Socially efficient discounting under ambiguity aversion

Christian Gollier
Toulouse School of Economics (LERNA and EIF)

Johannes Gierlinger[1]
Toulouse School of Economics (LERNA)

November 3, 2008

[1]This research benefitted from the financial support of the Chair "Sustainable finance and responsible investment" at TSE. **Correspondence**: Toulouse School of Economics, Manufacture des Tabacs, Aile J.-J. Laffont, MF007, 21 allée de Brienne, 31000 Toulouse, France. Email: johannes.gierlinger@univ-tlse1.fr

Abstract

We consider an economy with an ambiguity-averse representative agent who faces an uncertain consumption growth. We examine the condition under which ambiguity aversion reduces the socially efficient discount rate. We show that ambiguity aversion affects the interest rate in two ways. The first effect is an ambiguity prudence effect similar to the prudence effect that prevails in the expected utility model, but which requires decreasing ambiguity aversion to be signed. Aversion to ambiguity also entails more pessimism. But this pessimistic shift in beliefs generally has an ambiguous effect on the interest rate. We provide sufficient conditions under which ambiguity aversion does indeed decrease the socially efficient discount rate. The calibration of the model tells us that the effect of ambiguity aversion on the way we should discount distant cash flows is potentially large.

Keywords: Decreasing ambiguity aversion, ambiguity prudence, Ramsey rule, sustainable development.

1 Introduction

The emergence of public policy problems associated with the sustainability of our development has raised a considerable interest for the determination of a socially efficient discount rate. This debate has recently culminated in the publication of two reports about the evaluation of different public investments. On one side, the Copenhagen Consensus (Lomborg (2004)) put top priority to public programs yielding immediate benefits (fighting malaria and AIDS, improving water supply,...), and rejected the idea to invest much in the prevention of global warming. On the other side, the Stern Review (Stern (2007)) put tremendous pressure on acting quickly and heavily against global warming. Because global warming will really affect our economies in a relatively distant time horizon, the choice of the rate at which these costs are discounted plays a key role in reaching either conclusion. While Stern applies an implicit rate of 1.4% per year, the Copenhagen Consensus argues that an efficient rate should be around 5%. For the sake of illustrating the power of discounting, consider a project which yields its benefits in t years time. For a horizon $t = 100$ the Copenhagen Consensus would require a rate-of-return already 36 times higher than Stern.

As stated by the well-known Ramsey rule (Ramsey (1928)), the socially efficient discount rate (net of the rate of pure preference for the present) is equal to the product of relative risk aversion and the growth rate of consumption. The basic idea is that, given the assumption that one will be wealthier in the future, one is willing to improve future wealth by sacrificing current wealth only if the return of this investment is large enough to compensate for the increased intertemporal inequality that this investment generates. If we assume that the growth rate of wealth is 2% and relative risk aversion equals 2, this yields a discount rate of 4%.

However, if one wants to use this kind of idea to value investments affecting distant generations, it is crucial to take into account of the riskiness affecting the long term growth of consumption. Hansen and Singleton (1983), Gollier (2002) and Weitzman (2007a), among others, have extended the Ramsey rule by assuming an exogenously given stochastic process for the growth of the economy. This adds a precautionary term to the Ramsey rule which tends to reduce the discount rate in order to induce more investment for the future. The convexity of the prudent representative agent's marginal utility implies that the uncertainty about future consumption raises the expected marginal utility, i.e. the willingness to save for the future (Leland (1968), Drèze and Modigliani (1972)). This reduces the interest rate.

The present paper goes one step further in this analysis by recognizing that there is some degree of uncertainty on the stochastic process affecting the long-term growth of the economy. Such parameter uncertainty on priors is typically referred to as statistical ambiguity or Knightian uncertainty. We believe that this assumption is a realistic one, especially for long-term forecasts. Departing from the standard Subjective Expected Utility (SEU, Savage (1954)), we also assume that the representative agent is ambiguity-averse, i.e., that she dislikes mean-preserving spreads over prior beliefs. Indeed, starting with the pioneering work by Ellsberg (1961), ample evidence in favor of this hypothesis has been accrued.[1] All of which suggests that it is behaviorally meaningful to distinguish lotteries over prior distributions from lotteries over final outcomes. In what follows, we will consider a representative agent who displays "smooth ambiguity preferences", as recently proposed by Klibanoff, Marinacci and Mukerji (KMM, 2005, 2007), which entails the max-min criterion as a particular case. In the KMM model, the agent computes the expected utility of future consumption conditional to each possible value of the uncertain parameter. She then evaluates her future felicity by computing the certainty equivalent of these conditional expected utilities using an increasing and concave function ϕ. The concavity of this function implies that she dislikes any mean-preserving spread in the set of plausible beliefs, i.e. that she is ambiguity-averse.

In this paper, we address the question of how does ambiguity aversion affect the socially efficient discount rate. Intuitively, we might expect that it should raise the agent's willingness to save in order to compensate for the adverse effect of ambiguity on future welfare. It turns out, however, that this is not true in general: ambiguity aversion may increase the socially efficient discount rate. This is connected to two, possibly opposing, effects of ambiguity aversion on marginal utilities. On the one hand, there is an ambiguity prudence effect similar to the prudence effect in the expected utility framework. We show that the mere uncertainty on the conditional expected utility reduces its ϕ-certainty equivalent if and only if ϕ exhibits decreasing absolute (ambiguity) aversion (DAAA). This is more demanding than just the convexity of ϕ', because the future felicity is measured by the ϕ-certainty equivalent rather than by the expectation of ϕ.

On the other hand, as observed by KMM (2005, 2007), ambiguity aversion yields an implicit pessimism effect, which acts as if probability weights were shifted towards more unfavorable prior distributions, in the sense of the Monotone Likelihood Ratio order (MLR). However, this shift in beliefs does not in general imply a reduction of the interest rate. We derive pairs of conditions on the risk attitude and on the stochastic ordering of plausible distributions which guarantee that, under DAAA, the socially efficient discount rate is lower than in the ambiguity-neutral benchmark.

This paper is related to Weitzman (2007a) and Gollier (2007b), who also recognize the uncertainty affecting the growth of the economy as an important feature of the discounting problem. Weitzman (2007a) shows that the uncertainty affecting the volatility of the growth process may yield a term structure

[1] The *Ellsberg-Paradox* refers to the outcome of an experiment (Ellsberg (1961)). In an urn containing 90 balls there were 30 red balls, and the remaining were either black or yellow in unknown proportions. Participants had to bet on the color of the ball drawn, receiving a prize of \$100, in case of a successful bet. A large group preferred to bet on drawing *red* vs. betting on *black*. However, in a second stage they preferred to bet on *not* drawing *red* vs. betting on *not* drawing *black*. This choice pattern contradicts the hypothesis that participants associated unique subjective probabilities to each outcome of a draw, as required in the SEU framework. Note that betting on (or against) *red* is indeed an unambiguous act with well-defined winning probabilities, while betting on (or against) *black* is not. For a survey of the literature consult e.g. Camerer and Weber (1992).

of the discount rate that tends to minus infinity for very long time horizons. Gollier (2007b) provides a typology of more general structures for the parametric uncertainty and shows that the sign of the third or fourth derivative of the utility function are necessary to sign the effect of this uncertainty on the efficient discount rate, depending upon its type. We depart strongly from these works based on a SEU approach by introducing ambiguity aversion in the preferences of the representative agent.

Jouini, Napp and Marin (2007) and Gollier (2007a) consider the related question of how to aggregate diverging beliefs in a SEU framework. Jouini, Napp and Marin show that an aggregation bias might cause a richer evolution of the discount rate than in the representative agent models. In particular, the discount rate might be first increasing and only then approach its limit, namely the smallest individual rate.

The most active branch of the literature on ambiguous processes deals with asset pricing. Clearly, the underlying mechanisms are very similar to the ones we will study below. Methodologically, our paper is most closely related to Gollier (2006). He investigates comparative statics results of an increase in ambiguity aversion on the demand for risky assets. It turns out that, in general, omitting ambiguity aversion cannot be corrected for by assuming a higher degree of risk aversion.

The remainder of the paper is organized as follows. Section 2 introduces the basic model and presents the equilibrium pricing formula. In Section 3 an analytical example yields an adapted Ramsey-rule for the interest rate under ambiguity. We decompose the effect of ambiguity aversion into its two components in Section 4, whereas Sections 5 and 6 are devoted to respectively the ambiguity prudence effect and the pessimism effect. Section 7 investigates under which conditions our findings extend to any increase in ambiguity aversion. Finally, before concluding, we calibrate the model using two different specifications in Section 8.

