb{R}}$ is defined by (1). Let $S \in \mathcal{S}$ be un upper bounded set without maximum and let $k := \sup S$. Since σ satisfies (NEM) there is an $s \in \sigma(S)$. Clearly $s < k$. For every $n \in \mathbb{N}$ we have, by (CL+) and (TI),

$$-(1 - n^{-1})k + S \in \mathcal{S}$$

and

$$-(1 - n^{-1})k + s \in \sigma(-(1 - n^{-1})k + S).$$

Moreover, by (CL*) and (MI), we get

$$-(n - 1)k + nS = n(-(1 - n^{-1})k + S) \in \mathcal{S}$$

and

$$-(n-1)k+ns=n(-(1-n^{-1})k+s)\in\sigma(-(n-1)k+nS).$$

Since $\sup(-(n-1)k+nS)=k$ and $\lim_{n\to\infty}(-(n-1)k+ns)=-\infty$ we get $a(k)=-\infty$. By (TI) we infer that $a(l)=-\infty$ for every $l\in I\!R$. We know already, from proposition 4.3, that $a(+\infty)=-\infty$. ■

Let us notice that, in the context of decision making under risk, u and v are von Neumann-Morgenstern utility functions representing the same preferences iff $v=cu+d$, with $c>0$ and $d\in I\!R$. So, if one wants to stress the point of view that only preferences have a true meaning, one should use a "solution rule" for optimization problems that takes into account this fact. But proposition 4.5 just shows that it is impossible to do this in a non-trivial way. Otherwise stated: for von Neumann-Morgenstern preferences there is no sensible concept of approximate optimum! If one wants to talk in a meaningful way of approximate optimization, an escape route could be the addition of further details that allow for some "absolute" reference point (e.g.: how do we decide whether the oscillations of last week at the New York Stock Exchange were wild or not? Maybe we refer to the previous history as a benchmark). The interesting question is whether it can be done in a consistent way, without resorting to an "absolute" utility function.

5 Characterizations under IIA

Proposition 4.3 provides a nice characterization of solutions σ on collections of *upper bounded* subsets of $I\!R$: either σ selects the maximum of $S\in\mathcal{S}$ (if all upper bounded $S\in\mathcal{S}$ have a maximum) or there is an $\varepsilon\in(0,+\infty]$ such that σ selects all 'ε-optimal' elements. In order to get a nice characterization, which takes also the *unbounded* subsets of $\mathcal{S}$ into account, we have to replace (TI) by (IIA). Since (IIA) deals with inclusions, an appropriate condition upon $\mathcal{S}$ would be either to be *closed under taking subsets* (CL$\subset$) or to be *closed under taking supersets* (CL$\supset$). The following example shows that a nice characterization with (IIA), is not possible if $\mathcal{S}$ satisfies (CL$\subset$).

Example 5.1 Consider the class $\mathcal{S}$ of all non-empty and upper bounded subsets S which satisfy the condition that there exists a $t\in[0,1)$ such that $S\subseteq t+\mathbf{Z}$. Define the solution σ_a by

$$a(k):=\begin{cases}k-22 & \text{if } k\in\mathbf{Z}\\ k-37 & \text{otherwise}\end{cases}.$$

Clearly σ_a satisfies (SAC) and (NEM). It also satisfies (IIA): this is due to the fact that for $S, T \in \mathcal{S}$ with $S \subseteq T$ both are contained in the same $t + \mathbf{Z}$ for some $t \in [0,1)$. In fact, one can prove that any σ_a, with a feasible function a which is non-decreasing on $t + \mathbf{Z}$ for every $t \in [0,1)$, satisfies (SAC), (NEM) and (IIA).

The reason for strange examples as above lies in the fact that the collection $\mathcal{S}$ is too poor: $\mathcal{S}$ can be partitioned into several subcollections such that sets belonging to different subcollections are not related by inclusion. These problems do not occur if $\mathcal{S}$ satisfies (CL$\supset$).

