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Dynamical systems forced by shot noise as a new paradigm in the interest rate modeling

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Abstract. In this paper we give a generalized model of the interest rates term structure including Nelson-Siegel and Svensson structure. For that we introduce a continuous *m*-factor exponential-polynomial form of forward interest rates and demonstrate its considerably better performance in a fitting of the zero-coupon curves in comparison with the well known Nelson-Siegel and Svensson ones. In the sequel we transform the model into a dynamic model for interest rates by designing a switching dynamical system of the considerably reduced dimension $n < m$ generating the forward rate curves in form a càdlàg function. A system is described by *n*-th order linear differential equation driven by a stochastic or chaotic shot noise. From fitted forward rates we specify the parameters of the switching system and discuss perspectives of our models to produce term-structure forecasts at both short and long horizons.

Keywords: forward interest rates, shot noise processes, switching dynamical systems, chaotic Brownian subordination, chaotic maps

JEL classification: C13, C20 and C22

Disclaimer: The ideas presented below reflect the personal view of the author and are not necessarily identical to the official methodology used at WestLB AG

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1. Exponential-polynomial models of interest rates

We introduce an exponential-polynomial term structure model of interest rates by

$$f(t,Z) = c_l + f_0(t,Z), \quad f_0(t,Z) = \sum_{i=1}^{l-1} \varphi_i(t) \cdot \exp(-\gamma_i t) \tag{1}$$

where $z := \{c_1^{(1)},...,c_{p_{l-1}}^{(l-1)}, \gamma_1,...,\gamma_{l-1}\} \in Z \subset R^m$, $t > 0$ denotes time to maturity T, $\varphi_i(t) = \sum_{k=1}^{p_i} c_k^{(i)} t^{k-1}$ are polynomials of degree $p_i - 1$ with coefficients $c_k^{(i)}$; $k = 1,..., p_i$ ($p_i \in \{1,2,3,...\}$) and γ_i are positive real numbers. The total number of parameters in (1) is $m = \sum_{i=1}^{l-1} p_i + l$.

We note that the widely used Nelson-Siegel (N-S) [Nelson 1987] and Svensson (SV) [Svensson 1994] families of $f(t)$ can be easily derived from (1). Assuming $l = 2$ and $p_1 = 2$, $p_2 = 1$, i.e. dimensionality m=4, (1) leads to N-S forward rate curve

$$f_{NS}(t) = (c_1^{(1)} + c_2^{(1)} t) \cdot \exp(-\gamma_1 t) + c_2 \tag{2}$$

as well as the SV curve is given by

$$f_{SV}(t) = (c_1^{(1)} + c_2^{(1)} t) \cdot \exp(-\gamma_1 t) + c_1^{(2)} \cdot \exp(-\gamma_2 t) + c_3 \tag{3}$$

with $l = 3$, $p_1 = 2$ and $p_2 = 1$, $p_3 = 1$ i.e. m = 6.

Another special case of (1) is a curve of the exponentials mixture under $p_i = 1$, $\forall i$, i.e.

$$f_{EXP}(t) = \sum_{i=1}^{l-1} c_i \cdot \exp(-\gamma_i t) + c_l \tag{4}$$

We show that a performance of a new term structure (1) in a fitting of the yields is considerably higher of the well known Nelson-Siegel and Svensson ones. By other words a today's choice of the parameters in (1) is to be not limited by the state space $Z \subset R^4$ as for the N-S curve or $Z \subset R^6$ in the Svensson model, i.e. $Z \subset R^n$, $n \geq 6$.

The corresponding term structure of the bond prices will be then given by

$$B(t, Z) := \exp\left(-\int_0^t f(x, Z) dx \right) \text{ at } Z \subset R^n$$

2. ODE for the interest rate models

We establish that dynamics of the interest rates $f(t)$ in a model (1) follows a n-th order ODE

$$f^{(n)} + \beta_{n-1} f^{(n-1)} + \beta_{n-2} f^{(n-2)} + ... + \beta_1 f^{(1)} + \beta_0 f = 0 \tag{5}$$

where $n=\sum_{i=1}^{l-1} p_i + 1$, p_i is the multiplicity of the root γ_i, $i = 1,...,l$

of the corresponding characteristic polynomial

$$D(\gamma) = \gamma^n + \sum_{i=1}^{n-1} \beta_{n-i} \gamma^{n-i} \tag{6}$$

The coefficients of the ODE are given by the Vieta formula

$$\beta_{n-j} = (-1)^j \sum_{i_1<i_2<...<i_j}^{n} \gamma_{\tilde{i_1}} \gamma_{\tilde{i_2}} \cdot ... \cdot \gamma_{\tilde{i_j}} \tag{7}$$

with $\gamma_{\tilde{1}} = \gamma_{\tilde{2}} = ... = \gamma_{\tilde{p_i}} = \gamma_1$, $\gamma_{\widetilde{p_1+1}} = \gamma_{\widetilde{p_1+2}} = ... = \gamma_{\widetilde{p_1+p_2}} = \gamma_2$ and so on.

Example (Nelson-Siegel).

Recall that the Nelson-Siegel model corresponds to the case $l = 2$ and $p_1 = 2$. Then $n = p_1 + 1 = 3$, $\gamma_{\tilde{1}} = \gamma_{\tilde{2}} = \gamma_1$ and $\gamma_{\tilde{3}} = \gamma_2 = 0$. It follows that

$$\begin{aligned}
&(j=1): \beta_2 = (-1)^1 \left(\gamma_{\tilde{1}} + \gamma_{\tilde{2}} + \gamma_{\tilde{3}}\right) = -2\gamma_1 \\
&(j=2): \beta_1 = (-1)^2 \left(\gamma_{\tilde{1}}\gamma_{\tilde{2}} + \gamma_{\tilde{1}}\gamma_{\tilde{3}} + \gamma_{\tilde{2}}\gamma_{\tilde{3}}\right) = \gamma_1^2 \\
&(j=3): \beta_0 = (-1)^3 \gamma_{\tilde{1}}\gamma_{\tilde{2}}\gamma_{\tilde{3}} = 0
\end{aligned}$$

Thus, the corresponding differential equation is

$$f''' - 2\gamma_1 f'' + \gamma_1^2 f' = 0.$$

It is easy to check that the Nelson-Siegel curve $f(t) = c_1 + (c_2 + c_3 t) \cdot \exp(-\gamma_1 t)$

is a general solution of the ODE as a combination of the two obvious particular solutions of $r' = 0$ and $r'' - 2\gamma_1 r' + \gamma_1^2 r = 0$, such that

$$c_1 = f(0) + \frac{2f'(0)}{\gamma_1} + \frac{f''(0)}{\gamma_1^2}, c_2 = -2\frac{f'(0)}{\gamma_1} - \frac{f''(0)}{\gamma_1^2}, c_3 = -f'(0) - \frac{f''(0)}{\gamma_1} \quad . \tag{8}$$

Example (Svensson).

By analogy to the previous example the SV curve follows a 4th order ODE in the form

$$f^{(IV)} - (2\gamma_1 + \gamma_2) f''' + (\gamma_1^2 + 2\gamma_1\gamma_2) f'' - \gamma_1^2\gamma_2 f' = 0$$

Generally ODE (5) can be presented in a matrix form

$$\frac{d}{dt}\mathbf{f} = \mathbf{F}(\boldsymbol{\gamma})\cdot\mathbf{f},\ 0 \le t \le T \tag{9}$$

where f $= \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}$, $\boldsymbol{\gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_l \end{pmatrix}$, F $= \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -\beta_0(\boldsymbol{\gamma}) & -\beta_1(\boldsymbol{\gamma}) & \cdots & \cdots & \cdots & -\beta_{n-1}(\boldsymbol{\gamma}) \end{pmatrix}$ and

$f_i \equiv \frac{d^i}{dt^i} f(t)$.

Eq. (9) has a vector solution $\mathbf{f}(t) = e^{\mathbf{F}(\gamma)\cdot t}\cdot\mathbf{f}(0)$.

Define by

$$r(t,\tau) := f(t, Z_\tau) \tag{10}$$

the instantaneous forward rate at time *t* for date τ and by

$$r(0,\tau) = \lim_{t\to 0} r(t,\tau) = f(0, Z_\tau) \tag{11}$$

the short rates, where stochastic process Z_τ with values in $Z \subset R^m$ contains two groups of processes $\{c_1^{(1)}(\tau),\dots,c_{p_{l-1}}^{(l-1)}(\tau)\}$ and $\{\gamma_1(\tau),\dots,\gamma_{l-1}(\tau)\}$ such that the processes $c_i^j(\tau)$ depend on the stochastic processes $\{\gamma_1(\tau),\dots,\gamma_{l-1}(\tau)\}$ and the short rates (initial conditions) as shown by (8) for a Nelson-Siegel term structure.

Taking into account (9), (10) the evolution of the forward rates on the date τ is given by

$$\frac{d}{dt}\mathbf{r} = \mathbf{F}(\boldsymbol{\gamma}_\tau)\cdot\mathbf{r} \tag{12}$$

with the given vector $\boldsymbol{\gamma}_\tau$ and initial state vector $\mathbf{r}(0,\tau)$.

Assuming Z_τ is given τ – evolution of $r\,(t, \tau)$ can be described by two different ways. First one presents τ – evolution of $r\,(t, \tau)$ as a train of curves (12) for dates τ = 1, 2,… in a form (see Fig. 1)

$$\frac{d}{dt}\mathbf{r} = \mathbf{F}(\gamma_\tau)\cdot\mathbf{r} + \sum_i \xi_i \delta(t - i\cdot T), \quad 0 \le t \le \infty$$

or equivalently

$$\frac{d}{dt}\mathbf{r} = \mathbf{F}(\gamma_\tau)\cdot\mathbf{r} + \xi_\tau \delta(t - \tau\cdot T), \tag{13}$$

where $\tau = \left\lfloor \frac{t}{T} \right\rfloor + 1$ is a counting process ($\lfloor t \rfloor$ denotes the floor function (largest integer smaller or equal to t) and

$$\xi_\tau = \mathbf{f}(0, \mathbf{Z}_{\tau+1}) - \mathbf{f}(T, \mathbf{Z}_\tau). \tag{14}$$

We call the vector ξ_i as the stochastic amplitude of the impulse perturbation, which acts on system (13) at times $t = \tau T$, τ = 1,2,3, …. such that

$$r(\tau\cdot T + 0, \tau) = r(\tau\cdot T - 0, \tau) + \xi_\tau .$$

Thus r is a cadlag function, i.e. a right continuous function $r(\tau\cdot T + 0, \tau) \equiv r(\tau\cdot T, \tau)$, defined on R^n and has a left limit.

Between kicks iT and $(i+1)T$ a state vector is governed by the homogeneous system of linear differential equations (12) at $\tau = i+1$.

Fig. 1 illustrates the system (13) with a jump $\xi_1 \equiv f(0, Z_2) - f(T, Z_1)$ at $t = T$

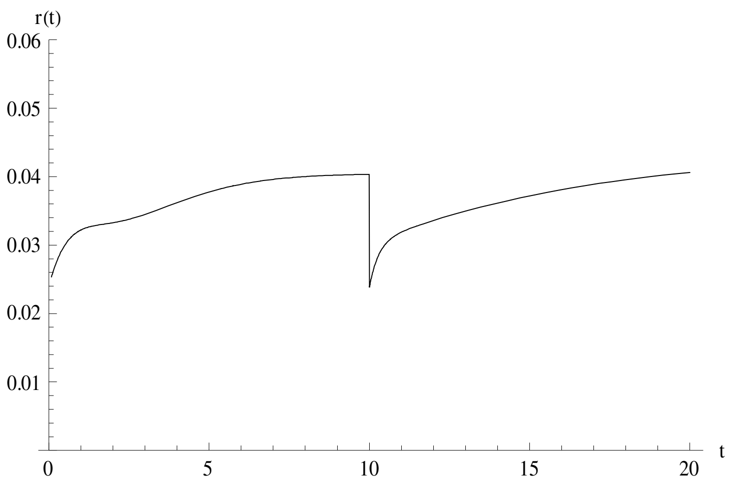


Fig.1 Train of the two first curves with T = 10 years.

The solution to (13) is explicitly given by

$$r = e^{F(\gamma_\tau)\cdot(t-(\tau-1)T)} \cdot \left[\sum_{i=1}^{\tau} \exp\left(T \cdot \sum_{j=i}^{\tau} F(\gamma_j) \right) \times \xi_{i-1} \right]$$

where we denote $\xi_0 \equiv r(0)$.

The second approach is based on a simple idea to express a random variable $\mathbf{f}(t, \mathbf{Z}_\tau)$ by $\mathbf{f}\left(t, \mathbf{Z}_{\tau_{fixed}}\right) + \zeta(t)$ in the interval $0 \le t \le T$. By other words we assume that dynamics of the interest rates r (t, τ) can be modelled by

$$d\mathbf{r} = \mathbf{F} \cdot \mathbf{r}\, dt + d\boldsymbol{\zeta}, \quad 0 \le t \le T, \forall \tau \qquad (15)$$

where $\mathbf{F}$ is a n x n matrix with the constant coefficients $\beta_i \equiv \beta_i\left(\boldsymbol{\gamma}_{fixed}\right)$ for any τ . The model (15) generates a predicted term structure, whose exponential-polynomial shape depends on the model parameters and the initial short rate. One can show that (15) is more general and includes a class of equilibrium models such as Vasicek, CIR, lognormal models.

