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# FUNCTIONAL PROPERTIES OF THROUGHPUT IN TANDEM LINES WITH UNRELIABLE SERVERS AND FINITE BUFFERS

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# Functional Properties of Throughput in Tandem Lines with Unreliable Servers and Finite Buffers

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**Abstract:** We explore functional properties of throughput in tandem production lines with unreliable servers, finite buffers, and arbitrary failure and repair times. We provide a mathematical framework that makes use of a function space construction to model the dependence of throughput on buffer capacities and maximum flow rates of machines. Using this framework we prove various structural properties of throughput and mention how these properties can be used to guarantee almost-sure convergence of sample-path optimization, a simulation-based optimization method, when applied to the optimal buffer allocation problem. Our exposition demonstrates the utility of using multifunctions in the modeling, analysis, and optimization of discrete event dynamic systems.

Among the properties established, monotonicity in buffer capacities and in machine flow rates are especially important. Although monotonicity results of this nature have appeared in the literature for *discrete* tandem lines, as far as we are aware the kind of analysis we present here has not yet been done for *continuous* tandem lines.

**Key Words:** multifunctions, stochastic optimization, discrete event dynamic systems, tandem queues, throughput, unreliable servers, finite buffer spaces, continuity, monotonicity, concavity

[1]This work has been done when both authors were at the Department of Industrial Engineering, University of Wisconsin-Madison.

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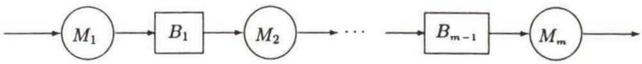


Figure 1: The tandem production line

# 1 Tandem Lines and Our Main Results

Investigating functional properties of performance measures, such as continuity, monotonicity, or convexity, is an important part of optimal design and control of stochastic systems. In this paper we are concerned with exploring some properties of the throughput of a tandem queue.

A tandem queue consists of a number of servers in series. There may be buffers of finite sizes between the servers. Jobs start at the first server, pass through each server in sequence, and finally leave the system after being served by the last server. These queues have been widely used to model a single line of multistage automated assembly lines or virtual paths in communication networks; see Buzacott and Shanthikumar (1992), and Yamashita and Önvural (1994) and references therein.

We focus on a particular tandem queue where service rates are deterministic and the servers are subject to random breakdowns with associated random repair times. It is common to use this type of queues to model tandem production lines in which machines are the servers. In a tandem production line there are $m$ processing machines $(M_1, \ldots, M_m)$ connected by $m-1$ buffers $(B_1, \ldots, B_{m-1})$. The material processed may be discrete entities (e.g. assemblies in an automobile factory), in which case we speak of a discrete tandem (DT) line, or it may be continuous (e.g. chemical production), in which case we refer to a continuous tandem (CT) line. The time it takes a machine to process one unit of product is called the cycle time. Notice that in a CT line the natural description for processing rate of a machine is the flow rate which is the reciprocal of cycle time.

The tandem lines we focus on have the following additional features:

- There is infinite supply to the first machine and infinite demand from the last machine.
- There is no transfer delay from machines to buffers, within buffers, or from buffers to machines.
- A machine may fail while it is processing and it may take some time to be repaired; it can fail only when it is operational. The amount of product processed by each machine between its failures, i.e. the operating quantity to failure for each machine is a random variable. Similarly the repair time for each machine is a random variable.

In the DT line we add:

- Cycle times of machines are deterministic.
- Machines are blocked via "manufacturing blocking" (Altiok and Stidham (1982)); that is, if a buffer becomes full, the machine upstream of it may begin to work on the next piece, but if it finishes its cycle and the buffer is still full, then it will be blocked.

In the CT line analogous to this DT line:

- Each machine has a deterministic maximum flow rate $C_i$, so machine $i$ can work at a rate anywhere between 0 and $C_i$.

Since many real world systems can be modeled by DT lines, they have received a lot of attention in the literature; see Suri and Fu (1994) and the references therein. One approach to model and analyze DT lines is to approximate them by CT lines. A translation of various input parameters and performance measures between the CT and DT lines can be found in Suri and Fu (1994). Furthermore, in Theorem 5.6 of Fu (1996), it is shown that the continuous production case is the limit of the discrete production case as the piece size approaches zero while the production rate remains constant.

There are several reasons why the CT line approximation approach might be attractive. Using CT lines instead of DT lines brings considerable increase in computational efficiency. Extensive numerical results on the substantial time advantage of CT simulations over DT simulations are reported in Suri and Fu (1994); they also present numerical experiments which show that such approximations are quite accurate. Using CT lines is beneficial from optimization point of view as well. When dealing with continuous parameters there is the possibility of obtaining gradient estimates. Furthermore, techniques for continuous parameter optimization are much more advanced than those for discrete parameter optimization.

We adopt the failure model of Suri and Fu (1994) in which the next failure of a machine is determined by the *quantity produced* since the last failure (as opposed to being determined by the *time of operation* since the last failure); see Suri and Fu (1994) for a discussion on why this is a natural failure model for CT lines that are approximations for DT lines. All of our results would go through however, with slight modifications, if we used the failure model that is based on the time of operation.

The characteristics of the line cause various interactions between the machines. If the buffer between the $ith$ and $(i+1)st$ machines is full, then the $ith$ machine cannot produce at a rate larger than the current rate of the $(i+1)st$ machine. This phenomenon is called *blocking*. A similar event is *starvation*; if the buffer between the $ith$ and $(i+1)st$ machines is empty, then the $(i+1)st$ machine cannot produce at a rate larger than the current rate of the $ith$ machine. In any case, whenever a machine is operational, it is operated at maximum possible rate. The main performance measure for a tandem line is its throughput, the amount of production completed by the last machine in unit time. In this paper, we are mainly concerned with investigating continuity, and first and second order stochastic properties of throughput as a function of buffer capacities.

Our interest in such results are twofold. First, they provide various qualitative guidelines for analysis and optimal control. By enabling one to compare different systems and determine which one performs superior without evaluating their performance individually, monotonicity results provide qualitative guidelines for design improvement. Furthermore, second order properties along with reversibility type of arguments may be used to provide guidelines about the optimal allocations in symmetric systems. Continuity/differentiability type results are likely to have impact on our choice of optimization method.