2 The model

We consider an economy à la Lucas (1978). Each agent in the economy is endowed with a tree which produces $\widetilde{c}_t$ fruits at date t, $t = 0, 1, 2, ...$ There is a market for zero-coupon bonds at date 0 in which agents may exchange the delivery of one fruit today against the delivery of $e^{r_t t}$ fruits for sure at date t. Thus, the real interest rate associated to maturity t is r_t. The distribution of $\widetilde{c}_t$ is a function of a parameter θ that can take values $1, 2, ..., n$. This parametric uncertainty takes the form of a random variable $\widetilde{\theta}$ whose probability distribution is a vector $q = (q_1, ..., q_n)$, where q_θ is the probability that $\widetilde{\theta}$ takes value θ. The cumulative distribution function of $\widetilde{c}_t$ conditional to θ is denoted $F_{t\theta}$. The crop conditional to θ is denoted $\widetilde{c}_{t\theta}$. An ambiguous environment for $\widetilde{c}_t$ is thus fully described by $\widetilde{c}_t \sim (\widetilde{c}_{t1}, q_1, ; ...; \widetilde{c}_{tn}, q_n)$. Conditional to θ, the expected utility of an agent who purchases α zero-coupon bonds with maturity t equals

$$U_t(\alpha, \theta) = Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t}) = \int u(c + \alpha e^{r_t t}) dF_{t\theta}(c).$$

We assume that u is three times differentiable, increasing and concave, so that $U(., \theta)$ is concave in the investment α, for all θ.

Following Klibanoff, Marinacci and Mukerji (2005) and its recursive generalization (Klibanoff, Marinacci and Mukerji (2007)), we assume that the preferences of the representative agent exhibit smooth ambiguity aversion. Ex ante, for a given investment α, the welfare of the agent is measured by $V_t(\alpha)$, which is the certainty equivalent of the conditional expected utilities:

$$\phi(V_t(\alpha)) = \sum_{\theta=1}^{n} q_\theta \phi(U_t(\alpha, \theta)) = \sum_{\theta=1}^{n} q_\theta \phi\left(Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})\right). \tag{1}$$

Function ϕ describes the investor's attitude towards ambiguity (or parameter uncertainty). It is assumed to be three times differentiable, increasing and concave. A linear function ϕ means that the investor is neutral to ambiguity. In such a case, the decision maker is indifferent to any mean-preserving spread of $U_t(\alpha, \widetilde{\theta})$, and $V_t(\alpha)$ can be represented by a subjective expected utility functional $V_t^{SEU}(\alpha) = Eu(\widetilde{c}_t + \alpha e^{r_t t})$. On the contrary, a concave ϕ is synonymous of ambiguity aversion in the sense that one dislikes any mean-preserving spread of the conditional expected utility $U_t(\alpha, \widetilde{\theta})$. An interesting particular case arises when absolute ambiguity aversion $A(U) = -\phi''(U)/\phi'(U)$ is constant, so that $\phi(U) = -A^{-1}\exp(-AU)$. As proved by Klibanoff, Marinacci and Mukerji (2005), the ex-ante welfare $V(\alpha)$ tends to maxmin expected utility functional $V_t^{MEU}(\alpha) = \min_\theta Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})$ when the degree of absolute ambiguity aversion ϕ tends to infinity. Thus, the maxmin criterion à la Gilboa and Schmeidler (1989) is a special case of this model.

The optimal investment α^* maximizes the intertemporal welfare of the investor, which is written as

$$\alpha^* \in \arg\max_{\alpha} \qquad u(c_0 - \alpha) + e^{-\delta t} V_t(\alpha). \tag{2}$$

where parameter δ is the rate of pure preference for the present. If ϕ and u are strictly concave, the objective function is concave in α and the solution to program (2), when it exists, is unique. The necessary and sufficient condition of program (2) is written as

$$u'(c_0 - \alpha^*) = e^{-\delta t} V_t'(\alpha^*)$$

Fully differentiating equation (1) with respect to α yields

$$V_t'(\alpha) = e^{r_t t} \frac{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})\right) Eu'(\widetilde{c}_{t\theta} + \alpha e^{r_t t})}{\phi'(V_t(\alpha))}.$$

Because we assume that all agents have the same preferences and the same stochastic endowment, the equilibrium condition on the market for the zero-coupon bond associated to maturity t is $\alpha^* = 0$. Combining the above two equations implies the following equilibrium condition:

$$r_t = \delta - \frac{1}{t} \ln \left[\frac{\sum_{\theta=1}^n q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right) Eu'(\widetilde{c}_{t\theta})}{u'(c_0)\phi'(V_t(0))} \right]. \tag{3}$$

This is also the socially efficient rate at which sure benefits and costs occurring at date t must be discounted in any cost-benefit analysis at date 0.

As a benchmark, consider the case of an ambiguity neutral representative agent. In that case, we get the standard bond pricing formula $r_t = \delta - t^{-1} \ln \left[Eu'(\widetilde{c}_t)/u'(c_0) \right]$.[2] In this special case, we see that the riskiness of future consumption reduces the socially efficient discount rate if and only if $Eu'(\widetilde{c}_t)$ is larger than $u'(E\widetilde{c}_t)$, i.e., if and only if u' is convex, or if the representative agent is *prudent* (Leland (1968), Drèze and Modigliani (1972), Kimball (1990)).

Our goal in this paper is to determine the conditions under which ambiguity aversion reduces the discount rate. An ambiguous environment $(\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$ is said to be acceptable if the supports of the $\widetilde{c}_{t\theta}$ are in the domain of u, and if all $Eu'(\widetilde{c}_{t\theta})$ are in the domain of ϕ. The set of acceptable ambiguous environments is denoted Ψ.

3 An analytical solution

Let us consider the following specification:

- The plausible distributions of $\ln \widetilde{c}_{t\theta}$ are all normal with the same variance $\sigma^2 t$, and with mean $\ln c_0 + \theta t$.[3]
- The parameter θ is normally distributed with mean μ and variance σ_0^2.[4]
- The representative agent's preferences exhibit constant relative risk aversion $\gamma = -cu''(c)/u'(c)$, i.e., $u(c) = c^{1-\gamma}/(1-\gamma)$.
- The representative agent's preferences exhibit constant relative ambiguity aversion $\eta = -\left| u \right| \phi''(u)/\phi'(u) \geq 0$. This means that $\phi(U) = k(kU)^{1-\eta k}/(1-\eta k)$, where $k = sign(1-\gamma)$ is the sign of u.

As is well-known, the Arrow-Pratt approximation is exact under CRRA and lognormally distributed consumption. Therefore, conditional to each θ, we have that

$$Eu(\widetilde{c}_{t\theta}) = (1-\gamma)^{-1} \exp(1-\gamma)(\ln c_0 + \theta t + 0.5(1-\gamma)\sigma^2 t).$$

[2] See for example Cochrane (2001).

[3] In continuous time, this would mean that the consumption process is a geometric brownian motion $d \ln c_t = \theta dt + \sigma dw$.

[4] We consider the natural continuous extension of our model with a discrete distribution for $\widetilde{\theta}$.

We can again use the same trick to compute the ϕ-certainty equivalent V, since $\phi(Eu(\widetilde{c}_{t\theta}))$ is an exponential function and the random variable $\widetilde{\theta}$ is normal, which is another case where the Arrow-Pratt approximation is exact. It yields

$$V_t(0) = (1-\gamma)^{-1}\exp(1-\gamma)(\ln c_0 + \mu t + 0.5(1-\gamma)\sigma^2 t + 0.5(1-\gamma)(1-k\eta)\sigma_0^2)t^2).$$

Similarly, we have that

$$\begin{aligned} E\phi'\left(Eu(\widetilde{c}_{t\theta})\right)Eu'(\widetilde{c}_{t\theta}) &= \exp -(\gamma+(1-\gamma)k\eta)(\ln c_0 + \mu t - 0.5(\gamma+(1-\gamma)k\eta)\sigma_0^2 t^2) \\ &\quad \times (1-\gamma)^{k\eta}\exp 0.5(\gamma^2 - k\eta(1-\gamma)^2))\sigma^2 t. \end{aligned}$$

Combining these analytical expressions for the expectations in equation (3) yields:

$$r_t = \delta + \gamma\mu - \frac{1}{2}\gamma^2(\sigma^2+\sigma_0^2 t) - \frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2 t. \tag{4}$$

Let us define g as the expected growth rate of consumption. It is easy to check that $g = \mu + 0.5(\sigma^2+\sigma_0^2 t)$. It implies that the above equation can be rewritten as

$$r_t = \delta + \gamma g - \frac{1}{2}\gamma(\gamma+1)(\sigma^2+\sigma_0^2 t) - \frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2 t. \tag{5}$$

The first two terms on the right-hand side of this equation correspond to the classical Ramsey rule. The interest rate is increasing in the expected growth rate of consumption g. When g is positive, decreasing marginal utility implies that the marginal utility of consumption is expected to be smaller in the future than it is today. This yields a positive interest rate. The third term expresses prudence. Because the riskiness of future consumption increases the expected marginal utility $Eu'(\widetilde{c}_t)$ under prudence, this has a negative impact on the discount rate.[5] Notice that the variance of consumption at date t equals $\sigma^2 t + \sigma_0^2 t^2$, so that it increases at an increasing rate with respect to the time horizon. There, the precautionary effect has a relatively larger impact on the discount rate for longer horizons. This argument has been developed in Weitzman (2007a) and Gollier (2007b) to justify a decreasing discount rate in an expected utility framework.