3. Approach I – demonstrating example: from estimating the yield curve to its dynamic modelling

First we empirically estimate process Z_τ. For that we are going to fit the default-free yield spreads, downloaded from the Reuters database. The observable time period is 23.02. 2006 – 14.01. 2008, i.e. contains q = 478 dates. In framework of the above exponential-polynomial approach we introduce a term structure model of yields curves by

$$Y(t, Z) := \frac{1}{t} \int_0^t f(x, Z) dx \qquad (16)$$

(16) can be easily done in a closed form. For this we need the relation [Prudnikov 1981]

$$\int_0^t x^k e^{-\gamma x} dx = -M_k(t) e^{\gamma t} + \frac{k!}{\gamma^{k+1}}, k = 0,1,2,..., \text{where } M_k(t) = \sum_{i=0}^{k} \frac{k!}{(k-i)!} \frac{t^{k-i}}{\gamma^{i+1}},\ \gamma < 0 .$$

Substituting the model (1) into (16) leads to

$$Y(t,Z) = c_l + \frac{1}{t} \sum_{i=1}^{l-1} \sum_{k=1}^{p(i)} c_k^{(i)} \left[-M_{k-1}(t) + \frac{k-1!}{\gamma^k} \right] \tag{17}$$

We introduce a minimization criterion as:

$$\rho(\tau) = \frac{1}{N} \sum_i \left(s_i(\tau) - Y_\tau(t_i, Z) \right)^2 \to \min_Z \tag{18}$$

where $\{s_i(\tau),\ i = 1,2,...,N_\tau\}$ are quotes of the yields on the base date τ. The cost function $\rho(\tau)$ is to be minimized by the appropriate choice of the $m = \sum_{i=1}^{l-1} p_i + l$ parameters of the state space Z. It is clear that in the fitting problem the following restriction

$$m = \dim(Z) = \sum_{i=1}^{l-1} p_i + l \le \min(\{N_\tau, \tau = 1,...,q\}) \tag{19}$$

is to be provided.

Given the number l of the parameters γ_l, $i = 1,...,l$ and distribution of their multiplicities p_l, $i = 1,...,l$ such that the condition (19) holds the nonlinear regression technique for the least squares criterion (18) leads to the minimum of the cost function with the optimal parameters $\{c_1^{(1)},...,c_{p_{l-1}}^{(l-1)}, \gamma_1,...,\gamma_{l-1}\} \in Z \subset R^m$.

We extend the criterion (7) by

$$\overline{\rho} = \frac{1}{q} \sum_{\tau=1}^{q} \rho(\tau) \to \min_{l, p_1,...,p_l} \tag{20}$$

with the obvious restrictions for the parameters

$$l,\ p_1, \ldots, p_l \in Z^+ = \{1,2,3,...\}. \tag{21}$$

We note that (20) according to a law of large numbers/ Birkhoff ergodic theorem approaches the mean of the stochastic/ chaotic cost function with $q \to \infty$.

The problem (18)-(21) for euro swap rates has the following twofold solution

1) $l=5,\ p_l=1,\ \ p_i=3,\ i=1,...,l-1$ and $\{c_1^{(1)},...,c_{p_{l-1}}^{(l-1)},\gamma_1,...,\gamma_{l-1}\}\in Z\subset R^{17}$ (m =17, $N_\tau=60,\ \forall\tau$) for a "laminar" period of the observed yields quotes on the bond market: 23.02.06 $(\tau=1)$ – 17.07.07$(\tau=355)$

2) $l=5,\ p_l=1,\ \ p_{l-1}=1, p_i=3,\ i=1,...,l-2$ and $\{c_1^{(1)},...,c_{p_{l-1}}^{(l-1)},\gamma_1,...,\gamma_{l-1}\}\in Z\subset R^{15}$ (m=15) for a "turbulence" period 18.07.07$(\tau=356)$–14.01.08 $(\tau=478)$.

The above calculated $m+l+1$ parameters specify a general term structure model of interest rates by the exponential-quadratic curves (1) ($p_l=3$) as well as a general term structure model of yields by the exponential-cubic curves (17).

Fig. 2 demonstrates a performance of the general model for the Euro swap rates by comparison of its cost function (18) with the cost functions of the conventional Nelson-Siegel and Svensson models and exponential model (4) with $l=6$.

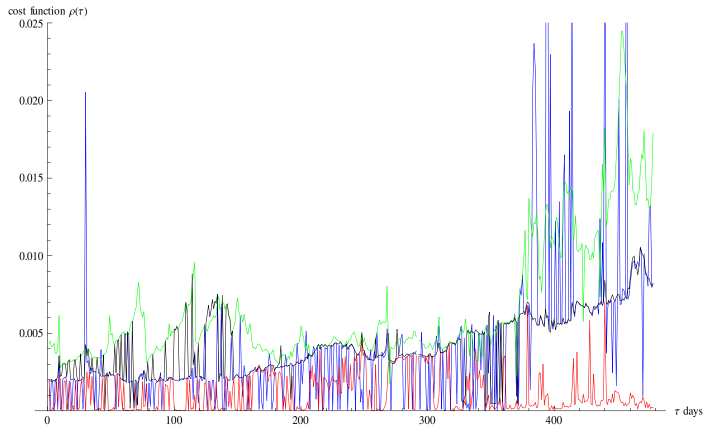


Fig. 2 Comparison of the cost functions for the observable time period: 23.02. 2006 – 14.01. 2008. N-S is a green curve, SV is a black, the exponential is a blue, and the general model is a red curve.

Moreover the ratios

$$\frac{\bar{\rho}_{NS}}{\bar{\rho}_{GEN}}=\frac{0.00653394}{0.00074}\approx 9,\quad \frac{\bar{\rho}_{SV}}{\bar{\rho}_{GEN}}=\frac{0.00419495}{0.00074}\approx 6 \text{ and } \frac{\bar{\rho}_{EXP}}{\bar{\rho}_{GEN}}=\frac{0.0039557}{0.00074}\approx 5 \quad (22)$$

quantify the performance of the general model with the derived optimal exponential-cubic curve. Thus our yield curve fitting is about 9 and 6 times better than conventional N-S and SV one, as well as 5 times better than an exponential curve (4) with $l = 6$, respectively.

The mean of the cost function (20) for the exponential model (4) has a local minimum at l = 6 in value $\overline{\rho}_{EXP} = 0.0039$ and at l =5 in value $\overline{\rho}_{GEN} = 0.00074$ for a general model as shown in Fig. 3.

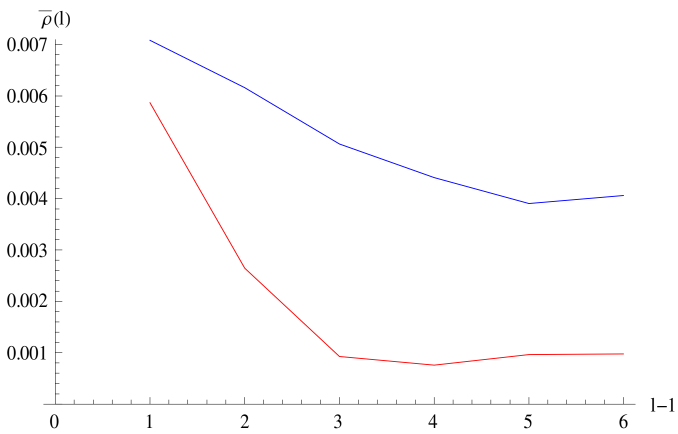


Fig. 3. The means of the cost functions: blue curve – exponential and red one – general model

Fig. 4 collects all base curves fitted to the available data on the date 25.03.08 .

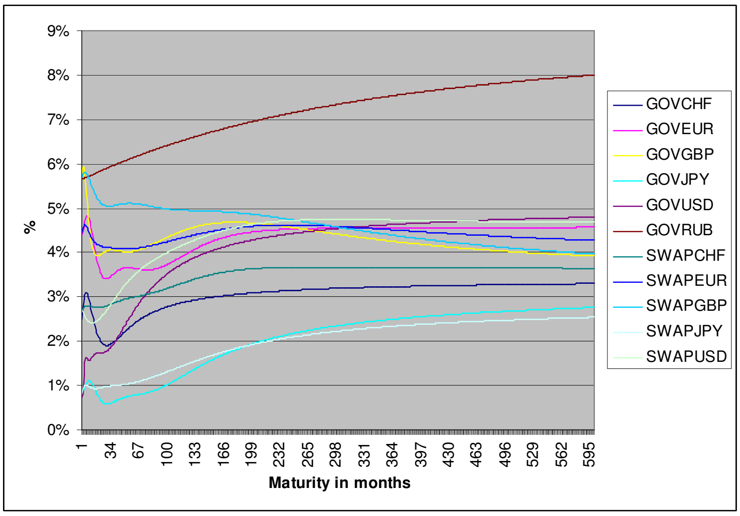


Fig. 4 Fitted zero curves on the spot date 25.03.08

Repeating a fitting procedure to the another date, let's say τ +1 = 26.03.08, we get a similar set of the base curves (16) which can be described by the system (13), where the impulse perturbation ξ_τ (14) is to be predetermined.

As an example we design a dynamical system for N-S instantaneous forward rate curve $r(t,\tau) = c_1(\tau) + (c_2(\tau) + c_3(\tau)t) \cdot \exp(-\gamma_\tau t)$ described by the following low-dimensional ODE

$$r''' - 2\gamma_\tau\, r'' + \gamma_\tau^2\, r' = \xi_\tau\, \delta(t - \tau \cdot T) \qquad (23)$$

where

$$\xi_\tau = c_1(\tau+1) + c_2(\tau+1) - c_1(\tau) - (c_2(\tau) + c_3(\tau)T) \cdot \exp(-\gamma_\tau T)\,. \qquad (24)$$

according to (10) and (14).

From the output of the above fitting procedure we retrieve the time series $\{c_1(\tau), c_2(\tau), c_3(\tau), \gamma_\tau | \tau = 1,2,\ldots, q = 478\}$ for the observable time period.

Applying (24) we immediately get time series of stochastic perturbation.

Let us introduce the *k*-th order increments for both processes ξ_τ, γ_τ by

$$\Delta^{k+1}\xi_\tau = \Delta^k \xi_{\tau+1} - \Delta^k \xi_\tau \qquad (25)$$

$$\Delta^{k+1}\gamma_\tau = \Delta^k \gamma_{\tau+1} - \Delta^k \gamma_\tau\,, \qquad (26)$$

where $\Delta^0 \equiv 1$, $k = 0, 1, 2, \ldots$.

We are now able to do an elementary statistical analysis of both processes ξ_τ, γ_τ.

Table 1 contains the histograms of the processes and Table 2 collects the histograms of the increments.

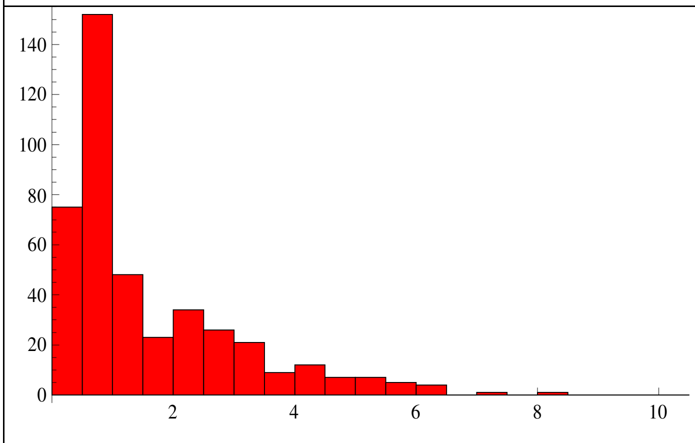
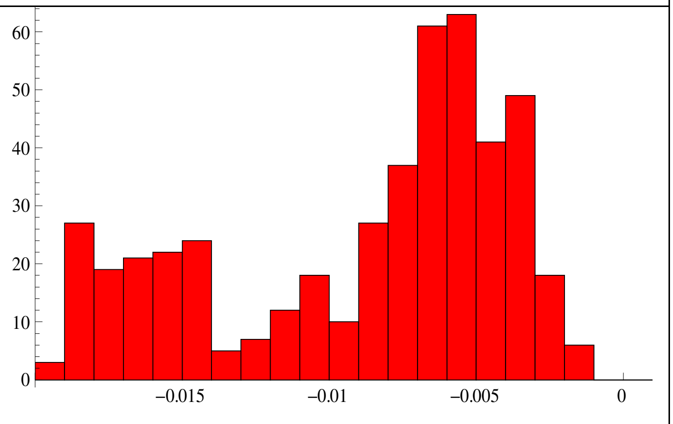
Histogram of γ_τ	Histogram of ξ_τ
	

Table 1.

