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ABSTRACT

Quiggin (1982) introduced anticipated ("rank-dependent") utility theory into decision making under risk. Questions have been raised about mathematical aspects in Quiggin's (1982) analysis. This paper settles these questions, and shows that a minor modification of Quiggin's axioms leads to a useful and correct result, with features not found in other recent axiomatizations.

1. INTRODUCTION

This note discusses mathematical aspects of Quiggin (1982), the paper that introduced anticipated utility into risk theory. Other terms are "rank-dependent utility", or, less tractable," expected utility with rank-dependent probabilities". We shall use the term anticipated utility for the special case of rank-dependent utility where the probability transformation function assigns value 1/2 to probability 1/2. The rank-dependent stream is currently the most popular one in nonexpected utility. Independently from Quiggin (1982, first version 1979), essentially the same form was developed by Schmeidler (1989, first version 1982), Yaari (1987, first version 1984), and Allais (1988, first version 1986). The special case considered by Yaari (with linear utility) had been developed and axiomatized before for welfare theory in Weymark (1981). The importance of the form is based on the possibility to express risk attitudes by ways to deal with probabilities, without violating basic requirements such as stochastic dominance or transitivity. Tversky & Kahneman (1990) adopted the form to obtain a new version of prospect theory.

Given the historical importance of Quiggin (1982), a new study of the mathematics in the paper seems appropriate. Examples A7-A9 below show some complications for that mathematics. There have been some discussions and misunderstandings about Quiggin's main theorem, and this note aims to clarify the issues. As we shall see, only a very minor modification of the axioms is needed. Yaari (1987, p.113) already suggested that Quiggin's axiom 2 should be strengthened. Indeed, it suffices to strengthen Quiggin's Axiom 2 to stochastic dominance, or, as we shall do, to a weaker version that only considers two-outcome prospects. The proof of the result will be entirely rewritten, and will not invoke continuity with respect to outcomes. Recently, variations on the axiomatization of Quiggin have been developed. Chew (1989) generalized Quiggin's model by deleting the restriction that the probability transformation assign value 1/2 to probability 1/2; he still required continuity both in outcomes and in probabilities. In Wakker's (1990) axiomatization it is possible that the probability transformation is not continuous, while Nakamura (1992) deleted the requirement that the utility function be continuous. So in a structural sense these results are more general than Quiggin's (1982). Still, in a logical sense none of these results is a complete generalization of Quiggin's. First, Quiggin's independence Axiom 4 only involves 1/2-1/2 prospects, and does not use other probabilities. Second, remarkably, Quiggin's dominance Axiom 2 and continuity Axiom 3 need only be imposed on two-outcome prospects. So Quiggin's result still stands as a useful axiomatization. An additional advantage of Quiggin's result is that concavity of utility can be characterized as easily as in expected utility: For 1/2-1/2 prospects the model coincides with expected utility. Hence, given the usual continuity conditions, preference of expected values over 1/2-1/2 prospects is necessary and sufficient for concavity of utility, as it is in expected utility.

2. DEFINITIONS

The notations and terminology of this paper will as much as possible follow Quiggin (1982). X is a set of *outcomes*, and may at this stage be any general set. We shall see, at the end of Appendix A2, that X is isomorphic to a connected topological space; the analysis in Quiggin (1982) implicitly used continuity with respect to a connected topology on X at several places. Our analysis will not use such an assumption, and the isomorphism to a connected topological space will be a consequence of the other assumptions rather than a presupposition. By Y we denote the set of *prospects*, i.e., of probability distributions over X with finite support, and by $\succcurlyeq$ we denote a binary ("preference") relation on Y. Outcomes x are identified with degenerate prospects. This induces a binary relation $\succcurlyeq$ on the outcome set X through the degenerate prospects. By $\{(x_1,...,x_n),(p_1,...,p_n)\}$ we denote the prospect assigning probability p_j to outcome x_j, $j=1,...,n$. Of course, the p_j's are nonnegative, and sum to one; $p_j=0$ is permitted. We write $\mathbf{x}$ for $(x_1,...,x_n)$, and $\mathbf{p}$ for $(p_1,...,p_n)$. In all results in this paper, $\succcurlyeq$ will be a weak order. So we can, and do, assume without further mentioning that $x_1 \preccurlyeq \ldots \preccurlyeq x_n$, i.e., the outcomes are rank-ordered. Let us emphasize that this assumption is essential to the analysis; the rank-ordering of outcomes is central in rank-dependent utility.

There have been many misunderstandings in the literature concerning equalities $x_j=x_{j+1}$, as permitted in our notation. Hence a detailed discussion is appropriate. This discussion, dealing with a seemingly irrelevant issue as a convention of notation, will automatically settle the essential issues in the paper. Example A9 below shows that the rank-dependent utility form cannot even be derived from our axioms, and cannot be distinguished from alternative forms that violate dominance, if $x_i=x_{i+1}$ is not allowed in the notation. Proposition A3 shows that under this alternative convention of notation, additional continuity and monotonicity conditions do exclude the alternative forms after all.

Appendix A1 shows that only those preferences and functionals satisfy natural continuity and monotonicity conditions, for which the convention of notation is irrelevant. In a way, this is exactly what common sense suggests, at least for normative applications. If a convention of notation is decisive for decision making, then something must be wrong. Indeed, for the form $\sum\phi(p_j)U(x_j)$ with nonlinear ϕ, the notational convention is relevant; it is well-known nowadays that the form violates monotonicity and continuity. This was, to the best of our knowledge, first discovered by Fishburn

(1978). For rank-dependent utility, the notational convention is irrelevant, and rank-dependent utility does satisfy monotonicity and continuity. Also rank-dependent utility can readily be extended to nonsimple prospects. The form $\sum\phi(p_j)U(x_j)$ cannot be extended to nonsimple prospects; this is another heuristic indication of its problematic nature. For descriptive applications, the latter form may be useful nevertheless, and the notational convention may be relevant, as the collapsing of outcomes may affect the perception of subjects.

Given our notation, prospects can be written as 2n tuples in several ways, e.g., the prospect $\{(1);(x)\}$ can also be written as $\{(\frac{1}{2},\frac{1}{2});(x,x)\}$. This means that functionals on Y can be expressed through 2n tuples if and only if they are invariant under different equivalent notations; we call this invariance *reduction-robustness*. For example, the often-studied functional $\{(x_1,...,x_n),(p_1,...,p_n)\} \mapsto \sum\phi(p_j)U(x_j)$, for $\phi:[0,1]\to[0,1]$ with $\phi(0)=0$, $\phi(1)=1$, is well-defined (identical for equivalent notations) only if ϕ is the identity, so that this form cannot deviate from expected utility! This is shown in Corollary A2. Hence, in studies where this form is used to deviate from expected utility, equalities $x_j=x_{j+1}$ must be excluded from the notation. For natural forms such as rank-dependent utility, the topic of this paper, the notational issue is irrelevant. Hence we chose the notation that is most convenient for the purposes of this paper.