The last term in the right–hand side of equation (5) characterizes the effect of ambiguity. Observe that it always tends to reduce the discount rate under positive ambiguity aversion ($\eta > 0$). This effect is increasing in the degree of ambiguity aversion η, in the degree of uncertainty σ_0, and in the time horizon t. This implies that more efforts will be made to improve the ambiguous future.

Observe, that in our example, in the absence of ambiguity (i.e. $\sigma_0^2 = 0$), the term structure is flat. The mere presence of ambiguity (i.e. $\sigma_0^2 > 0$ but $\eta = 0$) causes the rates to decrease linearly over time. Introducing ambiguity aversion steepens this decline.

The following sections investigate whether it is true in general, that ambiguity aversion decreases the socially efficient discount rate for any maturity. Contrary to the example presented above, the next section reveals that ambiguity aversion might even decrease the willingness to save.

[5] This precautionary effect is equivalent to reducing the growth rate of consumption g by the precautionary premium (Kimball (1990)) $0.5(\gamma+1)(\sigma^2+\sigma_0^2 t)$. Indeed, $\gamma+1 = -cu'''(c)/u''(c)$ is the index of relative prudence of the representative agent.

4 The two effects of ambiguity aversion

In this section, we decompose the effect of the introduction of ambiguity aversion into two components: an ambiguity prudence effect and a pessimism effect. The benchmark is when the representative agent is neutral to ambiguity, in which case the discount rate equals

$$r_t = \delta - \frac{1}{t} \ln \left[\frac{Eu'(\widetilde{c}_t)}{u'(c_0)} \right], \tag{6}$$

where $\widetilde{c}_t$ describes future consumption, which is distributed as $(\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$. Under ambiguity aversion, the pricing formula (3) can be rewritten in a similar fashion as

$$r_t = \delta - \frac{1}{t} \ln \left[a \frac{Eu'(\overset{\circ}{\widetilde{c}_t})}{u'(c_0)} \right], \tag{7}$$

where the constant a is defined as

$$a = \frac{\sum_{\theta=1}^{n} q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right)}{\phi'(V_t(0))}, \tag{8}$$

and where $\overset{\circ}{\widetilde{c}_t}$ is a distorted probability distribution $(\widetilde{c}_{t1}, \overset{\circ}{q}_1; ...; \widetilde{c}_{tn}, \overset{\circ}{q}_n)$ of future consumption, with

$$\overset{\circ}{q}_\theta = \frac{q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right)}{\sum_{\tau=1}^{n} q_\tau \phi' \left(Eu(\widetilde{c}_{t\tau}) \right)}, \tag{9}$$

for $\theta = 1, ..., n$. Thus, ambiguity aversion reduces the discount rate if

$$aEu'(\overset{\circ}{\widetilde{c}_t}) \geq Eu'(\widetilde{c}_t). \tag{10}$$

Notice that this condition simplifies to $a \geq 1$ when the agent is risk neutral. Because we don't constrain the risk attitude in any way except risk aversion, condition $a \geq 1$ is necessary to guarantee that ambiguity aversion reduces the discount rate. For reasons that will be clarified in the next section, we will refer to $a \geq 1$ as the ambiguity prudence effect.

In the absence of an ambiguity prudence effect ($a = 1$), condition (10) becomes $Eu'(\overset{\circ}{\widetilde{c}_t}) \geq Eu'(\widetilde{c}_t)$, which is referred to as the pessimism effect. At this stage, it is enough to say that it comes from a distortion of the beliefs $(q_1, ..., q_n)$ on the likelihood of the different plausible probability distributions $(\widetilde{c}_1, ..., \widetilde{c}_n)$.

5 The ambiguity prudence effect

In this section, we focus on whether the constant a defined by equation (8) is larger than unity, which is necessary to guarantee that the discount rate is reduced by the introduction of ambiguity aversion. This condition becomes necessary and sufficient in the special case of risk-neutral consumers. Notice that, in this special case, a can be interpreted as the sensitiveness of the $\phi-$certainty equivalent of $\overline{c}_{\widetilde{\theta}} = E\left[\widetilde{c}_{t\widetilde{\theta}} \mid \widetilde{\theta}\right]$ to an increase in saving.[6] The problem is thus to determine whether one more dollar saved yields an increase in the $\phi-$certainty equivalent future consumption. More generally, condition $a \geq 1$ can be rewritten as

$$\sum_{\theta=1}^{n} q_\theta \phi'\left(u_\theta\right) \geq \phi'(V_t) \text{ whenever } \Sigma_\theta q_\theta \phi(u_\theta) = \phi(V_t). \tag{11}$$

In words, do expected-utility-preserving risks raise expected marginal utility, where the utility function referred here is the ϕ function? The answer to this question is well-known in expected utility theory (see e.g. Gollier (2001, section 2.5)). This is true if and only if ϕ exhibits decreasing absolute aversion. Indeed, defining function ψ such that $\psi(\phi(U)) = \phi'(U)$ for all U, the above condition can be rewritten as

$$\sum_{\theta=1}^{n} q_\theta \psi\left(\phi_\theta\right) \geq \psi(\Sigma_\theta q_\theta \phi_\theta),$$

where $\phi_\theta = \phi(u_\theta)$ for all θ. This is true for all distributions of $(\phi_1, q_1; ...; \phi_n, q_n)$ if and only if ψ is convex. Because $\psi'(\phi(U)) = \phi''(U)/\phi'(U)$, this is true iff $A(U) = -\phi''(U)/\phi'(U)$, which is the index of absolute ambiguity aversion, be non-increasing. This proves the following results.

Lemma 1 *$a \geq 1$ (resp. $a \leq 1$) for all acceptable ambiguous environments $\widetilde{c} \in \Psi$ if and only if absolute ambiguity aversion is non-increasing (resp. non-decreasing).*

Proposition 1 *Suppose that the representative agent is risk neutral. The socially efficient discount rate is smaller (resp. larger) than under ambiguity neutrality for all acceptable ambiguous environments $\widetilde{c} \in \Psi$ if and only if ϕ exhibits non increasing (resp. non decreasing) absolute ambiguity aversion.*

Under risk neutrality, the driving force for the impact of ambiguity on the interest rate is not ambiguity aversion itself, but rather whether the degree of ambiguity aversion is increasing or decreasing with the level of conditional expected utility U. In the limit case with risk neutrality and constant absolute ambiguity aversion, ambiguity has *no* effect on the equilibrium interest rate. The intuition of these results is easy to derive from the observation that the period-t felicity V_t, which the certainty equivalent of the conditional expectations of future consumption, is approximately equal to expected consumption minus the ambiguity premium, which is proportional to ambiguity aversion A. It implies that the willingness to save is decreasing in A'. Thus, ambiguity aversion raises the willingness to save – and therefore reduces the equilibrium interest rate – if absolute ambiguity aversion is decreasing.

Exactly as decreasing absolute risk aversion is unanimously accepted as a natural assumption for risk preferences, we believe that decreasing absolute ambiguity aversion (DAAA) is a reasonable property of uncertainty preferences.

[6] Define $V(s, \overline{c}_{\widetilde{\theta}})$ such that $\phi(s+V) = E\phi(s+\overline{c}_{\widetilde{\theta}})$. We have that $a = \partial V(s, \overline{c}_{\widetilde{\theta}})/\partial s$ at $s = 0$.

It means that a local mean-preserving spread in conditional expected utility has an impact on welfare that is decreasing with the level of utility where this spread is realized.

