Optimal Auction Design with Risk Aversion and Correlated Information

by

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Abstract

In this paper, we develop a general auction model in which buyers and seller are risk averse and private information is multidimensional and correlated, and in this setting we examine the problem of optimal auction design. In particular, we consider the problem faced by someone who has an object to sell but who does not know how much prospective buyers might be willing to pay, and allowing for risk aversion and correlated information on the part of buyers, we demonstrate the existence of an auction procedure that yields the risk averse seller the highest expected utility among all the auction procedures that are rational and Bayesian incentive compatible.

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1. Introduction

Myerson (1981), building on the work of Vickrey (1961) and Harsanyi (1967/68), studied auctions as noncooperative games with incomplete information. Viewing the auction problem in this way, Myerson reduced the seller's problem to one of designing a revelation game, to be played by the potential buyers, having an equilibrium that maximized the seller's expected payoff. In his model, Myerson assumed that buyers and the seller were risk neutral and that the private information of buyers was independently distributed. In this paper, we develop a general auction model in which buyers and seller are risk averse and private information is multidimensional and correlated, and in this setting we examine the problem of optimal auction design. In particular, we consider the problem faced by someone who has an object to sell but who does not know how much prospective buyers might be willing to pay, and allowing for risk aversion and correlated information on the part of buyers, we demonstrate the existence of an auction procedure that yields the risk averse seller the highest expected utility among all the auction procedures that are rational and Bayesian incentive compatible.

Auctions in which the seller and the buyers are risk neutral and private information is independently distributed have been intensely studied in the literature (in addition to Myerson (1981), see for example Riley and Samuelson (1991) and Harris and Raviv (1981)). Two main conclusions emerge from this work. First, the four most common forms of auctions (Dutch, first-price, second-price, and English)[1] generate the same expected revenue for the seller. Second, for many common distributions of private information (including the normal, exponential, and uniform distributions) the four standard auction forms with suitably chosen reserve prices or entry fees are optimal from the perspective of the seller. These conclusions, however, are not robust with respect to changes in the assumption that the seller or the buyers are risk neutral *or* with respect to a change in the assumption that private information is independently distributed. For example, Maskin and Riley (1984) show that if the private information of buyers is independently distributed but buyers are risk averse, then from the seller's viewpoint first-price and English auctions are not revenue equivalent - nor are they optimal. Alternatively, Milgrom and Weber (1982) show that if buyers and the seller are risk neutral but private information is dependent - and in particular affiliated - then English auctions generate the

[1]The Dutch auction is conducted by an auctioneer who initially calls for a high price and then continuously lowers the price until some buyer stops the auction and claims the object at that price. In an English auction, the auctioneer begins by soliciting bids at a low price and then gradually raises the price until only one willing buyer remains. A first-price auction is a sealed-bid auction in which the buyer making the highest bid wins and pays the amount of his bid for the object. A second-price auction is also a sealed-bid auction in which the buyer making the highest bid wins and pays the amount of the second highest bid.

highest expected revenue to the seller, followed by the second-price auction, and finally the Dutch and first-price auctions. These studies indicate the importance of analyzing auction environments in which *both* risk aversion and correlated private information are present. While such auction environments are common in practice, they have not been examined in the existing literature. They are, however, the focus of this paper.

We formulate the seller's problem as one of designing a revelation game, to be played by risk-averse buyers, having an equilibrium that maximizes the risk-averse seller's expected utility.[2] As in Myerson (1981) each buyer's private information is represented by his type, and the seller's incomplete information problem arises due to his lack of knowledge concerning each buyer's type. The auction model we develop covers as special cases many of the auction environments analyzed in the existing literature - including the Milgrom-Weber (1982) model with affiliated private information and the Maskin-Riley (1984) model with risk-averse buyers. In contrast to much of the existing literature, our auction model allows multidimensional buyer type descriptions (i.e., each buyers private information is allowed to be multi-dimensional) and vector-valued payoffs. Thus, our model as well as our results can be applied to multidimensional bidding situations in which the participants are risk averse and information is correlated. These types of auctions are common in government procurement contracting (e.g., a defense contractor may bid on price and quality in the production of a weapons system - see Che (1993) and Johnson (1994)). Moreover, in our model we assume that each buyer's utility depends not only upon his own type (i.e., private information) but also upon the types of the other buyers. Thus, our model allows for informational externalities. Finally, in addition to assuming that auction payoffs are vector-valued, we assume that each buyer's utility depends not only upon his own auction payoff but upon the payoffs of others. Thus, our model allows for payoff externalities.

For the seller, the problem of game design reduces to one of mechanism choice. In particular, given the seller's probability beliefs concerning buyer types, the seller's problem is to choose a function, defined on the set of buyer types taking values in the set of probability measures defined over winner-payoff vector pairs, that maximizes the seller's expected utility. In choosing this function, or mechanism, the seller faces two constraints: (1) the mechanism must be such that no buyer has an incentive to report his type dishonestly (i.e., the mechanism must be Bayesian incentive compatible or BIC)[3], and (2) the mechanism must be such that each buyer has an incentive to participate in the auction in

[2]Thus, the auction problem can be viewed as a Stackelberg game in which the seller moves first, designing a revelation game to be played by the potential buyers who act as the followers.
[3]Here we follow Myerson (1981) in assuming that unless a buyer is given an incentive to misreport his type, he will report honestly. In a one-shot game such as an auction, this assumption is sensible.

the first place (i.e., the mechanism must be individually rational for each buyer). Besides being rational and BIC, the only other requirement we impose is that the mechanism be measurable. Thus, in our model the selection of an auction mechanism is governed by economic considerations rather than exogenous technical restrictions such as continuity and differentiability.

Because the existence problem is infinite dimensional, novel existence arguments are required. We base our resolution of the existence problem upon the notion of K-convergence almost everywhere and a result due to Balder (1990) on sequential compactness (with respect to K-convergence) in spaces of transition probabilities. Essentially, K-compactness provides a subsequence extraction principle that is analytically similar to sequential compactness for the topology of pointwise convergence. Given the pointwise nature (i.e., the type-dependent nature) of the rationality and Bayesian incentive compatibility constraints, this subsequence principle is precisely what is needed in order to begin to establish the existence of an optimal BIC auction mechanism. K-compactness, however, takes us only part of the way there. In order to finally establish existence, we must show that the set of rational, BIC auction mechanisms is K-closed. In particular, we must show that any equivalence class of auction mechanisms determined by the K-limit of a K-convergent sequence of rational, BIC auction mechanisms contains at least one rational, BIC auction mechanism.[4] *In general, not all the mechanisms contained in the equivalence class determined by the K-limit of a K-convergent sequence of rational, BIC auction mechanisms are BIC*. Moreover, in an auction model where private information is correlated and several buyers hold heterogeneous probability beliefs conditioned on private information, showing that such an equivalence class contains a BIC mechanism is a delicate matter. Here, we accomplish this by simply going through the task of constructing a BIC mechanism contained in such an equivalence class.