To avoid any misunderstanding, let us repeat that in this paper a functional V, defined on Y, will automatically satisfy equalities such as $V(\{(x_1,...,x_{n-1},x_n),(p_1,...,p_{n-1},p_n)\})$ $= V(\{(x_1,...,x_{n-1}),(p_1,...,p_{n-1}+p_n)\})$ for $x_{n-1}=x_n$. This is not an assumption, but a logical necessity, the two arguments of V being identical.

3. THE MAIN THEOREM

This section presents the modification of Quiggin's axiomatization of rank-dependent utility. Let us repeat that we denote preferences by $\succcurlyeq$, rather than by P as in Quiggin (1982). We write $\succ$ for strict preferences, and $\sim$ for equivalences. A *weak order* is *complete* (for all $\{\mathbf{x};\mathbf{p}\}$ and $\{\mathbf{x}';\mathbf{p}'\}$, $\{\mathbf{x};\mathbf{p}\}\succcurlyeq\{\mathbf{x}';\mathbf{p}'\}$ or $\{\mathbf{x}';\mathbf{p}'\}\succcurlyeq\{\mathbf{x};\mathbf{p}\}$) and transitive; completeness implies reflexivity.

A functional $V:Y\to\mathbb{R}$ *represents* $\succcurlyeq$ if $\{\mathbf{x};\mathbf{p}\}\succcurlyeq\{\mathbf{x}';\mathbf{p}'\} \Leftrightarrow V\{\mathbf{x};\mathbf{p}\} \geq V\{\mathbf{x}';\mathbf{p}'\}$. *Rank-dependent utility* holds if there exists a representing functional V of the form

$$V(\{(x_1,...,x_n);(p_1,...,p_n)\}) = \sum_{i=1}^{n}\left(f\left(\sum_{j=1}^{i}p_j\right) - f\left(\sum_{j=1}^{i-1}p_j\right)\right)U(x_i) \qquad (3.1)$$

for a function $U:X \to \mathbb{R}$, and a nondecreasing function $f: [0,1] \to [0,1]$ with $f(0)=0$, $f(1)=1$. *Anticipated utility* (*AU*) is the special case where $f(1/2)=1/2$.

Quiggin's (1982) Assumption R.1., on p.332 there, settles the notation that we discussed in the previous section. We shall also use the following structural assumption of Quiggin, ensuring that for each prospect there exists a "certainty equivalent":

R.2. For each prospect $\{\mathbf{x};\mathbf{p}\}$ there exists an outcome x such that $x \sim \{\mathbf{x};\mathbf{p}\}$.

Now we turn to the axioms:

AXIOM 1. The binary relation $\succcurlyeq$ is a weak order.

The dominance axiom of Quiggin will be adapted as follows. Both axioms below are implied by strict stochastic dominance when restricted to two-outcome prospects. The first imposes weak monotonicity with respect to probabilities, the other strict monotonicity with respect to outcomes for fixed probabilities $\frac{1}{2}$, $\frac{1}{2}$.

AXIOM 2'a. If $x_2 \succcurlyeq x_1$, and $p' \geq p$, then $\{(x_1,x_2);(1-p',p')\} \succcurlyeq \{(x_1,x_2);(1-p,p)\}$.

AXIOM 2'b. $\{(x_1',x_2');(\frac{1}{2},\frac{1}{2})\} \succcurlyeq \{(x_1,x_2);(\frac{1}{2},\frac{1}{2})\}$ whenever $x_2' \succcurlyeq x_2, x_1' \succcurlyeq x_1$, where the former preference is strict if one of the latter two is strict.

For the sake of comparison, we give Quiggin's (1982) Axiom 2, which is the restriction of Axiom 2'a to the case $p'=1$:

AXIOM 2Q. If $x_2 \succcurlyeq x_1$, then $x_2 \succcurlyeq \{(x_1,x_2);(1-p,p)\}$ for all p.

AXIOM 3 (Continuity). If $x_1,x_2,x_3 \in X$, $x_1 \preccurlyeq x_2 \preccurlyeq x_3$, then there exists p^* such that

$$x_2 \sim \{(x_1,x_3);(1-p^*,p^*)\}.$$

Note that under AU, with $f(\frac{1}{2})=\frac{1}{2}$, in $(\frac{1}{2},\frac{1}{2})$ prospects it does not matter which outcome is substituted first in the form (3.1), since each outcome has weight $\frac{1}{2}$. This suggests that for $(\frac{1}{2},\frac{1}{2})$ prospects the rank-ordering of outcomes is immaterial. We introduce an additional notation for $(\frac{1}{2},\frac{1}{2})$ prospects: $\{(x,,x');(\frac{1}{2},\frac{1}{2})\}$ denotes the prospect $\{(x,x');(\frac{1}{2},\frac{1}{2})\}$ if $x \preccurlyeq x'$, and $\{(x',x);(\frac{1}{2},\frac{1}{2})\}$ if $x' \preccurlyeq x$. The notation is useful in Axiom 4, where the rank-ordering of each pair x_i,x_i', and of x and x', is undetermined. Also the notation will be useful in proofs below.

AXIOM 4 (Independence); see Figure 1. Whenever $x \sim \{\mathbf{x};\mathbf{p}\}$, $x' \sim \{\mathbf{x}';\mathbf{p}\}$, $c_i \sim \{(x_i,,x_i');(\frac{1}{2},\frac{1}{2})\}$ for all i, then $\{\mathbf{c};\mathbf{p}\} \sim \{(x,,x');(\frac{1}{2},\frac{1}{2})\}$.

$$\{(x_1, \dots, x_i, \dots, x_n); (p_1, \dots, p_i, \dots, p_n)\} \sim x$$

for all i: $c_i \sim \{(x_i,,x_i'); (1/2,1/2)\}$

$$\{(c_1, \dots, c_i, \dots, c_n); (p_1, \dots, p_i, \dots, p_n)\} \boldsymbol{\sim} c \qquad c \sim \{(x,,x');(1/2,1/2)\}$$

$$\{(x_1', \dots, x_i', \dots, x_n'); (p_1, \dots, p_i, \dots, p_n)\} \sim x'$$

implied by independence

FIGURE 1 (*Independence*). Every c-outcome is a "midpoint" between the x-outcome above, and the x'-outcome below. The bold-printed equivalence is implied by the other equivalences.
In other words: Midpoints can as well be taken before, as after, the taking of certainty equivalents.

