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PERFORMANCE OF DELTA-HEDGING STRATEGIES IN INTERVAL MODELS - A ROBUSTNESS STUDY

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Performance of Delta-hedging strategies in Interval Models - A robustness study

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Abstract

In this paper we study the pricing of financial derivatives in a risky discrete-time world. We assume that price changes of the underlying asset may take any value in an interval, rather than just two values as in the binary tree model. Arbitrage arguments are used to derive an upper and lower bound for the option price, and the well-known Stop-loss and Delta-hedging strategies are given particular interpretations in this context. A robustness study is performed to analyze the effect of a misspecification of the interval bounds on the worst-case costs that may arise.

Keywords: Option Pricing; Limited Volatility; Delta-hedging; Binary tree; Martingale measure.

JEL Classification: G11;G13.

1 Introduction

Arbitrage pricing of financial derivatives is based on the fact that, *within* a given model representing a complete market, all risk can be eliminated by the selection of a suitable hedging strategy. Risk, however, returns through doubts on the model assumptions themselves. This motivates *robustness studies*, in which the sensitivity of results with respect to changes in the model assumptions is analyzed. In particular the effects of alternative assumptions on the volatility have been investigated extensively [e.g. Duffie and Skiadas (1994), Karoui and Quenez (1995), McEneaney (1997)].

In this paper we take the well-known binary tree model [Cox et al. (1979)] as our starting point. We represent possibly time-varying uncertainty about volatility by means of *interval models*, in which the proportional price changes of the underlying asset are allowed to take any value in an interval, rather than just two values as in the binary tree model. Of course the resulting model is not complete and so there is no uniquely determined option price; we will show, however, that arbitrage arguments still imply an upper and lower bound for the option price. Interval models have been used before in a sequence of papers [Howe et al. (1994, 1996, 1997)] in which optimal 'minimax' algorithms have been derived. In these papers the optimization takes place however over just one or two time steps; here we refrain from optimization but rather derive minimax bounds for profits and costs that are valid over the full life time of the derivative.

In this way we translate uncertainty about the Black-Scholes model assumptions, which underlie a binary tree model, to a *single* interval model that contains a substantially wider

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variety of price paths. This is in contrast with the more usual approach of parameter variation, in which also the *types* of price paths is extended, but not their *number* in a single model.
Part of our motivation for considering interval models comes from control theory. In recent years an extensive theory of robust control has been developed (see for instance [Doyle et al. (1989), Vidyasagar and Kimura (1986), Zhou et al. (1996)], and [Caravani (1995)] for an application in economics), which addresses the problem of finding feedback strategies that have acceptable performance over a range of possible models. In this context, disturbances and perturbations are often modeled as unknown but bounded. Although the robustness analysis in the present paper is restricted to the Delta-hedging strategies that are standard in the financial industry, on a longer term the authors aim at designing robust hedging strategies on the basis of methods that take inspiration from the robust control theory.
The results of this paper can be summarized as follows. For a given interval model, we compute the arbitrage-free interval of option prices. The well-known Stop-loss and Delta-hedging strategies are given particular interpretations within the interval model. We then proceed to a robustness investigation of Delta-hedging strategies, by comparing the performance of such strategies on a range of interval models. Delta-hedging as in a binary model with jumps that are at least as large as those in the interval model is found to lead to a safe position and to a rather high option price quote. Lower price quotes are obtained from Delta-hedging based on a binary model whose jumps are included in the interval allowed by the interval model; however, the position now has risk. It turns out that the risk of such under-hedging strategies as measured in an interval model can be substantially higher than would be inferred from a tree model corresponding to the interval end points. In other words, not always are the extreme jumps the ones that hurt the most.

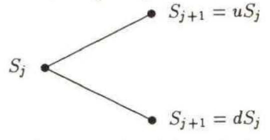
2 Preliminaries

2.1 Binary Tree Models

Our starting point is the well-known binary tree model for option pricing [Cox et al. (1979), Hull (1993)]. We take discrete time points $t_j = j\frac{T}{N}$ where T denotes the time to expiry. We consider a single underlying asset S; the price of this asset at time t_j will be denoted by S_j. An *asset price path* is a sequence

(1) $$S = \{S_0, \ldots, S_N\}.$$

The initial price S_0 will be fixed throughout the discussion. The binary tree model postulates the following structure in price paths:



Here u and d are the proportional jump factors. The set of all price paths in a binary tree are denoted by

(2) $$\mathbb{B}^{u,d} := \{S | S_{j+1} \in \{dS_j, uS_j\} \text{ for } j = 0, \ldots, N-1\},$$

where it is understood that all paths start at one and the same initial price S_0.
Under the assumption of frictionless markets and constant interest rates, the binary tree model gives sufficient information for option pricing by a no-arbitrage argument. Throughout the paper we shall consider the pricing of European call options; this is just to be specific, and the analysis could be carried out for other European-style derivatives.

The binary tree model shares many parameters with the standard Black-Scholes model based on the continuous time geometric Brownian motion (GBM) model [e.g. Hull (1993)]. Common parameters include the initial price S_0, the time to expiry T, the exercise price X and the interest rate r. However, the role of volatility parameter σ in the GBM model is assumed in the binary tree model by the jump factors u and d, together with the number of time steps N. In the discussion below we shall keep N fixed, and enforce a one-to-one relationship between jump factors and option price by imposing the constraint $d = 1/u$. Moreover, we prefer to parametrize tree models in terms of implied volatility σ rather than directly in terms of the jump factors. Therefore we define

(3) $\mathbb{B}^\sigma$ denotes the symmetric binary tree model that yields exactly the same arbitrage-free price for a European call option as is produced by the continuous time Black-Scholes model with volatility σ. This price is denoted by f_σ. The corresponding up- and downward jump factors in the binary tree model are denoted by resp. u_σ and $d_\sigma := 1/u_\sigma$. In particular, $u_0 = 1$ and $f_0 = [S_0 - X]^+$.