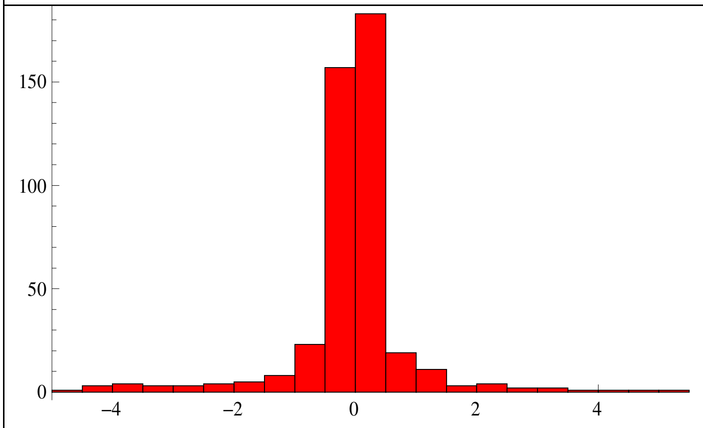
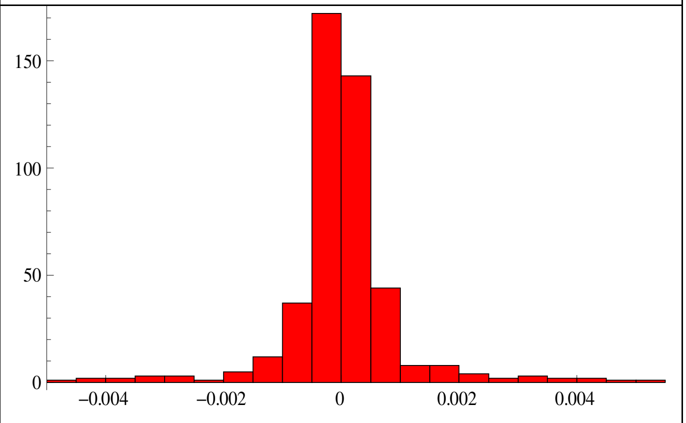
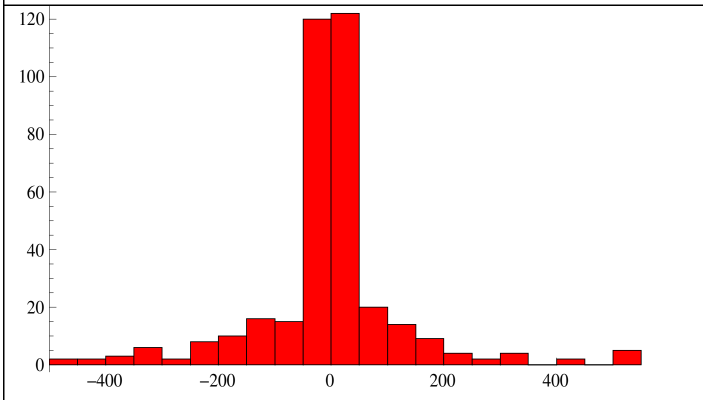
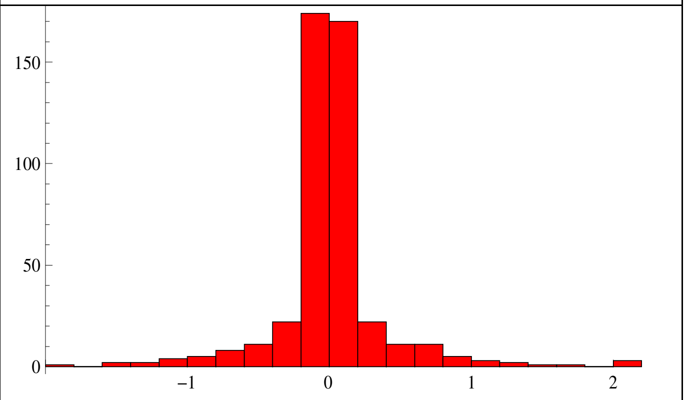
Histograms of the increments $\Delta^k \gamma_\tau$, $k = 1, 10$	Histograms of the increments $\Delta^k \xi_\tau$, $k = 1, 10$
	
	

Table 2.

We note that the both processes are diffusion processes characterized by symmetrical bell-like but not-Gaussian distributions of their increments. A χ^2 Pearson's test with the confidence level 0.95 rejects a hypothesis of the independence of the increments for both forced signal ξ_τ and γ_τ.

The means of $\Delta^k \gamma_\tau$ and $\Delta^k \xi_\tau$, $\forall k \geq 1$ are zeros. The sample variances $\sigma^2_{\Delta^k \gamma}, \sigma^2_{\Delta^k \xi}$ grow exponentially with the order k as shown in Fig. in a log scale

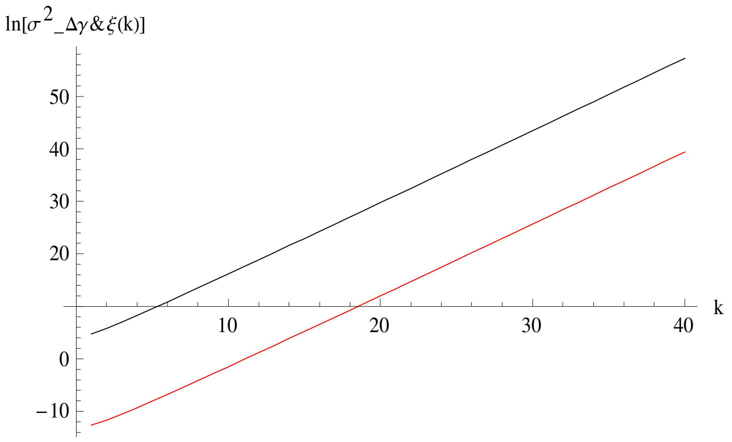


Fig.5. Red line is $\ln\sigma^2_{\Delta^k\xi}$, black line is $\ln\sigma^2_{\Delta^k\gamma}$

A distance between the above two lines does not remain constant with *k* but grows slowly with rate $\frac{d}{dk}\ln\frac{\sigma^2_{\Delta^k\gamma}}{\sigma^2_{\Delta^k\xi}}\underset{k\to\infty}{\to}0.00594878$. It means that a variance of the $\Delta^k\gamma_\tau$ grows a bit quicker than a variance of $\Delta^k\xi_\tau$, $\forall k\geq 1$.

The sample autocorrelation functions

$$C_{\Delta^k\gamma}(j)=\frac{\left(\sigma^2_{\Delta^k\gamma}\right)^{-1}}{q-k-j}\sum_{\tau=1}^{q-k-j}\Delta^k\gamma_\tau\Delta^k\gamma_{\tau+j},\ j=0,1,2,... \tag{27}$$

$$C_{\Delta^k\xi}(j)=\frac{\left(\sigma^2_{\Delta^k\xi}\right)^{-1}}{q-k-j}\sum_{\tau=1}^{q-k-j}\Delta^k\xi_\tau\Delta^k\xi_{\tau+j},\ j=0,1,2,.. \tag{28}$$

are presented in Table 3.

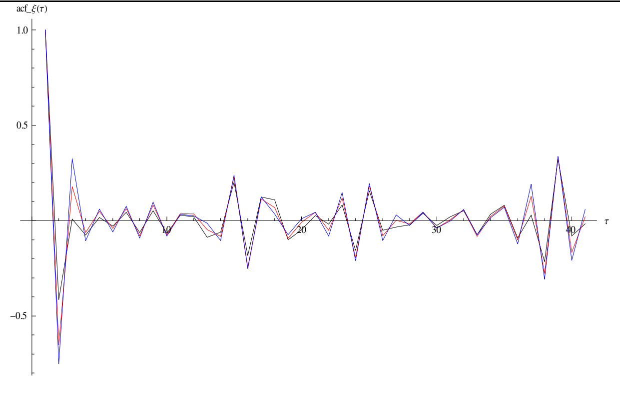
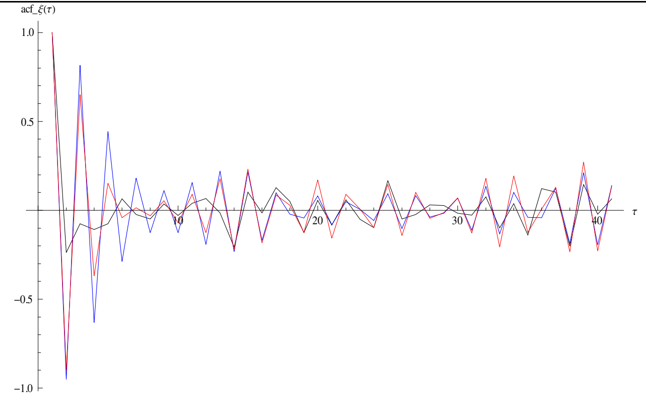
Autocorrelations of $\Delta^k \gamma_\tau$, $k = 1, 10, 20$	Autocorrelations of $\Delta^k \xi_\tau$, $k = 1, 10, 20$
	

Table 3.

To estimate a mutual correlation of ξ_τ and γ_τ and their increments we introduce the Pearson product-moment correlation coefficient [Norman L. Johnson 1995]

$$R(j,k) = \frac{\sigma^{-1}_{\Delta^k \gamma} \sigma^{-1}_{\Delta^k \xi}}{q-k-j} \sum_{\tau=1}^{q-k-j} \Delta^k \xi_\tau \cdot \Delta^k \gamma_{\tau+j}; \; j = 0,1,...; \; k = 0,1,... \qquad (29)$$

depicted in Figures 6 – 8.

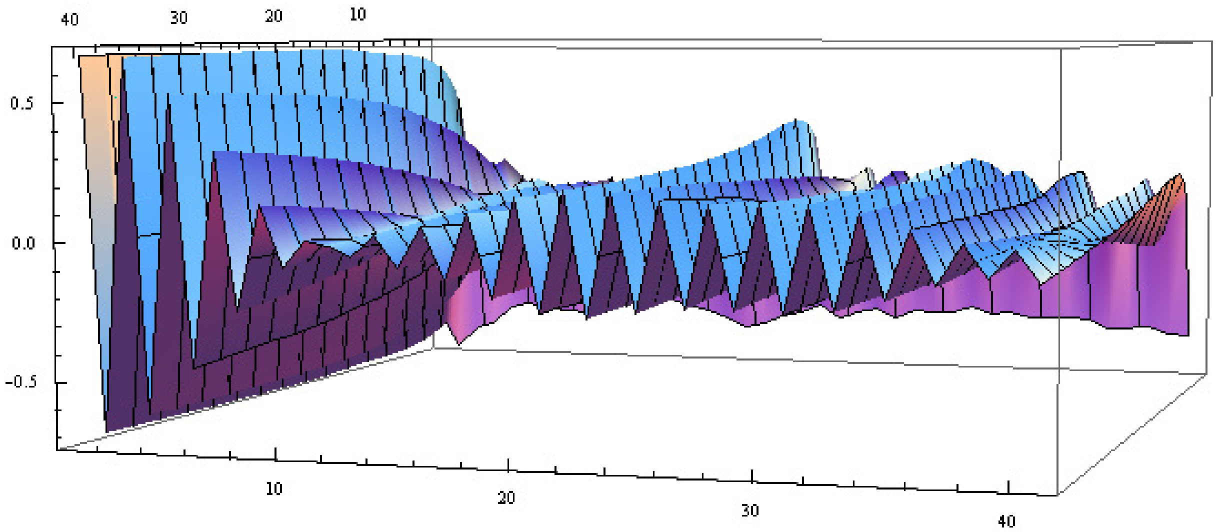


Fig. 6 . Geometric interpretation of the matrix R of dimension 40x40

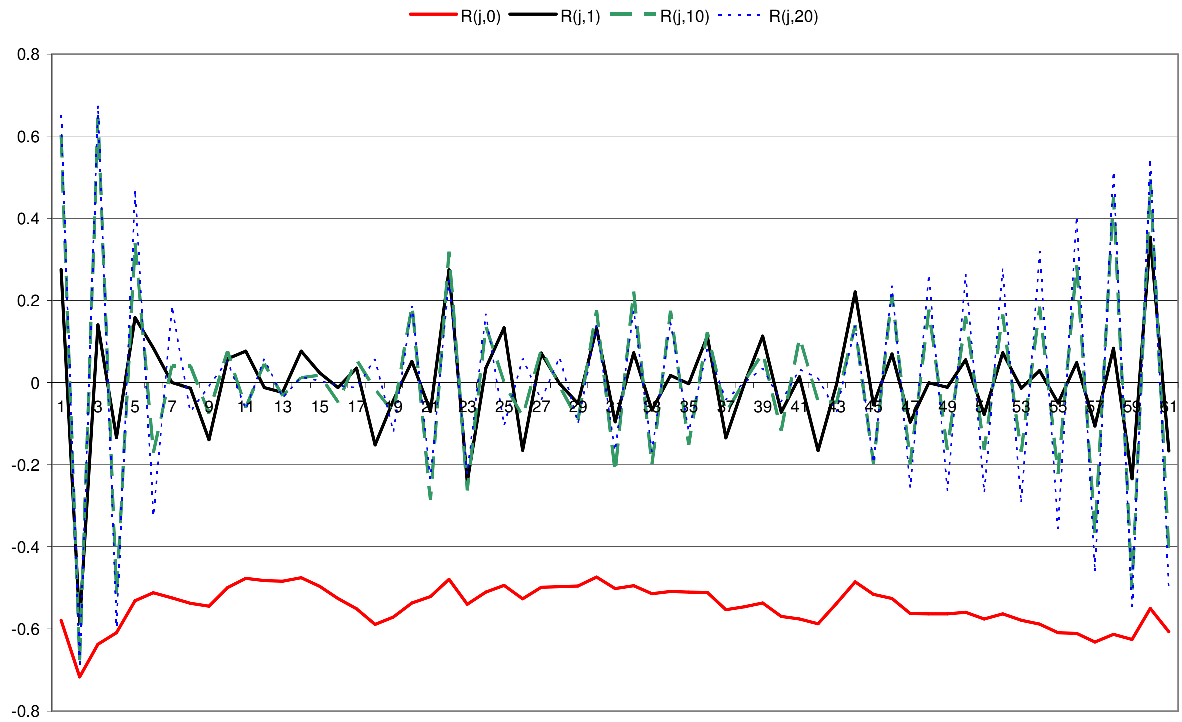


Fig. 7 The Pearson's correlation coefficient at $j = 0,1,\ldots,60$; $k = 0,\ 1,\ 10,\ 20$