The result of the lemma below is implied by Quiggin's (1982, top of p. 327) assumption that equivalent outcomes would not be distinguished.

LEMMA 3.1. If R.2 holds, as well as Axioms 1,2'b, and 4, then $\{\mathbf{x};\mathbf{p}\} \sim \{\mathbf{x}';\mathbf{p}\}$ whenever $x_i \sim x_i'$ for all i.

PROOF. By Axiom 2'b, $c_i \sim \{(x_i,,x_i');(\frac{1}{2},\frac{1}{2})\}$ both for $c_i=x_i$ and $c_i=x_i'$. By R.2, x and x' as in Axiom 4 exist. Now apply Axiom 4 both with $\mathbf{c}=\mathbf{x}$, and with $\mathbf{c}=\mathbf{x}'$. □

The following modification of Quiggin's (1982) characterization of AU is the main result of the paper.

THEOREM 3.2. Let $\succcurlyeq$ be a binary relation on the set Y of prospects. Then the following two statements are equivalent:

(i) Condition R.2, and Axioms 1, 2'a, 2'b, 3, and 4, are satisfied.
(ii) AU applies (so $f(\frac{1}{2})=\frac{1}{2}$), where f is continuous and nondecreasing, and the range of U is an interval.

Further, f in (ii) above is uniquely determined, and U is unique up to scale and location.
□

Note that, if X is an interval in the above theorem and U is nondecreasing, as it will be under traditional dominance, then U must be continuous as its range is an interval.

4. CONCLUSION

This paper has shown that the stochastic dominance Axiom 2 in Quiggin (1982) should be strengthened to obtain a characterization of anticipated utility. The main restriction of this characterization in comparison to later characterizations by Chew (1989) and Nakamura (1992) is that the probability transformation function f should assign value 1/2 to probability 1/2. This restriction, however, gives much in return. First, the independence Axiom 4 need only be imposed for the fixed probability 1/2, and need not be extended to other probabilities. Second, a remarkable, and to date still original, feature is that the dominance Axioms 2', as well as the continuity Axiom 3, need only be imposed on two-outcome prospects. Because of this, Quiggin's axiomatization continues to be of interest to date, and still offers features not found in other axiomatizations.

APPENDIX A1.

This Appendix formally studies and justifies the notational conventions used in the main body of the paper. It can be considered a formalization of Quiggin's (1982) Section 2. We adopt here a notation that deviates from the main body of the paper. Y now consists of abstract 2n tuples, denoted by $\{(x_1,\ldots,x_n),(p_1,\ldots,p_n)\}$, where the p_j's are still supposed to be nonnegative and to sum to one, the x_j's are still rank-ordered outcomes $x_1 \preccurlyeq \ldots \preccurlyeq x_n$, and n may be any natural number. The 2n tuples are no longer identified with prospects. Obviously, for $x_{j-1}=x_j$, the identity

$$\{(x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_n),(p_1,\ldots,p_{j-1},p_j,p_{j+1},\ldots,p_n)\} =$$
$$\{(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n),(p_1,\ldots,p_{j-1}+p_j,p_{j+1},\ldots,p_n)\} \tag{A1}$$

does not hold anymore. So for a functional $V:Y\rightarrow\mathrm{IR}$, and $x_{j-1}=x_j$, the equation

$$V(\{(x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_n),(p_1,\ldots,p_{j-1},p_j,p_{j+1},\ldots,p_n)\}) =$$
$$V(\{(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n),(p_1,\ldots,p_{j-1}+p_j,p_{j+1},\ldots,p_n)\}) \tag{A2}$$

need not hold. We call V **reduction-robust** if it does satisfy (A2). Note that this is necessary and sufficient for the possibility to identify V with a functional on prospects. To illustrate the restrictive nature of the equation, we give the following result. It is a small generalization of Allais (1988, Appendix A.1), and shows how the only possible reduction-robust form of the general functional in (A3) is rank-dependent utility.

PROPOSITION A1. Let $V:Y \to \mathbb{R}$ be of the form

$$V : \{(x_1,\dots,x_n),(p_1,\dots,p_n)\} \mapsto \sum_{j=1}^{n} h_j(p_1,\dots,p_n)U(x_j), \qquad \text{(A3)}$$

for a nonconstant function $U:X \to \mathbb{R}$, where each h_j depends only on the vector of ("rank-ordered-outcome") probabilities $(p_1,\dots,p_n)$. Suppose V is reduction-robust. Define $f(p) = h_1(p,1-p)$ on $[0,1]$. Then $h_i(p_1,\dots,p_n) = f(\sum_{j=1}^{i} p_j) - f(\sum_{j=1}^{i-1} p_j)$ for all $i,(p_1,\dots,p_n)$.

PROOF. Say $U(\alpha) < U(\beta)$. By reduction-robustness, and comparison of the prospects $\{\mathbf{p},(\alpha,\dots,\alpha)\}$, $\{\mathbf{p},(\beta,\dots,\beta)\}$ with the prospects $\{1,\alpha\}$ and $\{1,\beta\}$, we see that both $\sum_{j=1}^{n} h_j(\mathbf{p})U(\alpha)$ and $\sum_{j=1}^{n} h_j(\mathbf{p})U(\beta)$ must be independent of $\mathbf{p}$; as $U(\alpha)$ and $U(\beta)$ cannot both be zero, it follows that $\sum_{j=1}^{n} h_j(\mathbf{p})$ is independent of $\mathbf{p}$. We may assume the sum is 1, e.g., by multiplying U by an appropriate constant. We use below the equality $h_2(p,1-p) = 1-f(p)$. Again by reduction-robustness, and comparison of the prospect $\{\mathbf{p};\mathbf{x}\}$ for which $x_1=\dots=x_i=\alpha$, $x_{i+1}=\dots=x_n=\beta$ with the prospect $\{(p,1-p);(\alpha,\beta)\}$ where $p:=\sum_{j=1}^{i} p_j$, we get

$$\sum_{j=1}^{i} h_j(\mathbf{p})U(\alpha) \; + \; (1-\sum_{j=1}^{i} h_j(\mathbf{p}))U(\beta) \; = \; f(p)U(\alpha) + (1-f(p))U(\beta).$$

We conclude that $\sum_{j=1}^{i} h_j(\mathbf{p}) = f(\sum_{j=1}^{i} p_j)$. □

Note that the above result only used the nontriviality assumption that U be nonconstant. No continuity or monotonicity conditions were used. This shows the strength of reduction-robustness. The form in (A3) had already been proposed by Allais (1979, first version in 1952; see Formula IV in Section 41). Before that, Allais had proposed a very general form in (1) in Section 40, a form that can describe any transitive relation in many ways, so is not predictive. Allais subsequently proposed many ways to restrict this general form, among them the Formula IV. He permitted different outcomes in the notation to be identical, and apparently at that time did not realize the implication of the above proposition. Actually, Allais then did not even require that the outcomes should be rank-ordered, which in fact implies that the form must reduce to expected utility! Only in Allais (1988, Formula (1)), rank-ordering was imposed on the outcomes, that were again allowed to be identical, and the above result was derived. His form (5) is equivalent to (A3) (by the substitution $h_i = \phi_i - \phi_{i+1}$ for all $i<n$, and $h_n=\phi_n$).

