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# A GENERAL SOLUTION TO KING SOLOMON'S DILEMMA

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# A General Solution To King Solomon's Dilemma*

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## 1. Introduction

A classical situation involving disagreement between two parties is "King Solomon's Dilemma". The nature of Solomon's dilemma begins with a dispute between two women in which each claims to be the mother of a certain child. Of course, Solomon wishes to give the child to the rightful mother at no cost to her. The difficulty is that although Solomon knows that one of them is the mother, he does not know which one.

In one of the most convincing recent applications of the theory of implementation, Glazer and Ma (1989) provide a strikingly simple and elegant solution to Solomon's dilemma for the case in which it is commonly known among all three parties that the rightful mother values possession of the child at a dollars, and the impostor values possession of the child at b dollars, where $a>b$. Moore (1992) modified the Glazer-Ma mechanism to nicely accommodate the more general case in which it is commonly known that only the women know the values of a and b, while Solomon knows only that the true mother values possession of the child strictly more than the impostor does. Although slightly more complex than Glazer and Ma's mechanism, Moore's remains remarkably simple given the task at hand.

Our objective here is to complete the task of resolving Solomon's dilemma by removing the remaining informational restriction while maintaining simplicity in the implementing mechanism. We shall assume only that it is commonly known both that the women know who the rightful mother is and that she values possession of the child strictly more than does the impostor. It is shown that a second-price sealed-bid all-pay auction with the winner having an ex-post option to quit solves King Solomon's dilemma in iteratively undominated strategies. Indeed, four rounds of elimination suffice.

The remaining two sections present the model and solution. For ease of exposition, the formal details are kept to a minimum. The interested reader may wish to consult the appendix for a formal model of knowledge.

## 2. The Model

There are two agents, A and B[1]. A single object is to be allocated at no cost to the agent who values it most. We maintain the following assumptions, each of which is common knowledge between A and B: (i) the agents' values are positive and are distinct, (ii) each agent knows which of them has the higher value and each agent knows his own value, (iii) neither agent rules out the true value of the other agent, (iv) the low value agent places a finite upper bound on the other agent's value[2], and (v) each agent's payoff of obtaining the object at price p when

[1] For ease of exposition we describe the case involving two agents. The mechanism works in the same way for the n agent case when the identity of the agent having the highest value is common knowledge among the agents.

[2] One can dispense with this assumption by modifying the mechanism slightly.

its value to that agent is v, is v-p, while the payoff associated with paying p and not receiving the object is -p[3].

The agents will participate in a second-price sealed-bid all-pay auction *with an option*. The option is that after the bids are revealed to the agents, the winner (highest bidder) can either choose to stick with the auction (in which case he receives the object and both bidders pay the second-highest bid), or he can choose to *quit* and give the object to the other agent in which case no payments are made by either agent. If the two bids are identical, then the object is sold to one of them (determined by the toss of a fair coin) at a price equal to the common bid. In this case the other agent pays nothing.

## 3. The Solution

We now show that this mechanism implements the desired outcome in iteratively (weakly) undominated strategies.[4,5] Without loss of generality, we assume that agent A values the object more than B. Throughout the analysis below, p denotes a bid by A and q denotes a bid by B.

**Round 1**: For each agent, eliminate every strategy such that given the agent's value and the bid then specified by the strategy, the strategy also specifies quitting (buying the object) if the agent's bid is winning and the second highest bid is below (above) his value.

**Round 2**: Eliminate all strategies for A in which he bids above his value, a. All such bids are weakly dominated by bidding his value. To demonstrate this, we consider below all possible cases. It is useful to recall that by assumption

(*) A knows that his own value, a, strictly exceeds B's value.