We call this the *ambiguity prudence effect*, because it emerges as a consequence of the fact that the future conditional expected utility is uncertain. This raises the willingness to save exactly as the risk on future income raises savings in the standard expected utility model under "risk prudence". But contrary to risk prudence which is characterized by $u''' \geq 0$, ambiguity prudence is described by decreasing absolute uncertainty aversion, which is weaker than $\phi''' \geq 0$. This is because, in the intertemporal KMM model, the future felicity is represented by the $\phi-$ certainty equivalent of the conditional expected utilities, rather than by the expected $\phi-$ valuation of the conditional expected utilities. If we would have used this alternative model, ϕ' convex would have been the necessary and sufficient condition to sign the ambiguity prudence effect.

However, once we allow for risk aversion, another effect emerges, and non increasing ambiguity aversion is not sufficient anymore to unambiguously sign the effect of ambiguity on the discount rate. This is shown by the following counter-example.

> **Counter-example 1.** Let c_0 equal 2. We assume that $\widetilde{c}_t$ has two plausible distributions, $\widetilde{c}_{t1} \sim (1, 1/3; 4, 1/3; 7, 1/3)$ and $\widetilde{c}_{t2} \sim (3, 2/3; 4, 1/3)$. We assume that these two distributions are equally likely to be the true one, i.e., $q_1 = q_2 = 1/2$. We assume that the agent exhibits constant relative risk aversion (CRRA) with $\gamma = 2$, i.e., $u(c) = -c^{-1}$. We assume that the rate of pure preference for the present δ equals zero. It is easy to check that the interest rate equals 9.24% in that economy if the representative agent would be neutral to ambiguity. Suppose alternatively that she has constant absolute ambiguity aversion (CAAA) with $A = 2.11$, i.e., $\phi(U) = -\exp(-2.11U)$. Then, tedious computations lead to the conclusion that the socially efficient discount rate should be exactly zero in that economy: $r_t = 0$! Thus, this example demonstrates that DAAA is not enough to guarantee that ambiguity about future consumption reduces the discount rate. ■

6 The pessimism effect

This counter-example can be explained by the presence of a second effect, the pessimism effect. In the pricing formula (7), the expected marginal utility is computed by using the distorted random variable $\widetilde{c}_t^{\circ}$ rather than the original $\widetilde{c}_t$. The distortion of these implicit beliefs depends upon the degree of ambiguity aversion and is governed by rule (9). This section is devoted to characterize how the distortion affects the discount rate.

If this distortion is pessimistic in the sense of deteriorating the implicit distribution according to FSD, this pessimism effect would go with the prudence effect to reduce the socially efficient discount rate. To examine this specific question, we begin with the comparison of the distorted probabilities $\overset{\circ}{q} = (\overset{\circ}{q}_1, ..., \overset{\circ}{q}_n)$ to the original probabilities $q = (q_1, ..., q_n)$.

Suppose that priors are ranked in such a way that $Eu(\widetilde{c}_{t1}) \leq Eu(\widetilde{c}_{t2}) \leq ... \leq Eu(\widetilde{c}_{tn})$, i.e. in such a way that the agent always prefers a larger θ. We hereafter show that ambiguity aversion is equivalent to a distortion of the prior belief on parameter $\widetilde{\theta}$ in the sense of the Monotone Likelihood Ratio Order (MLR). By definition, a shift of beliefs from q to $\overset{\circ}{q}$ entails a deterioration in the sense of the *monotone likelihood ratio ordering* (MLR) if $q^{\circ}_{\widetilde{\theta}}/q_{\widetilde{\theta}}$ and $\tilde{\theta}$ are anti-comonotonic. Observe from (9) that $q^{\circ}_{\theta}/q_{\theta}$ is proportional to $\phi'(Eu(\widetilde{c}_{t\theta}))$. Thus, since ϕ' is decreasing, we know that $q^{\circ}_{\widetilde{\theta}}/q_{\widetilde{\theta}}$ and $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ are anti-comonotonic. By transitivity, we can state the following.

Lemma 2 *The subsequent conditions are equivalent:*

Corollary 1 *1. Beliefs q° are dominated by q in the sense of the monotone likelihood ratio order for any set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ such that $Eu(\widetilde{c}_{t1}) \leq Eu(\widetilde{c}_{t2}) \leq ... \leq Eu(\widetilde{c}_{tn})$.*

2. ϕ is concave.

This result has a very intuitive interpretation. Ambiguity aversion is characterized by an MLR-dominated shift in the prior beliefs. In other words, it biases beliefs by favoring the worse marginals in a very specific sense: if the agent prefers marginal $\widetilde{c}_{t\theta}$ to marginal $\widetilde{c}_{t\theta'}$, then, the ambiguity-averse representative agent increases the implicit prior probability q°_{θ} relatively more than the implicit prior probability $q^{\circ}_{\theta'}$. This gives some flesh to our terminology in which we refer to a pessimism effect for the distortion of implicit beliefs. This result generalizes – and build a bridge with – the maxmin case where all the weight is transferred to the worse θ.

Intuitively, this worsening of the future risk should induce the representative consumer to raise his saving. However, the MLR deterioration in the distribution $\widetilde{\theta}$ of the priors is not enough to ensure an unambiguously negative pessimism effect on the socially efficient discount rate, as counterexample 1 tells us. This would require that the probability distortion raises the unconditional expected marginal utility, which would be the case if it would overweight the scenarios that yield the larger conditional expected marginal utility. The above lemma says something different: it states that the probability distortion overweight the scenarios that yield the smaller expected utility. To solve this problem, we need that the conditional Eu and Eu' be ranked in opposite directions.

Lemma 3 *The following two conditions are equivalent:*

1. *The pessimism effect reduces the discount rate, i.e. $Eu'(\overset{\circ}{\widetilde{c}}_t) \geq Eu'(\widetilde{c}_t)$, for all ϕ increasing and concave;*

2. *$E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are anti-comonotonic.*

Proof: To prove that $2 \Rightarrow 1$, suppose that $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ be anti-comonotonic. Since ϕ' is decreasing, our assumption implies that $\phi'(E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right])$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are comonotonic. By the covariance rule, it implies that

$$
\begin{aligned}
Eu'(\overset{\circ}{\widetilde{c}}_t) &= \frac{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right) Eu'(\widetilde{c}_{t\theta})}{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)} \\
&\geq \frac{\left[\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)\right]\left[\sum_{\theta=1}^{n} q_\theta Eu'(\widetilde{c}_{t\theta})\right]}{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)} \\
&= \sum_{\theta=1}^{n} q_\theta Eu'(\widetilde{c}_{t\theta}) = Eu'(\widetilde{c}_t),
\end{aligned}
$$

In order to prove that $1 \Longrightarrow 2$, suppose by contradiction that $Eu(\widetilde{c}_{t1}) < Eu(\widetilde{c}_{t2}) < ... < Eu(\widetilde{c}_{tn})$, but there exists $\theta \in [1, n-1]$ such that $Eu'(\widetilde{c}_{t\theta}) \leq Eu'(\widetilde{c}_{t\theta+1})$. Then, consider any increasing and concave ϕ that is locally linear for all $U \leq Eu(\widetilde{c}_{t\theta})$ and for all $U \geq Eu(\widetilde{c}_{t\theta+1})$, and has a strictly negative derivative in between these bounds. For any such function ϕ, we have that $\phi'(E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right])$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are anti-comonotonic. Using the covariance rule as above, that implies that $Eu'(\overset{\circ}{\widetilde{c}}_t) < Eu'(\widetilde{c}_t)$, a contradiction. ■

It seems natural that if u is increasing and u' is decreasing, their expectations should rank lotteries in opposite direction. The theory of stochastic dominance tells us that this is not so easy! We are looking for families of lotteries $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ and families of utility functions u such that u and $-u'$ "agree" on a ranking of these lotteries. The simplest case is when $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to first-degree stochastic dominance (FSD), i.e. when $Ef(\widetilde{c}_{t\theta})$ is increasing in θ for all increasing functions f. Taking $f = u$ and $f = -u'$, two increasing functions, directly implies that condition 2 in Lemma 3 is satisfied under that condition. However, ranking the priors according to FSD is quite restrictive, so it would be better to extend this result to a weaker stochastic order, as the second-degree stochastic dominance order (SSD). This means that $Ef(\widetilde{c}_{t\theta})$ is increasing in θ for all increasing and concave functions f. If we assume that u has a convex derivative, that is, assuming that the representative agent is prudent, implies that $f = -u'$ is increasing and concave. Thus, condition 2 in Lemma 3 is again satisfied in that case. This yields the following proposition.