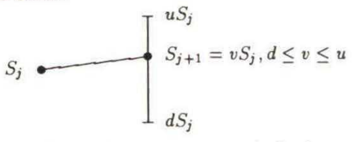
We remark that for small time step h, u_σ is approximately given by the well-known formula $u_\sigma \approx e^{\sigma\sqrt{h}}$.
We take the interest rate r equal to zero for simplifying the presentation; with some increase of complexity of notation, the development below can be carried out also for nonzero (but constant) interest rate.

2.2 Interval Models

The interval model is a modification of the binary tree model in which the relative price change in one step is restricted to an interval rather than to just two values. In other words, for a given asset S, with some given initial price S_0, the model restricts its price paths to

(4) $$\mathbb{I}^{u,d} := \{S | S_{j+1} \in [dS_j, uS_j] \text{ for } j = 0, \ldots, N-1\}.$$

This is depicted below.



The model parameters u and d denote respectively the maximal and minimal growth factor over one time step. Again we focus on symmetric models, with $d = 1/u$, and in analogy with (3) we define $\mathbb{I}^\sigma$ as the interval model with parameters $u_\sigma, 1/u_\sigma$, i.e.,

(5) $$\mathbb{I}^\sigma := \mathbb{I}^{u_\sigma, 1/u_\sigma}$$

Here σ can be interpreted as the maximal volatility. Notice that an interval model contains all price paths of a binary tree with the same parameters, and in addition all *interior* paths. These interior paths contain all binary tree paths corresponding to smaller σ[1], but also paths with time varying jumps.
We would like to stress that interval models have a much more convincing intrinsic motivation than binary trees, which are often viewed as mere computational tools in the continuous time Black-Scholes theory. The hypothesis that tomorrow's prices are in some (well-chosen) interval can be taken seriously on its own, unlike the claim that there are just two possible

[1] It is obvious and easily proved that u_σ is a strictly increasing function of σ.

outcomes as in binary trees. Therefore there is no strict need to 'let the stepsize go to zero' (which is anyway not straightforward in interval models), and we will even consider 'one step' interval models.

3 Hedging and Arbitrage in Interval Models

3.1 Hedging Strategies

Uncertainty about the future asset prices induces risk for the writer of an option. In complete markets this uncertainty can be squeezed out by hedging strategies, leading to a fixed arbitrage-free option price. In this section we analyze the effect of hedging in incomplete interval models.

Since we consider (European) call options C with exercise price X at time T, the final value of the option will be

$$[S_N - X]^+ \tag{6}$$

We consider *portfolios* consisting of one call option and a certain (in fact negative) fraction of assets, $\mathsf{P} = \mathsf{C} - \gamma \mathsf{S}$. A *strategy* is a rule for determining these fractions for future time instants on the basis of asset price levels. Formally we define a strategy g as a sequence of functions $\{g_0, \ldots, g_{N-1}\}$ that are *causal*[2] in S, assigning to each price path a sequence of portfolios $P_j = \mathsf{C} - \gamma_j \mathsf{S}$, with $\gamma_j := g_j(S_0, \ldots, S_j)$ for $j = 0, \ldots, N-1$. Causality is quite essential for the theory, and an obvious restriction in practice.

Trivial examples of strategies are taking a 'naked' or 'covered' position, which amounts to taking respectively $g_j = 0$ and $g_j = 1$ during the complete contract period. The corresponding costs are given by $[S_N - X]^+$ and $[S_N - X]^+ - (S_N - S_0)$, which is in most situations considered as an unacceptable risk.

Much better results can be obtained by smarter strategies that depend on prices and time. Two strategies will play a central role in the sequel. The first is the so-called Stop-loss strategy g^{SL}, which takes

$$\begin{aligned} g_j^{SL}(S_j) &= 0 \text{ if } S_j \leq X \\ g_j^{SL}(S_j) &= 1 \text{ if } S_j > X, \end{aligned} \tag{7}$$

which amounts to a covered position as soon as the option is in the money. Notice that this simple strategy does not depend on the model parameters.

The second strategy is the Delta-hedging strategy, which has already been mentioned in the context of binary trees and GBM's. It is designed in such a way that it yields certain outcomes for all price paths in binary trees. The strategy is most easily expressed in terms of backward recursions. For easy reference we use the symbol Δ for the strategy function.

$$\Delta_{N-1}(S_{N-1}) = \frac{[uS_{N-1} - X]^+ - [dS_{N-1} - X]^+}{(u-d)S_{N-1}} \tag{8}$$

$$\Delta_j(S_j) = \lambda \Delta_{j+1}(uS_j) + (1-\lambda)\Delta_{j+1}(dS_j) \tag{9}$$

with $\lambda := \frac{u(1-d)}{u-d}$. It is a matter of straightforward calculation that this strategy indeed yields the same outcome of costs along all paths in the binary tree $\mathbb{B}^{u,d}$.

Notice that on the trivial range of prices that cannot cross the exercise level anymore,

$$S_{N-j} \leq X/u^j \text{ or } S_{N-j} \geq X/d^j, \tag{10}$$

Delta-hedging coincides with the Stop-loss strategy, while within these boundaries the outcome of Delta-hedging is in between 0 and 1.

[2] So g_j may only depend on information that is available at time t_j, such as the realized asset prices $S_0, \ldots, S_j$, the time to maturity $T - t_j$, and the exercise price X. It is independent of $S_{j+1}, \ldots, S_N$.