(a) $q>p>a$: B wins the auction whether A bids p or a, and in both cases B exercises the option to quit. Hence, by (*), A knows this and so is indifferent between bidding p and a.
(b) $q=p>a$: By bidding p, agent A, with probability one-half, obtains the object for a price of $p>a$. However, a bid equal to a would render B the winner. B would then

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[3] Assumption (iii) might strike the reader as being very strong. In our view this is not the case. For to violate (iii) an agent must rule out the truth. But this is tantamount to drawing a definite conclusion when no such conclusion can possible be (definitively) drawn. In any event, the present informational assumptions (ii)-(iv) are substantially weaker than those in Glazer and Ma (1989) and Moore (1991).
[4] A strategy is a function from an agent's value to a non negative bid and a decision function. The decision function provides for each pair of bids in which the agent bid is winning, a decision to either quit or not. We define weak dominance as follows. A strategy s for A *weakly dominates* s' against a subset, Y, of B's strategies, if for every t in Y, every $a>0$, and every value $b>0$ of player B that player A of value a cannot rule out, s(a) yields at least as high a payoff as s'(a) against t(b), and a strictly higher payoff for at least one such triple t, a,b.
[5] The full force of iterative dominance is not needed here. Only four rounds of elimination are required. Earlier rounds of elimination can be justified more easily than later ones since each round can be justified only if the players if the players know that the previous eliminations have been made. Consequently, fewer rounds of elimination correspond to less stringent assumptions about the players' mutual knowledge and therefore render the solution more compelling.

take the option to quit giving A the object for free. By (*), A knows this so that A strictly prefers the bid a over p.
(c) p>q>a: If A bids p, then A wins the auction and takes the option to quit. But if A bids a, then B wins the auction and takes the option to quit. By (*) A knows this so that bidding a is strictly better for A than is bidding p.
(d) p>a=q: If A bids p, then A wins the auction and obtains a payoff of zero whether or not he chooses to quit. If A bids a, then with probability one-half he is the winner and again receives a payoff of zero, and with probability one-half B is the winner in which case A neither pays any money nor receives the object. Hence, A is indifferent between bidding a and p.
(e) p>a>q: A wins the auction whether he bids a or p. Hence A is indifferent between bidding a and p.

We conclude that it is weakly dominant for A to submit a bid less than or equal to his value.

**Round 3**: Eliminate all remaining strategies for B except those in which he chooses a bid that he knows is strictly above A's value. (Call such bids *conservative* ones for B.) That B places some upper bound on A's value is guaranteed by assumption (and this is common knowledge). In order to demonstrate that these strategies are weakly dominated, we consider all possibilities.
(a) q<p≤a: If B bids q, then A wins the auction and chooses to buy the object at price q. B must then also pay q obtaining a non positive payoff. If instead B bids conservatively, then B is guaranteed to win the auction (since from Round 2 A's bid is not above his value) and so is guaranteed a non negative payoff (he can always subsequently choose the option to quit). Moreover, for values of p below B's value of the object, B strictly prefers to bid conservatively since he will obtain a strictly positive payoff by purchasing the object after winning the auction.
(b) q=p<a: If B bids q, then with probability one-half B must buy the object at price p. Bidding conservatively guarantees B the option of buying the object at price p. The latter is strictly better for B whenever p differs from B's value and equally good otherwise.
(c) q=p=a: If B bids q, then with probability one-half B must buy the object at price p =a which is above his value, while a conservative bid guarantees B a non negative payoff.
(d) q>p, p≤a: B wins the auction whether he bids conservatively or bids q. Hence B is indifferent between the two.

We conclude that B submits a bid that he knows is strictly above A's value.

**Round 4**: Eliminate all remaining strategies for A except those in which he chooses a bid that he knows is above B's value. (Call these bids conservative ones for A. In particular then, conservative bids for A do not exceed A's value since such bids were eliminated in Round 2. Also, note that bidding his own value constitutes a conservative bid for A.) The reason that these strategies are dominated follows.

From Round 3, we have that B's bid exceeds A's value. Since at this stage A's bid does not exceed his own value, A knows that B will win the auction.