Proposition 2 *The pessimism effect reduces the socially efficient discount rate if*

- *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to FSD.*
- *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to SSD and* u *exhibits prudence.*

The second sufficient condition relax the constraint on the structure of the ambiguity, but it constrains the set of acceptable risk attitudes which must satisfy prudence in addition to risk aversion. In the absence of ambiguity, being prudent means that the agent would like to save more if a zero-mean risk is added to her wealth (Leland (1968), Drèze and Modigliani (1972)).

In the following proposition, we propose a third pair of sufficient conditions. Compared to the SSD/prudence condition, this third pair relax the condition that the priors must be ranked according to SSD, but it is more restrictive on the set of acceptable utility function, since prudence is replaced by the stronger DARA condition. We use a stochastic order introduced by Jewitt (1989).

Definition 1 *We say that $\widetilde{c}_{\theta'}$ dominates $\widetilde{c}_{\theta}$ in the sense of Jewitt if the following condition is satisfied: for all increasing and concave u, if agent u prefers $\widetilde{c}_{\theta'}$ to $\widetilde{c}_{\theta}$, then all agents more risk-averse than u also prefer $\widetilde{c}_{\theta'}$ to $\widetilde{c}_{\theta}$.*

Of course, from the definition itself, if $\widetilde{c}_{\theta'}$ dominates $\widetilde{c}_{\theta}$ in the sense of SSD, this preference order also holds in the sense of Jewitt, thereby showing that this order is weaker than SSD. Jewitt (1989) shows that distribution function $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of Jewitt if and only if the following condition holds: there exists some w in their joint support $[a, b]$, such that

$$\int_a^x (F_{t\theta'}(z) - F_{t\theta}(z))dz \geq 0 \qquad \text{for all} \qquad x \in [a, w], \tag{12}$$

$$\int_a^w (F_{t\theta'}(z) - F_{t\theta}(z))dz = 0 \tag{13}$$

$$\int_a^x (F_{t\theta'}(z) - F_{t\theta}(z))dz \qquad \text{is non-increasing} \quad \text{on } [w, b]. \tag{14}$$

Two random variables fulfill Definition **??** if there exists a consumption level w in their support such that, conditional on the outcome being lower than w, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of SSD, whereas conditional on the outcome being higher than w, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of FSD. Observe that second-degree stochastic dominance is indeed stronger than Jewitt's ordering, since SSD is contained in Definition **??** as a special case when we pick $w = b$.

Proposition 3 *The pessimism effect reduces the socially efficient discount rate if the set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to Jewitt's stochastic order and u exhibits decreasing absolute risk aversion.*

Proof: Decreasing absolute risk aversion means that $v = -u'$ is more concave than u in the sense of Arrow-Pratt. By definition of the Jewitt's stochastic order, it implies that $Eu(\widetilde{c}_{t\theta'}) \geq Eu(\widetilde{c}_{t\theta})$ implies that $Ev(\widetilde{c}_{t\theta'}) \geq Ev(\widetilde{c}_{t\theta})$, or equivalently, that $Eu'(\widetilde{c}_{t\theta'}) \leq Eu'(\widetilde{c}_{t\theta})$. Thus Eu and Eu' are anti-comonotonic. Using Lemma 3 concludes the proof. ■

Combing Lemma 1 with Propositions 2 and 3 yields our main result.

Proposition 4 *Suppose that the representative agent exhibits non increasing absolute ambiguity aversion (DAAA). Then, ambiguity aversion reduces the socially efficient discount rate if one of the following conditions holds:*

1. *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to FSD and* u *is increasing and concave.*

2. *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to SSD and* u *is increasing, concave, and exhibits prudence.*

3. *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to Jewitt (1989) and* u *is increasing and concave, and exhibits DARA.*

Observe that the result in our analytical example in Section 3 fits into condition *1*. A mere translation in the distribution constitutes a first-degree stochastic dominance. Yet, in many circumstances, the degrees of riskiness also differ across the plausible distributions, usually implying that the plausible prior distributions cannot be ranked according to FSD. Condition *2* provides a sufficient condition on risk attitudes if marginals can only be ranked according to second-degree stochastic dominance, which contains Rothschild-Stiglitz's increases in risk as a particular case. It turns out that in this case, in addition to risk-aversion, the representative agent should also be prudent. Note that even the weaker Jewitt-ordering from *3* only requires decreasing absolute risk aversion. This property is widely accepted in the economic literature and it is in particular compatible with the observation that more wealthy individuals tend to take more portfolio risk.[7]

7 The comparative statics of an increase in ambiguity aversion

Our results up to now characterize the effect of smooth ambiguity aversion on the equilibrium interest rate, starting from the ambiguity-neutral benchmark. This section is devoted to characterizing the effect of any increase in ambiguity aversion. For this purpose, consider two economies, $i = 1, 2$, which are identical up to the level of ambiguity aversion of the respective representative agent. In particular, suppose that the one in economy 2 is more ambiguity-averse, which means that $\phi_2(U) = k(\phi_1(U))$ for all U, with $k(\cdot)$ increasing and concave. According to the adjusted pricing formula in (7) an increase in ambiguity aversion decreases the social discount rate if and only if

$$a_2 Eu'(\tilde{c}_t^2) \geq a_1 Eu'(\tilde{c}_t^1), \tag{15}$$

[7] Notice that counter-example 1, the two random variables $\widetilde{c}_{t1}$ and $\widetilde{c}_{t2}$ cannot be ranked according to SSD. This is why we obtain that ambiguity aversion raises the interest rate in spite of the fact that $u'(c) = c^{-2}$ is convex.

where a_i is defined as in (8) with ϕ being replaced by ϕ_i, and where $\tilde{c}_t^i$ is the implicit consumption distorted by weights q_θ^i, as in (9). Naturally, taking ϕ_1 linear, we retrieve condition (10) from the SEU benchmark.

At the outset, we are able to generalize our findings about the pessimism effect to any increase in ambiguity aversion.

Lemma 4 *The following two conditions are equivalent:*

1. *Beliefs q^2 are dominated by q^1 in the sense of the monotone likelihood ratio order for any set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ such that $Eu(\widetilde{c}_{t1}) \leq ... \leq Eu(\widetilde{c}_{tn})$.*
2. *$\phi_2 = k(\phi_1)$ is more ambiguity-averse than ϕ_1, meaning that k is increasing and concave.*

Proof: Note that we need to find that $q_{\tilde{\theta}}^2/q_{\tilde{\theta}}^1$ and $\tilde{\theta}$ are anti-comonotonic. Using (9), we can rewrite the ratio as

$$\frac{q_\theta^2}{q_\theta^1} = k'\left(\phi_1(Eu(\widetilde{c}_{t\theta}))\right) \frac{\sum_{\tau=1}^n q_\tau \phi_1'\left(Eu(\widetilde{c}_{t\tau})\right)}{\sum_{\tau=1}^n q_\tau \phi_2'\left(Eu(\widetilde{c}_{t\tau})\right)}.$$

The fraction on the right hand side does not change with θ. Furthermore, k' is decreasing in its argument. Finally, since the argument $\phi_1(Eu(\widetilde{c}_{t\theta}))$ is itself increasing with θ by assumption, we get the desired result. ■

This implies that under the stochastic order conditions of Proposition 4, more ambiguity aversion reinforces the pessimism effect, thereby tending to reduce the interest rate. However, it is clear from section 5 that an increase of ambiguity aversion has an ambiguous impact on the ambiguity *prudence* effect, i.e., on a_i. In particular, introducing *increasing* absolute ambiguity aversion will rather raise the interest rate if the representative agent is risk neutral. For small degrees of ambiguity, the impact of a change in ϕ on a depends upon its impact of the speed at which absolute ambiguity aversion decreases, as stated in the following lemma.

Lemma 5 *Consider a family of ambiguous environment parametrized by $k \in R$ and a vector $(u_1, ..., u_n) \in R^n$ such that $Eu(\widetilde{c}_{t\theta}(k)) = u_0 + ku_\theta$ for all θ. Let us define $a(k) = \Sigma_\theta q_\theta \phi'(Eu(\widetilde{c}_{t\theta}(k)))/\phi'(V(k))$, where $\phi(V(k)) = \Sigma_\theta q_\theta \phi(Eu(\widetilde{c}_{t\theta}(k)))$. We have that*

$$a(k) = 1 - \frac{1}{2}Var(ku_{\widetilde{\theta}})\frac{\partial}{\partial u_0}\left(\frac{-\phi''(u_0)}{\phi'(u_0)}\right) + o(k^2), \tag{16}$$

where $\lim_{k\to 0} o(k^2)/k^2 = 0$.