Delta-hedging for the binary tree $\mathbb{B}^\sigma$ is denoted as Δ^σ, so

(11) $$\boxed{\Delta^\sigma \text{ is defined as in (9), with parameters } u, d \text{ equal to } u_\sigma, 1/u_\sigma}$$

3.2 Cost Intervals

A strategy determines for each asset price path $S = \{S_0, \ldots, S_N\}$ a sequence of portfolios $\mathsf{P}_j = \mathsf{C} - \gamma_j \mathsf{S}$. Notice that γ_j is the *outcome* of the strategy for an outcome of price paths, hence comparable to a yet unknown realization of a stochastic variable. The corresponding (also unknown) cost is the final pay-off plus the cost of trading. For a fixed strategy g this is given by

(12) $$Q^g(S) := [S_N - X]^+ - \Sigma_{j=0}^{N-1} \gamma_j (S_{j+1} - S_j)$$

The first term denotes the cost of the call option (without compensating premium, which is still to be determined), and the latter term is due to hedging.
Now the *cost range* of a strategy is simply the set of all possible outcomes of the costs for a given initial price,

(13) $$I^g := \bigcup_{S \in \mathbf{I}^{u,d}} Q^g(S).$$

It turns out that the cost range is always an interval, no matter which strategy is chosen.

PROPOSITION 3.1
The cost range I^g is a (not necessarily closed) interval for all interval models under all strategies. For strategies that are continuous (in price paths) the cost interval is closed.

For a proof we refer to the Appendix.
A worst/best case price path is a price path for which the maximum/ minimum costs are achieved. Discontinuous strategies may have no such price paths, but in numerical simulations this does not matter too much as there still exist worst and best cases for every desired level of finite precision.
Notice that if (i) no other information is available on asset prices than the interval model restrictions, and (ii) costs are the only criterion, then the cost of worst and best cases give full information on the performance of a strategy.
This means a substantial simplification with respect to binary tree models, which have in general a cost range consisting of N isolated points.

3.3 Arbitrage Intervals

Once a model for underlying asset prices has been adopted, arbitrage arguments put hard bounds on option prices. The assumption that prices follow a GBM, or follow paths in a binary tree, even pins down the price to just one value f_σ, with σ the volatility in the GBM and f_σ defined as in (3).
For interval models the arbitrage argument is equally convincing, but less powerful, and turns out to leave room for an *interval* of arbitrage-free prices. In this section we describe this *arbitrage interval* and the corresponding strategies. In Section 3.4 we give an interpretation in terms of martingale measures, as they have become standard in arbitrage theory.
An arbitrage opportunity is the possibility of making a sure profit[3]. This results in the following definition.

DEFINITION 3.2
The arbitrage-free interval of an interval model is the intersection of cost intervals over all strategies, $\bigcap_g I^g$.

[3] or, equivalently, a sure loss, as this always implies a sure profit for the counterparty.

This can be interpreted as the interval of arbitrage-free option premiums. Indeed, if there would exist a strategy g for which an option premium lies outside the cost interval I^g, this strategy leads to a sure gain (for all price paths in the model) for either the writer or the holder of the option.
The following main result describes the arbitrage bounds and the corresponding strategies.

THEOREM 3.3
Let f denote the premium for a European call option on an underlying asset that follows price paths in the interval model $\mathbb{I}^\sigma$.

1. *The unique strategy with lowest worst-case costs is the extreme Delta-hedging strategy Δ^σ, cf. (11). This lowest upper bound is the Black-Scholes price f_σ of the option as defined in (3), and is achieved for all price paths with extreme jumps over each time step, i.e., for all price paths in $\mathbb{B}^\sigma$.*

2. *The unique[4] strategy with highest best-case costs is the Stop-loss strategy (7). This highest lower bound is $f_0 = [S_0 - X]^+$ and is achieved for all price paths that do not cross the exercise price X.*

3. *The premium f is arbitrage free if and only if $f \in [f_0, f_\sigma]$.*

For a proof we refer to the Appendix.
We remark that the results also apply to asymmetric interval models (with $d \neq 1/u$), if f_σ is replaced by the arbitrage-free price in the binary tree with the parameter values u and d. The proof is completely analogous.
The result can be interpreted as follows. Interval models limit volatility (without assuming it to be constant). Hedging under the assumption that the maximum volatility will occur determines the maximum arbitrage-free price, and hedging based on zero volatility (then Stop-loss and Delta-hedging coincide) gives the minimum arbitrage-free price. Notice that for each single strategy the cost interval exceeds the arbitrage interval, so every strategy involves extra uncertainty besides the arbitrage interval. Further observe that arbitrage free prices must consist of the *intrinsic value* of the option (f_0) plus a fraction of its *time value* ($f_\sigma - f_0$).
While in the GBM and binary trees arbitrage arguments force one price and one strategy (at least theoretically), arbitrage in interval models only yield limits for the price and does not dictate one strategy. This is not only the weakness of the approach but also its strength. The freedom left by arbitrage opens the way to account for other obvious elements in option pricing, such as risk-attitudes and the difference between a long and short position. This can be done *within* the context of arbitrage, and not only after denying the original model assumptions, as is the case for Black-Scholes and binary tree models.

3.4 Martingale measures

Arbitrage theory is nowadays often developed on the basis of martingale measures. It may be clarifying to interpret the previous results from this perspective. In the absence of interest rates, martingale measures relate to stochastic price models in which the expectation of future prices is simply the current price[5]. Interval models can be colored with probability by assigning a distribution for price jump factors in the interval $[d, u]$ (which may depend on time and past and current price levels). A martingale measure $\mathbf{Q}$ has the property that for positive k, $E^{\mathbf{Q}}(S_{j+k}|S_j, S_{j-1}, \ldots, S_0) = S_j$. For a given interval model we only consider martingale measures that assign probability one to the class of paths that belong to the model.