In previous work by the author (e.g., Page (1989, 1994)) the problem of existence of optimal *dominant strategy incentive compatible* (DSIC) mechanisms has been analyzed in various principal-agent settings with adverse selection and moral hazard. Besides fundamental differences in the settings (i.e., the auction setting versus the principal-agent setting), there are also fundamental differences in the nature of the existence problems that arise in analyzing DSIC mechanisms versus BIC mechanisms - most notably with regard to the nature and resolution of the K-closure problem. As in the BIC case, in order to establish the existence of an optimal DSIC mechanism it must be shown that any equivalence class of mechanisms determined by the K-limit of a K-convergent sequence of rational, DSIC mechanisms contains at least one rational, DSIC mechanism. However, in

[4]Equivalence classes are identified with respect to a particular dominating measure.

the case of DSIC mechanisms this can be accomplished via a relatively straightforward application of measurable selection techniques. These techniques, however, cannot be used in the BIC case to resolve the K-closure problem.

The analysis here also continues an analysis of BIC mechanisms begun in Page (1992). There, Stackelberg games with incomplete information are considered in which each follower's payoff depends only upon his own type, his own action, and the leader's action (i.e., values are private and there is moral hazard as well as adverse selection). Besides focusing on the auction problem (i.e., a screening problem with adverse selection only), here we examine a screening problem with informational externalities: each buyer's payoff depends not only on his own type but on the types of the other buyers.

In Section 2, we present the basic ingredients of the auction model. In Section 3, we define what is meant by an auction mechanism, we define K-compactness, and we present our basic results on K-compactness in the set of auction mechanisms. In section 4, we define what is meant by a rational, BIC auction mechanism for an auction model with risk aversion and correlated types. In Section 5, we state our main results. Proofs are given in Section 6.

2. The Model

2.1 Basic Elements

[A-1]: We will assume that the basic elements of the auction are known to the seller as well as the buyers.

[A-2]:

I	=	{0,1,2, . . . , h}. The elements of I, denoted by i or j, will index the players in the auction, with 0 denoting the seller and i = 1,2, . . . h, denoting the buyers. Equip I with the metric ρ, defined as follows: $\rho(i, i') = 1$ if $i \neq i'$. (I, ρ) is a compact metric space.
X	=	a compact subset of R^h with elements denoted by $x = (x_1, x_2, ..., x_h)$. Each x is a vector of payments. If $x_i > 0$, then the seller makes a payment of x_i to the ith buyer, and if $x_i < 0$ then the ith buyer makes a payment of x_i to the seller.
T_i	=	the set of ith buyer types with elements denoted by t_i, equipped with the σ-field Σ_i.
T	=	$T_1 \times T_2 \times \cdots \times T_h$, with elements denoted by $t = (t_1, t_2, ..., t_h)$, equipped with the product σ-field $\Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_h$.
T_{-i}	=	$T_1 \times \cdots \times T_{i-1} \times T_{i+1} \times \cdots \times T_h$, with elements denoted by $t_{-i} = (t_1, ..., t_{i-1}, t_{i+1}, ..., t_h)$, equipped with the product σ-field $\Sigma_{-i} = \Sigma_1 \times \cdots \times \Sigma_{i-1} \times \Sigma_{i+1} \times \cdots \times \Sigma_h$.
(t_i, t_{-i})	=	$(t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_h) = t$.
p_i	=	for i = 0,1,2, . . . ,h, a probability measure defined on (T, Σ) representing the ith player's prior probability beliefs concerning buyer types.

2.2 Conditional Probability Beliefs

[A-3]: (1) For i = 1,2, . . . ,h, we will assume that $q_i(\cdot|\cdot)$ is a version of the regular conditional probability of p_i with respect to the marginal m_i of p_i defined on (T_i, Σ_i).

Thus, under [A-3] (1), for each $E \in \Sigma_{-i}$, $q_i(E|\cdot)$ is a real-valued, Σ_i-measurable function defined on T_i specifying for each of the ith buyer's types, the probability weight the ith buyer assigns to the subset E, and for any $S \in \Sigma_i$, $p_i(S \times E) = \int_S q_i(E|t_i) m_i(dt_i)$. If the sets, T_i, are Borel spaces and the σ-fields, Σ_i, Borel σ-fields, then by Dellacherie and

Meyer (1975, III, pp. 69-73), for each probability measure p_i defined on T, there exists regular conditional probabilities.

We will also assume that there is a product measure $\mu = \mu_1 \times \cdots \times \mu_h$ defined on (T,Σ), with each μ_i σ-finite on Σ_i, such that

[A-3]: (2) the ith buyer's probability measure p_i defined on the set T of buyer types is absolutely continuous with respect to μ (denoted $p_i << \mu$), and

(3) for each i and $t_i \in T$, the ith buyer's conditional probability measure $q_i(\cdot \mid t_i)$ defined on the set T_{-i} of other buyer types is absolutely continuous with respect to μ_{-i} (denoted $q_i(\cdot \mid t_i) << \mu_{-i}$), where μ_{-i} is the product measure $\mu_1 \times \cdots \times \mu_{i-1} \times \mu_{i+1} \times \cdots \times \mu_h$.

We will refer to μ as the dominating measure.

Remarks 1

(1) [A-3] is satisfied in any auction model in which each participant's probability distribution over types has a density function (i.e., Lebesgue measure serves as the dominating measure). Thus, in most auction models in the literature [A-3] is satisfied (see for example Riley and Samuelson (1981), Myerson (1981), Milgrom and Weber (1982), Maskin and Riley (1984), and Che (1993)).

