# Optimal positive thinking and decisions under risk

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**Abstract**

We examine a static one-riskfree-one-risky asset portfolio choice when the investor's well-being is affected by the anticipatory feelings associated to potential capital gains and losses. These feelings can be manipulated by the choice of subjective beliefs on the distribution of returns. However, the bias of these endogeneous subjective beliefs induces the choice of a portfolio that is suboptimal with respect to the objective expected utility of final wealth. We characterize the structure of these optimal beliefs. We show that the subjective probability distribution must be degenerated at the lower and upper bounds of feasible returns. When the intensity of anticipatory feelings is small, the formation of beliefs must be biased in favor of optimism, which implies an increase in the equilibrium demand for the risky asset. We also show that the optimal beliefs are approximately independent of the investor's degree of risk aversion.

**Keywords**: anticipatory feelings, portfolio choice, overconfidence, method Coué.

"Supervise especially your thoughts, because it is they which determine your life." Old Testament

# 1 Introduction

In the late XIXth century, Emile Coué, a french psychologist at the University of Nancy, promoted the idea that learning to control our thoughts can do much to improve well-being. Positive thinking does improve the quality of life of patients with a life-threatening disease by inducing a them to reduce their subjective probability of dying. It also seems to generate a positive placebo effect on the objective health risk. Nowadays, we hear medical practitioners telling their patients how important it is to have a positive attitude towards their health hazard. The so-called "method Coué" has however an important undesirable effect. By artificially downgrading the risk, the patient may in consequence prefer to spend less effort to fight the illness. Psychotherapists are well aware of the problem, as most of them forcefully claim that the method does never replace the medical treatment.

In this paper, we want to apply these ideas to other choice problems under uncertainty. In particular, we examine the portfolio choice problem of risk-averse consumers.[1] Of course, they cannot be any placebo effect in this case, since asset returns are totally out of control of individual investors. As in Brunnermeier and Parker (2003), we add two ingredients in the standard portfolio choice problem. First, we recognize that current felicity is affected by the anticipation of future pleasures and displeasures. As a consequence, controlling our thoughts about the likelihood of these events has a direct effect on welfare. In a portfolio context, positive thinking implies a mental manipulation of the objective probability distribution of assets return. If the investor has a positive demand for stocks, method Coué means increasing the subjective probability of a positive excess return. The undesirable effect of positive thinking is that this manipulation of beliefs is likely to affect the asset allocation of the investor. This in turn affects negatively the investor's

[1] Alternative interpretations of our choice problem can be found in insurance economics and in the theory of investment. A consumer faces a risk of loss for which there exists an insurance market offering proportional insurance contracts with an actuarially unfair tariff. The problem of the consumer is to select the rate of insurance coverage for the risk. In the theory of investment, a risk-averse entrepreneur with a linear technology must determine the optimal capacity of production under uncertainty about the output price.

future felicity. We assume that the investor selects subjective beliefs in order to maximize his lifetime well-being which is an increasing function of both current and future felicities. Because positive thinking raises current felicity but reduces future felicity, the problem of method Coué is to determine the best compromise between these two opposite forces.

This work departs from the long tradition in economics to measure an individual's lifetime utility has a discounted sum of his flow of felicity, as described for example by Samuelson (1937). This tradition is incompatible with the idea that happiness is extracted not only from the immediate consumption of goods and services, but also from thoughts. This is particularly the case for thoughts related to savoring the possibility of future pleasant events, or to fearing anxiously the consequences of adverse ones. Imagination is one of the principal forces of human beings. Anticipatory feelings have been incorporated in preferences by Caplin and Leahy (2001) who considered belief-dependent felicity functions. In the economic literature, Akerlof and Dickens (1982) are probably the first to assume that subjective beliefs are derived from a welfare-maximizing process.

The distortion of beliefs affects the individual's decision process. There is an important literature on the effect of a change in the distribution of risk on the optimal exposure to it. In the case of the one-riskfree-one-risky portfolio choice problem that we examine in this paper, Rothschild and Stiglitz (1971) have shown that an increase in risk in the distribution of returns of the risky asset does not necessarily reduces the demand for the risky asset. In the same fashion, Fishburn and Porter (1976) have shown that a first-order stochastically favorable shift in this distribution can reduce the demand for the risky asset by some risk-averse investors. More recently, Abel (2002) considered the effect of distorted beliefs on the equilibrium asset prices. Abel defined optimism by using very specific first-order stochastic dominant shifts in the subjective distribution of the risky asset's payoffs. He showed that optimism raises the demand for this asset, thereby reducing the equity premium. This observation is particularly important in our framework as we will show that risk-averse agents optimally distort the distribution of the risky asset in an optimistic way.

Our model is a two-date version of the dynamic model examined by Brunnermeier and Parker (2003), hereafter denoted BP. Because consumption takes place only at date 2 in our model, we are not able to examine the effect of method Coué on savings and consumption. Whereas BP assume that individuals extract as much lifetime utility from the date-1 anticipatory feelings than from the date-2 consumption, we assume more generally that the consumer's lifetime utility is a weighted sum of the date-1 felicity extracted from savoring and of the date-2 felicity of consumption. The weight measures the intensity of anticipatory feelings, anxiety and savoring. This parameter of preferences can take any value between 0 and 1, whereas BP assume that it's equal to 1/2. Assuming without loss of generality that the objective expected excess return of the risky asset is positive, any risk-averse investor with a zero intensity of anticipatory feeling will have a positive demand $\alpha^*$ for the risky asset. The main result of BP is to show that risk-averse investors with anticipatory feelings will distort beliefs in such a way either to increase their demand of the risky asset above $\alpha^*$, or to go short on the risky asset. Our work goes into more details in the description of optimal beliefs and on the demand for the risky asset that they induce.

We first exploit the linearity of expected utility with respect to state probabilities to prove that the optimal subjective probability distribution must be degenerated with at most two atoms, i.e., optimal beliefs are binary. This result is true for any von Neumann-Morgenstern preference functional, any intensity of anticipatory feelings, and any objective distribution of the risky asset. In a second step, we show under weak restrictions on the utility function that investors select the two atoms that are at the bounds of the set of possible asset returns. In other words, optimally controlling thoughts lead individual to believe that only the smallest possible return and the largest possible return can have a positive probability to occur. This strong result is compatible with the idea introduced by Tversky and Kahneman (1992) that the worst and best outcomes receive particular attention of decision-makers. The cumulative prospect theory takes this into account by assuming an S-shaped transformation function of the objective cumulative distribution function. This is equivalent to transferring the probability mass from the interior of the support of the distribution to its lower and upper bounds.[2] We show in this paper that this distortion of probabilities can be explained by a welfare-optimizing process of human beings with von Neumann-Morgenstern preferences.

Given the fact that optimal beliefs are degenerated at the extreme events, the only remaining problem is to determine the subjective probability of the best state. When the intensity of anticipatory feelings is small, we show that the demand for the risky asset is larger than the demand that is optimal

[2] For more details, see for example Tversky and Wakker (1995) and Abdellaoui (2000).

under the objective distribution of excess returns. Thus, we eliminate the possibility allowed by BP that risk-averse investors go short on the risky asset. Moreover, we show that the optimal subjective probability of the large return and the demand for the risky asset are increasing in the intensity of anticipatory feelings.