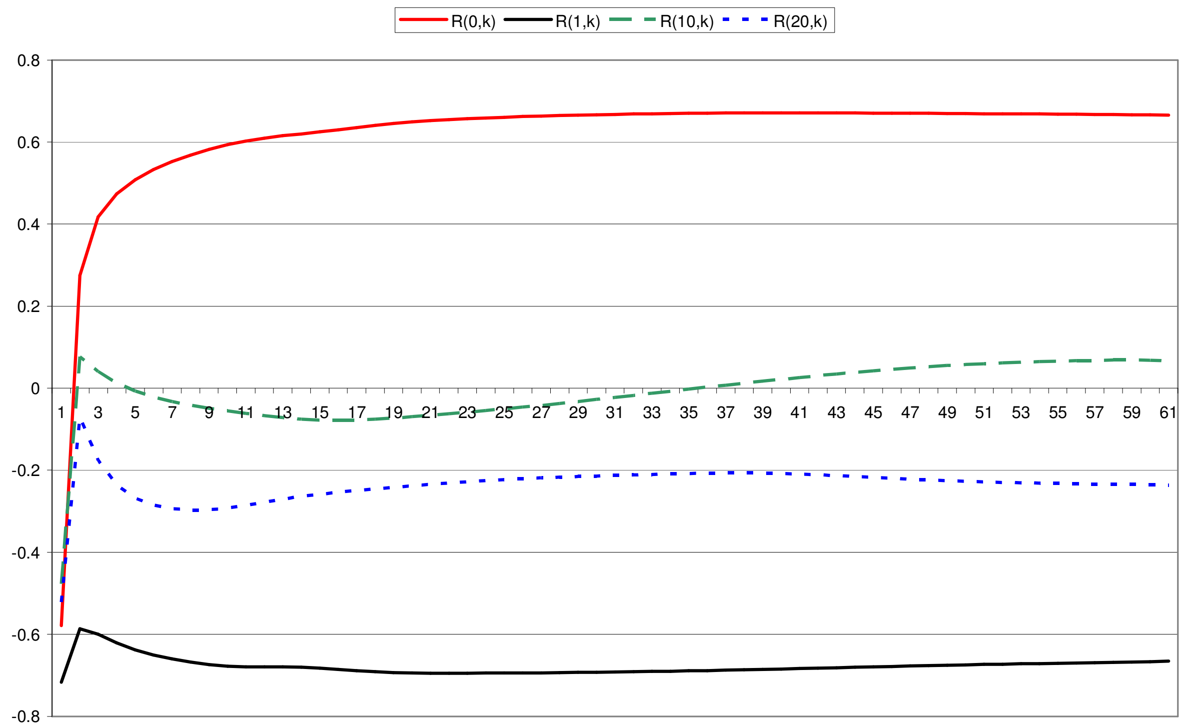


Fig. 8 The Pearson's correlation coefficient at $j = 0,1,10,20$; $k = 0,\ 1,\ \ldots,\ 60$

4. Approach II – forcing signal as a shot noise

We note that applying Vieta formula (7) at $j = n$ the coefficient β_0 is equal to zero since $\gamma_l = 0$. It implies that a general solution of the ODE (5) is a combination of the obvious solution of $f' = 0$ and the particular solution $\eta(t)$ of

$$f^{(n-1)} + \beta_{n-1} f^{(n-2)} + \beta_{n-2} f^{(n-3)} + ... + \beta_1 f = 0 \tag{30}$$

i.e. $f = r \equiv \eta + c$.

We are specifically interested in the behaviour of the system (30) with a shot noise as a chaotic/stochastic perturbation, i.e.

$$\frac{d}{dt}\boldsymbol{\eta} = \mathbf{F} \cdot \boldsymbol{\eta} + \sum_{i}^{N(t)} \mathbf{A}_i \delta(t - t_i), \quad 0 \le t \le T \tag{31}$$

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-2} \end{pmatrix}, \; \mathbf{F} = \begin{pmatrix} 0 & 1 & & \cdots & 0 & 0 \\ 0 & 0 & & 0 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -\beta_1 & & \cdots & \cdots & -\beta_{n-2} & -\beta_{n-1} \end{pmatrix}.$$

where t_k are successive occurrence or arrival times of δ-impulses,

$$t_0 (= 0) < t_1 < ... < t_k < ... < T$$

$N(t) = \max\{k : t_k \le t\}$ is a counting process.

The impulse perturbation acts on system (31) at times $t = t_k$, $k = 1,2,3, ….$ such that

$$\boldsymbol{\eta}(t_k + 0) = \boldsymbol{\eta}(t_k - 0) + \mathbf{A}_k . \tag{32}$$

In sequel we assume that η is a càdlàg function.

We introduce the positive inter-arrival times T_k such that

$$t_k = t_{k-1} + T_k = \sum_{i=1}^{k} T_i$$

Between kicks a state vector is governed by the following homogeneous system of linear differential equations

$$\frac{d}{dt}\boldsymbol{\eta} = \mathbf{F} \cdot \boldsymbol{\eta}$$

and the initial condition of the system $\boldsymbol{\eta}_0 \equiv \boldsymbol{\eta}(t_0 + 0)$ defines an evolution of a state vector (Cauchy theorem).

4.1. One-dimensional case

We consider a special case of (31) in a form of one-dimensional ODE:

$$\frac{d}{d\,t}\eta = \sum_{i}^{N(t)} A_i \delta(t - t_i), \quad 0 \le t \le T \tag{33}$$

4.1.1. Response to a Shot Noise. The generalized Wiener process

Integrating Eq. (33) we immediately get a solution

$$\eta(t) = \sum_{k=1}^{N(t)} A_k \equiv \eta_{N(t)} \quad (\eta(0) = 0) \tag{34}$$

A plot of the process η is depicted in Fig.

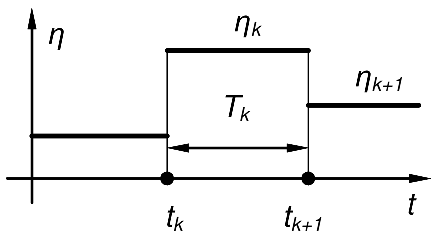


Fig. 9. A solution of Eq. (1)

Thus the process $\eta(t)$ (Fig. 9) is a rectangular signal with step heights η_k satisfying the relation:

$$\eta_k = \eta_{k-1} + A_k \tag{35}$$

Probability density function

We first establish that the distribution function of $\eta(t)$ is

$$P\left(\eta(t)\le z\right)=\sum_{k=1}^{\infty}P_k(t)P\left(\eta_k\le z\right),$$

where $P_k(t)=P\left(t_{k-1}\le t<t_k\right)=P\left(N(t)=k\right)$.

We assume zero mean for the magnitudes A_k . It immediately implies that $\eta(t)$ is a martingale with a zero mean

$$E(\eta)=E\left(N(t)\right)E(A)=0\ \ \left(\bar{A}=0\right)\ as\ E\left(\eta\mid N(t)\right)=N(t)E(A) \tag{36}$$

To calculate a variance of η (t) we use a law of total variance

$$D(\eta)=E\left[D\left(\eta\mid N(t)\right)\right]+D\left[E\left(\eta\mid N(t)\right)\right]$$

One can show that the conditional variance is

$$D\left(\eta\mid N(t)\right)=N(t)D(A)+2\sum_{n=1}^{N(t)-1}(N(t)-n)c_A(n) \tag{37}$$

where $c_A(k)=E\left(A_iA_{i+k}\right)$ is the autocorrelation function (acf) and $\sigma_A{}^2\equiv D(A)$ is the variance of the A_k.

Hence, for i.i.d. random or uncorrelated chaotic magnitudes A_k ($c_A(k)$=0) we have

$$D(\eta)\big|_{c_A(n)=0}=D(A)E\left(N(t)\right)+D\left(N(t)\right)E^2(A)=D(A)\bar{N}_t \tag{38}$$

where $\bar{N}_t\equiv H(t)=\sum_k kP\left\{N(t)=k\right\}=\sum_k P\left\{t_k<t\right\}\underset{t>>\bar{t}}{\simeq}\frac{t}{E\left(t_k\right)}$ is the intensity function.

For correlated random/chaotic A_k we assume that the first moment of the autocorrelation function $c_A(k)$ is finite

$$\sum_{k=1}^{\infty}k\left|c_A(k)\right|<\infty \tag{39}$$

and then the variance is given by

$$\sigma_{\eta_k}^2=k\sigma_A^2+o(1). \tag{40}$$

Let us introduce a new variable $\varepsilon_k = \dfrac{\eta_k - \overline{\eta}_k}{\sigma_{\eta_k}}$ with $E(\varepsilon_k)=0$ and $E(\varepsilon_k{}^2)=1$. It can be shown that ε_k converges in distribution to the standard normal law, i.e. the central limit theorem holds both with i.i.d.random [Feller] and chaotic magnitudes A_k [Chernov 1995]. In [Baranovski 2003], authors have presented the analytical expressions for the characteristic functions of the chaotic partial sums η_k of the magnitudes A_k generated by PWL onto maps and shown their fast convergence to the limit $\exp(-\omega^2/2)$.

We consider a piecewise constant function $W_k(t)$ on $t \in [0,1]$ such that

$$W_L(t) = \frac{\eta_{\lfloor kt \rfloor}}{\sqrt{D(A)}\sqrt{L}} = \frac{1}{\sqrt{D(A)}\sqrt{L}} \sum_{i=1}^{\lfloor kt \rfloor} A_i, \quad t \in [0,1], k = 0,1,\ldots,L \tag{41}$$

where $\lfloor x \rfloor$ is the floor function (it gives the greatest integer less than or equal to *x*).

Then for any k $\{W_k\}$ induces a measure on the space of continuous functions on [0,1] . According to the invariance principle this measure converges weakly, as $k \to \infty$, to the Wiener process W [Chernov 1995] Fig. 10 depicts examples of functions $\{W_k\}$ for different *k* when the magnitudes A_k are chaotic variables generated by a tent map on [-1,1]:

$$A_{n+1} = 1 - 2|A_n|, n = 1,2,\ldots \tag{42}$$

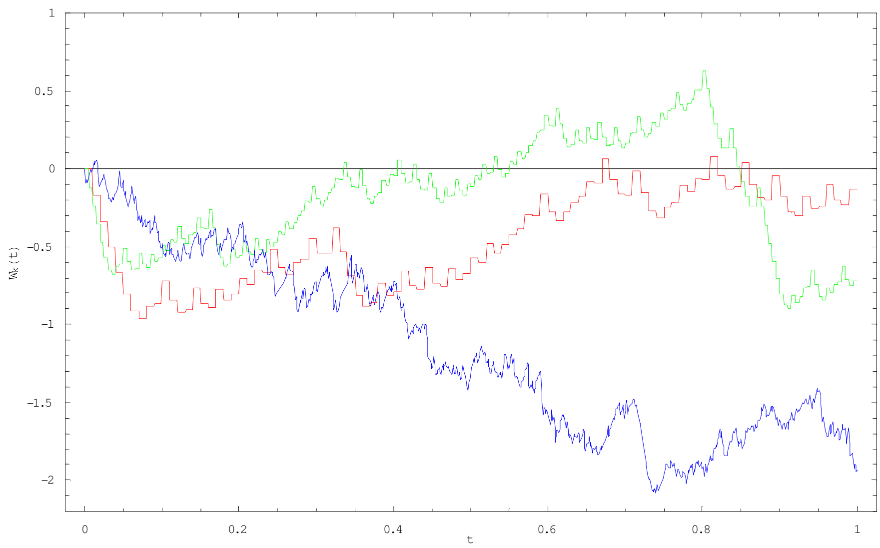


Fig. 10 Three realizations of the process *W* for *k* =100,300 and 10000 (red, green and blue line)

The weak invariance principle known also as the functional central limit theorem provides an approximation deterministic dynamical systems by a Brownian motion on large space and time scales.

Thus the distribution of $\eta(t)$ tends to the Gaussian law with the mean (36) and variance (38).

This confirms the diffusion character of $\eta(t)$. It follows that the Eq. (33) can be used for stochastic and chaotic modeling of the Wiener process.

Example 1. Valuation of the European call option.

The underlying asset of the European option is assumed to grow at the constant risk-free rate r perturbed by a stochastic/chaotic marked point process $\eta(t)$. Thus an asset price is modeled as

$$\frac{dS}{S} = rdt + d\eta \tag{43}$$

Properties:

1) Markov property: the next asset price ($S+dS$) depends solely on today's price

2) The next value for S is higher than the old by an amount

$$E(dS) = rSdt \ (as \ E(d\eta) = 0)$$

3) Variance of dS is

$$D(dS) = E(dS^2) - E^2(dS) = E\left(S^2(d\eta)^2\right) = S^2 D(d\eta) = S^2 D(A) dH(t)$$

We want to price a call option, i.e.

$$C(t,K) = e^{-r\cdot t} E\left[(S-K)^+\right] = e^{-r\cdot t} \sum_k P(N(t)=k) \cdot E\left[(S-K)^+ \mid N(t)=k\right], \tag{44}$$

where K is a strike price. We calculate a conditional expectation

$$\begin{aligned}
&C_k(t,K) = E\left[(S-K)^+ \mid N(t)=k\right] = \int_0^\infty (x-K)^+ \partial P\left(S \le x \middle| N(t)=k\right) = \\
&\int_K^\infty (x-K)\cdot \partial_x P\left(S_0 e^{\tilde{r}\cdot t+\eta(t)} \le x \middle| N(t)=k\right) = \int_K^\infty (x-K)\cdot \partial_x P\left(\eta_k \le \ln\left(\frac{x}{S_0}\right) - \tilde{r}\cdot t\right) = \\
&\int_K^\infty (x-K)\cdot p_{\eta_k}\left(\ln\left(\frac{x}{S_0}\right) - \tilde{r}\cdot t\right)\frac{dx}{x}
\end{aligned} \tag{45}$$

where $\tilde{r} = r - \frac{\sigma^2}{2}, \sigma^2 = \frac{D(A)}{\bar{T}}$.