(2) In the analysis to follow, we will use the dominating measure μ to define equivalence classes of auction mechanisms and to identify and keep track of the relevant sets of measure zero. Given that buyers hold heterogeneous probability beliefs conditioned on private information, it would be a difficult task to keep track of the sets of measure zero without such a dominating measure. We will also use μ to construct a method of detecting dishonest reporting by buyers (this is done in the proof of Theorem 5.1). In order to see that [A-3] is a mild assumption, consider the following example:

Example

The following example satisfies all the conditions given in [A-3]. Suppose there are two buyers, i = 1,2, such that for each i, $T_i = [0,\infty)$, so that $T = [0,\infty) \times [0,\infty)$, and let Σ be the Borel product σ-field, $B[0,\infty) \times B[0,\infty)$, in T. Equip (T,Σ) with the Lebesgue product measure $\mu = \mu_1 \times \mu_2$. Suppose now that the ith buyer's probability beliefs are given via a joint density function $h_i(\cdot)$, defined on T, so that for any $E \in \Sigma$,

$$p_i(E) = \int_E h_i(t)\mu(dt).$$

Thus, $p_i << \mu$. Now let $r_i(\cdot \mid t_i)$ denote the conditional density corresponding to the joint density $h_i(\cdot)$, so that for any $S \in \Sigma_{-i}$,

$$q_i(S \mid t_i) = \int_S r_i(t_{-i} \mid t_i)\mu_{-i}(dt_{-i}).$$

Thus, $q_i(\cdot \mid t_i) << \mu_{-i}$ for all $t_i \in T_i$.

2.3 Payoffs

$v_i(\cdot,\cdot,\cdot)$ = for i = 1,2,...,h, the ith buyer's real-valued payoff function defined on $T \times I \times X$. Thus, $v_i(t, j, x)$ is the payoff to the ith buyer if player j = 0,1,2,...,h wins the auction and the type and payment h-tuples are t and x respectively. We will assume that $v_i(t,\cdot,\cdot)$ is continuous on $I \times X$ for each $t \in T$, that $v_i(\cdot, j, x)$ is Σ-measurable on T for each $(j, x) \in I \times X$, and that $v_i(\cdot,\cdot,\cdot)$ is p_i- integrably bounded on $T \times I \times X$ (i.e., $|v_i(t, j, x)| \leq \xi_i(t)$ on $T \times I \times X$, where $\xi_i(\cdot)$ is a p_i- integrable function on T).

$u(\cdot,\cdot,\cdot)$ = the seller's real-valued payoff function defined on $T \times I \times X$. Thus, $u(t, j, x)$ is the payoff to the seller if player j = 0,1,2,...,h wins the auction and the type and payment h-tuples are t and x respectively. We will assume that $u(t,\cdot,\cdot)$ is upper-semicontinuous on $I \times X$ for each $t \in T$, that $u(\cdot,\cdot,\cdot)$ is $\Sigma \times B(I \times X)$- measurable on $T \times I \times X$, and that $u(\cdot,\cdot,\cdot)$ is p_0- integrably bounded from above on $T \times I \times X$ (i.e., $u(t, j, x) \leq \zeta(t)$ on $T \times I \times X$, where $\zeta(\cdot)$ is a p_0- integrable function on T). Here $B(I \times X)$ denotes the Borel σ-field in $I \times X$.

Note that if j = 0 "wins" (i.e., if the seller wins), then the seller keeps the object (i.e., the object is not sold).

3. Auction Mechanisms and K-compactness

Let $P(I \times X)$ denote the set of all probability measures defined on the Borel σ-field $B(I \times X)$ in $I \times X$, and equip $P(I \times X)$ with the topology of weak convergence of probability measures. Since $I \times X$ is a compact metric space, $P(I \times X)$ is compact and metrizable for the topology of weak convergence of measures (Parthasarathy (1967), Theorem 6.4). Elements of $P(I \times X)$ will be denoted by φ's, and we will write $\varphi_n \Rightarrow \varphi$ whenever the sequence $\{\varphi_n\}_n \subset P(I \times X)$ converges weakly to $\varphi \in P(I \times X)$.

We shall restrict attention to direct auction mechanisms. In a direct mechanism, the buyers simultaneously and confidentially make reports to the seller concerning their types and the seller then selects a winner and a vector of payments based on these reports (recall that if $x_i > 0$, then the seller makes a payment to the ith buyer; and if $x_i < 0$ then the ith buyer makes a payment to the seller). Thus, a direct mechanism is a function, $\varphi(\cdot): T \to P(I \times X)$ defined as follows: if $t = (t_1, t_2, \ldots, t_h)$ is the h-tuple of reported types, then the winner-payment pair, $(i, x) \in I \times X$, is selected according to the probability measure $\varphi(t) \in P(I \times X)$. By the Revelation Principle, we can restrict attention to direct mechanisms without loss of generality, as long as the mechanisms are incentive compatible (i.e., induce truthful reporting).

Now let $B(P(I \times X))$ denote the Borel σ-field in $P(I \times X)$ generated by the (metrizable) topology of weak convergence. A function $\varphi(\cdot): T \to P(I \times X)$ is said to be measurable if for any subset of probability measures $E \in B(P(I \times X))$

$$\varphi^{-1}(E) = \{t \in T : \varphi(t) \in E\} \in \Sigma.$$

Let $M(T, P(I \times X))$ denote the set of all measurable functions defined on T taking values in $P(I \times X)$. We shall assume throughout that the feasible set of auction mechanisms is given by $M(T, P(I \times X))$. Elements of $M(T, P(I \times X))$ will be denoted by $\varphi(\cdot|\cdot)$ and by $\varphi(\cdot)$ (i.e., $\varphi(d(j,x)|t)$ is the element in $P(I \times X)$ selected by the auction mechanism given reports t).

The notion of compactness we shall use in analyzing the seller's auction design problem is based on the notion of K-convergence almost everywhere.

3.1 Definition (K-convergence):

A sequence of mechanisms $\{\varphi_n(\cdot)\}_n \subset M(T, P(I \times X))$ is said to K-converge $[\mu]$ to a K-limit $\varphi(\cdot) \in M(T, P(I \times X))$, if for each subsequence, $\{\varphi_{n_k}(\cdot)\}_k$, there is a μ-null set $N \in \Sigma$ (i.e., $\mu(N) = 0$) such that for the sequence of averaged mechanisms, $\{\varphi^k(\cdot)\}_k$, where

$$\varphi^k(\cdot) = \frac{\varphi_{n_1}(\cdot) + \cdots + \varphi_{n_k}(\cdot)}{k},$$

$\varphi^k(t) \Rightarrow \varphi(t)$ for all $t \in T \setminus N$.

Thus, $\{\varphi_n(\cdot)\}_n$ K-converges a.e.$[\mu]$ to K-limit $\varphi(\cdot)$, if for each subsequence, $\{\varphi_{n_k}(\cdot)\}_k$, there is a set of h-tuples of buyer types $N \in \Sigma$ of μ-measure zero such that for every $t \in T \setminus N$, the sequence of probability measures, $\{\varphi^k(t)\}_k \subset P(I \times X)$, converges weakly to $\varphi(t) \in P(I \times X)$. Thus, for each $t \in T \setminus N$,

$$\lim_k \int_{I \times X} g(j,x)\varphi^k(d(j,x) \mid t) = \int_{I \times X} g(j,x)\varphi(d(j,x) \mid t),$$

for each real-valued continuous function g defined on the compact metric space $I \times X$.