[4] In fact there is some freedom for *at the money* situations (with $X = S_j$ for some j), as then every γ between 0 and 1 yields the same lower bound

[5] Accounting for interest rates is just a matter of a proper discounting of prices.

The crucial property of martingale measures is that the expectation of costs cannot be influenced by strategies. Indeed, with γ_j the outcome of an arbitrary (by definition non anticipating) strategy at time t_j, we have that for positive k, $E^{\mathbf{Q}}(\gamma_j(S_{j+1}-S_j)|S_j, S_{j-1}, \ldots, S_0) = 0$, so the expected costs of writing an option equals (cf. (12))

$$E^{\mathbf{Q}} Q^g(S) = E^{\mathbf{Q}}([S_N - X]^+), \tag{14}$$

which is just equal to the expected value of an uncovered option under the same martingale measure.
In complete markets there is a unique martingale measure, and under appropriate hedging costs are certain and (hence) equal to the expected value of the option. For example, the unique martingale measure for the binary tree $\mathbb{B}^{u,d}$ has independent jumps with probability $\frac{1-d}{u-d}$ for jump factor u and complementary probability $\frac{u-1}{u-d}$ for d.
In incomplete markets as represented by interval models, arbitrage-free prices still can be given the interpretation of expected values under martingale measures. These measures, however, are no longer unique and in general do not allow for a sure outcome of costs under any hedging strategy. A particularly simple martingale measure is the uniform distribution on jumps for interval models $\mathbb{I}^{u,d}$ with $d = 2 - u > 0$. For symmetric interval models $\mathbb{I}^{\sigma}$ with extreme jump factors $d = 1/u_{\sigma}, u = u_{\sigma}$ (cf. (5)), a family of martingale measures is obtained e.g. by assigning at time t_j probability $0 \leq m_j \leq 1$ to jump factor 1, and probabilities $\frac{(1-m_j)u}{1+u}$ and $\frac{1-m_j}{1+u}$ to respectively the extreme factors d and u, where m_j may depend on $(S_0, \ldots, S_j)$. A 'piecewise uniform' martingale measure on jump factors v is the one with densities $\frac{u^2}{u^2-1}$ and $\frac{1}{u^2-1}$ for respectively $d \leq v \leq 1$ and $1 < v \leq u$. The next theorem lists some basic results on the relation between martingale measures and arbitrage-free option prices in an interval model. We call a probability measure *degenerate* if it restricts price paths almost surely to a finite set.

THEOREM 3.4
Let be given an interval model $\mathbb{I}^{u,d}$ for asset prices, and let $[f_{\min}, f_{\max}]$ denote the corresponding interval of arbitrage-free prices for a European call option with exercise price X. In particular, $[f_{\min}, f_{\max}] = [f_0, f_{\sigma}]$ for symmetric models $\mathbb{I}^{\sigma}$, with $u = u_{\sigma}$, $d = 1/u_{\sigma}$ (cf. Theorem 3.3).

1. *Under every martingale measure, the expected option value is an arbitrage-free price.*

2. *Every arbitrage-free price is the expected option value under some martingale measure.*

3. *A martingale measure leads to expected option value $f_{\min}$ if and only if it prohibits the asset price to cross the exercise level X. Costs are certain then under the Stop-loss strategy. An example is given by the (degenerate) martingale measure $\mathbf{Q}^{\min}$ that assigns probability one to jump factor 1, implying constant price paths. This is also the unique martingale measure with expected option value $f_{\min}$ for at-the-money options.*

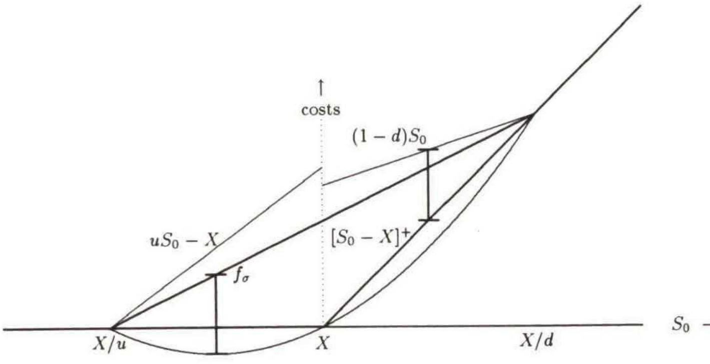
4. *There exists a unique martingale measure $\mathbf{Q}^{\max}$ for which $f_{\max}$ equals the expected option value. $\mathbf{Q}^{\max}$ is degenerate, assigns probability $\frac{1-d}{u-d}$ to jump factor u, and probability $\frac{u-1}{u-d}$ to d. Under $\mathbf{Q}^{max}$, $\mathbb{I}^{u,d}$ reduces to the binary tree $\mathbb{B}^{u,d}$, and the Delta-hedging strategy (9) yields certain costs $f_{\max}$.*

For a proof we refer to the Appendix.We remark that for expected costs $f \in (f_{\min}, f_{\max})$, martingale measures are highly non-unique, need not be degenerate, and may have positive continuous probability density for jumps in $[d, u]$.

3.5 One-step models

As an illustration of the previous results arbitrage intervals are depicted in Figure 1 for models with just one time step. A strategy for this simple case amounts to choosing a real

Figure 1: Arbitrage intervals in one-step models.