The corollary below considers the special case that has received most attention, in psychological papers starting around the fifties; see for instance Preston & Baratta (1948) or Edwards (1962). It leads to expected utility, instead of rank-dependent utility as obtained above.

COROLLARY A2. Let $V:Y\rightarrow\mathbb{R}$ be of the form

$$V : \{(x_1,\ldots,x_n),(p_1,\ldots,p_n)\} \mapsto \sum_{j=1}^{n}\phi(p_j)U(x_j) \qquad \text{(A4)}$$

for some nonconstant function $U:X\rightarrow\mathbb{R}$, and $\phi(0)=0$, $\phi(1)=1$. Suppose V is reduction-robust. Then ϕ is the identity.

PROOF. By Proposition A1, $\phi(p_j)$ must be of the form $f(\sum_{j=1}^{i}p_j) - f(\sum_{j=1}^{i-1}p_j)$, so that the latter difference depends only on p_j. This implies that f must be linear. Because $f(0)=0$, $f(1)=1$, f must be the identity. □

As follows from the above observations, to obtain a generalization of expected utility of the form as in (A4), a more subtle formulation must be chosen. The following formulation is usually chosen in the literature; we first give preparatory definitions. We call $\{\mathbf{x};\mathbf{p}\}$ *irreducible* (see for instance Fishburn, 1978) if $x_j \neq x_{j-1}$ for all j. For an arbitrary $\{\mathbf{x};\mathbf{p}\}$, the *reduced form* is the irreducible prospect obtained by collapsing all identical outcomes. In the literature usually the above formulas are applied to the reduced forms of prospects. So we define, for a general functional $V:Y\rightarrow\mathbb{R}$, the *reduced form* V' as the functional that assigns to each $\{\mathbf{x};\mathbf{p}\}$ the V value of $\{\mathbf{x}';\mathbf{p}'\}$. Note that V' is reduction-robust, and that V=V' if and only if V is reduction-robust. In the results below we use continuity conditions, which obviously cannot be defined for general outcome sets X. For simplicity, we shall assume that X is an interval. The following proposition shows that, under a continuity condition, functionals must be reduction-robust, so that the notational convention does not matter for these functionals.

PROPOSITION A3. Suppose that X is an interval, and that V and its reduced form V' are continuous in $\mathbf{x}$ for each fixed $\mathbf{p}$. Then V=V'. Consequently, V is reduction-robust.

PROOF. Each element $\{\mathbf{x};\mathbf{p}\}$ of Y can be approximated by irreducible elements $\{\mathbf{x}^k;\mathbf{p}\}$ of Y with the same $\mathbf{p}$ vector. For all these irreducible elements, V and V' coincide by the definition of V'. By continuity, V' and V coincide at $\{\mathbf{x};\mathbf{p}\}$. So V=V', and V must be reduction-robust. □

The corollary below considers the special case of the above proposition that is of interest in this paper. Together with Proposition A1 the corollary obtains rank-dependent utility, thus shows that the general form in Quiggin's (1982) Proposition 1 reduces to rank-dependent utility.

COROLLARY A4. Suppose X is an interval. Let $W:Y\rightarrow\mathbb{R}$ be of the form

$$W : \{\mathbf{x},\mathbf{p}\} \mapsto \sum_{j=1}^{n} h_j(\mathbf{p}')U(x_j'), \tag{A5}$$

where $\{\mathbf{x}';\mathbf{p}'\}$ is the reduced form associated with $\{\mathbf{x};\mathbf{p}\}$. Suppose that for each fixed $\mathbf{p}$, W is continuous in $\mathbf{x}$. Then

$$W(\{\mathbf{x},\mathbf{p}\}) = \sum_{j=1}^{n} h_j(\mathbf{p})U(x_j), \tag{A6}$$

for all $\{\mathbf{x},\mathbf{p}\}$, i.e., W is reduction-robust.

PROOF. $U = W(\{.;1\})$, so U is continuous. Define $V(\{\mathbf{x},\mathbf{p}\}) = \sum_{j=1}^{n} h_j(\mathbf{p})U(x_j)$. Continuity of U implies continuity of V in $\mathbf{x}$ for each fixed $\mathbf{p}$; by assumption W, the reduced form of V, satisfies the same continuity. Now Proposition A3 can be applied. □

The following corollary leads to expected utility instead of rank-dependent utility as obtained above.

COROLLARY A5. Suppose X is an interval. Let $W:Y\rightarrow\mathbb{R}$ be of the form

$$W : \{\mathbf{x},\mathbf{p}\} \mapsto \sum_{j=1}^{n} \phi(p_j')U(x_j') \tag{A7}$$

for a nonconstant function $U:X\rightarrow\mathbb{R}$, and $\phi(0)=0$, $\phi(1)=1$. Suppose that, for each fixed $\mathbf{p}$, W is continuous in $\mathbf{x}$. Then ϕ is the identity.

PROOF. This follows from Corollaries A4 and A2. □

Note that none of the above results has used any monotonicity condition. A monotonicity condition will be used below. To justify the notational convention in this paper, where $x_i=x_{i+1}$ is permitted, we should also provide a result, similar to Proposition

A3, for preference relations. This result, somewhat more complicated, should justify the notational convention when preferences rather than functionals are primitive. Again, we shall see below that for preference relations that satisfy natural continuity and monotonicity conditions, the notational convention is immaterial.

Let $\succcurlyeq$ be a binary relation on Y. Then $\succcurlyeq'$, the reduced form of $\succcurlyeq$, is defined as $\{\mathbf{x};\mathbf{p}\}\succcurlyeq'\{\overline{\mathbf{x}};\overline{\mathbf{p}}\}$ if and only if $\{\mathbf{x}';\mathbf{p}'\}\succcurlyeq\{\overline{\mathbf{x}}';\overline{\mathbf{p}}'\}$. We call $\succcurlyeq$ *reduction-robust* if $\succcurlyeq\, = \,\succcurlyeq'$. Note that a weak order $\succcurlyeq$ is reduction-robust if and only if every $\{\mathbf{x};\mathbf{p}\}$ is equivalent to its reduced form $\{\mathbf{x}';\mathbf{p}'\}$. If X is an interval then we call $\succcurlyeq$ *continuous in outcomes* if, for each $\mathbf{p}=(p_1,\ldots,p_n)$ and $\{\overline{\mathbf{x}};\overline{\mathbf{p}}\}$, the sets $\{\mathbf{x}: \{\mathbf{x};\mathbf{p}\}\succcurlyeq\{\overline{\mathbf{x}};\overline{\mathbf{p}}\}\}$ and $\{\mathbf{x}: \{\mathbf{x};\mathbf{p}\}\preccurlyeq\{\overline{\mathbf{x}};\overline{\mathbf{p}}\}\}$ are closed subsets of the set of rank-ordered n tuples. Note that this condition involves possibly different $\mathbf{p},\overline{\mathbf{p}}$, which makes it somewhat stronger than the continuity in $\mathbf{x}$ as used for the functionals above.