Second, the results we establish in this paper actually have quantitative merits. Finding optimal buffer allocations that maximize the throughput and do not violate various constraints has attracted a lot of interest. In Gürkan (1997), the sample-path optimization method is used to find optimal buffer allocations in tandem production lines with unreliable machines. Sample-path optimization is a recent simulation-based method to optimize performance functions of complex stochastic systems; see Gürkan *et al.* (1998) for a brief overview of the so-called "sample-path methods" that can be used for providing solutions for difficult stochastic optimization problems and stochastic variational inequalities. The idea is to observe a fixed sample path (by using the method of common random numbers from the simulation literature), solve the resulting deterministic problem using fast and effective methods from nonlinear programming, and then use the resulting solutions to infer information about the solution of the original stochastic problem. Clearly, effective and provably convergent optimization procedures, deterministic or stochastic, would require an adequate amount of regularity in the function to be optimized; Robinson (1996) contains a set of sufficient conditions that guarantee almost sure convergence of this approach in solving optimization problems. Using monotonicity and upper semicontinuity of throughput in buffer capacities, it is possible to show that the conditions (which appear in Robinson (1996)) that guarantee almost-sure convergence of sample-path optimization are satisfied; see Gürkan (1997). Therefore provided that a long enough sample-path is used, one can be confident about the closeness of the computed solution to a correct solution of the original problem; see Gürkan (1997) for a rigorous statement of these results, additional details,

Let $\Theta_T$ denote the throughput of a CT line up to time $T$, $b = (b_1, \dots, b_{m-1})$ the buffer capacities, and $C = (C_1, \dots, C_m)$ the cycle times of machines. Our main results could be summarized as follows:

- We prove that $\Theta_T$ is a non-decreasing function of $b$, for $T \in [0, \infty]$.
- For $T \in [0, \infty)$, $\Theta_T$ is discontinuous but an upper semicontinuous function of $b$; therefore it cannot be concave. On the other hand, empirically $\Theta_\infty$ appears to be a continuous and concave function.
- Although $\Theta_T$ for finite $T$ is not concave, in §4 we show that the number of departures from the system by time $t$, $D_t$, in the analogous DT line is a concave function of the buffer capacities. Notice that this gives us the concavity of the line throughput with respect to buffer capacities, since line throughput is just $t^{-1}D_t$. This is a well known result in the case of reliable machines with exponential service times, see for example Meester and Shanthikumar (1990), Anantharam and Tscoucas (1990), and Rajan and Agrawal (1994). We make a simple extension of this concavity result to cover the case of unreliable servers with deterministic cycle times and exponential failure and repair times.
- Aside from the results themselves, our way of analyzing the CT line is of interest in its own. We construct two multifunctions that model the dynamics of the system and explore the properties of these multifunctions. Using these properties and by making sample path comparisons, that is, fixing a sample path and comparing two processes constructed on a common probability space, we avoid making any distributional assumptions, except that all the random variables should have densities concentrated on $(0, \infty)$. We hope that our exposition demonstrates the utility of using multifunctions in the modeling, analysis, and optimization of discrete event dynamic systems.
- As a by-product our analysis , we also prove that $\Theta_T$ is a non-decreasing function of $C$, for $T \in [0, \infty]$.

The remainder of this paper is divided into four main section. At the end there are two appendices containing additional technical details. In §2, we provide a mathematical framework to model the dynamics of the CT line and develop necessary machinery for the technical analysis. In §3, we prove some functional properties of throughput, discuss some consequences of these results, and compare them to the known results from the literature. Finally, in §4 we show the concavity of throughput in buffer capacities in DT lines. §5 contains some concluding remarks.

# 2 Modeling the Dynamics via Multifunctions

In this section we provide a mathematical framework to model the dynamics of the tandem line. We construct two multifunctions and show some of their technical properties. In the next section, we use these to prove various properties of throughput in CT lines.

Let $T$ be the prespecified amount of time we observe the line and $q_i(t)$ be the amount produced by $M_i$ up to time $t$ for $i = 1, \ldots, m$. Then the line throughput can be defined as

$$\Theta_T = q_m(T)/T.$$

We define:
$b_j$ = buffer capacity of $B_j$,
$C_i$ = maximum flow rate of $M_i$,
$W_p^i$ = operating quantity between the $(p-1)st$ and the $pth$ failures at $M_i$,
$R_p^i$ = repair time of $M_i$ after the $pth$ failure.

For each $i$, $\{W_p^i\}_{p=1}^n$ and $\{R_p^i\}_{p=1}^n$ are random variables with distributions that are concentrated on $(0, \infty)$.

For a fixed sample path, i.e. for fixed sequences $\{W_p^i, i = 1, \ldots, m, p \geq 1\}$ and $\{R_p^i, i = 1, \ldots, m, p \geq 1\}$, let $f_{ij}$ be the quantity produced by the $ith$ machine up to its $jth$ failure. Then

$$f_{ij} = \sum_{p=1}^{j} W_p^i.$$

Fix $T$ (simulation time), let $\mathcal{C}([0,T], \mathbf{R}^m)$ be the space of continuous functions from $[0,T]$ to $\mathbf{R}^m$ with the sup-norm topology. That is, for $g \in \mathcal{C}([0,T], \mathbf{R}^m)$,

$$\|g\| = \sup\{|g_i(x)| : i = 1, \ldots, m, x \in [0,T]\}.$$

We next construct a multifunction $F : \mathbf{R}^{m-1} \to \mathcal{C}([0,T], \mathbf{R}^m)$ as follows. For any $b = (b_1, \ldots, b_{m-1}) \in \mathbf{R}_+^{m-1}$, we define $F(b)$ to be the set of continuous functions $g : [0,T] \to \mathbf{R}^m$ satisfying the following requirements:

$$\begin{aligned}
&g_1 \geq g_2 \geq \cdots \geq g_m \geq 0,\\
&g_i \text{ is non-decreasing for each } i = 1, \ldots, m,\\
&g(0) = 0,\\
&|g_i(x) - g_i(y)| \leq C_i|x - y| \text{ for any } x, y \in [0,T] \text{ and } i = 1, \ldots, m,\\
&g_i(x) - g_{i+1}(x) \leq b_i \text{ for any } x \in [0,T] \text{ and } i = 1, \ldots, m-1.
\end{aligned}$$

For $b \notin \mathbf{R}_+^{m-1}$, we let $F(b) = \emptyset$. Hence $\operatorname{dom} F = \mathbf{R}_+^{m-1}$. The graph of $F$ is defined as $gphF = \{(b,g) : g \in F(b)\}$. One should think of the functions $g \in F(b)$ as possible ways of operating the CT line. If we interpret $g_i(t)$ as the amount produced by machine $i$ up to time $t$, then functions in $F(b)$ obey the buffer capacity and maximum flow rate constraints:

(i) the amount produced by a machine cannot be less than the amount produced by the succeeding machine,

(ii) the amount produced by a machine does not decrease with time,

(iii) the line starts operating at time zero,

(iv) a machine cannot work at a rate higher than its maximum flow rate,

(v) the amount produced by a machine cannot exceed the amount produced by the succeeding machine plus the buffer capacity between them.

We define $A$ to be the following subset of $F(\infty)$:

$$A = \{g \in F(\infty) : \lambda(\{t : g_i(t) = f_{ij}\}) \geq R_j^i, \text{ for each } i = 1, \ldots, m \text{ and } j = 1, 2, \ldots\},$$

where $\lambda$ is the Lebesgue measure on $\mathbf{R}$. Again, if we think of functions in $A$ as possible ways of operating a CT line with unlimited buffer capacities between machines, then the condition

$\lambda(\{t : g_i(t) = f_{ij}\}) \geq R^i_j$ means that under any possible operating strategy the amount of time machine $i$ stays non-operational after its $jth$ failure is at least equal to its $jth$ repair time.