Things are much more complex when we allow for larger intensities of anticipatory feelings. It is well-known that the maximum subjective expected utility of the investor is a convex function of his subjective probability distribution. For example, this explains why the value of information is always positive, or why refining the information structure à la Blackwell (1951) makes the decision-maker better off. The convexity of the felicity extracted from anticipatory feelings with respect to the subjective probability distribution alerts us about an important difficulty of the selection of optimal beliefs, since the objective function does not need anymore to be concave in the decision variables. In the extreme case where only anticipatory feelings matter for lifetime well-being, optimal beliefs degenerate to subjective certainty at either the worse or best possible return, yielding an infinite demand for the risky asset and unbounded well-being. When the intensity of anticipatory feelings is large but smaller than unity, the Inada assumption that marginal utility tends to infinity when consumption tends to zero guarantees that the actual demand for the risky asset will be small enough to yield positive consumption in all states with a positive objective probability. This implies that optimal beliefs cannot degenerate to certainty.

The nonconcavity of the consumer's lifetime objective may generate various interesting results. For example, we show that Head-or-Tail games with a fair coin can make consumers mutually better off in spite of their risk aversion. This requires that anticipatory feelings count more than actual consumption in measuring lifetime well-being. This implies that it has two symmetric maximal subjective probabilities $p_1 > 1/2$ and $p_2 = 1 - p_1 < 1/2$ for the Head state. Therefore, there exists a competitive equilibrium where half of the population bets on Head and selects subjective probability $p_1$ of Head, whereas the other half of the population bets on Tail and selects subjective probability $p_2$ of Head. Because all stakes optimally selected by consumers are equal, the market for bets clear at a zero participation price. This competitive equilibrium makes all agents better off compared to an economy where no such gambling opportunity is offered. Notice that when the intensity of anticipatory feelings is less than 1/2, the lifetime objective function of consumers is globally concave and no such bifurcation occurs.

The optimal beliefs are equal to the objective ones in this case, thereby contradicting BP's Proposition 1(ii) that states that optimal beliefs are always distortions of the objective probability distributions.

# 2 The model

Our model is static, with a decision date $t = 0$ and a consumption date $t = 1$. At date 0, the consumer selects an asset portfolio. The portfolio is liquidated at date 1, and its value is consumed. We consider an economy with two assets. The first asset is riskfree and yields a return that is normalized to 0 over the period. The second asset is risky. It yields a random excess return $\widetilde{x}$ at date 1. It is assumed that the excess return of the risky asset is bounded downwards by $a < 0$ and upwards by $b > 0$. There is an objective cumulative probability distribution $Q \in X[a,b]$ for $\widetilde{x}$. $X[a,b]$ denotes the set of cumulative distribution function whose support is in $[a,b]$:

$$X[a,b] = \left\{ F : [a,b] \to [0,1] \mid dF(x) \geq 0 \ \forall x \in [a,b], \quad \int_a^b dF(x) = 1 \right\}$$

The consumer has a von Neumann-Morgenstern utility function $u$ that is assumed to be twice differentiable, increasing and concave. We assume that the Inada conditions are satisfied, with $\lim_{c \to 0_+} u'(c) = +\infty$ and $\lim_{c \to \infty} u'(c) = 0$. The decision problem of the agent at date $t = 0$ is to determine the size $\alpha$ of his investment in the risky asset. Because his initial wealth is $w_0$, he invests the remaining $w_0 - \alpha$ in the riskfree asset. His final wealth at date 1 in state $s$ is therefore equal to $w_0 + \alpha x_s$. At decision date $t = 0$, the beliefs of the consumer is characterized by a subjective cumulative probability distribution $P \in X[a,b]$ that may differ from the objective probability distribution $Q$. Given these beliefs, the consumer selects the portfolio $(\alpha, w_0 - \alpha)$ that maximizes his subjective expected utility. We obtain the following decision problem:

$$S(P) = \max_{\alpha} \quad E_P u(w_0 + \alpha \widetilde{x}) = \int_a^b u(w_0 + \alpha x) dP(x). \tag{1}$$

The expectation operator $E_P$ refers to the subjective probability distribution $P$. The optimal demand for the risky asset as a function of the beliefs is denoted $\alpha(P)$. It satisfies the following first-order condition:

$$E_P \widetilde{x} u'(w_0 + \alpha(P)\widetilde{x}) = 0. \tag{2}$$

Because $E_P u(w_0 + \alpha \widetilde{x})$ is concave in $\alpha$, this first-order condition is necessary and sufficient for optimality. By the Inada condition, it must be true that $w_0 + \alpha(P)x > 0$ for all $x$ with a positive subjective probability $dP(x)$.

Because of the potential bias in the subjective beliefs, the objective expected utility of the consumer at date 1 may differ from $S(P)$. The objective expected utility of a consumer with beliefs $P$ equals

$$O(P) = E_Q u(w_0 + \alpha(P)\widetilde{x}) = \int_a^b u(w_0 + \alpha(P)x)dQ(x). \tag{3}$$

It is important to observe that the consumer's objective expected utility depends upon the subjective probability distribution $P$ only through the choice of the portfolio allocation induced by $P$.

We now specify the lifetime well-being of the consumer with subjective beliefs $P$. At date $t = 0$, the consumer savors his subjective future utility, yielding savoring felicity $S(P)$ at that date. At date $t = 1$, the agent extracts felicity $O(P)$ from consuming his terminal wealth. His lifetime well-being $W$ is assumed to be a convex combination of his felicity at these two dates:

$$W(P) = kS(P) + (1-k)O(P). \tag{4}$$

Parameter $k$ measures the intensity of anticipatory feelings in lifetime utility. When $k = 0$, the consumer has no anticipatory feeling at date 0. When $k = 1$, he extracts felicity just from savoring future consumption flows, and current consumption has no impact on felicity. Brunnermeier and Parker (2003) consider the special case with $k = 1/2$.

As justified in the introduction, we assume that prior to date $t = 0$, the agent uses method Coué to control his thoughts. He selects the beliefs $P$ that maximizes his lifetime well-being:

$$P^* = \max_{P \in X[a,b]} W(P). \tag{5a}$$

The optimal demand for the risky asset is $\alpha^* = \alpha(P^*)$. The main objective of the paper is to compare $P^*$ to $Q$, and $\alpha^*$ to $\alpha(Q)$.

# 3 Some basic properties of optimal beliefs

It is noteworthy that date-1 felicity depends upon beliefs $P$ only through its effect on the choice of the optimal portfolio $\alpha = \alpha(P)$ at date $t = 0$.

In general, there are more than one probability distribution that yield that optimal portfolio $\alpha$. Let $B(\alpha) \subset X[a,b]$ be the set of subjective cumulative probability distributions that yield the same optimal portfolio choice $\alpha$:

$$B(\alpha) = \{P \in X[a,b] \mid \alpha(P) = \alpha\}. \tag{6}$$

It implies that $O(P) = O(P')$ for all $(P, P')$ in $B(\alpha)$.

This observation has an important consequence on the structure of optimal beliefs. Consider the optimal demand $\alpha^* = \alpha(P^*)$ that is induced by the optimal subjective beliefs $P^*$. From the various subjective probability distributions $P$ that yields this demand $\alpha^*$, the one that is selected by the consumer prior to date 0 must maximize the date-0 anticipatory felicity $S(P)$, since they all yields the same date-1 felicity $O(P^*)$. In other words, it must be true that

$$P^* \in \arg \max_{P \in B(\alpha^*)} S(P). \tag{7}$$

Observe that this property of optimal beliefs holds independent of the characteristics of the objective probability distribution $Q$. It allows us to derive the following useful properties of optimal beliefs.