PROPOSITION A6. Suppose X is an open interval. Let $\succcurlyeq$ and its reduced form $\succcurlyeq'$ be weak orders that are strictly increasing in each coordinate x_i of $\mathbf{x}$ for which $p_i>0$, and independent of coordinates x_i for which $p_i=0$, and continuous in outcomes. Then $\succcurlyeq\, =\, \succcurlyeq'$. Consequently, $\succcurlyeq$ is reduction-robust.

PROOF. Suppose, for contradiction, that not $\{\mathbf{x};\mathbf{p}\} \sim \{\mathbf{x}';\mathbf{p}'\}$. Say $\{\mathbf{x};\mathbf{p}\} \succ \{\mathbf{x}';\mathbf{p}'\}$. By strict increasingness of $\succcurlyeq$, independence of zero probabilities, openness of X, and continuity in outcomes, there exists an irreducible $\{\overline{\mathbf{x}};\mathbf{p}\}$ such that $\{\mathbf{x};\mathbf{p}\} \succ \{\overline{\mathbf{x}};\mathbf{p}\} \succ \{\mathbf{x}';\mathbf{p}'\}$, and such that all outcomes in $\mathbf{x}$ strictly dominate those in $\overline{\mathbf{x}}$. Also there exists an irreducible $\{\overline{\mathbf{x}}';\mathbf{p}\}$ such that $\{\overline{\mathbf{x}};\mathbf{p}\} \succ \{\overline{\mathbf{x}}';\mathbf{p}\} \succ \{\mathbf{x}';\mathbf{p}'\}$, and such that all outcomes in $\overline{\mathbf{x}}'$ strictly dominate those in $\mathbf{x}'$. Then $\{\mathbf{x};\mathbf{p}\} \succ' \{\overline{\mathbf{x}};\mathbf{p}\} \succ' \{\overline{\mathbf{x}}';\mathbf{p}\} \succ' \{\mathbf{x}';\mathbf{p}'\}$, the first preference by strict increasingness of $\succcurlyeq'$, the second by the definition of $\succ'$, and the third again by strict increasingness. This contradicts $\{\mathbf{x};\mathbf{p}\} \sim' \{\mathbf{x}';\mathbf{p}'\}$.

So always $\{\mathbf{x};\mathbf{p}\} \sim \{\mathbf{x}';\mathbf{p}'\}$, and $\succcurlyeq$ is truncation-robust. Obviously, then $\succcurlyeq\, =\, \succcurlyeq'$. □

In Quiggin (1982), in line 6 of Section 2, it is made explicit that the outcomes in the notation $\{(x_1,\ldots,x_n);(p_1,\ldots,p_n)\}$ are to be distinct at that moment; they are not yet rank-ordered at that stage. The motivation was to be able to discuss some functionals that we have called reduced forms. It is shown in Equations (1)-(5) there, for some special cases of (A4) above that have been proposed in the literature, that these violate monotonicity; in that reasoning, however, continuity is used implicitly. Above Equation (6) then the outcomes are assumed rank-ordered. Equation (6) defines the functional as in (A3) above (with $h(\frac{1}{2},\frac{1}{2})=(\frac{1}{2},\frac{1}{2})$). Essentially, the result of our Corollary A4 is then derived from monotonicity, where again continuity is used implicitly. Given that, the notational issue becomes irrelevant, and thus equalities $x_1=x_{i+1}$ in the notation of prospects can be permitted. This is done indeed in the remainder of Quiggin's paper. Formally, it was already permitted in the notation introduced above Equation (6); it is repeated in Assumption R.1. This also shows, similarly to Proposition A1, that the general form ((A3) above) as found in Quiggin's Proposition 1 is identical to the AU form ((3.1) above) as derived in Quiggin's proof in the Appendix there.

The following examples illustrate mathematical complications in Quiggin's (1982) analysis.

EXAMPLE A7. Let $X = \{x_1,x_2\}$, i.e., there are only two outcomes. Suppose $x_2 \succ x_1$ and $\{(x_1,x_2);(1-p,p)\} \sim x_1$ for all $p<1$. There does exist a rank-dependent utility representation for $\succcurlyeq$, with $U(x_2) = 1$, $U(x_1) = 0$, and $f(p) = 1$ for all $0<p<1$. Here f is uniquely determined. Thus there does not exist an AU model for $\succcurlyeq$ because $f(\frac{1}{2}) \neq \frac{1}{2}$. The preference relation satisfies all conditions in Statement (i) of Theorem 3.2, with the exception of Axiom 2b'. We only discuss Axiom 4. Nonequivalence in the conclusion can only occur if either $\{\mathbf{c};\mathbf{p}\}$ or $\{(x,,x');(\frac{1}{2},\frac{1}{2})\}$ is maximal, i.e., is x_2. But this straightforwardly implies that all other prospects are maximal, i.e., are x_2, as well. So Axiom 4 is satisfied. Note in particular that Axiom 2'a is satisfied, which for the special case p'=1 gives Axiom 2Q, i.e., Quiggin's Axiom 2. So all of Quiggin's conditions are satisfied, and formally this is a counterexample to Quiggin's Proposition 1.