Proof: Observe first that $V(0) = u_0$, $V'(0) = Eu_{\widetilde{\theta}}$, and $V''(0) = Var(u_{\widetilde{\theta}})\phi''(u_0)/\phi'(u_0)$. Notice also that $a(0) = 0$. We have in turn that

$$a'(k) = \frac{E\left[u_{\widetilde{\theta}}\phi''(u_0 + ku_{\widetilde{\theta}})\right]\phi'(V(k)) - E\left[\phi'(u_0 + ku_{\widetilde{\theta}})\right]\phi''(V(k))V'(k)}{\left(\phi'(V(k))\right)^2}.$$

It implies that $a'(0) = 0$. Differentiating again the above equality at $k = 0$ yields

$$\begin{aligned} {\phi'_0}^2 a''(0) &= E\left[u_{\tilde{\theta}}^2\right]\phi'_0\phi'''_0 + \left(Eu_{\tilde{\theta}}\right)^2 \phi''^2_0 - \left(Eu_{\tilde{\theta}}\right)^2 \phi''^2_0 - \left(Eu_{\tilde{\theta}}\right)^2 \phi'_0\phi'''_0 - \phi'_0\phi''_0 V''(0) \\ &= \left(E\left[u_{\tilde{\theta}}^2\right] - \left(Eu_{\tilde{\theta}}\right)^2\right)\left(\phi'_0\phi'''_0 - \phi''^2_0\right), \end{aligned}$$

where $\phi_0^{(i)} = \phi^{(i)}(u_0)$. This implies that

$$a''(0) = -Var(u_{\tilde{\theta}})\frac{\partial}{\partial u_0}\left(\frac{-\phi''(u_0)}{\phi'(u_0)}\right).$$

The Taylor expansion of a yields $a(k) = a(0) + ka'(0) + 0.5k^2 a''(0) + o(k^2)$. Collecting the successive derivatives of a above concludes the proof. ■

A direct consequence of the above lemma is that, for small degrees of ambiguity, a_2 is larger than a_1 if and only if

$$\frac{\partial}{\partial u_0}\left(\frac{-\phi''_2(u_0)}{\phi'_2(u_0)}\right) \geq \frac{\partial}{\partial u_0}\left(\frac{-\phi''_1(u_0)}{\phi'_1(u_0)}\right), \tag{17}$$

i.e., if absolute ambiguity aversion decreases more rapidly under ϕ_2 than under ϕ_1, locally at the ambiguity-free expected utility level u_0. Thus, for small degrees of ambiguity, a change in the attitude towards ambiguity from ϕ_1 to ϕ_2 yields an ambiguity prudence effect that tends to reduce the interest rate if condition (17) is satisfied.

Unfortunately, it is not true in general that condition (17) for all u_0 is sufficient for $a_2 \geq a_1$ for all ambiguous environments. To show this, let us consider the following counter-example.

> **Counter-example 2.** Let $\phi(U) = U^{1-\eta}/(1-\eta)$ defined on R^+. Observe that $-\phi''(U)/\phi'(U) = \eta/U$ is positive and decreasing in its domain. Moreover, an increase in η raises ambiguity aversion, and the speed at which absolute ambiguity aversion decreases with U. From Proposition 5, it implies that a is increasing in η when the risk on U is small. We show that this is not true for large degrees of ambiguity. Suppose that $u(c) = c$ and that there are $n = 2$ equally likely plausible probability distributions, with $\bar{c}_1 = 0.5$ and $\bar{c}_2 = 1.5$. Suppose also that $\delta = 0.25$. In Figure 1, we draw the socially efficient discount rate r_t for $t = 1$ as a function of the degree of relative ambiguity aversion η. As stated in Proposition 1, we see that the discount rate $r_1(\eta)$ under ambiguity aversion is always smaller than under ambiguity neutrality ($r(0)$). However, the relationship between the discount rate and the degree of ambiguity aversion is not monotone. For example, increasing relative ambiguity aversion from $\eta = 3$ to any larger level *raises* the discount rate.

With a counter-example based on the most common family of utility functions $\phi(U) = U^{1-\eta}/(1-\eta)$, there is no hope to get convincing sufficient conditions to guarantee that a marginal change in the attitude towards ambiguity

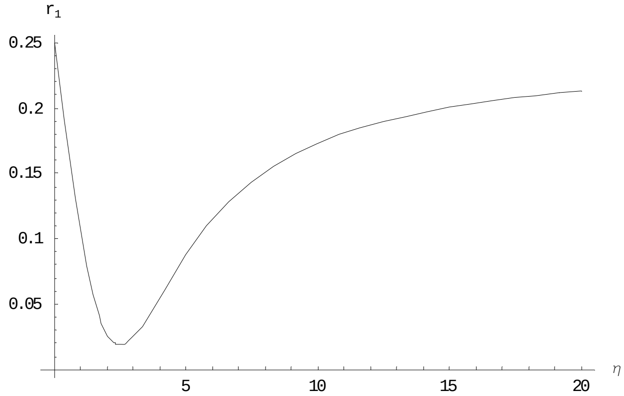

Figure 1: The discount rate as a function of relative ambiguity aversion. We assume that $\phi(U) = U^{1-\eta}/(1-\eta)$, $u(c) = c$, $\delta = 0.25$, $\overline{c}_1 = 0.5$, $\overline{c}_2 = 1.5$ and $p = 0.5$.

raises savings and reduces the equilibrium interest rate. We are left with three strategies to sign its effect on a:

- The degree of ambiguity aversion is small and condition (17) is satisfied;
- The initial degree of ambiguity aversion is small, so that Proposition 1 can be used as an approximation;
- The initial ϕ_1 function exhibits non decreasing ambiguity aversion, whereas the final ϕ_2 function exhibits non increasing ambiguity aversion. This implies that $a_1 \leq 1 \leq a_2$.

Any of these three conditions is sufficient to obtain that a_2 is larger than a_1. Combining it with any of the three conditions of Proposition 4 is sufficient to guarantee that a marginal increase in ambiguity aversion reduces the socially efficient discount rate.

8 Numerical illustrations

8.1 The power-power normal-normal case

As observed in Section 3, we can solve analytically for the socially efficient discount rate by taking a "power-power" specification. That is, CRRA risk preferences and CRAA ambiguity preferences allow for an exact solution if both ambiguity and the logarithm of consumption are normally distributed. In accordance with Weitzman (2007b) who considered a similar model under ambiguity neutrality, we will establish the following parameter values as a benchmark. Consider a "quartet of twos". Namely a rate of pure preference for the present

$\delta = 2\%$, a degree of relative risk aversion $\gamma = 2$, a mean growth rate of consumption $g = 2\%$, and standard deviation of growth $\sigma = 2\%$. We can rewrite the Ramsey rule 5 as

$$r_t = 5.88\% - 3\sigma_0^2 t(1 + \eta/2). \tag{18}$$

Hence, in the absence of ambiguity, the Ramsey rule prescribes a flat discount rate of 5.88%. We introduce ambiguity by assuming that the growth-trend has a normal distribution with standard deviation $\sigma_0 = 1\%$. In other words, consumers believe that with a 95% probability, the growth trend lies between 0% and 4%. Even in the absence of ambiguity aversion ($\eta = 0$), the introduction of ambiguous probabilities affects the term structure of discount rates, as shown by Weitzman (2007a) and Gollier (2007b). This is due to the fatter tails that this ambiguity yields for the distribution of future consumption. Indeed, ambiguity increases the volatility of log-consumption at date t by $\sigma_0^2 t^2$. Accordingly, the prudent agent wants to save more for more distant futures, and the interest rate should fall with the time-horizon.

If in addition the agent exhibits ambiguity aversion, the social discount rate decreases more quickly, as seen from equation (18). Ambiguity aversion has no effect on the short term interest rate in this specification.