A pdf of η_k can be found via its characteristic function. We note that

$$\psi_{\eta_k}(\omega) = E\left(i\,\omega\sum_{p=1}^{k} A_p\right) = \Theta_k(\omega,\omega,\dots\omega) \tag{46}$$

where

$$\Theta_k(\omega_1,\omega_2,...,\omega_k) = E\left(\exp\left(i\cdot\sum_{i=1}^{k}\omega_i\,A_i\right)\right) = \int_X\cdots\int \exp\left(i\cdot\sum_{i=1}^{k}\omega_i\cdot x_i\right)\cdot p_A(x_1,...,x_k)dx_1...dx_k \tag{47}$$

is a k-dimensional characteristic function of a sequence $\{A_1,\dots,A_k\}$ having a joint pdf $p_A(x_1,x_2,\dots x_k)$.

For a case of i.i.d. random values A_k (46) simplifies to

$$\psi_{\eta_k}(\omega) = \Theta_1^k(\omega) \tag{48}$$

where $\Theta_1(\omega) = \int_X e^{i\cdot\omega x} p_A(x)dx$ is the characteristic function of the distribution of A_k.

Here we focus on a special case of (46) when the magnitudes A_k are generated by a chaotic mapping

$$A_k = \varphi(A_{k-1}) \tag{49}$$

in an interval X.

A joint pdf does not factorize in this case and calculates as

$$p_A(x_1,x_2,...,x_k) = p_A(x_1)\cdot\prod_{i=1}^{k-1}\delta(x_{i+1}-\varphi^{(i)}(x_1))\,, \tag{50}$$

where $p_A(x)$ is the invariant density of the map φ.

The goal equation (46) simplifies for piece-wise linear onto maps

$$\varphi(x) = \{\varphi_i(x) = a_i\,x + b_i,\ x\in J_i, i = 1,2,...,m \tag{51}$$

such that $\forall i:\varphi\colon J_i \to X = (0,1)$.

We collect their main probabilistic properties:

- The invariant density is uniform with $\overline{A} = \frac{1}{2}$ and variance $\sigma_A^2 = \frac{1}{12}$
- The autocorrelation function is

$$c_A(k) = \sigma_A^2\,r^k,\ \ -1 < r = \sum_{i=1}^{m}\frac{1}{|a_i|\cdot a_i} < 1 \tag{52}$$

A property (39)-(40) can be easily illustrated with the exponentially decaying acf.

We next substitute the acf (52) into (37) and get

$$\sigma^2_{\eta_k} = k\sigma^2_A \frac{1-r}{1+r} - \sigma^2_A \frac{2r\left(1-r^k\right)}{\left(1-r\right)^2}$$

This confirms (39) at large k as $r^k \to 0$.

The characteristic function can be also calculated analytically. Substituting (50) into (47) for the inner integral we have

$$\int_X e^{i\cdot v_1 x_1} \cdot \delta(x_2 - \varphi(x_1)) \cdot \ldots \cdot \delta(x_k - \varphi^{(k-1)}(x_1))\, dx_1 = \sum_{l=1}^{m} \int_{J_l} e^{i\cdot v_1 x_1} \cdot \prod_{i=1}^{k-1} \delta(x_{i+1} - \varphi^{(i-1)}(\varphi_l(x_1))\, dx_1 \Big|_{\varphi_l(x_1)=z}$$

$$= \sum_{l=1}^{m} \frac{1}{|a_l|} \cdot \int_X e^{i\cdot v_1 \cdot \frac{z-b_l}{a_l}} \cdot \delta(x_2 - z) \cdot \ldots \cdot \delta\left(x_k - \varphi^{(k-2)}(z)\right) dz$$

$$= \sum_{l=1}^{m} \frac{1}{|a_l|} \cdot e^{i\cdot v_1 \cdot \frac{x_2 - b_l}{a_l}} \cdot \delta(x_3 - \varphi(x_2)) \cdot \ldots \cdot \delta\left(x_k - \varphi^{(k-2)}(x_2)\right).$$

Hence the following recurrence equation can be obtained

$$\Theta_k(\omega_1, \ldots, \omega_k) = \sum_{l=1}^{m} \frac{1}{|a_l|} \cdot e^{-i\cdot\omega_1 \cdot \frac{b_l}{a_l}} \cdot \Theta_{k-1}(\omega_2 + \frac{\omega_1}{a_l}, \omega_3, \ldots, \omega_k)$$

the solution of which is

$$\Theta_k(\omega_1, \ldots, \omega_k) = \sum_{i_1, i_2, \ldots, i_{k-1}=1}^{m} \prod_{n=1}^{k-1} \frac{1}{\left|a_{i_n}\right|} \cdot e^{-i\cdot \sum_{n=1}^{k-1} \omega_n \cdot \sum_{p=n}^{k-1} b_{i_p} \cdot \prod_{l=n}^{p} \frac{1}{a_{i_l}}} \Theta_1\left(\sum_{n=1}^{k-1} \omega_n \cdot \prod_{p=n}^{k-1} \frac{1}{a_{i_p}} + \omega_k \right), \tag{53}$$

where $\Theta_1(\omega) = \int_X e^{i\omega x} p_A(x) dx = \int_0^1 e^{i\omega x} dx = \dfrac{e^{i\omega} - 1}{i\omega}$ is the characteristic function of the uniform distribution. Setting $\omega_1 = \omega_2 = \ldots = \omega_k = \omega$ in (53) and substituting the result into (46) we first get a characteristic function and then a required pdf of η_k by use an inverse Fourier transform. In [Baranovski 2003] authors have shown a fast convergence of the characteristic function $\Theta_k(\omega, \ldots, \omega)$ of the cumulative sum $\eta_k = \sum_{p=1}^{k} A_p$ to $\cos\left(\overline{\eta}\, \omega\right) \exp\left(-\dfrac{\omega^2 \sigma^2_\eta}{2} \right)$, which is the characteristic function of a normal distribution with the mean $\overline{\eta}$ and variance σ^2_η.

For example, a tent map on the unit interval has the following characteristic function [Baranovski 2003]

$$\Theta_k(\omega) = \frac{1}{2^{k-1}} \sum_{j=1}^{2^{k-1}} e^{i\omega f(k,j)} \Theta_1\left(\frac{\omega}{2^{k-1}}(2j-1)\right),$$

where $f(k,j) = \begin{cases} f(k-1,j) + \frac{2j-1}{2^{k-1}}, & for \quad j = 1,2,...,2^{k-2} \\ f(k-1, j-2^{k-2}), & for \quad j = 2^{k-2}+1,...,2^{k-1} \end{cases}; f(1,1)=0$.

Then we get a price for the European call option

$$C(t,K) = e^{-r\cdot t} \sum_k P(N(t)=k) \cdot C_k(t,K),$$

where

$$C_k(t,K) = \frac{1}{2a} \sum_{i=1}^{2^{k-1}} \frac{1}{2i-1} \begin{cases} S_0 e^{\tilde{r}\cdot t}\left(e^{B_2} - e^{B_1}\right) - K \cdot 2a \cdot \frac{2i-1}{2^{k-1}}, if \;\; B(t) < B_1; \\ S_0 e^{\tilde{r}\cdot t}\left(e^{B_2} - e^{B(t)}\right) - K \cdot \left(B_2 - B(t)\right), \; if \;\; B_1 < B(t) < B_2; \\ 0, \; if \; B(t) > B_2. \end{cases}$$

$$B(t) = \ln\left(\frac{K}{S_0}\right) - \tilde{r} \cdot t, \; B_1 = 2a \cdot \left(f(k,i) - \frac{k}{2}\right), B_2 = B_1 + 2a \cdot \frac{2i-1}{2^{k-1}}$$

which converges to the Black-Scholes price as shown in Fig. 11.

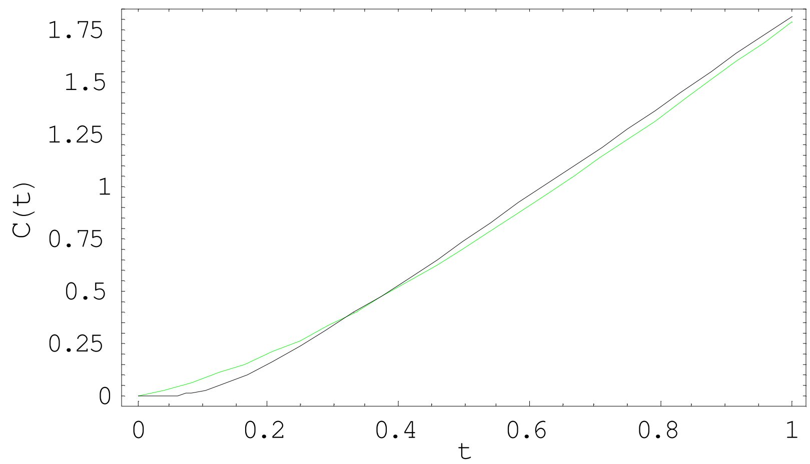


Fig.11. Chaotic price of the European call option (green curve) vs Black-Scholes price (black curve)

Example 2. Building hybrid (stochastic/chaotic) processes by Brownian subordination.

Here we calculate a price of the European call option if the underlying asset follows a Wiener process with a time driven by stochastic/chaotic marked point process

$$\eta(t)=\sum_{k=1}^{N(t)} A_k \tag{54}$$

Properties:

1) mean: $E(W(\eta))=E\left[E(W(\eta)\mid\eta)\right]=E(0)=0$

2) variance:

$$D(W(\eta))=E\left[D(W(\eta)\mid\eta)\right]+D\left[E(W(\eta)\mid\eta)\right]=E\left[\eta(t)\right]$$
$$=E\left[E(\eta\mid N(t))\right]=E\left[N(t)\cdot E(A)\right]=\bar{N}_t\cdot\bar{A}\Rightarrow\bar{A}>0$$

3) distribution function:

$$P\{W(\eta)<y\}=\sum_k P(N(t)=k)\cdot P\{W(\eta)<y\mid N(t)=k\}$$

Price of the European call option:

$$\begin{aligned}C(t,K)&=e^{-r\cdot t}E\left[\left(S_0\cdot e^{\tilde{r}t+W\left(\sum_{i=1}^{N(t)}A_i\right)}-K\right)^+\right]\\&=e^{-r\cdot t}\sum_k P(N(t)=k)\cdot\left\{S_0e^{\tilde{r}t}\frac{1}{2}e^{\frac{k\bar{A}}{2}}\left[1-erf\left(\frac{B(t)}{\sqrt{2k\bar{A}}}\right)-\sqrt{\frac{k\bar{A}}{2}}\right]-K\cdot\frac{1}{2}\left[1-erf\left(\frac{B(t)}{\sqrt{2k\bar{A}}}\right)\right]\right\}\end{aligned}\tag{55}$$

Comparison with Black-Sholes price:

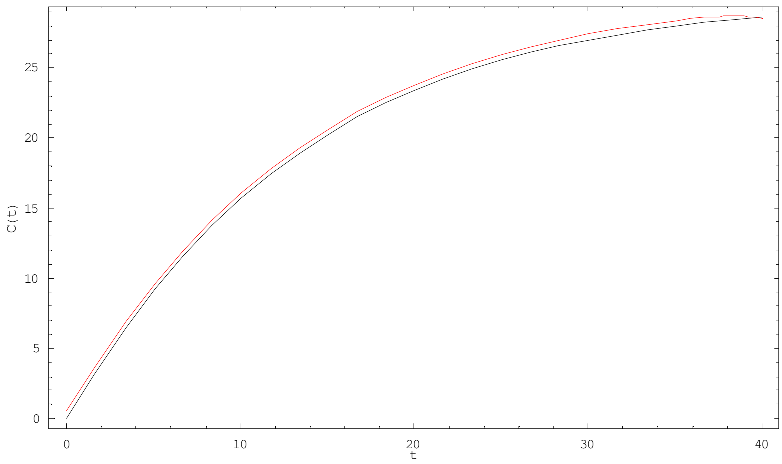


Fig. 12. Black curve is a Black-Sholes price; red one is $C(t,k)$

4.2. Case of simple real roots of a characteristic polynomial

We consider the case when the characteristic polynomial of the system (31) has simple real negative roots $\gamma_i, i = 1,2,...,n-1$. The Routh-Hurwitz theorem provides necessary conditions for that. By introducing the following two matrices

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 1 & ... & 1 \\ \gamma_1 & \gamma_2 & ... & \gamma_{n-1} \\ ... & ... & ... & ... \\ \gamma_1^{n-2} & \gamma_2^{n-2} & ... & \gamma_{n-1}^{n-2} \end{pmatrix}, \mathbf{E}(\mathrm{t}) = \begin{pmatrix} e^{\gamma_1 t} & 0 & 0 & 0 \\ 0 & e^{\gamma_2 t} & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & e^{\gamma_{n-1} t} \end{pmatrix} \quad (56)$$

The solution of (31) is then given by:

$$\boldsymbol{\eta}(t) = \mathbf{\Lambda}\mathbf{E}(t)\mathbf{D}_k, \quad at\ t_{k-1} \le t < t_k \quad (57)$$

where the vector of constants $\mathbf{D}_k = \left(D_1^{(k)}, \dots, D_{n-1}^{(k)}\right)^T$ can be specified from the initial conditions $\eta(t_{k-1})$ by

$$D_k = E(-t_{k-1}) \cdot \Lambda^{-1}\eta(t_{k-1}) \quad (58)$$

As η is a càdlàg we first get for $t = 0$ ($k = 1$)

$$\boldsymbol{\eta}(+0) \equiv \boldsymbol{\eta}_0 = \mathbf{\Lambda} \cdot \mathbf{E}(0) \cdot \mathbf{D}_1 \Rightarrow D_1 = \Lambda^{-1} \cdot \eta_0$$

then from (32) one can show that for $t = t_{k-1}$

$$\eta(t_{k-1}) \equiv \eta(t_{k-1}+0) = \Lambda \cdot E(t_{k-1}) \cdot D_k = \eta(t_{k-1}-0) + A_{k-1} \equiv \Lambda \cdot E(t_{k-1}) \cdot D_{k-1} + A_{k-1} \quad (59)$$

It leads to a recurrent equation:

$$\mathbf{D}_k = \mathbf{D}_{k-1} + \mathbf{E}(-t_k) \cdot \mathbf{\Lambda}^{-1}\mathbf{A}_{k-1}$$

with a solution

$$\mathbf{D}_k = \mathbf{\Lambda}^{-1}\boldsymbol{\eta}_0 + \sum_{\ell=1}^{k-1} \mathbf{E}(-\tau_l)\mathbf{\Lambda}^{-1}\mathbf{A}_\ell, \tag{60}$$

where $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}_0$.