3.2 Definition (K-compactness):
A subset Ψ of $M(T, P(I \times X))$ is said to be relatively K-compact [μ] if every sequence in Ψ contains a subsequence K-converging [μ] to some $\varphi(\cdot) \in M(T, P(I \times X))$. Ψ is said to be K-compact [μ] if every sequence in Ψ contains a subsequence K-converging to some $\varphi(\cdot) \in \Psi$.

The feasible set of auction mechanisms, $M(T, P(I \times X))$, can be viewed as a set of transition probabilities. In Balder (1990), the classical notion of tightness of probability measures (e.g., see Parthasarathy (1967)) has been generalized to cover the case of transition probabilities. This generalized notion of tightness is important for our purposes because it guarantees the K-compactness of the feasible set of auction mechanisms. For the moment assume that $I \times X$ is a complete, separable metric space (rather than a compact metric space as we have assumed here).

3.3 Definition (μ-tightness):
A subset Ψ of $M(T, P(I \times X))$ is said to be μ-tight if there exists a function $h: T \times I \times X \to [0, +\infty]$ such that

(i) $h(\cdot,\cdot,\cdot)$ is $\Sigma \times B(I \times X)$-measurable,

(ii) $h(t,\cdot,\cdot)$ is inf-compact on $I \times X$ for each t (i.e., $\{(j,x) \in I \times X : h(t,j,x) \le \gamma\}$ is compact for each $t \in T$ and $\gamma \in R$), and

(iii) $\sup_{\varphi(\cdot|\cdot) \in \Psi} \int_T \int_{I \times X} h(t,j,x)\varphi(d(j,x) \mid t)\mu(dt) < +\infty$.

The importance of μ-tightness is made clear by the following Theorem due to Balder (1990).[5] This result represents an extension of Komlos' Theorem (1967) to the function space $M(T, P(I \times X))$.

[5] In Balder (1990) a more general version of this result is given.

3.4 Theorem:
Let Ψ be a subset of $M(T, P(I \times X))$. Then the following are equivalent:

(i) Ψ is μ-tight.

(ii) Ψ is K-compact $[\mu]$.

Observe that since $I \times X$ is a compact metric space, any subset of auction mechanisms, $\Psi \subset M(T, P(I \times X))$, is automatically μ-tight (consider $h(\cdot,\cdot,\cdot)$ identically equal to zero). This observation leads immediately to the following:

3.5 Corollary: For $I \times X$ a compact metric space and (T, Σ, μ) a σ-finite measure space, any subset of $M(T, P(I \times X))$ is relatively K-compact $[\mu]$, and $M(T, P(I \times X))$ is K-compact $[\mu]$.

4. Rational and Bayesian Incentive Compatible Auction Mechanisms

For each $\varphi \in P(I \times X)$, let

$$V_i(t, \varphi) = \int_{I \times X} v_i(t, j, x)\varphi(d(j, x)).$$

It follows from the continuity of $v_i(t, \cdot, \cdot)$ on $I \times X$ for each $t \in T$ that $V_i(t, \cdot)$ is continuous on $P(I \times X)$ (with respect to the topology of weak convergence) for each t. Moreover, since $v_i(\cdot, j, x)$ is Σ-measurable on T for each $(j, x) \in I \times X$, $V_i(\cdot, \varphi)$ is Σ-measurable on T for each $\varphi \in P(I \times X)$. Thus, $V_i(\cdot, \cdot)$ is $\Sigma \times B(P(I \times X))$-measurable on $T \times P(I \times X)$ (Castaing and Valadier (1977), III.14), and thus, for any auction mechanism $\varphi(\cdot) \in M(T, P(I \times X))$, $V_i(t, \varphi(t))$ is Σ-measurable.

Under the mechanism $\varphi(\cdot) \in M(T, P(I \times X))$, if the h-tuple of reported types is $t = (t_1, t_2, ..., t_h)$, then the winner-payment pair (j,x) is selected according to the probability measure $\varphi(t) \in P(I \times X)$. In designing the auction mechanism, the seller faces two constraints: (1) the mechanism must be such that no buyer is given incentives to report his type dishonestly (i.e., the mechanism must be Bayesian incentive compatible or BIC), and (2) the mechanism must provide incentives for each buyer to participate in the auction in the first place (i.e., the mechanism must be individually rational for each buyer). Formally, these constraints can be stated as follows:

$\varphi(\cdot) \in M(T, P(I \times X))$ is rational and BIC if and only if for each i = 1,2,...,h,

$$\int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\varphi(d(j,x) \mid t_i, t_{-i}) q_i(dt_{-i} \mid t_i)$$
$$\geq \int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\varphi(d(j,x) \mid t_i', t_{-i}) q_i(dt_{-i} \mid t_i), \qquad (1)$$

for all $t_i \in T_i \setminus C_i$ and all $t_i' \in T_i$, where $C_i \in \Sigma_i$ and $m_i(C_i) = 0$; and

$$\int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\varphi(d(j,x) \mid t_i, t_{-i}) q_i(dt_{-i} \mid t_i) \geq 0, \qquad (2)$$

for all $t_i \in T_i \setminus Q_i$, where $Q_i \in \Sigma_i$ and $m_i(Q_i) = 0$.

Let B denote the subset of auction mechanisms in $M(T, P(I \times X))$ satisfying the BIC constraints (constraints (1)), and let Γ denote the subset of mechanisms satisfying the individual rationality constraints (constraints (2)). Thus, $B \cap \Gamma$ denotes the set of rational BIC auction mechanisms in $M(T, P(I \times X))$.

We shall assume that,

[A-4]: the set $B \cap \Gamma$ of all individually rational and Bayesian incentive compatible mechanisms is nonempty.