## 3.1 Optimal beliefs must be binary

**Proposition 1** *The optimal subjective probability distribution $P^*$ has at most two atoms: $dP^*(x) = 0$ for all $x \in [a,b]$ except at $x_- < 0$ and $x_+ > 0$ where $P^*$ has an upward discontinuity.*

Proof: We can rewrite problem (7) as follows:

$$dP^* \in \arg \max_{dP} \int_a^b u(w_0 + \alpha^* x) dP(x) \tag{8}$$

$$\begin{aligned} s.t. \quad \int_a^b x u'(w_0 + \alpha^* x) dP(x) &= 0 \\ \int_a^b dP(x) &= 1 \\ dP(x) &\geq 0 \quad \forall x \in [a,b]. \end{aligned}$$

The first constraint states that $P$ belongs to $B(\alpha^*)$, i.e., that beliefs $P$ yield the optimal risk exposure $\alpha^*$. The other two constraints define a cumulative probability distribution. Because the feasible set is compact, this problem has a solution. Observe that the above program is a linear programming problem with two equality constraints. As is well-known, the optimal solution has at most two atoms. In order to satisfy the first-order condition, it must be that $x_-$ and $x_+$ alternate in sign.$\blacksquare$

Thus, we conclude from this proposition that the optimal subjective beliefs take the form $P^* = (x_-, 1-p^*; x_+, p^*)$ for some pair $(x_-, x_+)$ and some scalar $p^*$ such that $a \leq x_- < 0 < x_+ \leq b$ and $p^* \in [0,1]$. It is linked to the optimal risk exposure $\alpha^*$ by the following rewriting of the first-order condition:

$$p^* x_+ u'(w_0 + \alpha^* x_+) + (1-p^*) x_- u'(w_0 + \alpha^* x_-) = 0 \tag{9}$$

Proposition 1 is useful because it replaces the problem of finding a probability distribution in the infinite dimensional space $X[a,b]$ into a problem of finding a triplet $(x_-, x_+, p)$ that maximizes $W(P)$. From the technique presented above, we can easily derive the following property of optimal beliefs: when there are $n$ independent assets in the economy, there must be at most $n$ states with a positive optimal subjective probability.

## 3.2 Only the extreme returns may have a positive subjective probability

In this section, we first show that at least one of the two subjectively possible returns must be at the bounds of interval $[a,b]$. We define $A(z) = -u''(z)/u'(z)$ as the Arrow-Pratt index of absolute risk aversion.

**Proposition 2** *The optimal subjective distribution $P^* = (x_-, 1-p^*; x_+, p^*) \in X[a,b]$ is such that either $x_- = a$ or $x_+ = b$.*

Proof: Suppose by contradiction that $x_- > a$ and $x_+ < b$. Consider a marginal change in $P$ such that the marginal increase in $x_+$ is compensated by a marginal reduction in $x_-$ in such a way that $\alpha^*$ is unaffected. Fully differentiating condition (9) yields

$$\left.\frac{dx_-}{dx_+}\right|_{\alpha^*} = -\frac{p^* u'(w_0 + \alpha^* x_+)}{(1-p^*) u'(w_0 + \alpha^* x_-)} \frac{1 - \alpha^* x_+ A(w_0 + \alpha^* x_+)}{1 - \alpha^* x_- A(w_0 + \alpha^* x_-)}.$$

The subjective expected utility equals

$$S = p^* u(w_0 + \alpha^* x_+) + (1 - p^*) u(w_0 + \alpha^* x_-).$$

Fully differentiating this equality yields

$$\left.\frac{dS}{dx_+}\right|_{\alpha^*} = p^* \alpha^* u'(w_0 + \alpha^* x_+) + (1 - p^*) \alpha^* u'(w_0 + \alpha^* x_-) \left.\frac{dx_-}{dx_+}\right|_{\alpha^*},$$

or, equivalently,

$$\left.\frac{dS}{dx_+}\right|_{\alpha^*} = p^* \alpha^{*2} u'(w_0 + \alpha^* x_+) \frac{x_+ A(w_0 + \alpha^* x_+) - x_- A(w_0 + \alpha^* x_-)}{1 - x_- A(w_0 + \alpha^* x_-)}.$$

Because $x_- < 0 < x_+$ and $A(.) > 0$, this is unambiguously positive. This change in beliefs increases the lifetime well-being of the consumer, which is a contradiction.■

This result states that at least one of the two possible returns must be an extreme return $a$ or $b$. In the next proposition, we claim that the two possible returns are extreme under some mild additional assumptions on the utility function. We define relative risk aversion as $R(z) = zA(z) = -zu''(z)/u'(z)$. It is weakly increasing if $R'(.)$ is uniformly non-negative.

**Proposition 3** *Suppose that absolute risk aversion is decreasing (DARA) and that relative risk aversion is weakly increasing (IRRA). Then, the optimal subjective distribution of returns has support $\{a, b\}$: $P^*$ is distributed as $(a, 1 - p^*; b, p^*)$.*

Proof: Suppose by contradiction that $x_- > a$ or $x_+ < b$. Suppose for example that $x_+$ is less than $b$. We consider a marginal increase in $x_+$ that is compensated by a change in $p$ in such a way that $\alpha^*$ be unaffected by the change. Fully differentiating equation (9) yields

$$\left.\frac{dp}{dx_+}\right|_{\alpha^*} = -\frac{p^* u'(w_0 + \alpha^* x_+) \left[1 - \alpha^* x_+ A(w_0 + \alpha^* x_+)\right]}{x_+ u'(w_0 + \alpha^* x_+) - x_- u'(w_0 + \alpha^* x_-)}. \tag{10}$$

By definition of the subjective expected utility, we have that

$$\left.\frac{dS}{dx_+}\right|_{\alpha^*} = p^* \alpha^* u'(w_0 + \alpha^* x_+) + \left[u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)\right] \left.\frac{dp}{dx_+}\right|_{\alpha^*}.$$

Using (10), it is positive if

$$\begin{aligned} K(x_+, x_-) &= \alpha^* x_+ u'(w_0 + \alpha^* x_+) - \alpha^* x_- u'(w_0 + \alpha^* x_-) \\ &\quad - [1 - \alpha^* x_+ A(w_0 + \alpha^* x_+)] [u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)] \end{aligned}$$

is positive. Observe that, by risk aversion,

$$K(0, x_-) = u(w_0 + \alpha^* x_-) - \alpha^* x_- u'(w_0 + \alpha^* x_-) - u(w_0)$$

is positive for all $x_-$. Notice also that

$$\begin{aligned} \frac{\partial K}{\partial x_+}(x_+, x_-) &= \alpha^* [u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)] [A(w_0 + \alpha^* x_+) + \alpha^* x_+ A'(w_0 + \alpha^* x_+)] \\ &= \alpha^* [u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)] [R'(w_0 + \alpha^* x_+) - w_0 A'(w_0 + \alpha^* x_+)] . \end{aligned}$$

Obviously, $\alpha^* [u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)]$ is positive. The second bracketed term in the right-hand side of the above equality is also positive since, by assumption, $R'$ is non-negative and $A'$ is negative. We conclude that $K$ is positive for all positive $x_+$. Therefore, this change in beliefs raises the lifetime well-being of the decision maker, a contradiction. A parallel proof can be made when $x_-$ is larger than $a$. ■

The familiar set of power utility functions $u(z) = z^{1-\gamma}/(1-\gamma)$ exhibits constant relative risk aversion and decreasing absolute risk aversion. Therefore, it satisfies the condition of the above proposition. More generally, decreasing absolute risk aversion is commonly accepted by the profession as a reasonable assumption. Nondecreasing relative risk aversion is compatible with the observation that, conditional to holding a portfolio, wealthier consumers invest a smaller share of their wealth in stocks.[3]

In the remaining of the paper, we will assume that the optimal subjective probability distribution is of the form $(a, 1-p^*; b, p^*)$. It remains to determine the only remaining degree of freedom, which is the probability $p^*$ of the state with the highest possible return $x = b$. Using an intuitive shortcut in notation, we can rewrite the problem of selecting subjective beliefs as

$$p^* \in \arg\max_p W(p; k) = kS(p) + (1-k)O(p) \tag{11}$$

with

$$S(p) = pu(w_0 + \alpha(p)a) + (1-p)u(w_0 + \alpha(p)a),$$

[3] See for example Guiso, Jappelli and Terlizzese (1996).

$$O(p) = E_Q u(w_0 + \alpha(p)\widetilde{x}),$$

and

$$pbu'(w_0 + \alpha(p)b) + (1-p)au'(w_0 + \alpha(p)a) = 0. \tag{12}$$

Before proceeding to characterize the optimal subjective probability of the high state, it is useful to determine the effect of an increase in this probability on the optimal demand for the risky asset. By the Inada conditions, $\alpha(p)$ tends to infinity when $p$ tends to unity, and it tends to minus infinity when $p$ tends to zero. In the next lemma, we show that an increase in the subjective probability of the high return state raises the demand for the risky asset.