Substituting (60) into (57) leads to a general solution of (31) as a mixture of the magnitudes of the all previous kicks in the system:

$$\boldsymbol{\eta}(t) = \mathbf{\Lambda}\mathbf{E}(t)\mathbf{\Lambda}^{-1}\boldsymbol{\eta}_0 + \sum_{\ell=1}^{k-1} \mathbf{\Lambda}\mathbf{E}(t-t_l)\mathbf{\Lambda}^{-1}\mathbf{A}_\ell, \quad at\ t_{k-1} \le t < t_k \tag{61}$$

From (59) and (58) we derive

$$\mathbf{A}_{k-1} = \boldsymbol{\eta}(t_{k-1}) - \mathbf{\Lambda} \cdot \mathbf{E}(T_k) \cdot \mathbf{\Lambda}^{-1} \cdot \boldsymbol{\eta}_{k-2} \tag{62}$$

Using (61) Eq. (62) transforms to

$$\mathbf{A}_{k-1} = \boldsymbol{\eta}(t_{k-1}) - \mathbf{\Lambda}\mathbf{E}(t_{k-1})\mathbf{\Lambda}^{-1}\boldsymbol{\eta}_0 - \sum_{l=1}^{k-2} \mathbf{\Lambda}\mathbf{E}(t_{k-1} - t_l)\mathbf{\Lambda}^{-1} \cdot \mathbf{A}_l \ .$$

The inverse matrix $\mathbf{\Lambda}^{-1}$ exists as the Vandermonde determinant $\det(\Lambda)$ does not equal to zero and for large time t

$$\mathbf{\Lambda}\begin{pmatrix} e^{\gamma_1 t} & & \\ & \cdots & \\ & & e^{\gamma_{n-1} t} \end{pmatrix}\mathbf{\Lambda}^{-1} \xrightarrow[t\to\infty]{} 0 \ .$$

A stationary mode is then established by the second term in (61).

4.2.1. Periodic perturbation: the inter-arrival times are equal to Ω

Now we start to analyze the statistical properties of the stationary mode of process $\boldsymbol{\eta}(t)$ of a system (31) forced by a periodic shot noise.

Let amplitudes $A_i^{(p)}$ be independent zero mean random values with the probability density function: $p_{A_i}(x,t),\ i = 1,2,...,n-1$ common for all p, i.e. t ; $A_i^{(k)}$ and $A_j^{(m)}$ are mutually independent values $\forall i \neq j, j \in \{1,...,n\text{-}1\}, \forall k, m$.

Then from (58) the stationary mode is given by

$$\eta_j(t) = \sum_{p=1}^{k-1} \sum_{i=1}^{n-1} A_i^{(p)} \cdot \alpha_{j,i}^{(p)}(t),\ j = 0,1,\ldots,n-2 \quad , \tag{63}$$

where

$$\alpha_{j,i}^{(p)}(t) = \sum_{m=1}^{n-1} s_{m,i} \cdot (\gamma_m)^j e^{\gamma_m \cdot (t - p \cdot \Omega)}\ , \tag{64}$$

$s_{m,i}$ being the elements of the inverse matrix $\mathbf{\Lambda}^{-1}$.

Then a vector $\overline{\eta}$ of the means and a vector σ_η^2 of the variances of the process $\eta(t)$ can be calculated from (63) as

$$\overline{\eta} = \nu \cdot \overline{A} \tag{65}$$

$$\sigma_\eta^2 = \vartheta \cdot \sigma_A^2 \tag{66}$$

where $\overline{A} = (\overline{A}_1, \ldots, \overline{A}_{n-1})^T$, $\sigma_A^2 = (\sigma_{A_1}^2, \ldots, \sigma_{A_{n-1}}^2)^T$,

$\mathbf{v} = \left(\nu_{j,i}(t)\right),\ j = 0,1,\ldots,n-2;\ i = 1,\ldots,n-1$ is a matrix (n-1)x(n-1) with the elements

$\nu_{j,i}(t) = \sum_{p=1}^{k-1} \alpha_{j,i}^{(p)}(t),\ j = 0,1,\ldots,n-2;\ i = 1,\ldots,n-1$.

$\boldsymbol{\upsilon} = \left(\upsilon_{j,i}(t)\right),\ j = 0,1,\ldots,n-2;\ i = 1,\ldots,n-1$

$\upsilon_{j,i}(t) = \sum_{p=1}^{k-1} \left[\alpha_{j,i}^{(p)}(t)\right]^2,\ j = 0,1,\ldots,n-2;\ i = 1,\ldots,n-1$.

4.2.2. Asymptotic properties of η(t)

We introduce the following notations: an exponential pulse shape $g_i(z) = e^{\gamma_i \Omega z}$, an amplitude

$\tilde{A}_k^{(i)} = A_k^{(i)} e^{-\gamma_i \Omega (k-1)}$.

Taking into account that $k-1 = \left\lfloor \frac{t}{\Omega} \right\rfloor$ for an arbitrary interval $(k-1)\Omega < t < k\Omega$, the process $\eta(t)$ can be rewritten in the following form

$$\eta(t) = \sum_{i=1}^{n-1} z_i(t),$$

where $z_i(t) = \sum_{k=1} \tilde{A}_i^{(k)} g_i\left(\frac{t}{\Omega} \bmod 1\right)$ is a sequence of pulses adjoining to each other with given form $g_i = g_i(z)$ at $0 < z \le 1$and random amplitudes distributed on $p_{A^{(i)}}(x), \forall i$ and fixed duration Ω.

The characteristic function of the process $\eta(t)$ factorizes:

$$\Psi(u,t) = E(e^{jx(t)u}) = \prod_{i=1}^{n} E(e^{jz_i(t)u}) = \prod_{i=1}^{n} \Psi_i(u,t)$$

and the distribution function:

$$F(y,t) = P(\eta(t) \le y)$$

can be easily calculated by use of the inverse theorem.

The mean value is given by

$$\int_{-\infty}^{\infty} y dF(y,t) = E\left(\eta(t)\right) = \sum_{i=1}^{n} E\left(A_k^{(i)}\right) g_i\left(\frac{t}{T} \bmod 1\right) = 0, \ (k-1)\Omega < t < k\Omega,$$

as $\frac{t-(k-1)\Omega}{\Omega} = \frac{t}{\Omega} \bmod 1$. The variance of $\eta(t)$ calculates as

$$E(\eta^2(t)) = E\left(\sum_{i=1}^{n-1} A_k^{(i)} g_i\left(\frac{t}{T} \bmod 1\right)\right)^2 = \sum_{i=1}^{n-1} E\left(A_k^{(i)} g_i\left(\frac{t}{T} \bmod 1\right)\right)^2 = \sum_{i=1}^{n-1} E(z_i^2(t)) = \sum_{i=1}^{n-1} b_i^2(t) = B_{n-1}^2(t)$$

We assume that the variance $b_i^2(t)$ of the elementary process $z_i(t)$ is finite entailing that for almost all t:

$$B_n^2(t) \underset{n\to\infty}{\longrightarrow} \infty.$$

Then one can show that a Lindeberg condition [Feller 1968] is satisfied:

$$\lim_{n\to\infty}\frac{1}{B_{n-1}^2(t)}\sum_{i=1}^{n-1}\int\limits_{|y|\ge\varepsilon\, B_{n-1}(t)} y^2 dF_i(y,t)=0,\ \forall\varepsilon>0\,, \tag{67}$$

where $F_i(y,t)$ is the distribution function of $z_i(t)$. Moreover

$$F_i(y,t)=P\left(z_i\left(t\right)<y\right)=F_{A_i}\left(\frac{y}{g_i\left(\frac{t}{T}\bmod 1\right)}\right)\Rightarrow F_i(-y,t)=1-F_i(y,t)$$

hence $\int\limits_{|y|\ge\varepsilon\, B_n(t)} y^2 dF_i(y,t)=0$. But at $n\to\infty$

$$\int\limits_{|y|<\varepsilon B_n(t)} y^2 dF_i(y,t)\to b_i^2(t),\ \varepsilon B_n(t)\to\infty\,.$$

Eq. (67) is a necessary and sufficient condition of convergence $F(y,t)$ to normal distribution with parameters $E(\eta(t))=0$ and $B_{n-1}^2(t)$, according to the Lindeberg and Feller theorem [Feller 1968].

4.2.3. An empirical model

Let us consider an exponential term structure of interest rates (4) with l = 6 which is characterized by a performance (22) . For that we fix a dimension n = 6 of a system (31) and set $T_k = \Omega$ as well as an arbitrary time t in a k-th interval $t\in\left[\left(k-1\right)\Omega,k\Omega\right)$.

According to the second approach (read discussion in section 2 and equation (15) take a spectrum of the eigenvalues by

$$\gamma_1=-4.21,\ \gamma_2=-2.68,\ \gamma_3=-1.94,\ \gamma_4=-3.83,\ \gamma_5=-1.59 \tag{68}$$

corresponding to the median fitted curve of the instantaneous forward rate on the date $\tau_{fixed}=255$ (22.02.2007) . From the eigenvalues we calculate the matrix $\mathbf{\Lambda}$ and its inverse $\mathbf{\Lambda}^{-1}$.

Above fitting procedure provides a sample of trajectories

$$\boldsymbol{\eta}_\tau\left(t\right)=\mathbf{f}\left(t,Z_\tau\right)-\mathbf{c}_{fix},\ \tau=1,...,q\,, \tag{69}$$

where $\mathbf{c}_{fix}=\left(c_l\left(\tau_{fix}\right),0,\ldots,0\right)^T$, $c_l\left(\tau_{fix}\right)\equiv c_6(255)\cong 0.043$

We rewrite Eq. (62) in the form

$$\mathbf{A}_k(\tau) = \boldsymbol{\eta}_\tau(k \cdot \Omega) - \boldsymbol{\Lambda} \cdot \mathbf{E}(\Omega) \cdot \boldsymbol{\Lambda}^{-1} \cdot \boldsymbol{\eta}_\tau((k-1) \cdot \Omega) \tag{70}$$

and calculate the sample central moments of the magnitudes $\mathbf{A}_k$

$$\boldsymbol{\mu}_i^{(k)} = \begin{bmatrix} \mu_{i,1}^{(k)} \\ \mu_{i,2}^{(k)} \\ \vdots \\ \mu_{i,n-1}^{(k)} \end{bmatrix} = \frac{1}{q} \sum_{\tau}^{q} \left(\mathbf{A}_k(\tau) - \overline{\mathbf{A}}_k \right)^i$$

where the vector $\overline{\mathbf{A}}_k = \left(\overline{A}_1^{(k)}, \overline{A}_2^{(k)}, \ldots, \overline{A}_{n-1}^{(k)} \right)^T$ has the components $\overline{A}_j^{(k)} = \frac{1}{q} \sum_{\tau=1}^{q} \overline{A}_j^{(k)}(\tau)$.

The central moments of the magnitudes $A_1^{(k)}$ are given by the following empirical relations:

$$\mu_{2i,1}^{(k)} = \sigma_{A_1^{(k)}}^{2i} \cdot e^{\beta(i-1)}, \; i = 1,2,\ldots \tag{71}$$

and

$$\mu_{2i-1,1}^{(k)} = \left(\sigma_{A_1^{(k)}}^{2} \right)^{i-\frac{1}{2}} \cdot \alpha \cdot e^{\beta(i-2)}, i = 2,3,\ldots, \tag{72}$$

where $\beta = 6.16751, \; \alpha = 21.7945$.