We shall also assume that

[A-5]: the set $P(I \times X)$ contains a probability measure φ' such that for each buyer i = 1,2,...,h,

$$\int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\varphi'(d(j,x)) q_i(t_{-i} \mid t_i) \leq 0 \text{ for all } t_i \in T_i.$$

Remarks 2

(1) [A-4] is nontriviality assumption: without [A-4] the auction design problem is uninteresting. [A-4] will be satisfied if, for example, the set $X \subset R^h$ of potential auction payoffs contains a vector x'' such that for each buyer i = 1,2,...,h,

$$v_i(t, j, x'') \geq 0 \text{ for all } (t, j) \in T \times I.$$

To see that this implies that $B \cap \Gamma$ is nonempty let $\varphi'' \in P(I \times X)$ be a probability measure such that $\varphi''(I \times \{x''\}) = 1$ and consider the auction mechanism that selects the probability measure φ'' for all h-tuples, t, of reported types. Such a mechanism is individually rational and Bayesian incentive compatible for each buyer. One candidate for the vector x'' is the zero vector. Note that if X does not contain the zero vector, then the zero vector can be attached to X, via union, without destroying the compactness of X, the upper semicontinuity of $u(t,\cdot,\cdot)$, or the continuity of $v_i(t,\cdot,\cdot)$ for $i = 1, 2, \ldots, h$ and $t \in T$.

(2) [A-5] guarantees that the seller has available a "penalty" mechanism. [A-5] will be satisfied if, for example, the set $X \subset R^h$ of potential auction payoffs contains a vector x' such that for each buyer i = 1,2,...,h,

$$v_i(t, j, x') \leq 0 \text{ for all } (t, j) \in T \times I.$$

Given the "penalty" vector x', any probability measure $\varphi' \in P(I \times X)$ such that $\varphi'(I \times \{x'\}) = 1$ satisfies [A-5]. Some possible candidates for "penalty" vectors are vectors with large negative components (i.e., payoff vectors that call for each buyer to make a large payments to the seller). Note that if X does not contain such a penalty vector, then such a penalty vector can be attached to X (via union) without destroying the compactness of X, the upper semicontinuity of $u(t,\cdot,\cdot)$, or the continuity of $v_i(\cdot, j, x)$ for $i = 1, 2, \ldots, h$ and $t \in T$.

5. Main Results

5.1 Theorem (*On the K-compactness of the set of rational, BIC auction mechanisms)*:
Suppose [A-1]-[A-5] hold. $B \cap \Gamma$ is nonempty, convex, and K-compact.

5.2 Corollary (*On the K-closure of the set of rational, BIC auction mechanisms)*:
The μ-equivalence class of auction mechanisms determined by a K-limit, $\hat{\varphi}(\cdot) \in M(T, P(I \times X))$, of a K-convergent sequence of rational, BIC auction mechanisms contains at least one rational, BIC auction mechanism.

5.3 Theorem (*On existence*):
Suppose [A-1]-[A-5] hold. The seller's auction design problem

$$\max_{\varphi(\cdot|\cdot)\in B\cap\Gamma} \int_T \int_{I\times X} u(t,j,x)\varphi(d(j,x)\,|\,t)p_0(dt), \tag{3}$$

has a solution.

6. Proofs

Proof of 5.1

From [A-4] we have that $B\cap\Gamma$ is nonempty. Convexity follows from the affinity of $V_i(t,\cdot)$ on $P(I\times X)$ for each i and t.

Consider a sequence of mechanisms $\{\varphi_n(\cdot)\}_n \subset B\cap\Gamma$. Since $M(T,P(I\times X))$ is K-compact [μ], we can assume without loss of generality that $\{\varphi_n(\cdot)\}_n$ K-converges to a K-limit $\hat{\varphi}(\cdot)\in M(T,P(I\times X))$. Thus, for some μ-null set $N\in\Sigma$, $\varphi^n(t)\Rightarrow\hat{\varphi}(t)$ for all $t\in T\setminus N$, where

$$\varphi^n(\cdot) = \frac{\varphi_1(\cdot)+\dots+\varphi_n(\cdot)}{n}.$$

Let

$$N(t_i) = \{t_{-i}\in T_{-i} : (t_i,t_{-i})\in N\}. \tag{4}$$

For each i, we have (see Ash (1972), section 2.6)

$$\mu(N) = \int_{T_i} \mu_{-i}(N(t_i))\mu_i(dt_i) = 0, \tag{5}$$

so that for some $N_i\in\Sigma_i$ with $\mu_i(N_i)=0$,

$$\mu_{-i}(N(t_i)) = 0 \text{ for all } t_i\in T_i\setminus N_i. \tag{6}$$

Since for each i, $\mu_i(N_i)\cdot\mu_{-i}(T_{-i}) = \mu(N_i\times T_{-i}) = 0$ and $p_i \ll \mu = \mu_i\times\mu_{-i}$, $p_i(N_i\times T_{-i}) = 0$, for each i = 0,1,2,...,h.

Now define

$$h(t) = \begin{cases} 1 & \text{if for some i, } \mu_{-i}(N(t_i)) > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

and consider the auction mechanism

$$\overline{\varphi}(t) = \hat{\varphi}(t) \cdot (1 - h(t)) + \varphi' \cdot h(t), \tag{8}$$

where $\varphi' \in P(I \times X)$ is the "penalty" measure given in [A-5]. Since $t_i \to \mu_{-i}(N(t_i))$ is Σ_i-measurable, $h(\cdot)$ is Σ-measurable, and thus $\overline{\varphi}(\cdot) \in M(T, P(I \times X))$. Note that since $\overline{\varphi}(t) = \hat{\varphi}(t)$ for all $t \in T \backslash (\cup_i N_i \times T_{-i})$, where the sets N_i are those given via (5) and (6), and since $\mu(\cup_i N_i \times T_{-i}) = 0$, $\overline{\varphi}(\cdot)$ and $\hat{\varphi}(\cdot)$ are in the same μ-equivalence class. Thus, $\overline{\varphi}(\cdot)$ is also a K-limit (with respect to the dominating measure μ) of the sequence $\{\varphi_n(\cdot)\}_n$. Note also that since $p_i << \mu$ for each i = 0,1,2,...,h, $\overline{\varphi}(\cdot)$ and $\hat{\varphi}(\cdot)$ are in the same p_i-equivalence class for each i.