**Lemma 1** *The demand for the risky asset is increasing in the subjective probability of the high return state:* $\partial\alpha/\partial p \geq 0$.

Proof: Fully differentiating condition (12) yields

$$\frac{\partial\alpha}{\partial p} = \frac{au'(w_0 + \alpha a) - bu'(w_0 + \alpha b)}{pb^2u''(w_0 + \alpha(p)b) + (1-p)a^2u''(w_0 + \alpha(p)a)}. \tag{13}$$

Both the numerator and the denominator are negative, which implies that $\partial\alpha/\partial p$ is positive.■

This result is linked to the literature on the relationship between the probability distribution of returns and the optimal demand for the risky asset. Gollier (1995) provides the necessary and sufficient condition on a change in distribution to raise the demand for the risky asset by all risk-averse investors. The change in distribution considered in Lemma 1 is a special case of a stochastic order named monotone probability ratio order by Eeckhoudt and Gollier (1995) and Athey (2002).

# 4 The case of small anticipatory feelings

In this section, we explore the special case of small intensities $k$ of anticipatory feelings. When $k$ vanishes, there is no anticipatory feeling at all, and the lifetime well-being $W(p; k = 0)$ equals the objective expected utility $O(p)$. It is obvious in this case that the agent selects the subjective probability

$p_0^*$ yielding the demand for the risky asset that is optimal for the objective probability distribution:

$$p_0^* b u'(w_0 + \alpha(Q)b) + (1 - p_0^*) a u'(w_0 + \alpha(Q)a) = 0. \tag{14}$$

It is easy to check that there exists a single probability $p_0^* \in [0,1]$ that satisfies equation (14). It is well-known that $\alpha(Q)$ has the same sign as the objective expected return $E_Q\widetilde{x}$.

We now examine the impact of introducing a small degree $k$ of anticipatory feelings on the optimal subjective probability $p^*(k)$ of the high return state. We know that it tends to $p_0^*$ when $k$ tends to zero. We determine the sign of $\partial p^*/\partial k$ at $k = 0$. In order to do this, we first establish the local concavity of the lifetime well-being with respect to the subjective probability of the high return state, when $k$ is small.

**Lemma 2** *Consider any probability distribution $P$ in $B(\alpha(Q)) \subset X[a,b]$. Consider any pair $(P_1, P_2)$ in $X^2[a,b]$ and any scalar $\lambda \in [0,1]$ such that*

$$P = \lambda P_1 + (1 - \lambda) P_2.$$

*It implies that*

$$O(P) \geq \lambda O(P_1) + (1 - \lambda) O(P_2).$$

Proof: Let $\alpha_i$ denote the optimal demand under beliefs $P_i : \alpha_i = \alpha(P_i)$. We have that

$$\begin{aligned} \lambda O(P_1) + (1-\lambda)O(P_2) &= \lambda E_Q u(w_0 + \alpha_1 \widetilde{x}) + (1-\lambda) E_Q u(w_0 + \alpha_2 \widetilde{x}) \\ &= E_Q \left[ \lambda u(w_0 + \alpha_1 \widetilde{x}) + (1-\lambda) u(w_0 + \alpha_2 \widetilde{x}) \right]. \end{aligned}$$

The concavity of $u$ implies that

$$\lambda u(w_0 + \alpha_1 x) + (1-\lambda) u(w_0 + \alpha_2 x) \leq u\left(w_0 + (\lambda \alpha_1 + (1-\lambda)\alpha_2) x\right)$$

for all $x$. It implies that

$$\lambda O(P_1) + (1-\lambda) O(P_2) \leq E_Q u\left(w_0 + (\lambda \alpha_1 + (1-\lambda)\alpha_2)\widetilde{x}\right).$$

We conclude that

$$\lambda O(P_1) + (1-\lambda) O(P_2) \leq \max_\alpha E_Q u(w_0 + \alpha \widetilde{x}) = O(Q).$$

Because $P$ belongs to $B(\alpha(Q))$, we know that $O(P) = O(Q)$. This concludes the proof.■

This lemma implies that the objective expected utility $O(P)$ is locally concave in the neighborhood of any subjective probability distribution $P$ belonging to $B(\alpha(Q))$. In Appendix A, we exhibit a numerical example showing that $O$ is not globally concave in $P$. However, because $E_Q u(w_0 + \alpha\widetilde{x})$ is concave in $\alpha$, and because $\alpha$ is increasing in the subjective probability $p$ of the high state as stated in Lemma 1, $O$ is single-peaked in $p$. It implies that the first-order condition of program (11) is necessary and sufficient when $k$ is small.

This first-order condition is written as

$$\begin{aligned} 0 &= \frac{\partial W}{\partial p}(p^*; k) = k\frac{\partial E_P u(w_0 + \alpha\widetilde{x})}{\partial \alpha}\frac{\partial \alpha}{\partial p} \\ &\quad + k\left[u(w_0 + \alpha b) - u(w_0 + \alpha a)\right] \\ &\quad + (1-k)\frac{\partial E_Q u(w_0 + \alpha\widetilde{x})}{\partial \alpha}\frac{\partial \alpha}{\partial p}. \end{aligned}$$

Because $\alpha$ maximizes $E_P u(w_0 + \alpha\widetilde{x})$, the first term in the right-hand side of this equality is zero. Using equation (13), we can thus rewrite the first-order condition to program (11) as follows:

$$0 = \frac{\partial W}{\partial p}(p^*; k) = k\left[u(w_0 + \alpha b) - u(w_0 + \alpha a)\right] \tag{15}$$

$$-(1-k)\frac{\left[bu'(w_0 + \alpha b) - au'(w_0 + \alpha a)\right] E_Q \widetilde{x} u'(w_0 + \alpha\widetilde{x})}{p^* b^2 u''(w_0 + \alpha(p)b) + (1-p^*)a^2 u''(w_0 + \alpha(p)a)}. \tag{16}$$

When $k = 0$, we verify that this condition simplifies to $E_Q \widetilde{x} u'(w_0 + \alpha\widetilde{x}) = 0$, which is true only if $\alpha = \alpha(Q)$. This yields in turn $p^* = p_0^*$ as defined by (14). Because $W$ is locally concave in $p$ around $p_0^*$, the optimal subjective probability $p^*$ is increasing in $k$ around $k = 0$ if and only if the cross-derivative of $W$ is positive when evaluated at $(p_0^*; k = 0)$. It is easy to check that

$$\frac{\partial^2 W}{\partial p \partial k}(p_0^*; 0) = u(w_0 + \alpha(Q)b) - u(w_0 + \alpha(Q)a).$$

The right-hand side of this equality has the same sign as $\alpha(Q)$. Thus the sign of $\partial\alpha/\partial k$ has the same sign as $\alpha(Q)$. Combining this result with Lemma 1 yields the following proposition. One can measure the degree of optimism by the difference between the subjective probability and the objective probability of the state that is more favorable to the agent's wealth. When $\alpha(Q)$, the favorable state is the high return state, and an increase in $p$ represents an increase in optimism. When $\alpha(Q)$ is negative, the investor goes short on the risky asset, and the favorable state is the low return state. The degree of optimism is inversely related to $p$ in that case.