$F(b)$ models the buffer capacity and maximum flow rate constraints, whereas $A$ models the failure and repair times of a CT line with unlimited buffer capacities between machines. So $F(b) \cap A$ can be thought of as the set of all possible ways of operating the CT line. Clearly, $q$ is in $F(b) \cap A$. Recall that among functions in $F(b) \cap A$, $q$ gives the amount produced using the strategy under which each machine is operated at maximum possible rate whenever it is operational; whenever we need, we will refer to this strategy as "strategy q". The pseudo-code developed in Fu (1996) prescribes a way of constructing such a strategy during a simulation. (His $v_i$ is the effective flow rate of $M_i$, $i = 1, \ldots, m$; at any time the pseudo-code prescribes how to set each one to its maximum possible value in a well-defined, non-circular way.)

Using this framework we can have the following three technical lemmas; their proofs are deferred to Appendix A.

**Lemma 1** *The multifunction $F$ has the following properties:*

*a.* $\operatorname{gph} F$ *is closed.*

*b.* $\operatorname{gph} F$ *is convex.*

*c.* $F$ *is compact-valued and* $F(b) \subset F(\infty)$ *for all* $b \in \mathbf{R}^{m-1}$.

*Proof.* See Appendix A.

In the next lemma we denote the *interior* of a set $S$ by $\operatorname{int} S$ and use the term Berge-usc for a multifunction, which we now define.

**Definition 1** *A multifunction $F$ from a topological space $Z$ to a topological space $Y$ is* Berge-usc *at a point $z_0$ of $Z$ if for each open set $U$ of $Y$ with $F(z_0) \subset U$ the set $\{z \in Z : F(z) \subset U\}$ is open. $F$ is Berge-usc in $Z$ if it is Berge-usc at every point of $Z$ and if $F(z)$ is compact for every $z \in Z$.*

Berge-usc is introduced in Berge (1963) under the name "upper semicontinuity"; see Rockafellar and Wets (1997) for a treatment of relationships between various semicontinuity and continuity notions for multifunctions. We thank the authors of that book for making the extracts of a draft version available to us.

We also need to define the Hausdorff distance between two sets. Let $S$ and $T$ be subsets of $\mathbf{R}^k$. We use the notation $e(S,T)$ for the *excess* of $S$ over $T$, defined by

$$e(S,T) = \sup_{s \in S} d(s,T); \qquad d(s,T) = \inf_{t \in T} \|s - t\|.$$

If $e(S,T)$ is small, then *each* point of $S$ is close to *some* point of $T$, though some points of $T$ might be far from any point of $S$. Such nonsymmetric behavior is not present in the *Hausdorff* distance between $S$ and $T$ that is defined by $h(S,T) = \max\{e(S,T), e(T,S)\}$.

**Lemma 2** *The multifunction $F$ is Berge-usc in $\mathbf{R}^{m-1}$ and $b \mapsto F(b)$ is a continuous mapping from $\operatorname{int}(\mathbf{R}^{m-1}_+)$ to compact subsets of $C([0,T], \mathbf{R}^m)$ with the metric topology induced by the Hausdorff distance.*

*Proof.* See Appendix A.

**Lemma 3** *$A$ is closed in $F(\infty)$.*

*Proof.* See Appendix A.

Now let $Q_T(b) = \sup\{g_m(T) : g \in F(b) \cap A\}$. In the next theorem, we show that the supremum in the definition of $Q_T(b)$ is actually attained and it is equal to the amount produced by the last machine up to time $T$ when each machine is operated at maximum possible rate whenever operational.

**Theorem 1** *Suppose that the event times have no cluster point. Then for each finite time $T$, $Q_T(b) = q_m(T)$.*

*Proof.* See Appendix B.

**Remark** Although, the assumption that the event times have no cluster point is fairly realistic and general, it excludes deterministic failure and repair times.

# 3 Properties of Throughput in CT lines

We now discuss some functional properties of $\Theta_T$ and $\Theta_\infty$. Below we use the term "non-decreasing" for a function $f : \mathbf{R}^m \to \mathbf{R}$, by which we mean that $f(x_1, \ldots, x_k) \geq f(y_1, \ldots, y_k)$ whenever $x_i \geq y_i$ for $i = 1, \ldots, k$.

**Theorem 2** *For $T \in [0, \infty]$, $\Theta_T$ is a non-decreasing function of $b$ with probability one.*

*Proof.* Observe that for $b' \leq b$, $F(b') \subset F(b)$. Hence $Q_T(b') \leq Q_T(b)$ and $\Theta_T$ is a non-decreasing function of $b$. ▮

The reader may compare this monotonicity result with Meester and Shanthikumar (1990). Their result is concerned with monotonicity of throughput as a function of buffer capacities of a *discrete tandem queue* with exponential service times and reliable servers, whereas we are concerned with monotonicity of throughput of a *continuous tandem line* with unreliable machines and deterministic flow rates. Furthermore, we do not make any distributional assumptions for the failure and repair times. Aside from these differences, our proof technique is very different. They use certain recursive equations to characterize the dynamics of the system, especially the number of departures from each server, and obtain the result by manipulating these equations inductively, whereas we provide a new function space representation to model the dynamics of the system and exploit this mathematical framework to obtain the result.

Meester and Shanthikumar (1990) and Anantharam and Tscoucas (1990) also show the concavity of sample throughput in buffer capacities. This result holds for the discrete analog of the system we are studying if failure and repair times are exponentially distributed, as shown in §4; however it fails to hold for CT lines; see Figure 2 and the discussion following Theorem 4.

We note that though it is not the main subject of the work reported here, one can also define a multifunction $\tilde{F}(C)$ from $\mathbf{R}^m$ to $\mathcal{C}([0,T], \mathbf{R}^m)$ by the same four conditions that we used to define F, where the variable is $C$, the vector of maximum flow rates. It is easy to see that $\tilde{F}(C') \subset \tilde{F}(C)$ if $C' \leq C$. Then by following the lines of proof of Theorem 2, we can show the monotonicity of throughput in flow rates.

**Theorem 3** *For $T \in [0, \infty]$, $\Theta_T$ is a non-decreasing function of $C$ with probability one.*

We should point out the difference between the monotonicity result of Theorem 3 and those of Shanthikumar and Yao (1989a); as in the previous result the difference is in the system studied and the proof technique employed. Theorem 3 is concerned with continuous tandem queues, whereas Shanthikumar and Yao study general discrete queueing networks for which the discrete tandem queue is a special case and use recursive equations to establish the monotonicity of throughput in the job service times. In addition to monotonicity, Shanthikumar and Yao (1989b) show that the reciprocal of throughput is a convex function of parameters of the external interarrival times and the machine service times, provided that these times themselves are convex functions of those parameters. A similar convexity result about discrete tandem queues with unreliable machines appears in Fu (1996). In addition, the convexity of reciprocal of throughput in maximum flow rates of machines in CT lines is proven in Fu (1996). We have pointed out to B.-R. Fu that by using the recursive equations for departure time process developed in Fu (1996), he can also show the monotonicity of throughput in maximum flow rates of machines; that would be an alternative way of proving Theorem 3.

**Remark** A GSMP representation is constructed in Suri and Fu (1994) to model CT lines. In Remark 2 of Gürkan (1997), it is shown that this GSMP is *not* non-interruptive (in the sense of Schassberger (1976)). Unfortunately, violation of the non-interruption condition rules out the applicability of the results, developed in Glasserman and Yao (1992a, 1992b), for checking the first and second order properties of stochastic systems that are modeled as non-interruptive GSMP's. Note that we are not ruling out the possibility of constructing a different GSMP representation (for this system) which is non-interruptive or modifying some of the results of Glasserman and Yao (1992a,1992b) so that they are applicable to interruptive GSMP's. However, both of these approaches would require further investigation which is not the subject of this paper.