Let $C_{in} \cup Q_{in} \in \Sigma_i$ denote the subset of ith buyer types (i = 1,2,...,h) such that $p_i((C_{in} \cup Q_{in}) \times T_{-i}) = 0$ and such that for ith buyer types $t_i \in C_{in} \cup Q_{in}$ rationality and/or incentive compatibility may fail to hold under the mechanism $\varphi_n(\cdot)$ (see (1) and (2)), and let

$$F_{i\infty} = [\cup_n (C_{in} \cup Q_{in})] \cup N_i, \tag{9}$$

where the sets N_i are given via (5) and (6). Since $p_i(F_{i\infty} \times T_{-i}) = 0$,

$$p_i(\cup_i (F_{i\infty} \times T_{-i})) = 0.$$

For each i and $t_i \in T_i \backslash F_{i\infty}$, we have the following observations:

(a) $\overline{\varphi}(t_i, t_{-i}) = \hat{\varphi}(t_i, t_{-i})$ for all $t_{-i} \in T_{-i} \backslash [(\cup_{j, j \neq i} N_j \times T_{-i,j}) \cup N(t_i)]$, where

$$T_{-i,j} = T_1 \times \ldots \times T_{i-1} \times T_{i+1} \times \ldots \times T_{j-1} \times T_{j+1} \times \ldots \times T_h.$$

Moreover, given [A-3], since $\mu_{-i}((\cup_{j, j \neq i} N_j \times T_{-i,j}) \cup N(t_i)) = 0$,

$$q_i((\cup_{j, j \neq i} N_j \times T_{-i,j}) \cup N(t_i) | t_i) = 0.$$

(b) Since $\int_{T_{-i}} \int_{I \times X} v_i(t_i, t_{-i}, j, x) \varphi^n(d(j,x) | t_i, t_{-i}) q_i(dt_{-i} \mid t_i) \geq 0$ for all n, and since $\varphi^n(t_i, t_{-i}) \Rightarrow \hat{\varphi}(t_i, t_{-i})$ for $t_{-i} \in T_{-i} \backslash [(\cup_{j, j \neq i} N_j \times T_{-i,j}) \cup N(t_i)]$, it follows from observation (a), the continuity of $V_i(t,\cdot)$ on $P(I\times X)$ for each i and t, and the Dominated Convergence Theorem (see Ash (1972)) that

$$\begin{aligned}\lim_n \int_{T_{-i}}\int_{I\times X} v_i(t_i,t_{-i},j,x)\varphi^n(d(j,x)|t_i,t_{-i})q_i(t_{-i}\mid t_i) \\ &= \int_{T_{-i}}\int_{I\times X} v_i(t_i,t_{-i},j,x)\hat{\varphi}(d(j,x)|t_i,t_{-i})q_i(t_{-i}\mid t_i) \\ &= \int_{T_{-i}}\int_{I\times X} v_i(t_i,t_{-i},j,x)\overline{\varphi}(d(j,x)|t_i,t_{-i})q_i(t_{-i}\mid t_i) \\ &\geq 0.\end{aligned}$$

Thus, by observations (a) and (b), $\overline{\varphi}(\cdot)\in\Gamma$.

In order to show that $\overline{\varphi}(\cdot)\in B$, we will show that for each i and $t_i\in T_i\setminus F_{i\infty}$

$$\begin{aligned}\int_{T_{-i}}\int_{I\times X} v_i(t_i,t_{-i},j,x)\overline{\varphi}(d(j,x)\,|t_i,t_{-i})q_i(dt_{-i}\mid t_i) \\ \geq \int_{T_{-i}}\int_{I\times X} v_i(t_i,t_{-i},j,x)\overline{\varphi}(d(j,x)\,|t_i',t_{-i})q_i(dt_{-i}\mid t_i)\end{aligned}$$

for all t_i' in T_i. (10)

There are two cases to consider.

Case 1: $\mu_{-i}(N_i(t_i'))>0$.

Under case 1, $\overline{\varphi}(t_i',t_{-i})=\varphi'\cdot h(t_i',t_{-i})=\varphi'\in P(I\times X)$ for all $t_{-i}\in T_{-i}$ (recall that $h(t_i',t_{-i})=1$ in this case -- see (7) and (8) above). Thus, on the RHS of (10) we have

$$\begin{aligned}\int_{T_{-i}}\int_{I\times X} v_i(t_i,t_{-i},j,x)\overline{\varphi}(d(j,x)\,|t_i',t_{-i})q_i(dt_{-i}\mid t_i) \\ = \int_{T_{-i}}\int_{I\times X} v_i(t_i,t_{-i},j,x)\varphi'(d(j,x))q_i(t_{-i}\mid t_i)\leq 0,\end{aligned}$$

and since $t_i\in T_i\setminus F_{i\infty}$, by observation (b), we have on the LHS of (10)

$$\int_{T_{-i}}\int_{I\times X} v_i(t_i,t_{-i},j,x)\overline{\varphi}(d(j,x)\,|t_i,t_{-i})q_i(dt_{-i}\mid t_i)\geq 0.$$

Thus, (10) holds for case 1.

Case 2: $\mu_{-i}(N_i(t_i'))=0$.

Under case 2, $\overline{\varphi}(t_i',t_{-i}) = \hat{\varphi}(t_i',t_{-i})$ for all $t_{-i} \in T_{-i} \setminus [(\cup_{j,j\neq i} N_j \times T_{-i,j}) \cup N(t_i')]$, and since $\mu_{-i}((\cup_{j,j\neq i} N_j \times T_{-i,j}) \cup N(t_i')) = 0$, $q_i((\cup_{j,j\neq i} N_j \times T_{-i,j}) \cup N(t_i') | t_i) = 0$. Also, since $\varphi^n(t_i',t_{-i}) \Rightarrow \hat{\varphi}(t_i',t_{-i})$ for all $t_{-i} \in T_{-i} \setminus [(\cup_{j,j\neq i} N_j \times T_{-i,j}) \cup N(t_i')]$, it follows from the continuity of $V_i(t,\cdot)$ on $P(I \times X)$ for each i and t, and the Dominated Convergence Theorem that

$$\begin{aligned}\lim_n \int_{T_{-i}} \int_{I\times X} & v_i(t_i,t_{-i},j,x)\varphi^n(d(j,x)|t_i',t_{-i})q_i(t_{-i} \mid t_i) \\ &= \int_{T_{-i}} \int_{I\times X} v_i(t_i,t_{-i},j,x)\hat{\varphi}(d(j,x)|t_i',t_{-i})q_i(t_{-i} \mid t_i) \\ &= \int_{T_{-i}} \int_{I\times X} v_i(t_i,t_{-i},j,x)\overline{\varphi}(d(j,x)|t_i',t_{-i})q_i(t_{-i} \mid t_i).\end{aligned}$$

By observations (a) and (b), since $t_i \in T_i \setminus F_{i\infty}$,

$$\begin{aligned}\lim_n \int_{T_{-i}} \int_{I\times X} & v_i(t_i,t_{-i},j,x)\varphi^n(d(j,x)|t_i,t_{-i})q_i(t_{-i} \mid t_i) \\ &= \int_{T_{-i}} \int_{I\times X} v_i(t_i,t_{-i},j,x)\hat{\varphi}(d(j,x)|t_i,t_{-i})q_i(t_{-i} \mid t_i) \\ &= \int_{T_{-i}} \int_{I\times X} v_i(t_i,t_{-i},j,x)\overline{\varphi}(d(j,x)|t_i,t_{-i})q_i(t_{-i} \mid t_i).\end{aligned}$$