The thick lines correspond to the arbitrage interval in the interval model $\mathbb{I}^{u,d}$, as function of the initial price S_0, with upper bound $f_\sigma = \frac{1-d}{u-d}(uS_0 - X)$ and lower bound $[S_0 - X]^+$. The thin lines with discontinuity in $S_0 = X$ denote the worst-case costs for the Stop-loss strategy, the curved line below denotes the best-case costs under the Delta-hedging strategy. Δ^σ, which are given by $\frac{(uS_0-X)(S_0-X)}{(u-d)S_0}$. In addition, for both strategies one cost interval is depicted, for Delta-hedging with an initial price below X, and for Stop-loss with an initial price $S_0 > X$

number γ in a portfolio $\mathsf{C} - \gamma\mathsf{S}$, independently of the outcome of S_1. The given formulas follow from simple calculations.
The left-most and right-most regions correspond to trivial situations in which it is certain that the option will not be exercised ($uS_0 \leq X$) or always will be exercised ($dS_0 \geq X$). Notice that then Delta-hedging and Stop-loss both have resp. $\gamma = 0$ or $\gamma = 1$. The corresponding arbitrage interval then reduces to one point, and this certain outcome of costs determines the option premium.
Worst cases for Delta-hedging are $S_1 = uS_0$ or $S_1 = dS_0$, while minimal costs are achieved for $S_1 = X$. For Stop-loss this depends on whether the option is in/at the money or out of the money. In the first case ($X \geq S_0$), the worst case has $S_1 = uS_0$, in the second case (with $X < S_0$), $S_1 = dS_0$. Best cases under Stop-loss are achieved for all price paths that do not cross X. Notice that $S_1 = X$ is a common worst case for both strategies, while either $S_1 = uS_0$ or $S_1 = dS_0$ is a common best case. This implies that weighted combinations of these strategies give weighted combinations of cost intervals. Generalization of this result to models with more than one step is under current investigation.

4 Robustness Analysis

Our starting point for the robustness analysis is a (European call) option on an underlying asset with volatility σ under corresponding Delta-hedging. So we first assume the asset price to follow paths in the binary tree $\mathbb{B}^\sigma$, and hedge against the costs of the written option according to strategy Δ^σ, cf. (11). Under these assumptions a certain outcome of costs is enforced equal to f_σ.
We analyze the sensitivity of these costs for a change of assumptions on the assets volatility, in three respects. First we relax the assumption of constant volatility and consider the

nominal volatility as just an upper bound. Secondly we consider over-hedging, when the actual bound on volatility is smaller. Finally we consider under-hedging, in which case the presumed maximum volatility is too low. This amounts to replacing the nominal model $\mathbb{B}^\sigma$ by

- $\mathbb{I}^\sigma$ in the 'limited volatility' case,
- $\mathbb{I}^\tau$ (and $\mathbb{B}^\tau$) with $\tau < \sigma$ for over-hedging,
- $\mathbb{I}^\tau$ (and $\mathbb{B}^\tau$) with $\tau > \sigma$ for under-hedging.

In the next three subsections we describe some general results on the sensitivity of costs with respect to these model changes, and we conclude the section by a rather extensive numerical example.

4.1 Limited Volatility

As the nominal situation we consider a binary tree $\mathbb{B}^\sigma$, a corresponding Delta-hedging strategy Δ^σ, and the resulting arbitrage price of the option f_σ. This price equals the costs for all paths in the binary tree $\mathbb{B}^\sigma$.
Now suppose the volatility may drop below σ and need not be constant over time. This is accounted for by considering, in addition to the binary tree paths in $\mathbb{B}^\sigma$, also interior paths in the interval model $\mathbb{I}^\sigma$, which may have smaller jumps at any moment. The outcome of costs for these interior paths need not be equal to f_σ, and the question arises how large this difference can be.
Notice that for models of one step it is easy to obtain an analytic formula for this effect, cf. Section 3.5. For $u = u_\sigma$ and $d = 1/u_\sigma$ the cost interval under Δ^σ is given by

(15)
$$I^{\Delta^\sigma} = [\frac{(u_\sigma S_0 - X)(S_0 - X)}{(u_\sigma - 1/u_\sigma)S_0}, f_\sigma].$$

Costs may fall to this lower bound in 'quiet' interior paths, with not all jumps at the limits. This fall is zero for $S_0 \leq X/u_\sigma$ and $S_0 \geq u_\sigma X$, and has maximum value $\frac{(u-1)(1-d)}{(u-d)}$ for $S_0 = X$. The best-case costs are even smaller than the minimum arbitrage price, and the difference is the largest for $S_0 = X/\sqrt{u}$ for out of the money options and $S_0 = X/\sqrt{d}$ for in the money options.
A second analytic result, valid for any number of steps, concerns the worst-case costs: they remain equal to f_σ, as a consequence of Theorem 3.3. So a (temporarily) fall of volatility leads to a fall of costs. For multi-step models it is hard to keep track of the analytic formula. Therefore we only give an impression of the cost interval by the numerical example in Section 4.4.

4.2 Over-hedging

We analyze the performance of the Delta-hedging strategy Δ^σ, assuming that the actual price paths are in $\mathbb{I}^\tau$ with $\tau < \sigma$. This means that the actual volatility is below the volatility for which the strategy is designed. We use the notation

(16)
$$I^{\Delta^\sigma} = [f_{\min}, f_{\max}]$$

for the corresponding worst- and best-case costs, and concentrate on $f_{\max}$.
First observe that

(17)
$$f_\sigma \leq f_{\max} < f_\tau,$$

as $\mathbb{I}^\tau \subset \mathbb{I}^\sigma$, while f_τ is the unique strategy with minimum worst-case costs in $\mathbb{I}^\tau$.
In fact the worst cases in $\mathbb{I}^\tau$ have constant maximum volatility τ.