The next result deals with the upper semicontinuity of sample throughput. This is important since the lack of upper semicontinuity in a function to be maximized may cause great difficulties when doing practical optimization.

**Theorem 4** *For $T \in [0, \infty)$, $\Theta_T$ is an upper semicontinuous function of $b$ with probability one.*

*Proof.* Let $T \in [0, \infty)$. We will show that $q_m(T)$ is an upper semicontinuous function of $b$ and the result will follow since $\Theta_T(b) = q_m(T)/T$. By Theorem 1 it is enough to show that $Q_T(b)$ is an upper semicontinuous function of $b$. Let $H : F(\infty) \to \mathbf{R}$ be defined by $H(g) = g_m(T)$. Then $H$ is continuous and attains its supremum over $F(b) \cap A$ since the set $F(b) \cap A$ is compact by Lemmas 1 and 3. Furthermore for any $y \in \mathbf{R}$, the set $S_y = \{g \in F(\infty) : H(g) < y\}$ is open. Then

$$\begin{aligned}
\{b : Q_T(b) < y\} &= \{b : g_m(T) < y \text{ for all } g \in F(b) \cap A\} \\
&= \{b : F(b) \cap A \subset S_y\} \\
&= \{b : F(b) \subset S_y \cup A^c\}.
\end{aligned}$$

So $\{b : Q_T(b) < y\}$ is an open set since $S_y \cup A^c$ is open and $F$ is Berge-usc. ∎

The reader may wonder whether the sample throughput, $\Theta_T$ for $T \in [0, \infty)$, is lower semicontinuous as well. In fact, $\Theta_T$ is a discontinuous function of buffer capacities for finite $T$; see Figure 2. This is due to the fact that if two events occur at the same time, an infinitesimal change in buffer capacities may cause the order of these events to change, as illustrated by a simple, numerical example in Gürkan (1996), p. 52-56. Of course, when the failure quantities and repair times for machines have continuous distributions, one may argue that the probability of a continuous random variable being equal to a specific value is zero; hence the probability that the time of two events coincides in a discrete event simulation is zero, as well. Therefore these types of phenomena cannot take place, in practice. On the other hand, it is clear from Figure 2 that once a sample path (a random number sequence $\omega$) is fixed, there are some buffer capacities at which this type of phenomenon does occur and results in discontinuities in throughput. In other words, at each $b$ the probability of throughput being discontinuous is zero; but the probability of throughput being discontinuous at *some* $b$ is not zero.

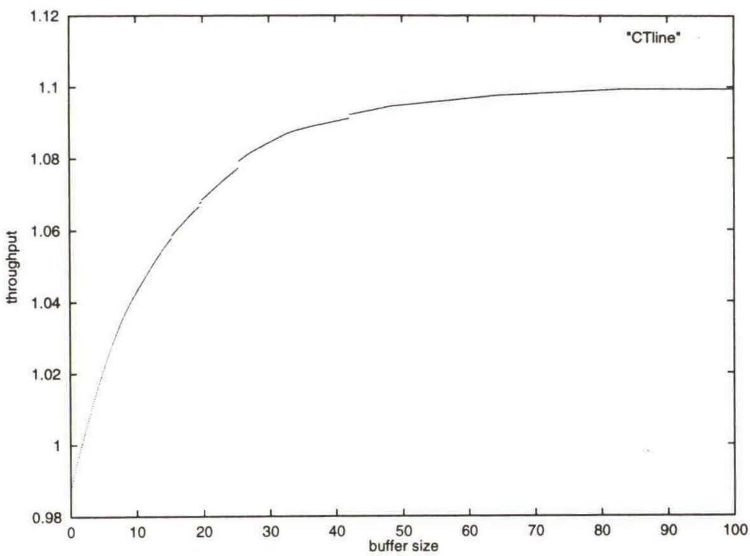


Figure 2: Simulation results for a 2-machine line with exponential failure and repair rates.

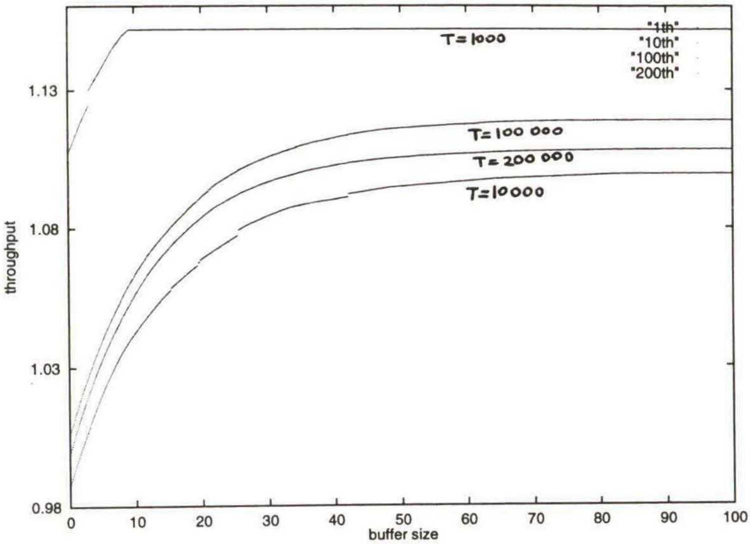


Figure 3: The throughput of a 2-machine CT line for different run lengths.

Using the upper semicontinuity and monotonicity of sample throughput, one can easily show that for any finite $T$, any $b$, and any $\epsilon > 0$, there exists $\delta > 0$ such that for every $j$, if $0 < \Delta b_j < \delta$ and $b' = b + \Delta b_j$ then $\Theta_T(b) \leq \Theta_T(b') < \Theta_T(b) + \epsilon$. This shows that the phenomenon described in that example cannot occur when buffer capacities are *increased* by an infinitesimal amount; it can only occur when they are decreased by an infinitesimal amount. Furthermore, this phenomenon may likewise occur when the operating time to failure (instead of operating quantity) is a random variable, see Remark 4.33 of Gürkan (1996) or if one chooses the $jth$ failure epoch of machine $M_i$ as the stopping time (instead of a fixed stopping time).

It is worth to mention that although $\Theta_T$ for $T \in [0, \infty)$ is discontinuous, $\Theta_\infty$ appears to be a continuous function. Intuitively, this is expected: the steady-state throughput of a line should not be very sensitive to arbitrarily small changes in the buffer capacities. In a 2-machine line, the continuity of steady-state throughput is provided by the analytical formula derived in Gershwin and Schick (1980). For longer lines we are not aware of results of this nature, although computational evidence strongly indicates that the steady-state throughput is indeed a *continuous* function of buffer capacities, see Figure 3. Figure 3 displays the throughput of a 2-machine CT line, where operating quantities to failures and repair times are exponentially distributed, for different run lengths $T$. In extensive numerical experiments (also for longer lines) we observed the same kind of behavior: a discontinuous function with frequent jumps of large sizes when $T$ is small, a smooth function when $T$ is large.