Finally, since for all n, $\varphi^n(\cdot) \in B$,

$$\begin{aligned}\int_{T_{-i}} \int_{I\times X} & v_i(t_i,t_{-i},j,x)\varphi^n(d(j,x)|t_i,t_{-i})q_i(t_{-i} \mid t_i) \\ &\geq \int_{T_{-i}} \int_{I\times X} v_i(t_i,t_{-i},j,x)\varphi^n(d(j,x)|t_i',t_{-i})q_i(t_{-i} \mid t_i),\end{aligned}$$

for all n. Taking limits on both sides of the inequality above,

$$\begin{aligned}\int_{T_{-i}} \int_{I\times X} & v_i(t_i,t_{-i},j,x)\overline{\varphi}(d(j,x)|t_i,t_{-i})q_i(t_{-i} \mid t_i) \\ &\geq \int_{T_{-i}} \int_{I\times X} v_i(t_i,t_{-i},j,x)\overline{\varphi}(d(j,x)|t_i',t_{-i})q_i(t_{-i} \mid t_i).\end{aligned}$$

Thus, (10) holds for case 2. Q.E.D.

Remarks 3

The mechanism, $\overline{\varphi}(\cdot)$, defined in (8) above imposes a penalty, $\varphi' \in P(I \times X)$, on all the buyers if any one buyer tells a mathematically inconvenient lie concerning his type. Assumption [A-5] guarantees that such a penalty is available, and assumptions [A-3] (2) and (3) (i.e., the assumptions concerning the dominating measure μ) guarantee that these mathematically inconvenient lies can be detected under the conditions of incomplete information prevailing in the auction.

Proof of 5.3

Let

$$\overline{U} = \sup_{\varphi(\cdot|\cdot)\in B\cap\Gamma} \int_T \int_{I\times X} u(t,j,x)\varphi(d(j,x)\,|\,t)p_0(dt).$$

Since the seller's payoff function, $u(\cdot,\cdot,\cdot)$, is p_0-integrably bounded, $\overline{U}$ is finite. Let $\{\varphi_n(\cdot)\}_n \subset B \cap \Gamma$ be a sequence of auction mechanisms such that

$$\int_T \int_{I\times X} u(t,j,x)\varphi_n(d(j,x)\,|\,t)p_0(dt) \to \overline{U}.$$

Since $B\cap\Gamma$ is K-compact $[\mu]$, we can assume without loss of generality that $\{\varphi_n(\cdot)\}_n$ K-converges to a K-limit $\overline{\varphi}(\cdot) \in B\cap\Gamma$, and since $p_0 << \mu$, we can conclude that $\varphi^n(\cdot) \Rightarrow \overline{\varphi}(\cdot)$ a.e. $[p_0]$, where

$$\varphi^n(\cdot) = \frac{\varphi_1(\cdot) + \ldots + \varphi_n(\cdot)}{n}.$$

Also, since

$$\int_T \int_{I\times X} u(t,j,x)\varphi_n(d(j,x)\,|\,t)p_0(dt) \to \overline{U},$$

$$\frac{1}{n}\sum_{k=1}^{n} \int_T \int_{I\times X} u(t,j,x)\varphi_k(d(j,x)\,|\,t)p_0(dt) \to \overline{U}.$$

Thus, $\int_T \int_{I\times X} u(t,j,x)\varphi^n(d(j,x)\,|\,t)p_0(dt) \to \overline{U}$.

For each $t \in T$ and $\varphi \in P(I \times X)$, let

$$U(t,\varphi) = \int_{I\times X} u(t,j,x)\varphi(d(j,x)). \tag{11}$$

By Lemma 1.5 of Nowak (1984), $U(t,\cdot)$ is upper semicontinuous on $P(I\times X)$ with respect to the (metrizable) topology of weak convergence of measures for each t, and by Lemma 1.6 of Nowak, $U(\cdot,\cdot)$ is $\Sigma\times B(P(I\times X))$-measurable (see also chapter III in Dellacherie and Meyer (1975)). Thus, for any mechanism $\varphi(\cdot)\in M(T,P(I\times X))$, the function $t\rightarrow U(t,\varphi(t))$ is Σ-measurable, where

$$U(t,\varphi(t))=\int_{I\times X}u(t,j,x)\varphi(d(j,x)|t). \tag{12}$$

Next, $\varphi^{n}(\cdot)\Rightarrow\overline{\varphi}(\cdot)$ a.e. $[p_0]$ implies via the upper semicontinuity of $U(t,\cdot)$ on $P(I\times X)$ that $\limsup_n U(t,\varphi^n(t))\le U(t,\overline{\varphi}(t))$ a.e. $[p_0]$. Since the seller's payoff function, $u(\cdot,\cdot,\cdot)$, is p_0-integrably bounded from above, it follows from Fatou's Lemma (see Ash (1972)) that

$$\limsup_n\int_T U(t,\varphi^n(t))p_0(dt)\le\int_T\limsup_n U(t,\varphi^n(t))p_0(dt).$$

Thus,

$$\limsup_n\int_T U(t,\varphi^n(t))p_0(dt)=\overline{U}\le\int_T U(t,\overline{\varphi}(t))p_0(dt),$$

and since $\overline{\varphi}(\cdot)\in B\cap\Gamma$,

$$\int_T U(t,\overline{\varphi}(t))p_0(dt)=\int_T\int_{I\times X}u(t,j,x)\overline{\varphi}(d(j,x)\,|\,t)p_0(dt)=\overline{U}. \qquad \text{Q.E.D.}$$

Remarks 4

(1) The seller will hold the auction if and only if it is rational for him to do so. This can be formally expressed as follows. Suppose $\overline{\varphi}(\cdot)\in B\cap\Gamma$ solves the seller's auction design problem given in expression (3) above. Then the seller will hold the auction if and only if the optimal mechanism $\overline{\varphi}(\cdot)$ is such that,

$$\int_T\int_{I\times X}u(t,j,x)\overline{\varphi}(d(j,x)\,|\,t)p_0(dt)\ge 0.$$

(2) In the introduction we stated that ... *not all the mechanisms contained in the equivalence class determined by the K-limit of a K-convergent sequence of rational, BIC auction mechanisms are BIC.* In order to see why this is the case, consider the following:

Let $\{\varphi_n(\cdot)\}_n \subset B \cap \Gamma$ be any sequence of mechanisms. Since $M(T, P(I \times X))$ is K-compact [μ], we can assume without loss of generality that $\{\varphi_n(\cdot)\}_n$ K-converges to a K-limit $\hat{\varphi}(\cdot) \in M(T, P(I \times X))$. Thus, for some μ-null set $N \in \Sigma$, $\varphi^n(t) \Rightarrow \hat{\varphi}(t)$ for all $t \in T \setminus N$, where

$$\varphi^n(\cdot) = \frac{\varphi_1(\cdot) + L + \varphi_n(\cdot)}{n}.$$

As in the proof of Theorem 5.1, let

$$N(t_i) = \{t_{-i} \in T_{-i} : (t_i, t_{-i}) \in N\}.$$

For each i, we have

$$\mu(N) = \int_{T_i} \mu_{-i}(N(t_i))\mu_i(dt_i) = 0,$$

so that for some $N_i \in \Sigma_i$ with $\mu_i(N_i) = 0$,

$$\mu_{-i}(N(t_i)) = 0 \text{ for all } t_i \in T_i \setminus N_i.$$

Since for each i, $\mu_i(N_i) \cdot \mu_{-i}(T_{-i}) = \mu(N_i \times T_{-i}) = 0$ and $p_i << \mu = \mu_i \times \mu_{-i}$, $p_i(N_i \times T_{-i}) = 0$, for each i = 0,1,2,...,h.

Now let $C_{in} \cup Q_{in} \in \Sigma_i$ denote the subset of ith buyer types (i = 1,2,...,h) such that $p_i((C_{in} \cup Q_{in}) \times T_{-i}) = 0$ and such that for ith buyer types $t_i \in C_{in} \cup Q_{in}$ rationality and/or incentive compatibility may fail to hold under the mechanism $\varphi_n(\cdot)$ (see (1) and (2)), and let

$$F_{i\infty} = [\cup_n (C_{in} \cup Q_{in})] \cup N_i,$$

where the sets N_i are given via (5) and (6). Since $p_i(F_{i\infty} \times T_{-i}) = 0$,

$$p_i(\cup_i(F_{i\infty} \times T_{-i})) = 0.$$

In determining whether or not the K-limit $\hat{\varphi}(\cdot)$ is Bayesian incentive compatible, a problem arises if for the ith buyer with true type $t_i \in T_i \setminus F_{i\infty}$, there is a type, t_i' in T_i, that the ith buyer can report such that $q_i(N_i(t_i') \mid t_i) > 0$. To see why there is a problem consider the following:

Since $\{\varphi^n(\cdot)\}_n \subset B \cap \Gamma$, we have for each n,

$$\int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\varphi^n(d(j,x)|t_i, t_{-i})q_i(t_{-i} \mid t_i)$$
$$\geq \int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\varphi^n(d(j,x)|t_i', t_{-i})q_i(t_{-i} \mid t_i). \qquad (*)$$

Moreover, since $t_i \in T_i \setminus F_{i\infty}$ taking the limit on the left hand side (LHS) of (*) we obtain

$$\lim_n \int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\varphi^n(d(j,x)|t_i, t_{-i})q_i(t_{-i} \mid t_i)$$
$$= \int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\hat{\varphi}(d(j,x)|t_i, t_{-i})q_i(t_{-i} \mid t_i).$$

However, because $q_i(N_i(t_i') \mid t_i) > 0$ and because K-convergence may fail to hold for types t_{-i} in $N_i(t_i') \subset T_{-i}$, the limit on the RHS of (*),

$$\lim_n \int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\varphi^n(d(j,x)|t_i', t_{-i})q_i(t_{-i} \mid t_i),$$

may not equal

$$\int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\hat{\varphi}(d(j,x)|t_i', t_{-i})q_i(t_{-i} \mid t_i).$$

Thus, we cannot conclude that

$$\int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\hat{\varphi}(d(j,x)|t_i, t_{-i})q_i(t_{-i} \mid t_i)$$
$$\geq \int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\hat{\varphi}(d(j,x)|t_i', t_{-i})q_i(t_{-i} \mid t_i),$$

and thus we cannot conclude that the K-limit mechanism $\hat{\varphi}(\cdot)$ is Bayesian incentive compatible. Note that if $q_i(N_i(t_i') \mid t_i) = 0$, there is no problem - we have

$$\lim_n \int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\varphi^n(d(j,x)|t_i', t_{-i})q_i(t_{-i} \mid t_i)$$
$$= \int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\hat{\varphi}(d(j,x)|t_i', t_{-i})q_i(t_{-i} \mid t_i).$$

Thus, taking limits on both sides of (*), we obtain

$$\int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\hat{\varphi}(d(j,x)|t_i, t_{-i})q_i(t_{-i} \mid t_i)$$
$$\geq \int_{T_{-i}} \int_{I\times X} v_i(t_i, t_{-i}, j, x)\hat{\varphi}(d(j,x)|t_i', t_{-i})q_i(t_{-i} \mid t_i),$$

and we can easily conclude that $\hat{\varphi}(\cdot)$ is Bayesian incentive compatible.

Note that if we use the K-limit mechanism $\hat{\varphi}(\cdot)$ and the penalty measure φ' to construct a new mechanism $\overline{\varphi}(\cdot)$ given by

$$\overline{\varphi}(t) = \hat{\varphi}(t)\cdot(1 - h(t)) + \varphi' \cdot h(t)$$

where

$$h(t) = \begin{cases} 1 & \text{if for some i, } \mu_{-i}(N(t_i)) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

(as we did in the proof of Theorem 5.1) then the resulting mechanism is contained in the μ-equivalence class determined by the K-limit $\hat{\varphi}(\cdot)$ and is rational and Bayesian incentive compatible. Thus, by constructing the mechanism $\overline{\varphi}(\cdot)$ we avoid altogether the problem caused by the possibility that for some buyer i with true type $t_i \in T_i \setminus F_{i\infty}$ there is a type t_i' such that $q_i(N_i(t_i') \mid t_i) > 0$.

(3) The auction model and the existence result presented here can easily be extended to other auction settings. For example, by replacing $X \subset R^h$ with a compact metric space of state-contingent contracts, we can conclude from our existence result that there exists an optimal Bayesian mechanism for contract auctions with risk averse participants. We can

also modify our model so as to treat auctions in which payoffs are awarded to coalitions. Finally, we can use our model to extend to an incomplete information setting the basic theory of all-pay auction (e.g., Baye, Kovenock, and De Vries (1993)).

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