Proposition 5.1 *Let $\mathcal{S}$ be a collection of non-empty subsets of $I\!R$ which satisfies (CL$\supset$) and let σ be a solution on $\mathcal{S}$. The solution σ satisfies (SAC), (NEM) and (IIA) if and only if $\sigma = \sigma_a$ for some non-decreasing function $a : I\!R^* \to \overline{I\!R}$ satisfying $a(k) \leq k$ for every $k \in I\!R^*$, with strict inequality for every $k \in I\!R^*$ for which there is an $S \in \mathcal{S}_k$ with $k \notin S$.*

Proof Again we only prove the only-if-part. Suppose σ satisfies (SAC), (NEM) and (IIA). By (SAC) we may conclude that $\sigma = \sigma_a$, where a is defined by (1). The only thing left to prove is that a is non-decreasing. Choose $a(k) = -\infty$ if $\mathcal{S}_k = \emptyset$. Notice that, if $\mathcal{S}_k = \emptyset$ for some $k \in I\!R^*$, (CL$\supset$) implies that $\mathcal{S}_l = \emptyset$ for every $l < k$. So let $k, l \in I\!R^*$ with $k < l$ and suppose that $\mathcal{S}_k \neq \emptyset$ and $\mathcal{S}_l \neq \emptyset$. By (CL$\supset$) we get $S := (-\infty, k] \in \mathcal{S}_k$ and $T := (-\infty, l] \in \mathcal{S}_l$. Therefore $\sigma(S) = [a(k), k]$ and $\sigma(T) = [a(l), l]$. As a consequence we get, by (IIA),

$$[a(l), k] = \sigma(T) \cap S \subseteq \sigma(S) = [a(k), k].$$

Therefore $a(k) \leq a(l)$ which finishes the proof. ■

An example of a solution, satisfying the requirements above, is given by $\hat{\sigma}_{\varepsilon,k}$, described in h) of example 3.3. The proposition above is not completely satisfactory, due to the fact that (CL$\supset$) is a very strong requirement on the class $\mathcal{S}$. However, example 5.1 showed that it is not easy to get rid of it. Another approach is that one asks for some strengthening of (SAC), instead of looking for too special classes $\mathcal{S}$.

A solution σ on $\mathcal{S}$ satisfies (SMAC) (*strong monotonic approximation consistency*) if for every $S, S_1, S_2, \ldots \in \mathcal{S}$ with $\sup S \leq \sup S_i$ for every $i \in I\!N$ the following statement is true:

> if $s_i \in \sigma(S_i)$ for every $i \in I\!N$ and $s \in S$ is such that $s \geq \liminf_{i\to\infty} s_i$ then $s \in \sigma(S)$.

It is obvious from the definitions that (SMAC) implies (SAC). Conversely, if $\mathcal{S}_\infty = \emptyset$, we have that (SAC) and (TI) imply (SMAC).
With respect to what has been said at the beginning of this section, about an unsatisfactory characterization of solutions for unbounded subsets, we want to point out the following: if we replace (SAC) and (TI) by (SMAC), which is a weaker condition on *bounded* subsets, it is possible to give a nice characterization which takes also *unbounded* sets into account.

Proposition 5.2 *Let $\mathcal{S}$ be a collection of non-empty subsets of $\mathbb{R}$ and let σ be a solution on $\mathcal{S}$. The solution σ satisfies (SMAC) and (NEM) if and only if $\sigma = \sigma_a$ for some non-decreasing function $a : \mathbb{R}^* \to \overline{\mathbb{R}}$ satisfying $a(k) \leq k$ for every $k \in \mathbb{R}^*$, with strict inequality for every $k \in \mathbb{R}^*$ for which there is an $S \in \mathcal{S}_k$ with $k \notin S$.*