Although $\Theta_T$ for finite $T$ is not concave in CT lines, in the next section we show that the number of departures from the system by time $t$, $D_t$, in the analogous DT line is a concave function of the buffer capacities. This immediately gives the concavity of the line throughput with respect to buffer capacities, since line throughput is just $t^{-1} D_t$. As mentioned earlier, this is a well known result in the case of reliable machines with exponential service times, see for example Meester and Shanthikumar (1990), Anantharam and Tscoucas (1990), and Rajan and Agrawal (1994); we make a simple extension of this result to cover the case of unreliable servers with deterministic cycle times and exponential failure and repair times.

## 4 Concavity of Throughput in DT lines

In this section, we show that the departure process of the DT line has a certain type of concavity property which will be made precise later, in the buffer capacities.

Let $r_n^i$ be the total repair time of $M_i$ for all failures that occur at $M_i$ while it is working on the *nth* job, and $S_n^i$ be the service time (including the repair time) of the *nth* job at $M_i$. Then it is easy to see the following relation:

$$S_n^i = C_i^{-1} + r_n^i.$$

Let $\omega = (\{S_n^1\}_{n=1}^{\infty}, \dots, \{S_n^m\}_{n=1}^{\infty})$ be the underlying service time sequence. Define $D_i(t,\omega)$ to be the number of jobs completed up to time $t$ by the *ith* machine and let $D(t,\omega) = (D_1(t,\omega), \dots, D_m(t,\omega))$. $D(t,\omega)$ is the departure process we are interested in. Also let

$$u_{ij} = \begin{cases} \sum_{k=i}^{j-1} b_k & \text{if } i < j \leq m \\ 0 & \text{if } 1 \leq j < i \\ \text{undefined} & \text{if } j = i \end{cases} \tag{4.1}$$

It is clear that $D(t,\omega)$ must satisfy:

$$D_i(t,\omega) \leq D_j(t,\omega) + u_{ij} \quad \text{for all} \quad (t,\omega) \in \mathbf{R}_+ \times \Omega, \qquad 1 \leq i \neq j \leq m. \tag{4.2}$$

Equation (4.2) basically says that the number of jobs completed by the *ith* machine up to time $t$ cannot exceed the number of jobs completed up to time $t$ by any preceding machine and by any succeeding machine plus the total buffer space between these two. To prove the main result we need to introduce a few concepts, namely those of TDES, CDES and a NBU random variable. We keep the exposition of TDES and CDES very brief; the interested reader is referred to Rajan and Agrawal (1993, 1994).

A *timed discrete event system* (TDES) (Rajan and Agrawal (1993, 1994)) has two components, the logical component and the temporal component. The logical component specifies all feasible sequences of successive event occurrences, i.e. deals with the order in which events can occur, whereas the temporal component selects one particular sequence of events and their occurrence times. A *constrained discrete event system* (CDES) (Rajan and Agrawal (1993, 1994)) is a subclass of TDES whose logical component is completely specified by a constraint function. Let $\mathbf{Z}$ be the set of integers. We now define CDES in mathematical terms.

**Definition 2** *Let $g : \mathbf{Z}_+^m \longrightarrow \mathbf{Z}_+^m$ be any function that satisfies $g(y) \leq g(z)$ whenever $y \leq z$ and $y, z \in \mathbf{Z}_+^m$, and $e_i$ be the ith unit vector in $\mathbf{Z}^m$. The CDES $\Delta$ with constraint function $g$ is a timed discrete event system $\Delta = (\Gamma, A, \delta, \psi, a)$ with state space $\Gamma = \{y \in \mathbf{Z}_+^m : y \leq g(y)\}$, event set $A = \{1, \dots, m\}$, enabling (multi)function $\delta(y) = \{i : y_i + 1 \leq g_i(y + e_i)\}$ for $y \in \Gamma$, transition function $\psi(y,i) = y + e_i$, and initial state $a = \vec{0}$.*

**Definition 3** *A real valued non-negative random variable $X$ is said to be new-better-than-used (NBU) if for each non-negative, non-decreasing, bounded and measurable function $h : \mathbf{R} \to \mathbf{R}_+$, one has*

$$E[h(X-t) | X \geq t] \leq E[h(X)],$$

*i.e., for any $t \geq 0$, the distribution of $X - t$, given that $X \geq t$, is stochastically smaller than the distribution of $X$.*

It is easy to show that a random variable with Erlang distribution is NBU and that adding a constant to an NBU random variable conserves this property. We will use these facts in the proof of Lemma 4 which we now state.

**Lemma 4** *Assume that for each $i$ and $p$, $W_p^i$ and $R_p^i$ are exponentially distributed random variables with means $w_i$ and $1/r_i$ respectively. Consider the sequence of random variables $\{S_n^i\}_{n=1}^\infty$ for $i = 1, \dots, m$. Then,*

*(a) For each $i$, $S^i = \{S_n^i\}_{n=1}^\infty$ is an i.i.d. sequence of NBU random variables.*

*(b) $\{S^1, \dots, S^m\}$ are independent.*

*Proof.* For (a): First we will show that $\{S_n^i\}_{n=1}^\infty$ is an i.i.d. sequence of random variables. Let $N^i(n)$ be the number of failures machine $M_i$ experiences while working on the $nth$ job. Note that $N^i(n)$ is a random variable that can take the value 0. Hence for each $n$, $S_n^i = C_i^{-1} + r_n^i$, where $C_i$ is a constant and $r_n^i$ is the sum of $N^i(n)$ many exponentially distributed random variables. Let $M^i(t) = \sup\{p : W_1^i + \ldots + W_p^i \leq t\}$. Since $W_k^i, k = 1, \ldots$ are exponentially distributed random variables, $M^i(t)$ is a Poisson process. Furthermore for $n \in \mathbf{Z}_+$, $M^i(n)$ is the index of the last failure occurring not later than the completion of the $nth$ job. Thus $N^i(n) = M^i(n) - M^i(n-1)$ and $r_n^i = \sum_{k=M^i(n-1)+1}^{M^i(n)} R_k^i$. If we let $X^i(t) = \sum_{k=1}^{M^i(t)} R_k^i$, then $X^i(t)$ is a compound Poisson process since $\{R_k^i\}_{k=1}^\infty$ is an i.i.d. sequence of exponential random variables which is independent of $\{M^i(t) \mid t \geq 0\}$. Independence of $\{R_k^i\}_{k=1}^\infty$ and $\{M^i(t) \mid t \geq 0\}$ follows from independence of $\{R_k^i\}_{k=1}^\infty$ and $\{W_k^i\}_{k=1}^\infty$. Since a compound Poisson process has independent and stationary increments the result follows by observing $r_n^i = X^i(n) - X^i(n-1)$.

Let $Y_k^i$ be the random variable $r_n^i$ conditioned on the event $\{N^i(n) = k\}$; then from the above discussion it is apparent that the distribution of $Y_k^i$ is an $Erlang(k, r_i)$.