EXAMPLE A8. Let $X = \mathbb{R}$, U is the identity, and AU holds, with one exception: the function $f:[0,1] \to \mathbb{R}$ is not necessarily nondecreasing; it does satisfy $f(0)=0$, $f(\frac{1}{2})=\frac{1}{2}$, and $f(1)=1$. Necessary and sufficient for verification of Axiom 2Q, i.e., Quiggin's (1982) Axiom 2, is that $f(p) \geq 0$ for all p. Necessary and sufficient for verification of Axiom 3, is that $f([0,1]) \supset [0,1]$. Condition R.2 and Axiom 4 are satisfied. Thus f does not have to be nondecreasing, and may even take values larger than 1. □

EXAMPLE A9. Yaari (1987) suggested, for $X=\mathbb{R}$ and U the identity, the form $\sum w(p_j)x_j$ with w continuous and $w(p)+w(1-p) \leq 1$, as a counterexample to Quiggin's (1982) characterization of AU in his Proposition 1. Yaari did not make explicit which notational conventions he followed. Under the notational conventions of this paper Yaari's form must be identical to expected value maximization (see Corollary A5), which obviously would not provide a counterexample to Quiggin's result. Hence let us assume that the reduced form of $\sum w(p_j)x_j$ should be taken. A critical question for verification of Quiggin's axioms then is which notational conventions should be adopted in the formulations of these axioms. If the axioms are taken exactly as in this paper, in full strength, then Axiom 4 will be violated by Yaari's form as soon as w is nonlinear, even if $w(1/2)=1/2$: Given continuity and nonlinearity of w, there must exist probabilities p_1,p_2

such that $w(p_1)+w(p_2) \neq w(p_1+p_2)$. Now, with $p_3 = 1-p_1-p_2$, any $x \sim \{(x_1,x_2,x_3);(p_1,p_2,p_3)\}$, $x' \sim \{(x'_1,x'_2,x'_3);(p_1,p_2,p_3)\}$, $c_i \sim \{(x_i,,x_i');(\frac{1}{2},\frac{1}{2})\}$ for all i, and further $0<x_1<x_2<x_3$, $0<x'_1 = x'_2<x'_3$, we get $\{(c_1,c_2,c_3);(p_1,p_2,p_3)\} \prec \{(x,,x');(\frac{1}{2},\frac{1}{2})\}$ if $w(p_1)+w(p_2) > w(p_1+p_2)$, and $\{(c_1,c_2,c_3);(p_1,p_2,p_3)\} \succ \{(x,,x');(\frac{1}{2},\frac{1}{2})\}$ if $w(p_1)+w(p_2) < w(p_1+p_2)$. I.e., Axiom 4 is violated. So also under these notational conventions Yaari's form does not provide a counterexample to Quiggin's result.

Finally, if the axioms are weakened to apply only to cases where all prospects in question are irreducible, then Yaari's form with $w(1/2)=1/2$ can be made to satisfy all of these *weakened versions* of Quiggin's axioms. If then however this same convention of notation is applied to the general form (see (A3) above) as provided in Quiggin's Proposition 1, then Yaari's form is a special case of Quiggin's form, so again, Yaari's form does not provide a counterexample to Quiggin's result. But it does then deviate from AU and rank-dependent utility, so that in this case AU is not characterized by the (weakened versions of) the axioms. This shows once more the importance of the notational issue. Yaari did suggest that Quiggin's Axiom 2 should be strengthened. This paper has proved that Yaari's suggestion is correct. □

APPENDIX A2. PROOF OF THEOREM 3.2.

Necessity of the conditions is straightforwardly verified; we only mention that R.2 is implied by the assumption that the range of U is an interval. So we assume the conditions of Statement (i) hold, and derive Statement (ii). In the major part of the proof we make the following assumption; only at the end of the proof, the Assumption will be relaxed.

ASSUMPTION A10. There exists a best outcome x^1, and a worst outcome x^0; $x^1 \succ x^0$.

STAGE 1 (Construction of binary values of U). Define $U(x^1)=1$, $U(x^0)=0$. By R.2, there exists $x^{1/2} \sim \{(x^0,x^1);(\frac{1}{2},\frac{1}{2})\}$. By Axiom 2'b, $x^1 \succ x^{1/2} \succ x^0$. Define $x^{1/4} \sim \{(x^0,x^{1/2});(\frac{1}{2},\frac{1}{2})\}$, $x^{3/4} \sim \{(x^{1/2},x^1);(\frac{1}{2},\frac{1}{2})\}$, and similarly define $x^{1/8}$, $x^{3/8}$, ..., and inductively every $x^{a/2^n}$. To each $x^{a/2^n}$ we assign U value $a/2^n$. By repeated application of Axiom 2'b and transitivity, U is representing on the set of all $x^{a/2^n}$.

STAGE 2 (An application of Axiom 4). We derive the following condition:

$$\{(x^{a2^{-n}},x^{b2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{(a2^{-n}+b2^{-m})/2}. \tag{A8}$$

By multiplying by a large $2^{m'}$, it suffices to derive the result only for n=m, and a−b even. The latter is derived by induction with respect to m. For m=1 it holds true. Suppose, as induction hypothesis, that it holds true for 1,...,m−1, where m≥2. We show, for all appropriate a,k:

$$\{(x^{(a-k)2^{-m}},x^{(a+k)2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{a2^{-m}}. \tag{A9}$$

Below each time Axiom 4 is applied. The equivalences needed for that always follow from the induction hypothesis (and the definition of the $x^{c2^{-n}}$). To verify that, it must be checked that several integers, and differences of these integers divided by two, are even. This will not be made explicit.

CASE 1: a and k are even. Then the equivalence follows from the induction hypothesis.
CASE 2: a is odd, k is even. Then, by Axiom 4, (A9) follows from the two equivalences below, where the left prospect in (A9) plays the role of $\{\mathbf{c};\mathbf{p}\}$ in Axiom 4.

$$\{(x^{(a-1-k)2^{-m}},x^{(a-1+k)2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{(a-1)2^{-m}}$$

$$\{(x^{(a+1-k)2^{-m}},x^{(a+1+k)2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{(a+1)2^{-m}}.$$

CASE 3: a is odd, k is odd. Then either a−k ≥ 2, or, if a=k, then a+k ≤ 2^m−2, given that m≥2 and a+k=2a is not a multiple of 4. If a−k ≥ 2 then, by Axiom 4, (A9) follows from the two equivalences below, where the left prospect in (A9) plays the role of $\{\mathbf{c};\mathbf{p}\}$ in Axiom 4.

$$\{(x^{(a-2-k)2^{-m}},x^{(a+k)2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{(a-1)2^{-m}}$$

$$\{(x^{(a+2-k)2^{-m}},x^{(a+k)2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{(a+1)2^{-m}}.$$

If a+k ≤ 2^m−2 then, by Axiom 4, (A9) follows from the two equivalences below, where the left prospect in (A9) plays the role of $\{\mathbf{c};\mathbf{p}\}$ in Axiom 4.

$$\{(x^{(a-k)2^{-m}},x^{(a-2+k)2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{(a-1)2^{-m}}$$

$$\{(x^{(a-k)2^{-m}},x^{(a+2+k)2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{(a+1)2^{-m}}.$$

CASE 4: a is even, k is odd. Then, by Axiom 4, (A9) follows from the two equivalences below, where the left prospect in (A9) plays the role of $\{\mathbf{c};\mathbf{p}\}$ in Axiom 4.