Proof First we prove the if-part: suppose $\sigma = \sigma_a$ for some a as mentioned above. By proposition 4.2 we infer that σ satisfies ((SAC) and) (NEM). Suppose $S, S_1, S_2, \ldots \in \mathcal{S}$ with $\sup S \leq \sup S_i$ for every $i \in \mathbb{N}$ and let $s \in S$, $s_i \in \sigma(S_i)$ for every $i \in \mathbb{N}$, be such that $s \geq \liminf_{i\to\infty} s_i$. Since $s_i \geq a(\sup S_i) \geq a(\sup S)$ for every $i \in \mathbb{N}$, we get $s \geq a(\sup S)$. Hence $s \in \sigma(S)$. So, σ satisfies (SMAC).
For the proof of the only-if-part suppose that σ satisfies (SMAC) and (NEM). Since (SMAC) induces (SAC) we get $\sigma = \sigma_a$, where a is defined by (1). Notice however, that a needs not be non-decreasing. Now define $\bar{a} : \mathbb{R}^* \to \overline{\mathbb{R}}$ by

$$\bar{a}(k) = \inf\{a(l) : l \text{ is such that } \mathcal{S}_l \neq \emptyset \text{ and } l \geq k\}$$

(with the usual convention that $\inf \emptyset = +\infty$). Clearly, $\bar{a}$ is non-decreasing. We prove that $\sigma_a = \sigma_{\bar{a}}$. So, let $S \in \mathcal{S}$ and $k := \sup S$ (which induces $\mathcal{S}_k \neq \emptyset$). Since $\bar{a}(k) \leq a(k)$ we have $\sigma_a(S) \subseteq \sigma_{\bar{a}}(S)$. Now let $s \in \sigma_{\bar{a}}(S)$. By definition of $\bar{a}(k)$ there is a sequence $l_1, l_2, l_3, \ldots$ with $\mathcal{S}_{l_i} \neq \emptyset$ and $l_i \geq k$ for every $i \in \mathbb{N}$ and $\bar{a}(k) = \lim_{i\to\infty} a(l_i)$. For every $i \in \mathbb{N}$ there is, by definition of $a(l_i)$, an $S_i \in \mathcal{S}_{l_i}$ and $s_i \in \sigma(S_i)$ such that $\lim_{i\to\infty} a(l_i) = \lim_{i\to\infty} s_i$. Since $s \geq \bar{a}(k) = \lim_{i\to\infty} a(l_i) = \lim_{i\to\infty} s_i$ we get by (SMAC): $s \in \sigma(S) = \sigma_a(S)$. So, $\sigma_{\bar{a}}(S) \subseteq \sigma_a(S)$. ∎

6 Approximation with sequences

A remarkable result of section 4 was that there was no solution, besides the trivial σ_{tot}, satisfying (SAC), (NEM), (TI) and (MI), in case $\mathcal{S}$ contains at least one upper bounded set without maximum. In this section we will get a positive result by considering generalized solutions. In order to do so we need some definitions.

Let $\mathcal{S}$ be a non-empty collection of non-empty subsets of $\mathbb{R}$. A *generalized solution* Σ on $\mathcal{S}$ is a map which assigns to every $S \in \mathcal{S}$ a subset $\Sigma(S)$ of $S_{\text{inc}}^{\mathbb{N}}$, the collection of non-decreasing sequences in S. A sequence $\underline{s} = s_1, s_2, \ldots$ in $S_{\text{inc}}^{\mathbb{N}}$ can be interpreted as a sequence of approximate optimal elements, where the degree of approximation gets better when indices are increasing. For generalized solutions we use the following axioms.
A solution Σ on $\mathcal{S}$ satisfies (AC) (*approximation consistency*) if for every $S_1, S_2 \in \mathcal{S}$ with $\sup S_1 = \sup S_2$ the following statement is true:

$$\text{if } \underline{s_2} \in \Sigma(S_2) \text{ and } \underline{s_1} \in S_1 \text{ is such that } \lim \underline{s_1} \geq \lim \underline{s_2} \text{ then } \underline{s_1} \in \Sigma(S_1).$$

The definitions of (NEM), (TI) and (MI) for generalized solutions are obvious.