Next we will show that $\{S_n^i\}_{n=1}^\infty$ is a sequence of NBU random variables. Obviously the $S_n^i$ are non-negative. For any non-decreasing, non-negative, bounded measurable function $h$ and any $t \geq 0$, we have:

$$\begin{aligned} E[[h(S_n^i - t) \mid S_n^i \geq t] \mid N^i(n) = k] &= E[h(Y_k^i + C_i^{-1} - t) \mid Y_k^i + C_i^{-1} \geq t] \\ &\leq E[h(Y_k^i + C_i^{-1})] \end{aligned}$$

The first equality follows from the independence of the repair times and the failure times whereas the second inequality is a consequence of $Y_k^i$ being NBU.
Thus

$$\begin{aligned} E[h(S_n^i - t) \mid S_n^i \geq t] &= E[E[[h(S_n^i - t) \mid S_n^i \geq t] \mid N^i(n)]] \\ &\leq \sum_{k=0}^{\infty} E[h(Y_k^i + C_i^{-1})] \cdot P\{N^i(n) = k\}. \end{aligned}$$

We also have

$$E[h(S_n^i)] = E[E[h(S_n^i) \mid N^i(n)]] = \sum_{k=0}^{\infty} E[h(Y_k^i + C_i^{-1})] \cdot P\{N^i(n) = k\}.$$

Hence we conclude $E[h(S_n^i - t) | S_n^i \geq t] \leq E[h(S_n^i)]$. So $\{S_n^i\}_{n=1}^\infty$ is a sequence of i.i.d. NBU random variables.

For (b): $S^i$ only depends on $C_i$, $\{R_n^i\}_{n=1}^\infty$, and $\{W_n^i\}_{n=1}^\infty$. But for $i \neq j$ the random number sequences $\{R_n^i\}_{n=1}^\infty$ and $\{R_n^j\}_{n=1}^\infty$ are independent, and so are $\{W_n^i\}_{n=1}^\infty$ and $\{W_n^j\}_{n=1}^\infty$. Hence $\{S^1, \dots, S^m\}$ are independent. ∎

We now proceed with the main result. We write $X\stackrel{d}{=}Y$ to denote the equivalence in distribution of the random variables $X$ and $Y$ and $\lceil x \rceil$ to represent the ceiling function at $x$, i.e. it is the smallest integer greater than or equal to $x$.

**Theorem 5** *Assume that for each $i$ and $p$, $W_p^i$ and $R_p^i$ are exponentially distributed random variables with means $w_i$ and $1/r_i$ respectively. Let $b^0, b^1, \ldots, b^r$ be different buffer configurations for the DT line defined above, where $b^0 = \lceil \sum_{p=1}^r \alpha^p \cdot b^p \rceil$, $\sum_{p=1}^r \alpha^p = 1$ and $\alpha^p \geq 0$ for $p = 1, \ldots, r$. Consider the family of CDES $\Delta^0, \Delta^1, \ldots, \Delta^r$ corresponding to the buffer configurations $b^0, b^1, \ldots, b^r$, with state spaces $\Gamma^0, \Gamma^1, \ldots, \Gamma^r$ and event counting processes $D^0, D^1, \ldots, D^r$ respectively. Let $\omega = (\{S_n^1\}_{n=1}^\infty, \ldots, \{S_n^m\}_{n=1}^\infty)$ be the underlying service time sequence. Then there exist a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and clock time sequences $\omega^p = (\{S_n^1(p)\}_{n=1}^\infty, \ldots, \{S_n^m(p)\}_{n=1}^\infty)$, $0 \leq p \leq r$, defined on $(\Omega, \mathcal{F}, \mathcal{P})$ such that*

$$\omega^p \stackrel{d}{=} \omega \text{ for } 0 \leq p \leq r$$

*and*

$$D^0(t, \omega^0) \geq \lceil \sum_{p=1}^r \alpha^p \cdot D^p(t, \omega^p) \rceil \quad \text{for all } t \geq 0, \text{ a.s.}$$

*Proof.* In Lemma 4, we have shown that in the clock sequence $\omega = (\{S_n^1\}_{n=1}^\infty, \ldots, \{S_n^m\}_{n=1}^\infty)$, $S^i = \{S_n^i\}_{n=1}^\infty$ is an i.i.d. sequence of NBU random variables for all $i$, and $\{S^1, \ldots, S^m\}$ are independent.

For $0 \leq p \leq r$, we define $g^p : \mathbf{Z}_+^m \to \mathbf{Z}_+^m$ by $g^p(x) = (g_1^p(x), \ldots, g_m^p(x))$ where

$$g_i^p(x) = \min_{\substack{1 \leq j \leq m \\ j \neq i}} \{x_j + u_{ij}^p\}$$

with $x = (x_1, \ldots, x_m) \in \mathbf{Z}_+^m$ and $u_{ij}^p$ defined as in (4.1) for buffer configuration $b^p$. Then for $0 \leq p \leq r$, $D^p(t, \omega) \leq g^p(D^p(t, \omega))$; this follows from (4.2).

With constraint function $g^p$, the tandem production line corresponding to the buffer configuration $b^p$ is modelled as a constrained discrete event system (CDES) $\Delta^p$ with state space $\Gamma^p = \{y \in \mathbf{Z}_+^m : y \leq g^p(y)\}$, for $0 \leq p \leq r$.

Take $y^p \in \Gamma^p$, $1 \leq p \leq r$. Let $y^0 = \lceil \sum_{p=1}^r \alpha^p y^p \rceil$. Then for $1 \leq j \leq m$,

$$\begin{aligned}
y_j^0 &= \lceil \sum_{p=1}^r \alpha^p y_j^p \rceil \\
&\leq \lceil \sum_{p=1}^r \alpha^p \min_{\substack{1 \leq i \leq m \\ i \neq j}} \{y_i^p + u_{ji}^p\} \rceil \\
&\leq \min_{\substack{1 \leq i \leq m \\ i \neq j}} \lceil \sum_{p=1}^r \alpha^p y_i^p + \sum_{p=1}^r \alpha^p u_{ji}^p \rceil \\
&\leq \min_{\substack{1 \leq i \leq m \\ i \neq j}} \{\lceil \sum_{p=1}^r \alpha^p y_i^p \rceil + \lceil \sum_{p=1}^r \alpha^p u_{ji}^p \rceil\} \\
&\leq \min_{\substack{1 \leq i \leq m \\ i \neq j}} \{y_i^0 + \sum_{k=j}^{i-1} b_k^0\}
\end{aligned}$$

Hence $y^0 \leq g^0(y^0)$, which implies $y^0 \in \Gamma^0$. We conclude that $\Gamma^0 \supseteq \lceil \sum_{p=1}^r \alpha^p \cdot \Gamma^p \rceil$.

The result follows by Theorem 4.5 of Rajan and Agrawal (1994). ∎

**Remark 1** We remark that the result would remain valid if one uses "operating time to failure" model as opposed to "operating quantity to failure" model; that is when the next failure of a machine is determined by the time of operation of the machine since the last failure. The only difference will be in Lemma 4: the relation between the two processes $N^i$ and $M^i$ will become $N^i(n) = M^i(nC_i) - M^i((n-1)C_i)$.