**Proposition 4** *Introducing small anticipatory feelings in the lifetime objective function of the consumer makes him more optimistic about his portfolio return:*

$$\alpha(Q) \left. \frac{dp^*}{dk} \right|_{k=0} \geq 0.$$

*It raises the optimal portfolio risk:*

$$\alpha(Q) \left. \frac{d\alpha(p^*)}{dk} \right|_{k=0} \geq 0.$$

*These inequalities are strict when the objective expected return is not zero.*

The intuition to this result is simple. Suppose that the objective expected return is positive, so that the optimal demand $\alpha(Q)$ for the risky asset is positive when there is no anticipatory feeling. It is sustained by the beliefs that the probability of the high return $b$ is $p_0^*$. Consider a marginal increase in the subjective probability of that state. It marginally increases the demand for the risky asset. But, by the envelope theorem, this marginal increase in the demand has no effect on the objective expected utility. To the contrary, it increases the subjective expected utility. Globally, when $k > 0$, it raises the lifetime well-being. This argument cannot be extended to consumers having a larger intensity of anticipatory feelings. Indeed, in this case, a marginal change in the subjective probability distribution would have an effect on the objective expected utility.

In Figures 1 and 2, we illustrate Proposition 4 by assuming that the agent has a power utility function with constant relative risk aversion $\gamma = 3$. The worst possible return is $a = -100\%$, whereas the best possible return is $b = +150\%$. The objective probability distribution is $Q \sim (-1, 1/2; +1.5, 1/2)$, yielding a positive expected excess return. In Figure 1, we have drawn the optimal subjective probability of the high return as a function of the intensity

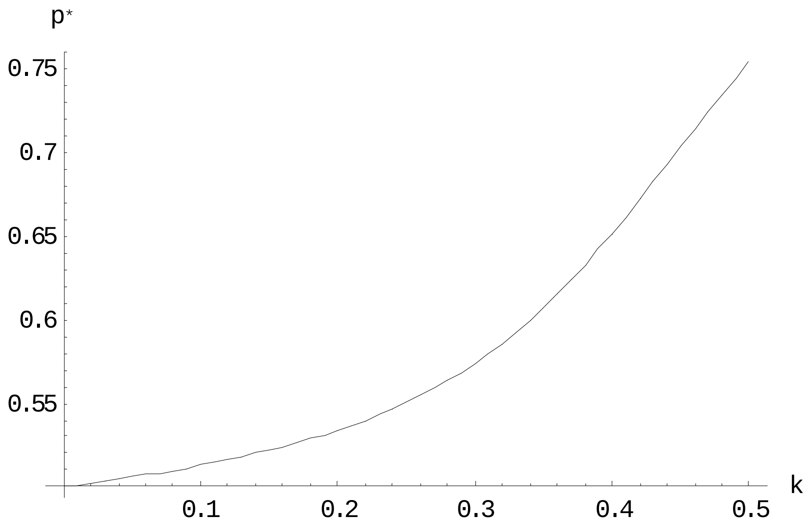


Figure 1: Optimal probability of the high return state, as a function of the intensity of anticipatory feelings. Parameter values: $\gamma = 3$, $Q \sim (-1, 1/2; +1.5, 1/2)$.

$k$ of anticipatory feelings. In Figure 2, we depicted the relationship between $k$ and the optimal share of wealth invested in the risky asset. As stated in Proposition 4, we get upward sloping curves. When there is no anticipatory feeling, the optimal share of wealth invested in the risky asset is equal 5.5%. When anticipatory feelings count as much as the objective future felicity ($k = 0.5$), this optimal share goes up to 21.0%.

# 5 The case of large anticipatory feelings

We have seen in the previous section that the lifetime well-being $W$ as a function of the subjective probability of success is locally concave and globally single-peaked when $k$ is small. This does not to be the case when anticipatory feelings play a more important role in the measurement of welfare. When $k$ tends to unity, $W(p;k)$ tends to the subjective expected utility $S(p)$. As seen in definition (1), $S(p)$ is the maximum of various linear functions of $p$. Therefore, $S(p)$ is a convex function of the subjective probability $p$ of the high return. Thus, when $k = 1$, the optimal probability $p_1^*$ must be either

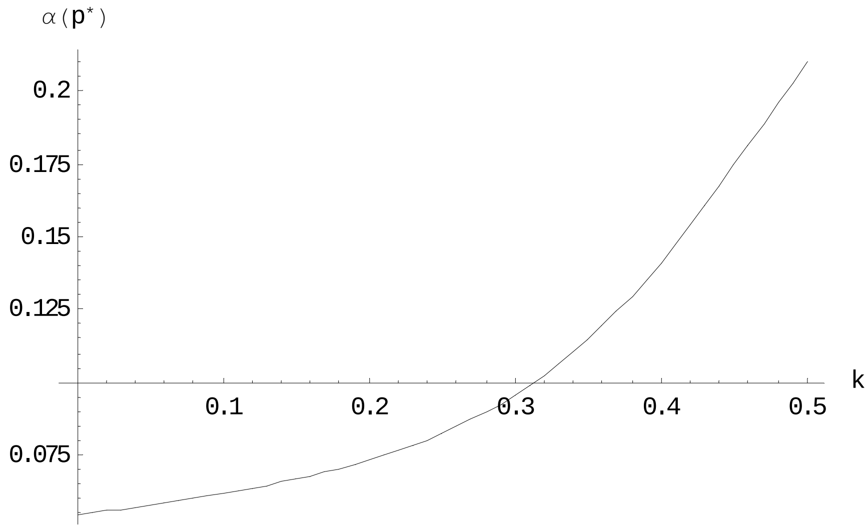


Figure 2: The demand for the risky asset, as a function of the intensity of anticipatory feelings. Parameter values: $\gamma = 3$, $Q \sim (-1, 1/2; +1, 1/2)$.

0 or 1. In both case, the subjective expected utility tends to $\bar{u}(+\infty)$. The optimal exposure to the portfolio risk is unbounded.

Suppose without loss of generality that the decision-maker with $k = 1$ selects $p_0^* = 1$. Of course, this solution is not feasible when $k$ is smaller than unity, since it yields a negative final wealth in all states with a negative excess return. It implies that the agent with $k < 1$ must reduce his subjective probability of the high return. This must be done in order to induce him to reduce his demand for the risky asset in such a way that $w_0 + \alpha(p)a$ be positive.