$$\{(x^{(a-1-k)2^{-m}},x^{(a+1+k)2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{a2^{-m}}$$

$$\{(x^{(a+1-k)2^{-m}},x^{(a-1+k)2^{-m}});(\frac{1}{2},\frac{1}{2})\} \sim x^{a2^{-m}}.$$

STAGE 3 (definition of U on entire X). We define

$$U : x \mapsto \sup\{U(x^{a2^{-m}}) : x^{a2^{-m}} \preccurlyeq x\}.$$

This is indeed a true extension of U, and it follows straightforwardly that

$$x' \succcurlyeq x \Rightarrow U(x') \geq U(x). \tag{A10}$$

This implies in particular that U is constant on ~ equivalence classes, which will be crucial for several definitions below. We can not conclude at this stage that U would represent $\succcurlyeq$ on outcomes, as the implication $x' \succ x \Rightarrow U(x) > U(x)$ has not yet been derived. This implication will only be established in the sequel, and its derivation will invoke the definition of f below, and Axioms 3 and 4. For a prospect $\{\mathbf{x};\mathbf{p}\}$, we define $V\{\mathbf{x};\mathbf{p}\}$ as the U value of an outcome x for which $x \sim \{\mathbf{x};\mathbf{p}\}$; note that by R.2 such an x exists, and, by constantness of U on ~ equivalence classes, $V\{\mathbf{x};\mathbf{p}\}$ is independent of the particular x that we choose. Obviously, by (A10),

$$\{\mathbf{x}';\mathbf{p}'\} \succcurlyeq \{\mathbf{x};\mathbf{p}\} \Rightarrow V(\{\mathbf{x}';\mathbf{p}'\}) \geq V(\{\mathbf{x};\mathbf{p}\}). \tag{A11}$$

Next we derive the following variation on (A9), for all $x_2 \succcurlyeq x_1$:

$$V\{(x_1,x_2);(\tfrac{1}{2},\tfrac{1}{2})\} = \tfrac{1}{2}U(x_1) + \tfrac{1}{2}U(x_2). \tag{A12}$$

To prove this, note that, by the implication $U(x') < U(x) \Rightarrow x' \prec x$ as following from (A10), the inequalities $a2^{-m} < U(x_1) < a'2^{-m}$ and $b2^{-m} < U(x_2) < b'2^{-m}$ imply the preferences $x^{a2^{-m}} \prec x_1 \prec x^{a'2^{-m}}$ and $x^{b2^{-m}} \prec x_2 \prec x^{b'2^{-m}}$. Hence, by Axiom 2'b, $\{(x^{a2^{-m}},x^{b2^{-m}});(\frac{1}{2},\frac{1}{2})\} \prec \{(x_1,x_2);(\frac{1}{2},\frac{1}{2})\} \prec \{(x^{a'2^{-m}},x^{b'2^{-m}});(\frac{1}{2},\frac{1}{2})\}$. By (A11), $V\{(x^{a2^{-m}},x^{b2^{-m}});(\frac{1}{2},\frac{1}{2})\} \leq V\{(x_1,x_2);(\frac{1}{2},\frac{1}{2})\} \leq V\{(x^{a'2^{-m}},x^{b'2^{-m}});(\frac{1}{2},\frac{1}{2})\}$. So, by (A8), $V\{(x_1,x_2);(\frac{1}{2},\frac{1}{2})\}$ is enclosed between $(a2^{-m}+b2^{-m})/2$ and $(a'2^{-m}+b'2^{-m})/2$ for all m,a,b,a',b' as above. This can only be if (A12) holds true.

STAGE 4 (construction of continuous and nondecreasing f). For every $0 \leq p \leq 1$ we define $f(p) := 1 - V\{(x^0,x^1);(p,1-p)\}$. Obviously, $f(0)=0$, $f(1)=1$, and, by the definition of $x^{1/2}$, $f(1/2) = 1/2$. Further f is nondecreasing, by Axiom 2'a and (A11). Also f is continuous: For every $x^{a/2^n}$ there exists, by Axiom 3, a p such that $\{(x_0,x_1);(p,1-p)\} \sim x^{a/2^n}$, i.e., $f(p) = 1 - a/2^n$. This shows that the range of f is dense in [0,1]. The nondecreasing f cannot make "jumps", and must be continuous.

STAGE 5 (surjectivity of U). We show now that $U(X)=[0,1]$. Let $\mu \in [0,1]$. Take p such that $1-\mu = f(1-p) = 1-V\{(x^0,x^1);(1-p,p)\}$; so $V\{(x^0,x^1);(1-p,p)\} = \mu$. Then, by R.2, there exists $x \sim \{(x^0,x^1);(1-p,p)\}$. By (A11), $V(x) = \mu$; so $U(x) = \mu$.

STAGE 6 (U and V are representing). The derivation in this stage will not be elementary. Of course, if U is representing for $\succcurlyeq$ on X, then V is representing for $\succcurlyeq$ on Y, so we only derive the former. By (A10), it suffices to assume that $x''\succ x'$ and $U(x'')=U(x')=:\mu$, and derive a contradiction.

Note that there does exist a function, say U', that represents $\succcurlyeq$ on X: Choose for each equivalence class $\{x'\in X: x'\sim x\}$ a *probability equivalent*, i.e., a p such that $\{(x^0,x^1);(1-p,p)\} \sim x$. By Axiom 3 there exists at least one such p. Then define $U'(x')=p$ for all x' from the equivalence class. By Axiom 2'a, $x''\succ x'$ must imply $U'(x'')>U'(x')$. This, and constantness of U' on $\sim$ equivalence classes, shows that U' is representing. The existence of a representing U' excludes the existence of an uncountable number of disjoint preference intervals $\{x\in X: x''\succcurlyeq x\succcurlyeq x'\}$ for $x''\succ x'$, as the latter would lead to uncountably many distinct rational numbers, from each interval $]U'(x'),U'(x'')[$ one. So it suffices, for contradiction, to derive an uncountable number of such preference intervals.

Either $\mu\neq 0$, or $\mu\neq 1$, say the latter. Take any $\mu<v<1$. By Stage 5, there exists x_v such that $U(x_v)=v$. Now, by (A12), $V\{(x'',x_v);\frac{1}{2},\frac{1}{2}\} = (\mu+v)/2 = V\{(x',x_v);\frac{1}{2},\frac{1}{2}\}$, whereas, by Axiom 2'b, $\{(x'',x_v);\frac{1}{2},\frac{1}{2}\} \succ \{(x',x_v);\frac{1}{2},\frac{1}{2}\}$. We take $x''_{(\mu+v)/2}\sim\{(x'',x_v);\frac{1}{2},\frac{1}{2}\}$ and $x'_{(\mu+v)/2}\sim\{(x',x_v);\frac{1}{2},\frac{1}{2}\}$. Then $x''_{(\mu+v)/2}\succ x'_{(\mu+v)/2}$, but $U(x''_{(\mu+v)/2}) = U(x'_{(\mu+v)/2})$. Such outcomes can be constructed for each v between μ and 1, and $\{\{x\in X: x'_{(\mu+v)/2}\preccurlyeq x\preccurlyeq x''_{(\mu+v)/2}\}\}_{v\in]\mu,1[}$ gives an uncountable number of mutually disjoint preference intervals.