# 5 Conclusion

We have explored functional properties of throughput in tandem production lines with unreliable machines and finite buffer capacities. For this purpose, we constructed two multifunctions to model the dynamics of this discrete event dynamic system and investigated their properties. Although some of our results have been part of the folklore, for example see Ho *et al.* (1983) about the monotonicity of throughput in buffer capacities in continuous tandem lines, as far as we are aware formal proofs have not appeared in the literature before. This type of results clearly provide qualitative insight and guidelines for planners. We have also given references to work reported elsewhere (Gürkan (1997)), where these results are actually used to show almost-sure convergence of a stochastic optimization algorithm, in a rigorous way.

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**Appendix A: Proofs of Lemmas from §2**

For $b \in \mathbf{R}_+^{m-1}$ write $F(b) = F_1(b) \cap F_2(b) \cap F_3(b) \cap F_4(b)$ where
$F_1(b) = \{g : g_1 \geq \ldots \geq g_m \geq 0\}$,
$F_2(b) = \{g : g_i \text{ is non-decreasing for each } i = 1, \ldots, m\}$,
$F_3(b) = \{g : g_i(0) = 0, |g_i(x) - g_i(y)| \leq C_i|x - y| \text{ for any } x, y \in [0, T] \text{ and } i = 1, \ldots, m\}$,
$F_4(b) = \{g : g_i(x) - g_{i+1}(x) \leq b_i \text{ for any } x \in [0, T], \text{ and } i = 1, \ldots, m-1\}$.

**Lemma 1** *The multifunction $F$ has the following properties:*
*a.* gph $F$ *is closed.*
*b.* gph $F$ *is convex.*
*c.* $F$ *is compact-valued and* $F(b) \subset F(\infty)$ *for all* $b \in \mathbf{R}^{m-1}$.

*Proof.* For (a), we take a sequence $\{(b^n, g^n)\}$ in gph $F$ that converges to a point $(b, g)$ and show that $(b, g) \in \text{gph} F$. Clearly, $g \in F_1(b) \cap F_2(b) \cap F_4(b)$. Take $\epsilon > 0$ and find a positive integer $N_\epsilon$ such that for all $n \geq N_\epsilon$, $t \in [0, T]$, and $i = 1, \ldots, m$, $\|g_i^n(t) - g_i(t)\| < \epsilon$. Then for all $x, y \in [0, T]$ and $i = 1, \ldots, m$,

$$\begin{aligned}
\|g_i(x) - g_i(y)\| &= \|g_i(x) - g_i^n(x) + g_i^n(y) - g_i(y) + g_i^n(x) - g_i^n(y)\| \\
&\leq \|g_i(x) - g_i^n(x)\| + \|g_i(y) - g_i^n(y)\| + \|g_i^n(x) - g_i^n(y)\| \\
&< 2\epsilon + C_i\|x - y\|.
\end{aligned}$$

Since $\epsilon$ can be made arbitrarily small, we must have $g \in F_3(b)$ as well. Hence gph $F$ is closed.
To prove (b), we take $(b, g), (a, h) \in \text{gph} F$ and $\lambda \in [0, 1]$. Clearly,

$$(1 - \lambda)g + \lambda h \in F_1((1 - \lambda)b + \lambda a) \cap F_2((1 - \lambda)b + \lambda a) \cap F_4((1 - \lambda)b + \lambda a).$$

For any $i = 1, \dots, m$ and $x, y \in [0,T]$,

$$\begin{aligned} \|[((1-\lambda)g_i(x) + \lambda h_i(x)] - [((1-\lambda)g_i(y) + \lambda h_i(y)]\| &\le (1-\lambda)\|g_i(x) - g_i(y)\| + \lambda\|h_i(x) - h_i(y)\| \\ &\le (1-\lambda)C_i\|x-y\| + \lambda C_i\|x-y\| \\ &= C_i\|x-y\|. \end{aligned}$$

Hence $(1-\lambda)g + \lambda h \in F_3((1-\lambda)b + \lambda a)$ as well.

Clearly, $F_1(b), F_2(b), F_3(b)$, and $F_4(b)$ are closed sets. Furthermore, for any $g \in F_3(b)$, any $x \in [0,T]$, and $i = 1, \dots, m$, $|g_i(x)| \le C_i|x| \le C_i T$. Hence for any $g \in F_3(b)$, $\|g\| \le \max_{i=1}^m C_i T$ and

$$\|g(x) - g(y)\| = \max_{i=1}^{m} |g_i(x) - g_i(y)| \le \max_{i=1}^{m} C_i\|x-y\| \text{ for any } x, y \in [0,T].$$

Then by the Arzelà–Ascoli theorem $F_3(b)$ is compact. Hence $F$ is compact-valued. Furthermore we have for all $b \in \mathbf{R}^{m-1}$, $F(b) \subset F(\infty)$. ▮

**Lemma 2** *The multifunction $F$ is Berge-usc in $\mathbf{R}^{m-1}$ and $b \mapsto F(b)$ is a continuous mapping from* $\operatorname{int}(\mathbf{R}_+^{m-1})$ *to compact subsets of $C([0,T], \mathbf{R}^m)$ with the metric topology induced by the Hausdorff distance.*

*Proof.* Since $\operatorname{gph} F$ is closed and for all $b \in \mathbf{R}^{m-1}$, $F(b)$ is a subset of the compact set $F(\infty)$, the multifunction $F$ is Berge-usc in $\mathbf{R}^{m-1}$ by the corollary to Theorem 7 in Section 7.1 of Berge (1963). Berge-usc implies that for any $\epsilon > 0$ and any $b \in \mathbf{R}^{m-1}$, there exists a $\delta > 0$ such that

$$e(F(b'), F(b)) < \epsilon \text{ for every } b' \text{ with } \|b' - b\| < \delta. \tag{5.1}$$

To see this, observe that $F(b) + \operatorname{int}(\epsilon\mathbf{B})$ is an open neighborhood of $F(b)$ and use the definition of Berge-usc.

Let $\epsilon > 0$ and take $b \in \operatorname{int}\operatorname{dom} F = \operatorname{int}(\mathbf{R}_+^{m-1})$ and $g \in F(b)$. By applying Theorem 1 of Robinson (1976) to the inverse multifunction $F^{-1}$, we can find $\delta(g) > 0$ such that $F^{-1}(g + \epsilon\mathbf{B}) \supset b + \delta(g)\epsilon\mathbf{B}$, i.e. if $\|b' - b\| < \epsilon\delta(g)$, then there exists $f \in F(b')$ with $\|g - f\| < \epsilon$. Notice that $\delta(g)$ depends on $g$; however for every $h \in F(b)$ one could always take $\delta(h) \ge \delta(g)(1 + \|h - g\|)^{-1}$, see p. 133 of Robinson (1976). If we let $K_g = \max_{h \in F(b)} \|h - g\|$ (which is attained since $F(b)$ is a compact set) and $\delta = \delta(g)(1 + K_g)^{-1}/\epsilon > 0$, then $\delta \le \delta(h)$ for all $h \in F(b)$. So for all $g \in F(b)$ and $b'$ with $\|b' - b\| < \delta$, there exists $f \in F(b')$ with $\|f - g\| < \epsilon$. This is equivalent to having $e(F(b), F(b')) < \epsilon$ if $\|b' - b\| < \delta$ which together with (5.1) gives the continuity of the mapping $b \mapsto F(b)$, for all $b \in \operatorname{int}(\mathbf{R}_+^{m-1})$ using the Hausdorff distance. ▮