Proposition 4.1
The worst-case price path in $\mathbb{I}^\tau$ under over-hedging Δ^σ with $\sigma > \tau$ is in $\mathbb{B}^\tau$.

Proof For $N = 1$ it is obvious that worst cases are at the boundary of $S_1 = [dS_0, uS_0]$ with $u = u_\tau$ and $d = 1/u$, and that these worst-case costs are convex in the initial price. Similar to the proof of Theorem 3.3.1, it can be proved by induction that worst cases have extreme jumps and remain convex in the initial price for any number of time steps. □

This implies that, in case of over-hedging, there is no extra loss in interval models as compared to the binary trees. In fact the analysis could take place entirely on the level of binary trees (or even GBM's), by considering worst-case cost of Delta-hedging based on a too high volatility.

4.3 Under-hedging

Now we consider the case that the hedge strategy underestimates the volatility of assets. So we analyze the performance of the Delta-hedging strategy Δ^σ, assuming that the actual price paths are in $\mathbb{I}^\tau$ with $\tau > \sigma$.
The analogue of the inequalities (17) is now

$$f_\sigma < f_\tau < f_{\max}, \tag{18}$$

as Δ^τ is the unique strategy with minimal worst-case costs in $\mathbb{I}^\tau$.
In contrast with over-hedged options, worst cases in $\mathbb{I}^\tau$ under Δ^σ need not have maximum constant volatility. The next example shows that interval models may cause a substantial increase of worst-case costs as compared to the corresponding binary tree.
Consider an at the money European call option with exercise price $X = S_0 = 100$ at $T = 2$. Nominal volatility σ is taken such that $u_\sigma = 6/5$ in a two step model, while the actual maximum volatility has $u_\tau = 5/4$. This corresponds to $\sigma = 0.16$ and $\tau = 0.19$. Nominal costs are given by $f_\sigma = \frac{100}{11}$, and the optimal strategy Δ^τ for the actual model would yield $f_\tau = \frac{100}{9}$.
Worst-case costs in $\mathbb{I}^\tau$ under strategy Δ^σ turns out to be $f_{\max} = 875/66$ for worst-case path $\{S_0, S_1, S_2\} = \{100, 500/6, 2500/24\}$. The worst-case costs in the binary tree $\mathbb{B}^\tau$ equal $f_{\text{bin}} = \frac{125}{11}$ for worst cases $\{100, 125, 100\}$ and $\{100, 125, 625/4\}$.
Note that the excess of nominal costs in the binary tree $\mathbb{B}^\tau$ is at most $f_{\text{bin}} - f_\sigma = \frac{25}{11}$, which may be nearly doubled in $\mathbb{I}^\tau$ to $f_{\max} - f_\sigma = \frac{25}{6}$. The key value in the worst-case path is $S_1 = 500/6$, corresponding to a non-extreme first jump in $\mathbb{I}^\tau$. This is the highest asset price that maneuvers the optimistic hedge Δ^σ into an uncovered position, thus preparing for large costs in the second step. This illustrates that replacing the assumption of constant volatility by limited volatility may increase sensitivity of costs for under-hedging considerably.

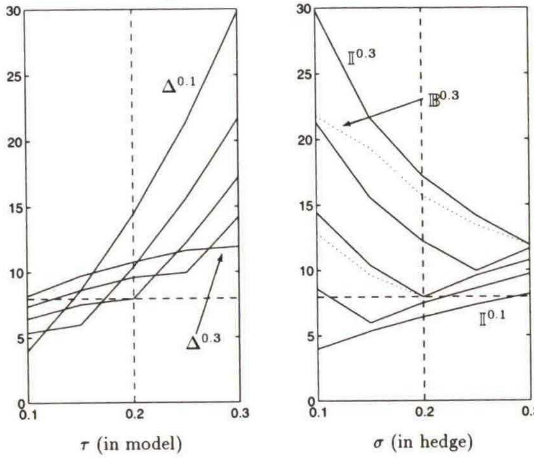
4.4 Numerical Example

We consider the worst-case costs for several combinations of nominal and actual volatilities. As before, nominal volatility is denoted by σ, and this determines the hedging strategy Δ^σ. The actual volatility is denoted by τ, and this determines the set of price paths under consideration, $\mathbb{I}^\tau$ (and for comparison also $\mathbb{B}^\tau$).
Global constants in the example are

initial price	$S_0 = 100$
exercise price	$X = 100$
exercise time	$T = 1$
interest rate	$r = 0$
time step	$h = 0.1$ so $N = 10$

Think of an option with exercise date one year off and adaption of the hedge portfolio every five weeks. As main reference point we take $\sigma^* = \tau^* = 0.2$, which means an annual variance of asset prices of 20%. We consider the worst-case costs of hedging strategies Δ^σ in the actual models $\mathbb{I}^\tau$ with σ and τ ranging over 0.1 to 0.3 (with a grid of 0.05). Worst cases are determined by a backward recursive algorithm, according to the principles of dynamic programming.

Figure 2: Worst-case costs for Δ^σ in $\mathbb{I}^\tau$.



In the left plot each line corresponds to worst-case costs under a fixed strategy Δ^σ for a range of interval models, in the right plot every line denotes the worst-case costs in a fixed interval model $\mathbb{I}^\tau$ for a range of hedging strategies. The dotted lines denote worst-case costs in the binary trees $\mathbb{B}^{0.2}$ and $\mathbb{B}^{0.3}$, which are not depicted in the left.