STAGE 7 (Jensen's equation). Fix $\mathbf{p}=(p_1,\ldots,p_n)$ in this step. Because U represents $\succcurlyeq$, and because of Lemma 3.1, we can write $V\{(x_1,\ldots,x_n);\mathbf{p}\} = W(U(x_1),\ldots,U(x_n))$ for a function W. For simplicity of notation, from now on we identify outcomes with their U values in this stage. The domain of W is the set $[0,1]^n_{\uparrow}$ of all $\mathbf{x} = (x_1,\ldots,x_n)\in [0,1]^n$ with $0\le x_1\le\ldots\le x_n\le 1$. We show that W satisfies Jensen's equation, i.e., for all $\mathbf{x},\mathbf{y}\in [0,1]^n_{\uparrow}$,

$$W\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) = \frac{W(\mathbf{x}) + W(\mathbf{y})}{2}.$$

Define $c_i \sim \{(x_i,y_i);(\frac{1}{2},\frac{1}{2})\}$ for all i. Then $c_i = (x_i+y_i)/2$ for all i. Note that $c_1\le\ldots\le c_n$, so that $\mathbf{c}\in [0,1]^n_{\uparrow}$. Let $x\sim\{\mathbf{x};\mathbf{p}\}$, $y\sim\{\mathbf{y};\mathbf{p}\}$. By Axiom 4, $\{\mathbf{c};\mathbf{p}\} \sim \{(x,y);(\frac{1}{2},\frac{1}{2})\}$. So, substituting (A12), we get $W(\mathbf{c}) = (U(x)+U(y))/2$. This implies Jensen's equation.

STAGE 8 (W is linear, and gives the AU form). By standard techniques it can be shown that W as obtained in Stage 7, must be linear. In general, solutions of Jensen's equation exist that are nonlinear, but these are very irregular. Axiom 2'a excludes all those nonlinear solutions. From the definition of f, it follows that the weights employed in the linear W, are exactly what they should be according to AU. AU can also be derived from

Proposition A1. The remainder of this stage gives a formal derivation of linearity of W for a fixed $(p_1,\ldots,p_n)$.

Define $e_1 := (1,\ldots,1), \ldots, e_2 := (0,1,\ldots,1), \ldots, e_n := (0,\ldots,0,1)$. On a rank-ordered cone it is convenient to take $e_1,\ldots,e_n$ as basis, because then standard results of Aczél (1966) can be applied literally. The details are as follows. Define $W':\mathbb{R}^n_+ \to \mathbb{R}$ in the following way. If $\sum_{i=1}^{n} y_i \le 1$, $W'(y) := W(\sum_{i=1}^{n} y_i e_i)$. On the domain covered so far, W' satisfies Jensen's equation, in particular, given $W'(0,\ldots,0) = 0$, $W'(x/2^m) = W'(x)/2^m$ for all m. For a general $(x_1,\ldots,x_n) \in \mathbb{R}^n_+$, find any 2^m large enough to ensure that, for $y_i := x_i/2^m$, $\sum_{i=1}^{n} y_i \le 1$. Next define $W'(x) := 2^m W'(y)$. From Jensen's equation on the domain covered before, it follows that the definition of W'(x) does not depend on the particular choice of m and y, and that in fact W' satisfies Jensen's equation throughout its domain. For the fixed **p** we get, by the definition of f, $f(p_1+\ldots+p_n) = W(e_1) = W'(1,0,\ldots,0)$, $f(p_2+\ldots+p_n) = W(e_2) = W'(0,1,\ldots,1)$, ..., $f(p_n) = W(e_n) = W'(0,\ldots,0,1)$. The proof is complete if a contradiction is derived from nonlinearity of W'.

By Aczél (1966, Section 2.1, extended in Section 5.1.1), W' can be nonlinear only if there exists an i, and an irrational λ_i, such that (with the ith coordinate 1 hereafter) $W'(\lambda_i(0,\ldots,0,1,0,\ldots,0)) \ne \lambda_i W'((0,\ldots,0,1,0,\ldots,0))$; $\lambda_i \le 1$ can always be taken, so that $W(\lambda_i(e_i)) \ne \lambda_i W(e_i)$. Say $W(\lambda_i(e_i)) > \lambda_i W(e_i)$. Again, by Aczél (1966), for all rational r_i, $W(r_i(e_i)) = r_i W(e_i)$. So there is a rational r_i such that $r_i > \lambda_i$, but $W(\lambda_i(e_i)) > W(r_i(e_i))$. This constitutes a violation of Axiom 2'a, for the prospects $\{(0,\lambda_i);(p_1+\ldots+p_{i-1},p_i+\ldots+p_n)\}$ and $\{(0,r_i);(p_1+\ldots+p_{i-1},p_i+\ldots+p_n)\}$. If $W(\lambda_i(e_i)) < \lambda_i W(e_i)$, then a rational $r_i < \lambda_i$ is found to reveal a violation of Axiom 2'a. This completes the proof of Stage 8.

A rereading of the proof, plus substitution of AU, shows that any choice of $U(0) = \sigma$, $U(1) = \tau$, for general $\tau > \sigma$ instead of $\tau = 1$, $\sigma = 0$, could be made, and would uniquely determine a positive affine transform of the function U as in the proof above, and that the function f is uniquely determined.

Finally, we relax Assumption A10. If all outcomes are equivalent, then by R.2, all prospects are equivalent, and the result is trivial. So let there be nonequivalent outcomes. We fix some $x^1 \succ x^0$. For each $y \succcurlyeq x^1 \succ x^0 \succcurlyeq z$, we can construct an AU representation for prospects with outcomes $\{x \in X: y \succcurlyeq x \succcurlyeq z\}$, similar to the construction under Assumption A10. By the uniqueness results for U and f as established above, this AU representation for outcomes $\{x \in X: y \succcurlyeq x \succcurlyeq z\}$ can be made to coincide with the AU representation established above, which uniquely determines the extended AU representation. As the outcomes involved in any prospect are finite, so bounded, the AU representation is uniquely determined for all prospects. This completes the proof of Theorem 3.2.

Note that the set of ~ equivalence classes of the outcome set is isomorphic to an interval; the set X, when endowed with the order topology, is a connected topological space.

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