In order to calibrate the model, one needs to evaluate the degree of relative ambiguity aversion η. To do this, let us consider the following thought experiment.[8] Suppose that the growth rate of the economy over the next 10 years is either 20% with probability π, otherwise it equals 0%. Suppose that the true value of π is unknown. Rather, it is uniformly distributed on $[0, 1]$, as in the Ellsberg game in which the player has no information about the proportion of black and white balls in the urn. Let us define the certainty equivalent growth rate $CE(\eta)$ as the sure growth rate of the economy that yields the same welfare than in the ambiguous risky environment described above. It is implicitly defined by the following condition:

$$\left(k\frac{(1+CE)^{1-\gamma}}{1-\gamma}\right)^{1-k\eta} = \int_0^1 \left(k\left(\pi\frac{1.2^{1-\gamma}}{1-\gamma} + (1-\pi)\frac{1^{1-\gamma}}{1-\gamma}\right)\right)^{1-k\eta} d\pi,$$

where γ is set at $\gamma = 2$. In Figure 2, we have drawn the certainty equivalent as a function of the degree of relative ambiguity aversion. In the absence of ambiguity aversion (or if π is known to be equal to 50%), the certainty equivalent growth rate equals $CE(0) = 9.1\%$. Surveying experimental studies based on the Ellsberg game, Camerer (1999) reports ambiguity premia $CE(0) - CE(\eta)$ that are in the order of magnitude of 10% of the expected of the expected value in this kind of Ellsberg-style uncertainty. In this environment, this yields a reasonable ambiguity premium of 10% of 10%, i.e., a 1% reduction in the growth rate. Thus, ambiguity aversion should reduce the certainty equivalent to 9.1% to around 8%. From Figure 2, this is compatible with a degree of relative ambiguity aversion between $\eta = 5$ and $\eta = 10$.

[8] It is based on a 10-year version of the calibration exercice performed by Collard, Mukerji, Sheppard and Tallon (2008), who considered a power-exponential specification.

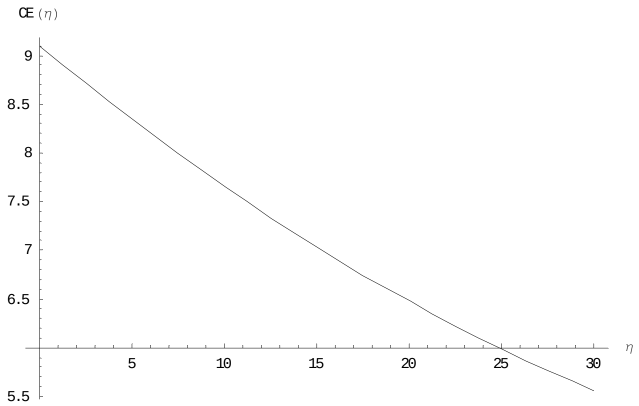

Figure 2: The certainty equivalent growth rate CE (in %) as a function of relative ambiguity aversion η. We assume that the growth rate is either 20% or 0% respectively with probability π and $1-\pi$, with $\pi \sim U(0,1)$. Relative risk aversion equals $\gamma = 2$.

Table 1 reports the values of efficient rates for projects with maturity 10 and 30 respectively.

Table 1: The social discount rate at the benchmark "quartet of twos", with $\sigma_0 = 1\%$.

t	$\boldsymbol{\eta = 0}$	$\boldsymbol{\eta = 5}$	$\boldsymbol{\eta = 10}$
10	5.58%	4.83%	4.08%
30	4.98%	2.73%	0.48%

Whereas the ambiguity aversion has no effect on the short term interest rate, its effect on the long rate is important. The discount rate for a cash flow occurring in 30 years is reduced from 4.98% to 2.73% when relative ambiguity aversion goes from $\eta = 0$ to $\eta = 5$.

The discrepancies between the settings call for an empirical separation between standard risk and ambiguity in an economy. While the former shifts the level of the yield curve, the latter determines its slope. A negative slope tends to increase the relative importance of long-term costs and benefits. We want to stress here the amplification potential of ambiguity aversion for the evaluation of long-term projects.

8.2 An AR(1) process for log consumption with an ambiguous long term trend

Clearly, while delivering simple expressions, our benchmark economy abstracts from rich consumption dynamics, notably any serial correlation. It is thus not surprising that our predictions do not fare well when confronted with the term structure of interest rates observed on financial markets. Thus, we will relax the assumption of uncorrelated growth rates and allow for persistence of shocks, as in Collard, Mukerji, Sheppard and Tallon (2008) and Gollier (2008). We hereafter show that this model can produce the desired non-linear term structure in the short run and the medium run. Still, in the limit, the model generates a linearly decreasing term structure in the long run.

Consider first an auto-regressive consumption process of order 1 à la Vasicek (1977), but in which the long term growth μ of log consumption around which the actual growth mean-reverts is uncertain:

$$\begin{aligned} \ln c_{t+1} &= \ln c_t + x_t \\ x_t &= \xi x_{t-1} + (1-\xi)\mu + \varepsilon_t \\ \varepsilon_t &\sim N(0,\sigma^2),\ \varepsilon_t \perp \varepsilon_{t'} \\ \mu &\sim N(\mu_0,\sigma_0^2), \end{aligned} \tag{19}$$

where $0 \le \xi \le 1$. That is, system (19) describes an AR(1) consumption process with unknown trend. The polar case without persistence ($\xi = 0$), amounts to the discrete time equivalent of the geometric Brownian motion considered in Section 3 and calibrated here above. In contrast, $\xi = 1$ describes shocks on the growth of log consumption that are fully persistent. We can follow the same lines as we did to obtain equation (4) to obtain the following generalization:

$$r_t = \delta + \gamma\frac{EX_t}{t} - \frac{1}{2}\gamma^2\frac{Var\left[X_t \mid \mu\right] + Var\left[E[X_t \mid \mu]\right]}{t} - \frac{1}{2}\eta\left|1-\gamma^2\right|\frac{Var\left[E[X_t \mid \mu]\right]}{t}, \tag{20}$$

where X_t is defined as

$$X_t = \ln c_t - \ln c_0 = \mu t + (x_{-1} - \mu)\frac{\xi(1-\xi^t)}{1-\xi} + \sum_{\tau=1}^{t}\frac{1-\xi^\tau}{1-\xi}\varepsilon_{t-\tau}.$$

It yields

$$\frac{EX_t}{t} = \mu_0 + (x_{-1} - \mu_0)\frac{\xi(1-\xi^t)}{t(1-\xi)},$$

$$\frac{Var\left[X_t \mid \mu\right]}{t} = \frac{\sigma^2}{(1-\xi)^2} + \sigma^2\frac{\xi(1-\xi^t)}{t(1-\xi)^3}\left[\frac{\xi(1+\xi^t)}{1+\xi} - 2\right],$$

and

$$\frac{Var\left[E[X_t \mid \mu]\right]}{t} = \frac{\sigma_0^2}{t}\left(t - \frac{\xi(1-\xi^t)}{1-\xi}\right)^2.$$

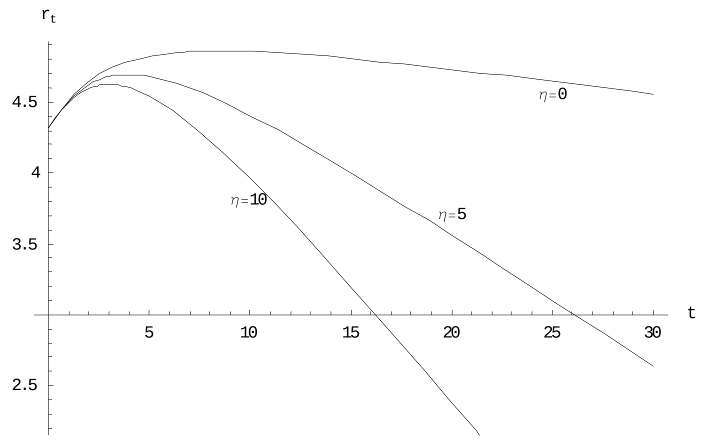

Figure 3: The term structure of discount rates in the case of an AR(1) with an ambiguous long term trend, with $\delta = 2\%$, $\gamma = 2$, $\mu_0 = 2\%$, $\sigma = 2\%$, $\sigma_0 = 1\%$, $x_{-1} = 1\%$, and $\xi = 0.7$.