The central moments of the magnitudes $A_j^{(k)}, j = 2,3,\ldots,n-1$ demonstrate also the following patterns:

the even moments

$$\mu_{2i,j}^{(k)} = \sigma_{A_j^{(k)}}^{2i} \cdot \phi_i^{(k)}, \; i = 1,2,\ldots \tag{73}$$

and the odd moments

$$\mu_{2i-1,j}^{(k)} = \left(\sigma_{A_j^{(k)}}^{2} \right)^{2i-1} \cdot \iota_i^{(k)}, i = 2,3,\ldots \tag{74}$$

where the constants $\phi_i^{(k)}, \iota_i^{(k)}$ can be tabulated.

4.2.4. A chaotic model

Here we discuss an inverse problem: how to design a dynamical system (31) forming process $\eta(t)$ with the given statistical properties. For that we need to provide a generator of the magnitudes with the prescribed properties discussed above.

We will consider a case when the vector of magnitudes $\mathbf{A}_\mathrm{k}$ is given by

$$\mathbf{A}_k = \left(0,0,\ldots,0,A_k\right)^T, \tag{75}$$

where the magnitudes are chaotic variables. It means that a just (n-2)-th derivative of a solution $\eta(t)$ changes by jump A_k, which is governed by a chaotic map $A_k = \varphi\left(A_{k-1}\right)$.

The system coefficients β_k are coupled with the eigenvalues $\left\{\gamma_1,\ldots,\gamma_{n-1}\right\}$ by Vieta's formula.

Without loss of generality we fix an arbitrary time t in a k-th interval $t \in \left[\left(k-1\right)\Omega, k\Omega\right)$. Then from (61) the stationary mode is given by

$$\eta_j\left(t\right) = \frac{d^j}{dt^j}\eta\left(t\right) \equiv \eta^{(j)}\left(t\right) = \sum_{p=1}^{k-1}\alpha_{p,j}\left(t\right)A_p,\ j = 0,1,\ldots,n-2 \tag{76}$$

where

$$\alpha_{p,j}\left(t\right) = \sum_{i=1}^{n-1} s_{i,n-1}\cdot\gamma_i^{\,j} e^{\gamma_i\left(\frac{t}{\Omega}-p\right)}, \tag{77}$$

$s_{i,n-1}$ being the elements of the inverse matrix $\mathbf{\Lambda}^{-1}$.

The mean and variance of the process $\eta(t)$ and its derivatives are

$$\overline{\eta}_j = E\left(\eta_j\left(t\right)\right) = \overline{A}\sum_{p=1}^{k-1}\alpha_{p,j}\left(t\right) \tag{78}$$

$$\sigma_{\eta_j}^2 = D\left(\eta_j\left(t\right)\right) = \sigma_A^2\sum_{i=1}^{k-1}\alpha_{i,j}^2\left(t\right) + 2\sum_{l=1}^{k-2}c_A\left(l\right)\sum_{i=1}^{k-l-1}\alpha_{i,j}\left(t\right)\alpha_{i+l,j}\left(t\right) \tag{79}$$

The distribution function of $\eta_j(t)$ is then defined by

$$F_j(y,t) = P(\eta_j(t) \le y) = P\left(\sum_{p=1}^{k-1}\alpha_{p,j}\left(t\right)A_p \le y\right) \tag{80}$$

and its characteristic function becomes

$$\psi_j(\omega,t)=E\left(i\,\omega\sum_{p=1}^{k-1}\alpha_{p,j}(t)\,A_p\right)=\Theta_{k-1}\left(\omega\alpha_{1,j}(t),\omega\alpha_{2,j}(t),\ldots,\omega\alpha_{k-1,j}(t)\right) \quad (81)$$

where $\Theta_k(\omega_1,\omega_2,...,\omega_k)$ is the k-dimensional characteristic function of the sequence $\{A_p$ $p=1,\ldots,k-1\}$.

Note that for a case of i.i.d. random values A_k (68) simplifies to

$$\Psi_j(\omega,t)=\prod_{i=1}^{k-1}\Theta_1\left(\omega\alpha_{i,j}(t)\right) \quad (82)$$

where $\Theta_1(\omega)=\int_X e^{i\cdot\omega x}p_A(x)dx$ is the characteristic function of the distribution of A_k .

As above shown a central limit theorem holds for $\sum_{p=1}^{k-1}A_p$ .Then a characteristic function (81) of η_j (t) approaches $\cos\left(\overline{\eta}_j\cdot\omega\right)\exp\left(-2^{-1}\omega^2\sigma^2_{\eta_j}\right)\ \forall j$ at large t or k. Thus the response of a linear system (31) forced by chaotic shot noise is normally distributed process.

From the eigenvalues (68) by use of Vieta's formulas we define the coefficients $\beta_i,\ i=0,1,2,3,4$ leading to the following 5th order differential equation

$$\eta^{(V)}+14.24\cdot\eta^{(IV)}+82.25\cdot\eta'''+208.81\cdot\eta''+267.94\cdot\eta'+132.77\cdot\eta=\sum_i A_i\delta(t-i\cdot\Omega) \quad (83)$$

The equation has a solution (76) at $j=0$.

We assume that the amplitudes A_k of the impulse perturbation are chaotic uncorrelated variables with zero mean. Then from (79) we get

$$\sigma^2_\eta=\sigma^2_A\cdot\mu(t) \quad (84)$$

where $\mu(t)=\sum_{i=1}^{k-1}\alpha^2_{i,0}(t)$.

Under the given spectrum of the eigenvalues $\gamma_i, i=1,...,5$ (68) and the following coefficients $s_{1,5}=0.29,\ s_{2,5}=0.7,\ s_{3,5}=-0.9,\ s_{4,5}=-0.54,\ s_{5,5}=0.44$ and, for example, Ω = 1/360 (one day) a component $\mu(t)$ can be easily calculated by (77) as shown in Fig. 13

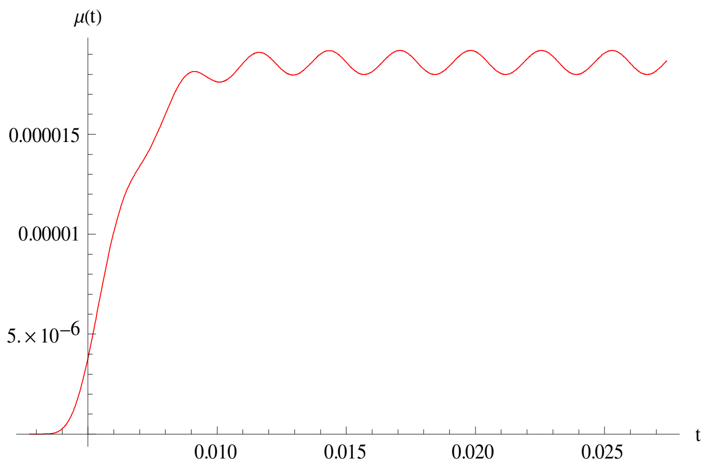


Fig. 13 $\mu(t)$ within $\Omega < t < 10\,\Omega$

The $\mu(t)$ quickly becomes a periodic function with a period Ω.

In Fig. 14 we plot a ratio $\dfrac{\sigma_\eta^2(t)}{\mu(t)}$ calculated from a sample variance of $\eta(t)$.

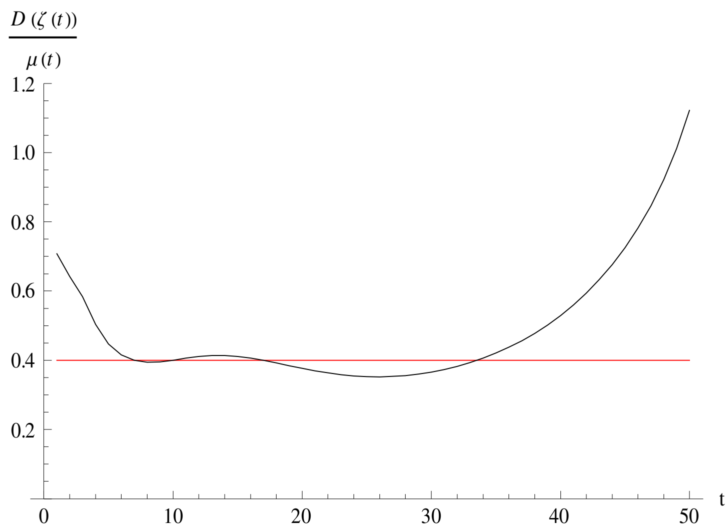


Fig. 14. A ratio $\dfrac{\sigma_\eta^2(t)}{\mu(t)}$ within $0 < t < 50$ years

From (84) and Fig. 18 we can approximate a variance of the magnitudes by

$$\sigma_A^2 = \frac{\sigma_\eta^2}{\mu(t)} \approx 0.4,\ \ 5 \le t \le 35 \qquad (85)$$

For a special case $t = 7$ (years) we calculate $\sigma_\eta^2 \approx 7.459 \cdot 10^{-6}$, $\mu(t) \approx 1.867 \cdot 10^{-5}$ and then

$\dfrac{\sigma_\eta^2(t)}{\mu(t)} = 0.3995$.

On the base of an approach [Baranovski&Daems 1995] we design a piece-wise linear map

$$A_{i+1} = \begin{cases} 2A_i + b, & -b \le A_i < 0 \\ -2A_i + b, & 0 \le A_i \le b \end{cases} \tag{86}$$

which is characterized by an uniform probabilistic measure on the interval [-*b*, *b* =1.095] with zero mean, the variance 0.4 and zero acf $c_A(n) = 0,\ \forall n \ge 1$.

Taking into account an analysis in section 4.1. and having a chaotic sequence $\{A_1, A_2, \ldots, A_{k-1}\}$ one can compute a solution $\eta(t)$ and its fourth derivative $\eta_j(t)$ on the time interval [0, *k Ω*] as shown in Figs. 15 - 17.

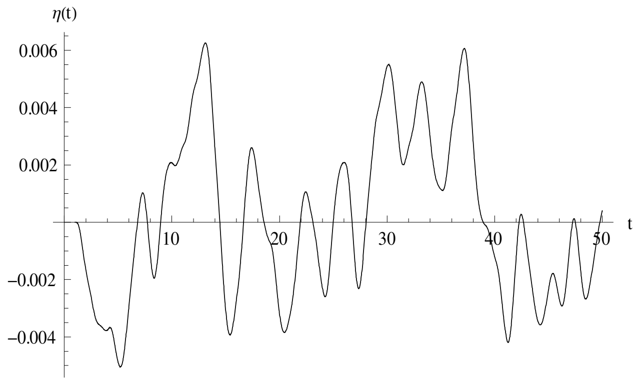


Fig. 15 a solution $\eta(t)$ on the time interval [*Ω*, 50 years] ; Ω = 1 (one year)

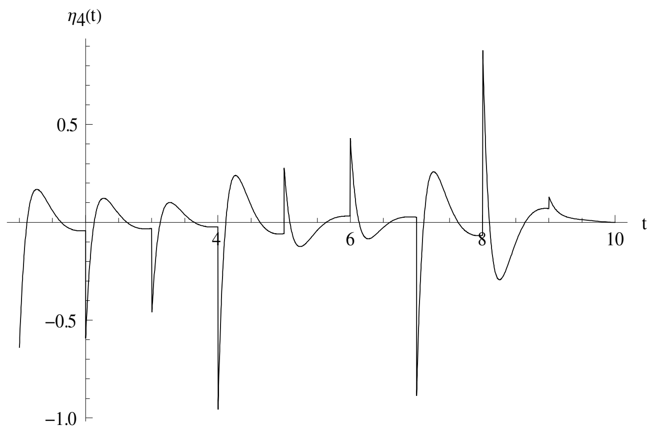


Fig. 16 a solution $\eta_4(t)$ on the time interval [*Ω*, 10 years] ; Ω = 1 (one year)

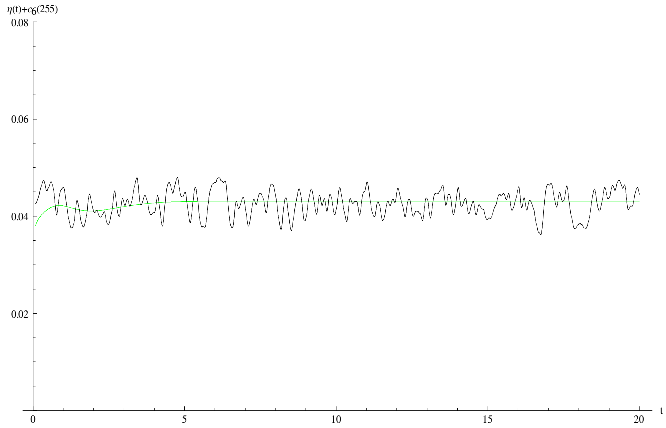


Fig. 17 . A path (black trajectory) of a solution of dynamical system with amplitudes A_k generated by tent map (70) in comparison to $f(t, Z_{255})$ (green curve): the instantaneous forward rate at time t for date 22.02.2007 ; Ω = 1/12 (one month)

Fig. 16 demonstrates a jump character of the last component of the vector solution $\eta(t)$ and confirms that the magnitudes of the jumps are $\boldsymbol{\eta}_4(k\Omega+0)-\boldsymbol{\eta}_4(k\Omega-0)\equiv \mathbf{A}_k$.

A phase portrait on Fig. 18 represents an attractor of the dynamical system.