**Lemma 3** *$A$ is closed in $F(\infty)$.*

*Proof.* Take a sequence $\{g^n\}$ in $A$ that converges to a function $g$ in $F(\infty)$. Assume that $g \notin A$. Then there exist $i$ and $j$ with $\lambda(\{t : g_i(t) = f_{ij}\}) < R_j^i$. Since $g \in F(\infty)$, each component of $g$ is continuous and non-decreasing. Therefore the set $\{t : g_i(t) = f_{ij}\}$ is actually an interval, say $[r,s]$. Choose $\delta > 0$ small enough so that $\lambda([r-\delta, s+\delta]) < R_j^i$, $g$ increases in $[r-\delta, r]$, and $g$ increases in $[s, s+\delta]$. Then $\epsilon := \min\{g_i(s+\delta) - f_{ij}, f_{ij} - g_i(r-\delta)\} > 0$. Since the $g^n \to g$ in the sup-norm, we have uniform convergence in each component. Hence there exists $N_\epsilon$ such that $|g_i^n(t) - g_i(t)| < \epsilon$ for all $n \ge N_\epsilon$ and $t \in [0,T]$. Take $t > s + \delta$, then for any $n \ge N_\epsilon$ we have

$$\begin{aligned} g_i^n(t) &> g_i(t) - \epsilon \\ &\ge g_i(t) - (g_i(s+\delta) - f_{ij}) \\ &\ge f_{ij}. \end{aligned}$$

Similarly, we can show that $g_i^n(t) < f_{ij}$ for any $t < r - \delta$ and $n \geq N_\epsilon$. So for any $n \geq N_\epsilon$, if $t \notin [r - \delta, s + \delta]$ then $g_i^n(t) \neq f_{ij}$. Therefore we have

$$\lambda(\{t : g_i^n(t) = f_{ij}\}) \leq \lambda([r - \delta, s + \delta]) < R_j^i,$$

by choice of $\delta$. This contradicts the fact that $g^n \in A$. ∎

## Appendix B: Proof of Theorem 1 from §2

**Theorem 1** *Suppose that the event times have no cluster point. Then for each finite time $T$, $Q_T(b) = q_m(T)$.*

*Proof.* Let $v_i(t)$ be the rate of machine $i$ at time $t$ under strategy $q$ and $v_i^g(t)$ be the rate of machine $i$ at time $t$ under strategy $g$. When $t$ is the time of an event, we take $v_i(t) = v_i(t^+)$.

Without loss of generality we assume $b_i > 0$ for each $i$ (otherwise we could combine two machines). Suppose there exists $g \in F(b) \cap A$ such that $g_m(T) > q_m(T)$. Let $\tau = \inf\{t : g_i(t) > q_i(t)$ for some $i\}$ where $\tau < T$. Suppose that $\{t_k\}$ is a sequence decreasing to $\tau$, such that for each $k$ there is an index $i_k$ with $g_{i_k}(t_k) > q_{i_k}(t_k)$. By using the pigeonhole principle we can find some $i$ such that for a subsequence $\{t_{k_j}\}$ we have $g_i(t_{k_j}) > q_i(t_{k_j})$ for each $j$. For simplicity, rename this sequence as $\{t_k\}$. Note that $g_i(\tau) = q_i(\tau)$ and $g_i(t) > q_i(t)$ for $t \in (\tau, \tau + \delta_0]$ for some $\delta_0 > 0$, by continuity of $g_i$ and $q_i$.

Under strategy $q$, machine $i$ cannot be under repair at time $\tau$. To see this, suppose it were not true; then under strategy $g$ machine $i$ must have finished the same repair by time $\tau$. So it must have begun the repair earlier, say at $t_0$, whereas under $q$ machine $i$ began its repair at time $t_1 > t_0$. But $q_i(t) < q_i(t_1)$ for $t < t_1$ (failures are operational only), so $g_i(t_0) = q_i(t_1) > q_i(t_0)$ which contradicts the definition of $\tau$.

By assumption, $\tau$ is not a cluster point of the event times. Since under $q$ the rate of machine $i$ changes only at an event time, there is $\delta_1 > 0$ such that in the interval $[\tau, \tau + \delta_1]$ that rate is constant, say $v_i^q$. We claim that $v_i^q < C_i$. To see this, observe that if it were not true, then we would have for all $\delta \in (0, \min\{\delta_0, \delta_1\})$

$$g_i(\tau + \delta) - q_i(\tau + \delta) = g_i(\tau) - q_i(\tau) + \int_\tau^{\tau+\delta} [v_i^g(t) - C_i]dt.$$

Since $g_i(\tau) = q_i(\tau)$, we would have $g_i(\tau + \delta) - q_i(\tau + \delta) \leq 0$ which contradicts the existence of $\delta_0$.

Therefore for small enough $\delta \in (0, \min\{\delta_0, \delta_1\})$ either

a) $q_i(t) = q_{i-1}(t)$ for $t \in [\tau, \tau + \delta]$

or

b) $q_i(t) = q_{i+1}(t) + b_i$ for $t \in [\tau, \tau + \delta]$;

since if neither (a) nor (b) occurs, then machine $i$ should be running at rate $C_i$ on $[\tau, \tau + \delta)$.

If (a) occurs, then for sufficiently large $k$

$$\begin{aligned} g_{i-1}(t_k) - q_{i-1}(t_k) &= g_{i-1}(t_k) - g_i(t_k) - [q_{i-1}(t_k) - q_i(t_k)] + g_i(t_k) - q_i(t_k) \\ &\geq g_i(t_k) - q_i(t_k) > 0. \end{aligned}$$

We get the first of these inequalities since $g_{i-1}(t_k) - g_i(t_k) \geq 0$ and $q_{i-1}(t_k) = q_i(t_k)$. The second inequality is a consequence of the choice of $\delta$. Now we can repeat the same argument for machine $i - 1$. Note that we must then have $g_{i-1}(\tau) = q_{i-1}(\tau)$ and this time we know that only (a) can occur. So we get the same property for $i - 2$, $i - 3, \ldots$. Eventually we reach machine 1 and a contradiction (since the first machine is never starved).

If (b) occurs, then for sufficiently large $k$

$$\begin{aligned} g_{i+1}(t_k) - q_{i+1}(t_k) &= g_{i+1}(t_k) - g_i(t_k) + q_i(t_k) - q_{i+1}(t_k) + g_i(t_k) - q_i(t_k) \\ &\geq g_i(t_k) - q_i(t_k) > 0. \end{aligned}$$

The first of these inequalities follows from $g_{i+1}(t_k) + b_i \geq g_i(t_k)$ and $q_i(t_k) - q_{i+1}(t_k) = b_i$. The second inequality is a consequence of the choice of $\delta$. Here again we must have $g_{i+1}(\tau) = q_{i+1}(\tau)$. Therefore we can repeat the above argument for machine $i+1$ and this time we know that (b) is the only possibility. So we get the same property for $i+2$, $i+3, \ldots$. Eventually we reach machine $m$ and a contradiction (since the last machine is never blocked). ∎

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