**Proposition 5** *The subjective expected utility $S(p)$ is a convex function of the subjective probability of the high return state. It implies that the optimal subjective probability is either 0 or 1 when only anticipatory feelings matter ($k = 1$). When $k$ is smaller than unity, $p^*$ is positive and less than unity.*

When $k$ is smaller than unity, $W$ is a convex combination of a convex function $S$ and of a single-peaked function $O$. The search for an optimal subjective probability may be complex in such an environment. To illustrate, let us consider the case of constant relative risk aversion $\gamma = 3$, $w_0 = 1$,

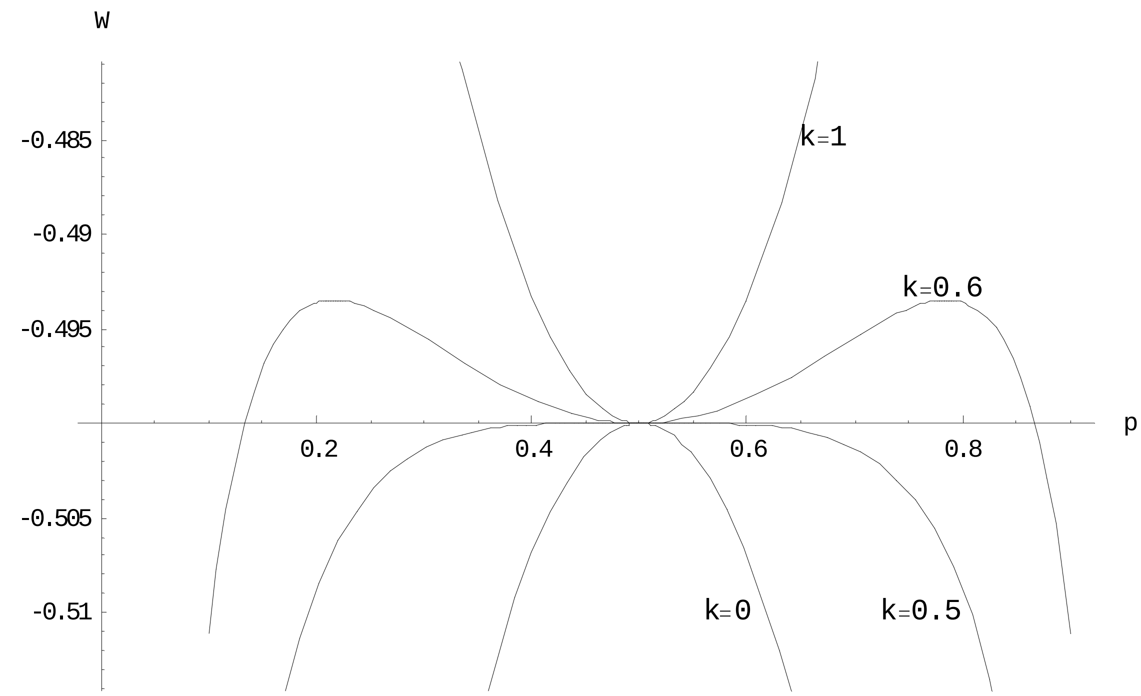


Figure 3: The lifetime well-being as a function of the subjective probability of the high return, for various intensities $k$ of anticipatory feelings. Parameter values: $\gamma = 3$, $Q \sim (-1, 1/2; +1, 1/2)$.

together with $a = -100\%$ and $b = +100\%$. We assume that the objective distribution of returns is $Q \sim (-1, 1/2; +1, 1/2)$. In Figure 3, we have drawn the lifetime well-being $W$ as a function of the subjective probability $p$ for various values of $k$. When $k$ is smaller than or equal to $1/2$, $W$ is globally single-peaked and the optimal subjective probability is $p^* = 1/2$, implying that investing only in the riskfree asset is optimal. This is an example where the optimal subjective probability distribution coincides with the objective ones.

When $k$ is in $]1/2, 1[$, function $W$ exhibits a convex-concave-convex shape, with two symmetric optimal beliefs. The optimal subjective probability that is larger than one-half is first constant and then increasing in $k$, as seen in Figure 4. Notice that the existence of two symmetric optima shows that providing zero-sum gambling opportunities can be helpful to improve welfare

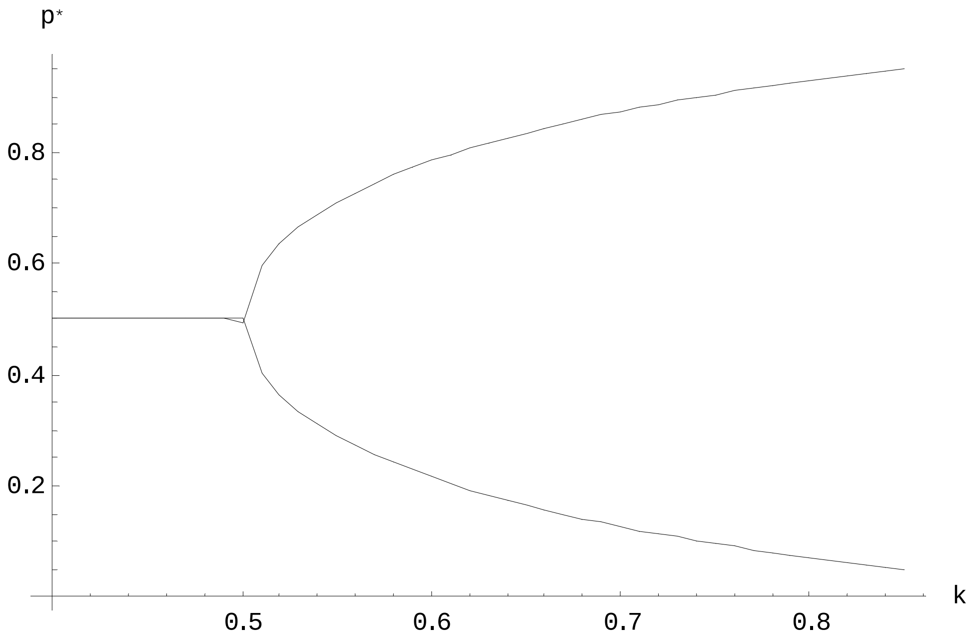


Figure 4: The optimal subjective probability $p^*$ as a function of the intensity $k$ of anticipatory feelings. Parameter values: $\gamma = 3$, $Q \sim (-1, 1/2; +1, 1/2)$.

in an homogeneous economy of risk-averse agents. Suppose that two agents with constant relative risk aversion $\gamma = 3$ and with an intensity $k = 0.6$ of anticipatory feelings are considering playing Head-or-Tail game with a fair coin. In this economy, there is a competitive equilibrium where each agent puts $\alpha(p^*) = 21.2\%$ of initial wealth at stake by betting on either Head or Tail, optimally subjectively believing to have a probability of success of $p^* = 78,43\%$.

# 6 Approximate solution

Suppose that $\mid \alpha \mid$ is small. It implies that we can approximate $u'(w_0 + \alpha x)$ by $u'(w_0) + \alpha x u''(w_0)$, which is equal to $u'(w_0)(1 - \alpha x A_0)$, where $A_0 = A(w_0)$. First-order condition (12) is thus approximated as

$$[pb + (1-p)a] - \alpha A_0 \left[pb^2 + (1-p)a^2\right] \simeq 0,$$

which implies that

$$\alpha(p) = \frac{1}{A_0}\frac{pb + (1-p)a}{pb^2 + (1-p)a^2}. \tag{17}$$

Using second-order Taylor approximations for $u(w_0 + \alpha x)$ yields in turn that

$$\begin{aligned} S(p) &= pu(w_0 + \alpha(p)a) + (1-p)u(w_0 + \alpha(p)a) \\ &\simeq u(w_0) + \alpha(p)\left[pb + (1-p)a\right]u'(w_0) + 0.5(\alpha(p))^2\left[pb^2 + (1-p)a^2\right]u''(w_0) \\ &= u(w_0) + 0.5\frac{u'(w_0)}{A(w_0)}\frac{\left[pb + (1-p)a\right]^2}{pb^2 + (1-p)a^2}. \end{aligned}$$