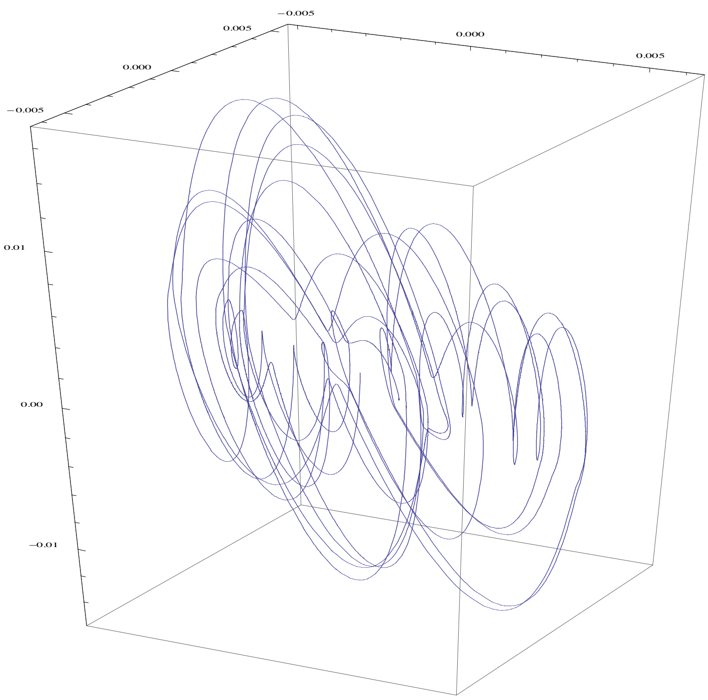


Fig. 18. Parametric plot of the vector $[\eta_0(t), \eta_1(t), \eta_2(t)]$ at $0 \le t \le 50$ and Ω = 1 (one year)

Fig. 19 shows plots $[\eta_0(t),\eta_1(t),\eta_2(t)]$ and $[\eta_1(t),\eta_2(t),\eta_3(t)]$ together.

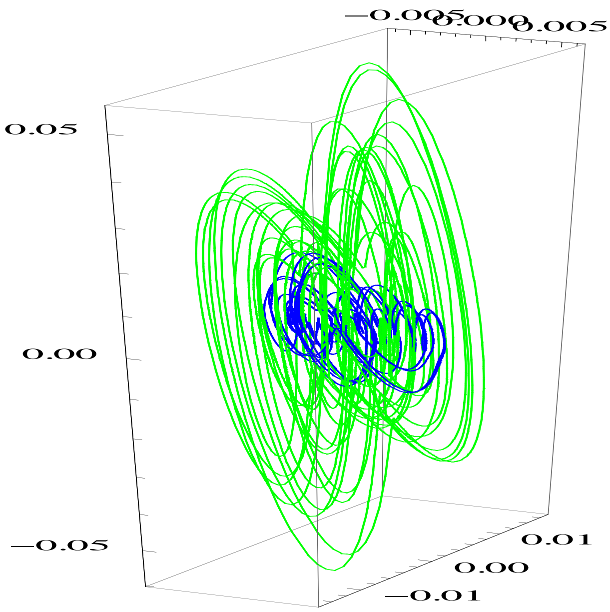


Fig. 19 Parametric plots $[\eta_0(t),\eta_1(t),\eta_2(t)]$ (blue curve) and $[\eta_1(t),\eta_2(t),\eta_3(t)]$ (green curve)

It is clear from Fig.23 that a dynamical system (70) demonstrates "expansion", characterized by exponential growth of the variances $\sigma^2_{\eta_j}$ with j, order of derivative of the solution $\eta(t)$.

A characteristic function of $\eta_j(t)$ can be analytically obtained. We first establish that the map (87) is topologically equivalent to a tent map (51) with the parameters $\{(a_1 = 2, b_1 = 0), (a_2 = -2, b_1 = 2)\}$. A corresponding homeomorphism is the following linear function $2b \cdot x - b;\ b = 1.095$ [Baranovski&Daems 1995] . It implies that a characteristic function (81) of the solution $\eta_j(t)$ in a k-th interval $t \in \left[(k-1)\Omega, k\Omega\right)$ is given by

$$\psi_j(\omega,t) = \Theta_{k-1}\left(2b\omega\alpha_{1,j}(t),\ 2b\omega\alpha_{2,j}(t),\ldots,2b\omega\alpha_{k-1,j}(t)\right)\cdot Exp\left(-i\cdot b\omega\sum_{p=1}^{k-1}\alpha_{p,j}(t)\right), \qquad (87)$$

where the (k-1) -dimensional characteristic function (66) of a tent map.

Fig. 20 shows the characteristic function (87) at t = 7 (years), i.e. $t = (k-1)\cdot\Omega,\ k = 8,\ \Omega = 1$ in comparison with $Exp\left(-\frac{\omega^2\sigma^2_\eta}{2}\right)$ as the characteristic function of a Gaussian distribution with zero mean and a variance $\sigma_\eta^2 = \sigma_A^2 \sum_{i=1}^{7} \alpha_{i,0}^2(t) \approx 7.467 \cdot 10^{-6}$ calculated from (79) at $c_A(n) = 0, \forall n \geq 1$.

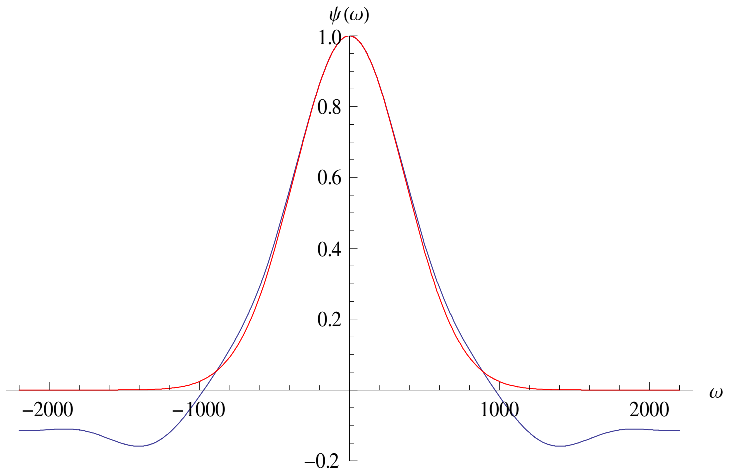


Fig. 20. Characteristic function of η (t) at t = 7 (years) (black curve) and Gaussian process (red curve)

A histogram of the 1000 paths of η (t) at t = 7 is presented in Fig. 21

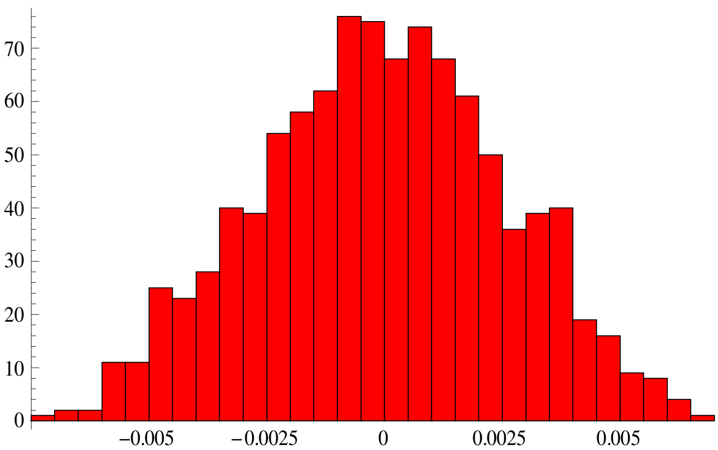


Fig. 21 Histogram of the solution $\eta(t)$ at t =7

A χ^2 Pearson's test with the confidence level 0.95 accepts a hypothesis on normal distribution of the solution $\eta(t)$ with zero mean and the variance 7.46 10^(-6).

In Table 6 we collect all remaining characteristic functions of η_j (t) , j = 1, 2, 3, 4 at t = 7 (years), i.e. $t = (k-1) \cdot \Omega,\ k = 8,\ \Omega = 1$ in comparison with the corresponding asymptotical curves $Exp\left(-\frac{\omega^2\sigma_{\eta_j}^2}{2}\right)$, $j=1,2,3,4$, the characteristic functions of a Gaussian distribution with zero mean and a variance $\sigma_{\eta_j}^2=\sigma_A^2\sum_{i=1}^{7}\alpha_{i,j}^2(t)$, $j=1,2,3,4$ (Fig. 26)

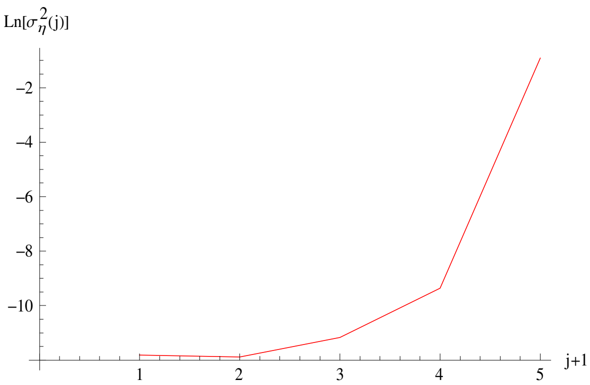
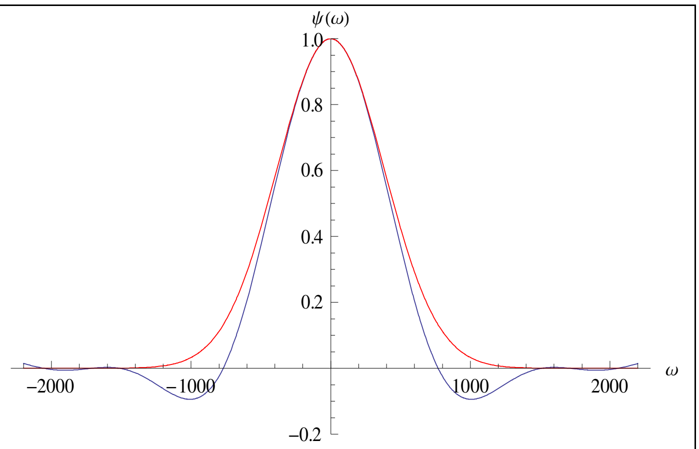
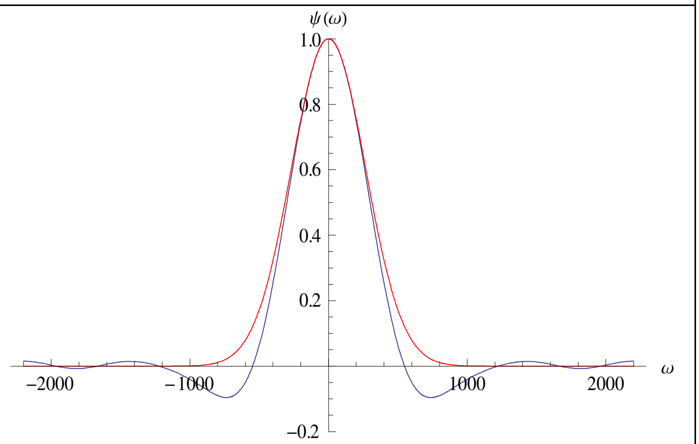


Fig. 22. Log of the variance

characteristic function of *η 1* (*t*)	
characteristic function of *η 2* (*t*)	

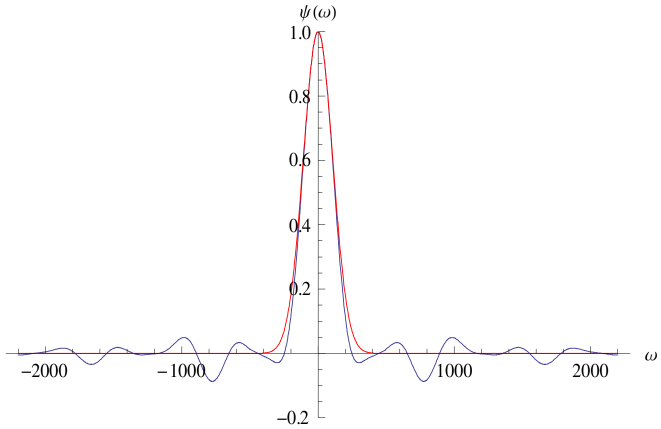
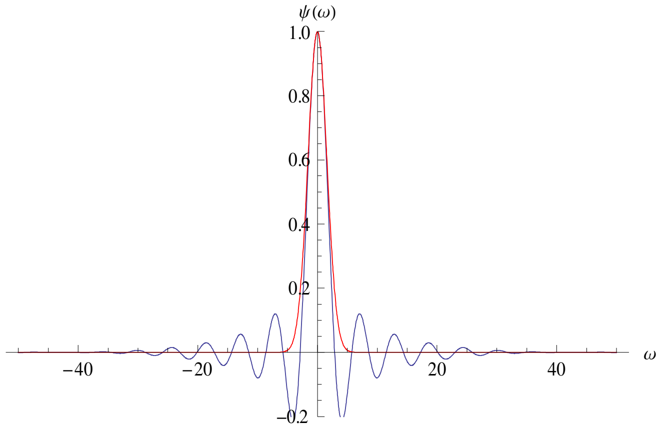
characteristic function of $\eta_3(t)$	
characteristic function of $\eta_4(t)$	

Table 6. Characteristic functions of the components of a solution η in comparison with the corresponding asymptotical characteristic functions of a Gaussian distribution

Conclusions

In this paper we have first proposed an exponential-polynomial model of the interest rates and then demonstrated its performance in a fitting of the zero-coupon curves. Capturing dynamic dependencies in the fitted curves we have in a second step designed a dynamical system forced by shot noise with chaotic/stochastic jumps. In our proposed class the mean-reversion speed of the diffusive and the jump part can be adjusted separately or jointly by a suitable design of chaotic maps with prescribed probabilistic properties.

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