Example 6.1 The following two generalized solutions, defined on a collection $\mathcal{S}$ obeying (CL+) and (CL*), all satisfy (AC), (NEM), (TI) and (MI).

a) Define Σ_{opt} by $\Sigma_{opt}(S) := \{\underline{s} \in S_{\text{inc}}^{\mathbb{N}} : \lim \underline{s} = \sup S\}$.

b) Define Σ_{tot} by $\Sigma_{tot}(S) := S_{\text{inc}}^{\mathbb{N}}$.

The following proposition shows that all generalized solutions, satisfying (AC), (NEM), (TI) and (MI), are mixtures of the solutions in example 6.1.

Proposition 6.1 *Let $\mathcal{S}$ be a collection of non-empty subsets of $\mathbb{R}$ and let Σ be a generalized solution on $\mathcal{S}$. The solution Σ satisfies (AC), (NEM), (TI) and (MI) if and only if Σ coincides with one of the solutions Σ_{opt} or Σ_{tot} on the collection of upper bounded subsets in $\mathcal{S}$ and if Σ coincides with one of these two solutions (but not necessarily the same) on $\mathcal{S}_{+\infty}$.*

Proof The if-part of the proof is left to the reader. For the only-if-part assume that Σ satisfies (AC), (NEM), (TI) and (MI). We only prove that Σ coincides with one of the solutions Σ_{opt} or Σ_{tot} on the collection of upper bounded subsets. The proof for $\mathcal{S}_{+\infty}$ is similar. We distinguish between two cases.

Case 1: $\Sigma(S) \subseteq \Sigma_{opt}(S)$ for every upper bounded $S \in \mathcal{S}$.

Let $S \in \mathcal{S}$ be upper bounded. By (NEM), there is a sequence $\underline{s} \in \Sigma(S)$. By (AC), comparison of every sequence $\underline{s}' \in \Sigma_{opt}(S)$ with $\underline{s}$, implies $\Sigma_{opt}(S) \subseteq \Sigma(S)$. Therefore, $\Sigma(S) = \Sigma_{opt}(S)$.

Case 2: there is an upper bounded $S' \in \mathcal{S}$ with $\Sigma(S') \not\subseteq \Sigma_{opt}(S')$.

Let $S' \in \mathcal{S}$ and $\underline{s}' \in \Sigma(S')$ be such that $\underline{s}' \notin \Sigma_{opt}(S')$, let $k' := \sup S'$ and $k'' := \lim \underline{s}'$. Since $\underline{s}' \notin \Sigma_{opt}(S')$ we have $k'' < k'$. Moreover, we have for every $n \in I\!N$ by (CL+) and (TI),

$$-(1 - n^{-1})k' + S' \in \mathcal{S}$$

and

$$-(1 - n^{-1})k' + \underline{s}' \in \Sigma(-(1 - n^{-1})k' + S').$$

Moreover, by (CL*) and (MI), we get

$$-(n - 1)k' + nS' = n(-(1 - n^{-1})k' + S') \in \mathcal{S}$$

and

$$-(n - 1)k' + n\underline{s}' = n(-(1 - n^{-1})k' + \underline{s}') \in \Sigma(-(n - 1)k' + nS').$$

Now let $\underline{s} \in \Sigma_{tot}(S')$ and let $l := \lim \underline{s}$. Choose $n \in I\!N$ such that $-(n - 1)k' + nk'' < l$. By (AC), comparison of the sequences $\underline{s}$ and $-(n-1)k' + n\underline{s}'$ yields $\underline{s} \in \Sigma(S')$. Therefore, $\Sigma(S') = \Sigma_{tot}(S')$. By (TI) one infers that $\Sigma(S) = \Sigma_{tot}(S)$ for every upper bounded $S \in \mathcal{S}$. ■

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