Let $m_i = E_Q\widetilde{x}^i$ denote the objective moment of order $i$ of $\widetilde{x}$. Using again second-order Taylor approximations yields

$$\begin{aligned} O(p) &= E_Q u(w_0 + \alpha(p)\widetilde{x}) \\ &\simeq u(w_0) + \alpha(p)m_1u'(w_0) + 0.5(\alpha(p))^2m_2u''(w_0) \\ &= u(w_0) + 0.5\frac{u'(w_0)}{A(w_0)}\frac{pb + (1-p)a}{pb^2 + (1-p)a^2}\left[2m_1 - \frac{pb + (1-p)a}{pb^2 + (1-p)a^2}m_2\right]. \end{aligned}$$

Combining these two observations implies that

$$\begin{aligned} W(p) &= kS(p) + (1-k)O(p) \\ &\simeq u(w_0) + 0.5\frac{u'(w_0)}{A(w_0)}F(p), \end{aligned} \tag{18}$$

with

$$F(p) = \frac{pb + (1-p)a}{pb^2 + (1-p)a^2}\left\{\frac{pb + (1-p)a}{pb^2 + (1-p)a^2}\left(k\left[pb^2 + (1-p)a^2\right] - (1-k)m_2\right) + 2(1-k)m_1\right\}. \tag{19}$$

It is noteworthy that this approximation is exact when $u$ is quadratic. We thus obtain the following interesting insight.

**Proposition 6** *When $u$ is quadratic in the relevant domain of wealth, the optimal subjective probability is independent of the consumer's attitude towards risk. It maximizes function $F$ defined by (19), where $m_1$ and $m_2$ are the objective first two moments of the excess return of the risky asset.*

The first-order condition associated to the maximization of $F(p)$ is equivalent to finding the roots of a third-degree polynomial. This is in line with the observation made in relation to Figure 3 that $\partial^2 W/\partial p^2$ can alternate twice in sign, with $W$ having a concave-convex-concave or convex-concave-convex shape. We check that in the special case with no anticipatory feeling ($k = 0$), $F$ is concave in $p$ with a maximum $p_0^*$ such that

$$\frac{p_0^* b + (1 - p_0^*)a}{p_0^* b^2 + (1 - p_0^*)a^2} = \frac{m_1}{m_2}.$$

This means that the subjective probability $p_0^*$ is selected in such a way that the objective and subjective Sharpe ratios be the same. It yields the same optimal portfolio than the one that is optimal under rational expectation.

When the utility function is not quadratic in the relevant domain, the solution presented in Proposition 6 is only an approximation of the optimal solution. This is a good approximation only when the optimal portfolio risk $\mid \alpha(p^*) \mid$ is small. This is the case for example when $m_1/m_2$ is small in absolute value and $k$ is small. The first condition implies that the absolute value of $\alpha(Q)$ is small, whereas the second condition means that $\alpha(p^*)$ is close to $\alpha(Q)$. To illustrate, consider again the case with $a = -100\%$, $b = +150\%$, $Q \sim (-1, 1/2; 1.5, 1/2)$, $w_0 = 1$, $k = 0.1$ together with a constant relative risk aversion equaling $\gamma = 3$. In Figure 5, we compare the true $W(p)$ and the approximated one specified in equation (18). The optimal subjective probability of the high return is equal to $p^* = 0.513$. It corresponds to an optimal share of wealth invested in stocks equaling $\alpha(p^*) = 6.17\%$, which is small. The approximate solution gives $p^* \simeq 0.515$. We see that the size of the error of the approximation is small for intermediate values of $p$. When $p$ goes closer to 0 or1, the induced portfolio risk becomes large, and the quality of the approximation deteriorates dramatically. This is because the quadratic utility functions do not satisfy the Inada conditions.

An important question is to determine whether the heterogeneity in risk aversion may explain the heterogeneity of subjective beliefs in the population. When preferences belong to the quadratic class, the optimal subjective probability distribution is independent of the degree of risk aversion of the investor. When the utility function is not quadratic, optimal beliefs are generally not independent of risk preferences. Brunnermeier and Parker (2003) conclude that the heterogeneity of risk aversion in the population could explain the heterogeneity of subjective beliefs. However, because smooth functions can

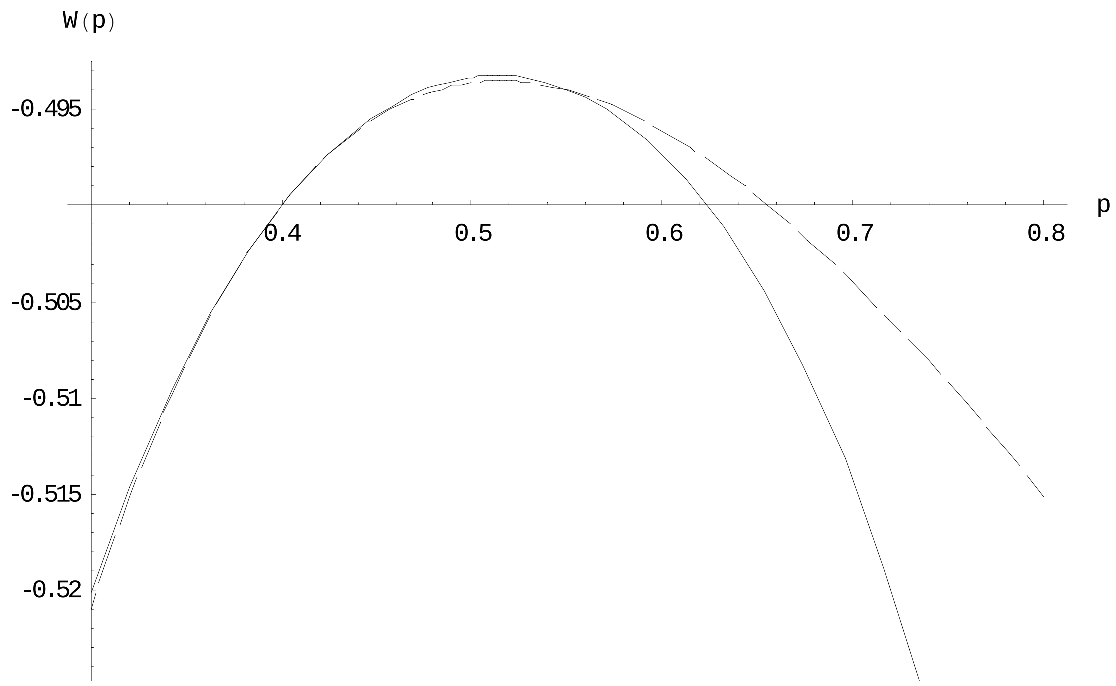


Figure 5: The true $W$-curve (plain) and the approximate $W$-curve (dashed) as a function of the subjective probability of the high return state. Parameter values: $\gamma = 3$, $Q \sim (-1, 1/2; +1.5, 1/2)$, $k = 0.1$.

always be well approximated by a quadratic utility function in a small domain, we should not expect to generate a lot of heterogeneity on beliefs in an economy with small portfolio risks at equilibrium. The assumption of small portfolio risks is compatible with the general tone of the literature on the equity premium puzzle. The puzzle is based on the observation that indeed actual portfolio risks are very small compared to the optimal risk computed on the basis of the large objective risk premium on financial markets. We illustrate the low sensitivity of optimal beliefs to changes in risk aversion by considering again the numerical example used above. We examine in particular the effect of a change in the relative risk aversion $\gamma$ on the optimal subjective probability of the high state. This relationship is described in Figure 6. The most striking aspect of this figure is the range of the vertical axe: as relative risk aversion varies from 0.5 to 10, the optimal subjective probability of the high state varies within interval $[0.5126, 0.5131]$.