To illustrate, suppose that $\delta = 2\%$, $\gamma = 2$, $\mu_0 = 2\%$, $\sigma = 2\%$, $\sigma_0 = 1\%$,and $x_{-1} = 1\%$. Following Backus, Foresi and Telmer (1998) for example, suppose also that $\xi = 0.7$ year^{-1}, such that a shock has a half-life of 3.2 years. In Figure 3, we have drawn the term structure of discount rates for 3 different degrees of ambiguity aversion: $\eta = 0$, 5, and 10. We can see that, as in the absence of persistence, the role of ambiguity aversion is to force a downward slope of the yield curve for long time horizons. This is confirmed by the following observation:

$$\lim_{t\to\infty} \frac{\partial r_t}{\partial t} = -\frac{1}{2}\eta \left|1-\gamma^2\right| \sigma_0^2.$$

8.3 An AR(1) process for log consumption with an ambiguous degree of mean reversion

Consider alternatively an auto-regressive consumption process of order 1 with a known long term trend, but in which there is some ambiguity about the coefficient of mean reversion:

$$\begin{aligned} \ln c_{t+1} &= \ln c_t + x_t \\ x_t &= \xi x_{t-1} + (1-\xi)\mu + \varepsilon_t \\ \varepsilon_t &\sim N(0,\sigma^2),\ \varepsilon_t \perp \varepsilon_{t'} \\ \xi &\sim U(\underline{\xi},\overline{\xi}). \end{aligned}$$

There is no analytical solution for the discount rate, which must be computed numerically by estimating the following two terms that appear in equation (3)

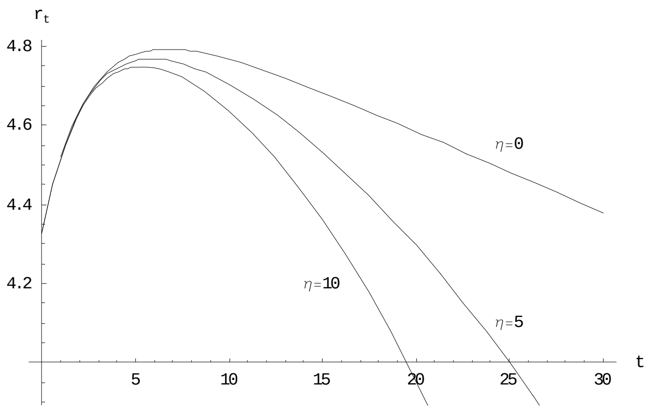

Figure 4: The term structure of discount rates in the case of an AR(1) with an ambiguous mean reversion coefficient, with $\delta = 2\%$, $\gamma = 2$, $\mu = 2\%$, $\sigma = 2\%$, $x_{-1} = 1\%$, and $\xi \sim U(0.5, 0.9)$.

(we normalized $c_0 = 1$):

$$\frac{E\phi'\left(Eu\right)Eu'}{u'(c_0)} = bE\exp\left(-(\gamma + k\eta(1-\gamma))E\left[X_t \mid \xi\right] + \frac{1}{2}\left(\gamma^2 - k\eta(1-\gamma)^2\right)Var\left[X_t \mid \xi\right]\right)$$

$$\phi'(V_t(0)) = b\left(E\exp\left((1-k\eta)(1-\gamma)E\left[X_t \mid \xi\right] + \frac{1}{2}(1-k\eta)(1-\gamma)^2 Var\left[X_t \mid \xi\right]\right)\right)^{\frac{-k\eta}{1-k\eta}}.$$

In Figure 4 we draw the term structure of the discount rate with the same parameter values as in the previous section, except that $\mu = 2\%$ and $\xi \sim U(0.5, 0.9)$. As before, longer time horizons yields more ambiguity in the set of plausible distributions of consumption, which implies that ambiguity aversion has a stronger negative impact on the discount rates associated to these longer durations.

9 Conclusion

The present paper has shown how ambiguity-aversion changes the way one should discount future costs and benefits of investment projects. In line with recent literature, our analysis suggests that parameter uncertainty might well be decisive in long-term policy appraisals. Nevertheless, we found that, in general, it is not true that ambiguity aversion always decreases the socially efficient discount rate. We have, however, identified moderate requirements on risk-attitudes and the statistical relation among prior distributions, such that decreasing ambiguity aversion should induce us to use a smaller discount rate. Our numerical illustrations indicate that the effect of ambiguity aversion on the discount rate is large, in particular for longer time horizons.

References

Backus, D.K., B.R. Routledge and S.E. Zin, (2004), Exotic preferences for Macroeconomists, *NBER Macroeconomics Annual 2004*, pp. 319-414.

Camerer, C., (1999), Ambiguity-aversion and non-additive probability: Experimental evidence, models and applications, in Uncertain decisions: Bridging theory and experiments, ed. by L. Luini, pp. 53-80, Kluwer Academic Publishers.

Camerer, C. and M. Weber, (1992), Recent developments in modelling preferences: uncertainty and ambiguity, *Journal of Risk and Uncertainty*, 5, 325–370.

Cochrane, J., (2001), *Asset Pricing*, Princeton University Press.

Collard, F., S. Mukerji, K. Sheppard and J.-M. Tallon, (2008), Ambiguity and the historical equity premium, mimeo, Toulouse School of Economics.

Drèze, J.H. and F. Modigliani, (1972), Consumption decisions under uncertainty, *Journal of Economic Theory*, 5, 308–335.

Ellsberg, D., (1961), Risk, ambiguity, and the savage axioms. *Quarterly Journal of Economics*, 75, 643-669.

Gilboa, I. and D. Schmeidler (1989), Maxmin expected utility with a non-unique prior. *Journal of Mathematical Economics*, 18, 141–153.

Gollier, C., and M.S. Kimball, (1996), Toward a systematic approach to the economic effects of uncertainty: characterizing utility functions, Discussion paper, University of Michigan.

Gollier, C., (2001), *The Economics of Risk and Time*, MIT Press, Cambridge.

—— (2002), Time horizon and the discount rate, *Journal of Economic Theory*, 107, 463-473.

—— (2006), Does ambiguity aversion reinforce risk aversion? Applications to portfolio choices and asset pricing, IDEI Working Paper, n. 357.

—— (2007a), Whom should we believe? Aggregation of heterogeneous beliefs, *Journal of Risk and Uncertainty*, 35, 107-127.

—— (2007b), The consumption-based determinants of the term structure of discount rates, *Mathematics and Financial Economics*, 2, 81–101.

—— (2008), Discounting with fat-tailed economic growth, *Journal of Risk and Uncertainty*, forthcoming.

Groom, B., P. Koundouri, E. Panopoulou and T. Pantelidis, (2004), Model selection for estimating certainty equivalent discount, mimeo, UCL, London.

Hansen, L. and K. Singleton, (1983), Stochastic consumption, risk aversion and the temporal behavior of assets returns, *Journal of Political Economy*, 91, 249-268.

Jewitt, I., (1989), Choosing between risky prospects: the characterization of comparative statics results, and location independent risk, *Management Science*, 35, 60–70.

Jouini, E., C. Napp and J.-M. Marin, (2007), Discounting and divergence of opinion, mimeo.

Hansen, L.P. and T.J. Sargent (2007a), Recursive robust estimation and control without commitment, *Journal of Economic Theory*, 136, 1–27.

—— (2007b), *Robustness*, Princeton University Press, Princeton.

Kimball, M.S. (1988), Farmers' cooperatives as behavior towards risk. *American Economic Review* 78(1), 224-232.

Kimball, M.S., (1990), Precautionary savings in the small and in the large, *Econometrica*, 58, 53–73.

Klibanoff, P., M. Marinacci, and S. Mukerji, (2005), A smooth model of decision making under ambiguity. *Econometrica*, 73(6), 1849–1892

—— (2007), Recursive smooth ambiguity preferences. mimeo.

Leland, H., (1968), Savings and uncertainty: The precautionary demand for savings, *Quarterly Journal of Economics*, 45, 621–36.

Lomborg, B., (2004, ed.), *Global Crises, Global Solutions. Copenhagen Consensus Challenge.* Cambridge University Press.

Lucas, R., (1978), Asset prices in an exchange economy, *Econometrica*, 46, 1429–1446.

Ramsey, F.P., (1928), A mathematical theory of savings, *The Economic Journal*, 38, 543-59.

Rothschild, M. and J. Stiglitz, (1970), Increasing risk: I. A definition, *Journal of Economic Theory*, 2, 225–243.

Savage, L.J., (1954), *The foundations of statistics*, New York: Wiley. Revised and Enlarged Edition, New York: Dover (1972).

Stern, N., (2007), *The Economics of climate change: The Stern review.* Cambridge University Press, Cambridge.

Vasicek, 0., (1977), An equilibrium characterization of the term structure, *Journal of Financial Economics*, 5, 177-188.

Weitzman, M.L., (1998), Why the far-distant future should be discounted at its lowest possible rate?, *Journal of Environmental Economics and Management*, 36, 201–208.

—— (2001), Gamma discounting, *American Economic Review*, 91, 260-271.

—— (2007a), Subjective expectations and asset-return puzzles, *American Economic Review*, 97, 1102–1130.

—— (2007b), The Stern review of the Economics of climate change, *Journal of Economic Literature*, forthcoming.