As interval models are nested for increasing volatility, worst-case costs for a fixed strategy must be increasing with τ in the left-hand plot. Both plots also show the optimality of Δ^σ for $\mathbb{I}^\tau$ in case $\tau = \sigma$, e.g. for σ^* at the intersection of the dashed lines. One of the striking aspects is the asymmetry in over- and under hedging: the loss of under-hedging by $\Delta^{0.1}$ in $\mathbb{I}^{0.3}$ is much larger than the loss of over-hedging by $\Delta^{0.3}$ in $\mathbb{I}^{0.1}$. The dotted lines in the right-hand plot again illustrate the extra risk of non-constant volatility, especially in case of severe under-hedging.

5 Conclusions

We analyzed the robustness of Delta-hedging strategies for varying assumptions on the actual volatility. We relaxed the assumption of constant volatility (as for price paths in binary trees), and allowed for time varying, limited volatility in interval models. Arbitrage-free option prices are not unique anymore, but may take any value between the intrinsic value of the option and its arbitrage price for constant maximum volatility. This result is given an interpretation in terms of martingale measures. Further we have shown that under Delta-hedging for constant maximum volatility, worst-case costs are still most effectively suppressed and remain at the corresponding arbitrage price, while best case costs may be much lower or even negative. If the actual limit on volatility is overestimated in Delta-hedging, costs are less than expected, but could have been even lower under optimal Delta-hedging. The most risky case turns out to be the one in which volatility is underestimated. Worst-case cost are then higher than expected. Moreover, this excess of worst-case costs due to under-hedging is really amplified if non-constant volatility is taken into account. In a numerical example we gave an impression of quantitative aspects of the robustness of Delta-hedging. Further work will concentrate on developing explicit robustness criteria, and the role of strategies different from Delta-hedging.

A Proofs

PROOF OF PROPOSITION 3.1 The proof is by induction on N. For $N = 1$ the costs of a strategy are given by $[S_1 - X]^+ - \gamma_0(S_1 - S_0)$ for some real number $\gamma_0 = g_0(S_0)$, *independent of* S_1. The costs are continuous (and piecewise linear) in S_1. Now for a fixed initial price S_0, S_1 is restricted to an interval by the model. Continuous functions map intervals to intervals, so I^g is an interval. In fact it is obvious that they are even closed intervals for $N = 1$.
Next assume that the proposition is true for models with less than N steps, and consider the total cost range I^g in an N step model for some fixed strategy g. First consider the costs of price paths $\{S_0, \ldots, S_N\}$ with $S_N = S_{N-1}$. It follows from the induction hypothesis that the cost range over all these paths form an interval, I' say. Now suppose that there is a reachable cost $p \in I^g$ that does not belong to this interval. By definition, this implies that this cost p is only achievable for some price path in the model with $v := S_N - S_{N-1} \neq 0$. Now this price paths with S_N replaced by $S_{N-1} + \alpha v$ belongs to the interval model for all $0 \leq \alpha \leq 1$. The corresponding costs are continuous in α, from which it follows that all costs between I' and p are feasible, which implies that I^g must be an interval.
If a strategy is continuous, it induces a cost function that is continuous in the price paths, and as the set $\mathbb{I}^{u,d}$ is compact, in that case the cost function must achieve its maximum and minimum value for some price paths in $\mathbb{I}^{u,d}$.
An example with a cost interval that is not closed requires $N > 1$ and a discontinuous strategy. Consider for an arbitrary interval model with two time steps and parameters (u, d), the strategy $g = \{g_0, g_1\}$ with $g_0 = 0$, and $g_1 = 0$ if $S_1 < uS_0$, and $g_1 = 1$ if $S_1 = uS_0$. Then $Q(S) = [S_2 - X]^+ - C_0$ for all price paths with $S_1 \neq uS_0$, and $Q(S) = [S_2 - X]^+ - C_0 - [S_2 - S_1]$ if $S_1 = uS_0$. With $u = 5/4$, $d = 4/5$, $S_0 = X = 100$, the cost range is given by the half-open interval $[0, \frac{900}{16})$. □

PROOF OF THEOREM 3.3 *1.* The proof is again by induction on the number of steps N. For one-step models ($N = 1$), the costs are $[S_1 - X]^+ - \gamma_0(S_1 - S_0)$, with γ_0 the outcome of the strategy, which must be independent of S_1. This is piecewise linear and convex in S_1 for every γ_0, hence achieves its maximum at the boundaries $S_1 = uS_0$ or $S_1 = dS_0$, with $u := u_\sigma$ and $d = 1/u$. In case $uS_0 \leq X$, it is optimal (i.e., it yields the lowest maximum costs) to take $\gamma_0 = 0$, and costs are 0 for every admissible S_1. In case $dS_0 \geq X$, it is optimal to take $\gamma_0 = 1$, and costs are $S_0 - X$ for every admissible S_1. In all other cases, the costs have opposite sensitivity in both boundary extremes, and hence it is optimal to choose γ_0 such that both boundary extremes coincide, i.e., $\gamma_0 = (uS_0 - X)/(u - d)S_0$, which is precisely the rule for Delta-hedging in binary trees. It follows that the worst-case costs are the costs under Delta-hedging in the binary tree for $N = 1$. Observe that the costs are convex in the initial price S_0.
Now assume that for all models with less steps than N, the statement is proved, and in addition, that the worst-case costs are convex in the initial price. In an N-step model, this implies that for a given 'initial' price S_1 at t_1, the strategy Δ^σ yields the lowest worst-case costs, and these costs are *convex in* S_1.
Then at $t_0 = 0$ a value for γ has to be found such that the maximum of costs $f(S_1) - \gamma(S_1 - S_0)$ is as small as possible, with $f(S_1)$ the worst-case costs for paths starting at t_1 in S_1. By induction hypothesis, $f(S_1)$ is convex in S_1, and hence the optimal value for γ corresponds to an equal maximum at the boundaries $S_1 = dS_0$ and $S_1 = uS_0$, which is precisely the characteristic of the Delta-hedging strategy. Hence the worst-case costs equal the costs of binary tree price paths under Delta-hedging, which must equal f_σ. In order to maintain the induction hypothesis, notice that f_σ is indeed convex in the initial price S_0.
2. The main lines are the same as in *1.* For $N = 1$ and the nontrivial case $dS_0 \leq X \leq uS_0$, the the best-case costs are achieved for $S_1 = X$, and these costs are maximal for $\gamma_0 = 0$ if $S_0 < X$ and $\gamma_0 = 1$ for $S_0 > X$. For $S_0 = X$ any value in $[0, 1]$ will maximize the best-case costs.
By induction hypothesis, assume that Stop-loss maximizes best-case costs for all models with less than N steps. In particular, for fixed S_1 best-case costs are maximized by Stop-loss, and hence equal $[S_1 - X]^+$. The problem hence reduces to the one-step situation, which has been proved.