# 7 Concluding remarks

We have shown that the selection of optimal beliefs in the one-riskfree-one-risky-asset portfolio problem is governed by very precise rules. First, we have shown that these beliefs must be degenerated at the worst and best possible returns, as suggested by the cumulative prospect theory. Second, when the intensity of anticipatory feelings is small, the problem of selecting beliefs is well-behaved and concave, yielding a unique optimal subjective probability of the best return. Except in the case of a zero objective expected excess return, this optimal beliefs always yield an increase in the optimal risk exposure when compared to the one that is optimal under the objective probability distribution. Moreover, investors with a larger intensity of anticipatory feelings raise their subjective probability of the good state together with their optimal risk exposure. Because the mental process of distorting beliefs in favor of savoring the prospect of large capital gains, the induced optimism of investors will not be helpful to solve the equity premium puzzle, quite the contrary. The problem is more complex when anticipatory feelings play a larger role in the measurement of well-being. In particular, we showed that the objective function may not be concave in the subjective probability distribution, thereby yielding potential bifurcation and multiple local maxima. When the optimal portfolio risk is small, we showed that optimal beliefs are almost insensitive to the degree of risk aversion of the investor.

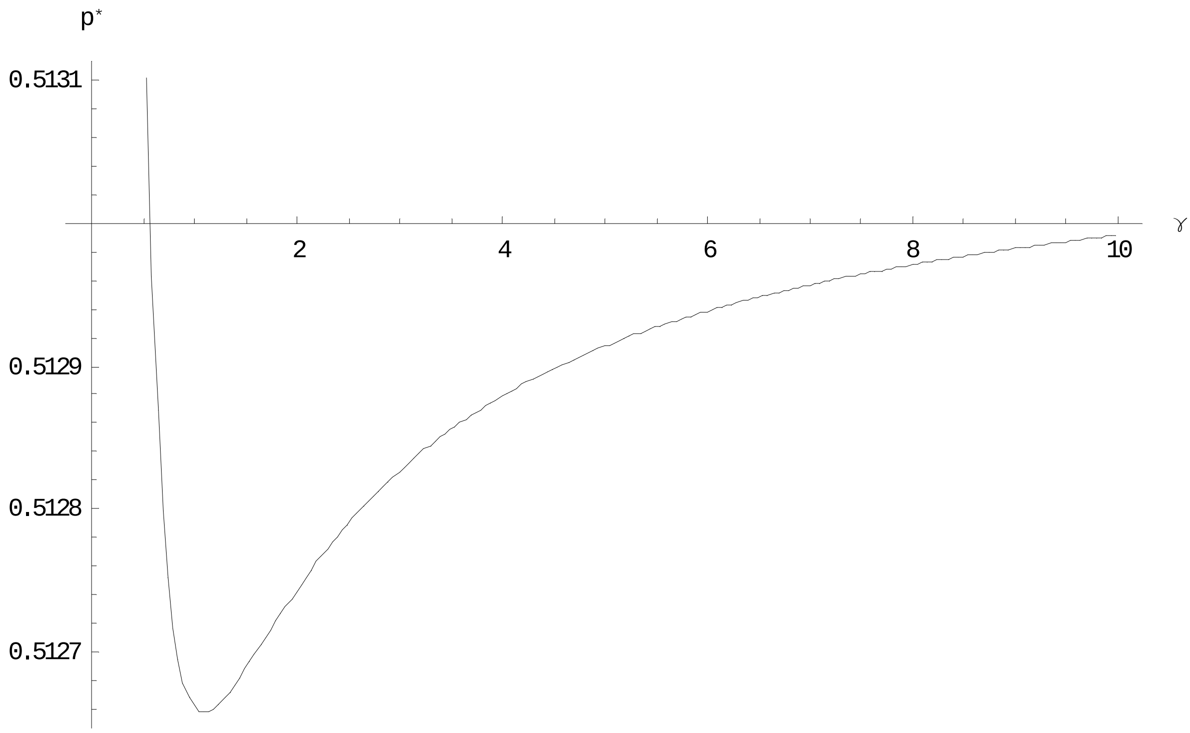


Figure 6: The impact of risk aversion on optimal beliefs. Parameter values: $Q \sim (-1, 1/2; +1.5, 1/2)$, $k = 0.1$.

This work calls for more investigations in several directions. First, it would be interesting to examine a more general model in which more risk-taking opportunities are available. This would be useful in order to examine the effect of anticipatory feelings on the optimal diversification of individual asset portfolios. Second, the current model does not take into account of the adverse effect of disappointment of the optimally optimistic investors when they will eventually be forced to recognize the objective performance of their asset portfolio. Third, this work suggests that delegating the selection of the individual asset portfolios to an independent agent can be efficient. This would neutralize the negative effect on portfolio choices of distorting individual beliefs.

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## Appendix A: The objective expected utility is not globally concave in the subjective probability distribution

In this Appendix, we show that $O$ needs not be globally concave in $P$. We consider the following counter-example. The consumer's relative risk aversion is a constant equaling $\gamma = 0.1$. We normalize initial wealth to unity. The extreme possible returns are $a = -1$ and $b = 1$. The objective probability distribution is $Q \sim (-1, 1/2; +1, 1/2)$. By Lemma 2, we know that $O$ is locally concave around $p_0^* = 1/2$. In Figure 7, we draw the objective expected utility $O(p)$ as a function of the probability of the high return.

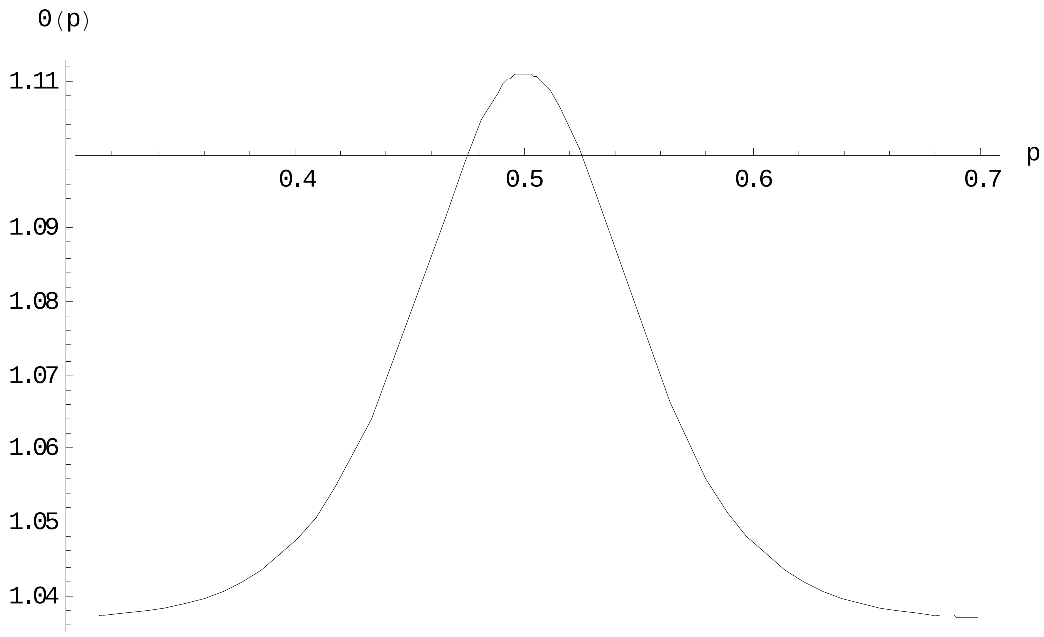


Figure 7: The objective expected utility $O$ as a function of the subjective probability $p$, when $Q \sim (-1, 1/2; +1, 1/2)$ and $u(z) = z^{0.9}/0.9$.