Consequently, by choosing a conservative bid, A guarantees himself the object for free, since A's conservative bid, being above B's value, is certain to induce B to quit after B wins the auction. On the other hand, among those bids remaining (i.e. those which do not exceed A's value) by choosing a non conservative one A runs the risk that his bid is less than or equal to B's value and then that B chooses to purchase the object. Agent A then would not obtain the object and would also pay the amount of his bid.

We conclude that A submits a conservative bid (i.e. one that is not above his own value and one that he knows is above B's value). Since at this point in the elimination process B also submits a conservative bid (i.e. one that he knows is above A's value), and neither player rules out the truth, no further elimination is possible. All remaining strategies yield the same outcome, namely that B wins the auction and chooses to exercise the option to quit. Thus A receives the object and neither agent makes any payment.

**Appendix**

In this appendix we provide a formal model of knowledge along the lines of Aumann (1976).

Let $\Omega$ denote the set of states of the world and let $\Pi_A$ and $\Pi_B$ denote agent A's and B's information partitions of $\Omega$ respectively. Let $\Pi_i(\omega)$ denote the element of $\Pi_i$ containig those states that i does not distinguish between (or rule out) when the true state is $\omega$. Also, let $\Psi:\Omega\to\mathbf{R}^2_{++}$ be a mapping taking states of the world into a value of the object for each agent. We maintain the following assumption for all $\omega$, $\omega\in\Omega$ and $i = A, B$:

1. $\omega\in\Pi_i(\omega)$

2. If $\Psi(\omega)= (x_A,x_B)$ then
   (a) $x_A\neq x_B$
   (b) $\omega'\in\Pi_i(\omega)\to\Psi_i(\omega)=x_i$
   (c) $x_A>x_B$, $\omega'\in\Pi_i(\omega)$ and $\Psi(\omega')=(x'_A,x'_B)\to x'_A>x'_B$.

3. $\Psi(\Pi_i(\omega)$ is bounded.

Assumptions 1-3 express, respectively that: 1. neither agent rules out the truth; 2(a). the agents' values are distinct, (b) each agent knows his own value, (c) each agent knows whose value is larger; and 3. each agent places a finite upper bound on the other 's value at each state of the world.

A strategy for an agent is a function from states of the world into a non negative bid and a decision function. The decision function specifies for each pair of bids in which the agent's bid is winning, a decision to either quit or not. The agent's strategy must be measurable with respect to his information partition.

Given strategies s and r for agents A and B respectively, let $u(s(\omega), r(\omega)|\omega)$ denote A's payoff from the auction when the state is $\omega \in \Omega$. Let s' be another strategy for A and let Y be a subset of strategies for B. Then we say that s *weakly dominates* s' for A against Y, if $u(s(\omega), r(\omega)|\omega) \geq u(s'(\omega), r(\omega)|\omega)$ for all $\omega \in \Omega$ and all $r \in Y$, with at least one such pair $\omega$ and r yielding a strict inequality. Weak dominance is similarly defined for agent B.

With these definitions, the steps taken in the main text can be applied equally well here and the result is the same. Our auction mechanism implements the desired outcome in iteratively undominated strategies whenever the situation can be modelled as above. We now give two such examples. The examples are also meant to illustrate the permissiveness of our informational assumptions.

Example 1 (Glazer and Ma (1989)): Fix $x>y$. Let $\Omega=\{(x,y), (y,x)\}$ $\Pi_A=\Pi_B=\{\{(x,y)\},\{(y,x)\}\}$ and $\Psi(\omega)=\omega$ for all $\omega \in \Omega$.

Example 2 (Moore (1991)): Let $\Omega=\{(a,b)\in \mathbf{R}^2_{++} | a\neq b\}$ $\Pi_A=\Pi_B=\{\{(a,b)\}|(a,b)\in\Omega\}$.

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