For all price paths that do not cross the discounted exercise price X, the outcome of the strategy is either holding constantly a naked position ($\gamma_j = 0$) or a fully covered option ($\gamma_j = 1$), and it is easily verified that in these cases the lower bound of the cost interval is achieved.
3. This is a consequence of *1* and *2*. □

PROOF OF THEOREM 3.4 *1.* Let f denote the expected value of the option. As f is the expected value under all strategies, it must belong to the cost interval I^g for any strategy g, cf. (13) and Proposition 3.1. Hence $f \in \bigcap_g I^g = [f_{\min}, f_{\max}]$, cf. Definition 3.2.

2. Let $\mathbf{Q}^\alpha$ be the unique martingale measure for the scaled binary tree $\mathbb{B}^{u,d}$ with parameters $u_\alpha := 1 + \alpha(u-1), d_\alpha := 1/u_\alpha$, so with $p(u_\sigma) = \frac{1-d_\sigma}{u_\sigma - d_\sigma}$. These are martingale measures on $\mathbb{I}^{u,d}$ for every $\alpha \in [0,1]$. The expected option value f_α is continuous in α, and $f_\alpha = f_{\min}$ for $\alpha = 0$ and $f_\alpha = f_{\max}$ for $\alpha = 1$. Hence every price $f \in [f_{\min}, f_{\max}]$ occurs as expected option value under some martingale measure.
3. If asset prices cannot cross the exercise level, the value of the option at exercise time effectively depends linearly on the value of the underlying asset at exercise time, and so the expected option value must equal $S_0 - X$ if $S_0 \geq X$, and 0 if $S_0 \leq X$, under any martingale measure, which is exactly $f_{\min}$. Clearly then the Stop-loss strategy (which now amounts to a constant naked or covered position) yields certain costs. Conversely, suppose a measure assigns positive probability to a crossing of the exercise level X. Let j^* denote the first time step in which a crossing is possible. A martingale measure cannot enforce a crossing in any step (it either enforces constant prices, or assigns positive probability to both an increase and decrease of prices), and hence assigns positive probability to the set of paths that cross X just once, in step j^*. Under stop-loss, the outcome of costs for all these paths is strictly larger than $f_{\min}$: for out-of-the-money options, costs are $S_{j^*+1} - X > 0 = f_{\min}$, and for in-the money options costs are $S_0 - S_{j^*+1} > S_0 - X = f_{\min}$. As $f_{\min}$ equals the minimal outcome of costs under Stop-loss for any price path in $\mathbb{I}^{u,d}$, expected costs must be strictly larger than $f_{\min}$, and hence also the expected option value.
Finally, the same argument shows that for at-the-money options, any martingale measure that allows for price changes yields an uncertain outcome under Stop-loss: then $j^* = 0$, the set of price paths with $S_j > X$ for $j > 0$ has positive probability, and each path has costs $S_1 - X > 0 = f_{\min}$. Hence the expected option value, which equals the expected costs under any strategy, is higher than $f_{\min}$. (Notice that, somewhat arbitrarily, $\gamma_0 = 0$ according to (7). For definitions with $0 < \gamma_j \geq 1$ in case $S_j = X$, the same argument could be applied to the set of all paths with prices remaining below X after the first step.)
4. $\mathbf{Q}^{\max}$ is the unique martingale measure on the binary tree $\mathbb{B}^{u,d}$, and by definition $f_{\max}$ is the (certain) outcome of costs in $\mathbb{B}^{u,d}$ under the corresponding Delta-hedging strategy, and hence expected costs under $\mathbf{Q}^{\max}$ (for any strategy) must equal $f_{\max}$. Uniqueness of $\mathbf{Q}^{\max}$ can be proved as follows. The Delta-hedging strategy according to parameters u, d (see (9)) has $f_{\max}$ as upper bound of costs, and if these are also expected costs, $f_{\max}$ must be the certain outcome of costs (with probability one). Now under the given Delta-hedging strategy, costs of all price paths in $\mathbb{I}^{u,d}$ that do not belong to $\mathbb{B}^{u,d}$ are strictly lower than $f_{\max}$, which can be proved straightforwardly by induction on the number of time steps. Indeed, in one-step models maximum costs are achieved only at both boundaries dS_0 and uS_0; the induction step relies on the fact that maximum costs under the given Delta-hedging strategy are strictly convex in initial prices, implying that maximum costs can only be achieved for an extreme jumps in the first step.
Hence the measure should assign probability one to the set of binary tree paths $\mathbb{B}^{u,d}$, and $\mathbf{Q}^{\max}$ is the only martingale measure with this property. □

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