Sequential Legislative Lobbying under Political Certainty*

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Abstract

In this paper, we analyse the equilibrium of a sequential game-theoretical model of lobbying, due to Groseclose and Snyder (1996), describing a legislature that vote over two alternatives, where two opposing lobbies, lobby 0 and lobby 1, compete by bidding for legislators' votes. In this model, the lobbyist moving first suffers from a second mover advantage and will make offers to legislators only if they deter any credible counter-reaction from his opponent, i.e. if he anticipates to win the battle. Our main focus is on the calculation of the smallest budget that he needs to win the game and on the distribution of this budget across the legislators. We study the impact of the key parameters of the game on these two variables and show the connection of this problem with the combinatorics of sets and notions from cooperative game theory.

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1 Introduction

In this paper, we consider a theoretical model of lobbying describing a legislature[1] that vote over two alternatives[2], and two opposing lobbies, lobby 0 and lobby 1, compete by bidding for legislators' votes[3]. We examine how the voting outcome and the bribes offered to the legislators depends on the lobbies's willingness to pay, legislators's preferences and the decision making process within the legislature.

There are many different ways to model the lobbying process. In this paper, we adopt the sequential model pioneered by Groseclose and Snyder (1996) and followed up by Banks (2000) and Diermeier and Myerson (1999). In their model, the competition between the two lobbies is described by a targeted offers game where each lobby gets to move only once, and in sequence. For most of the paper, lobby 1 is pro-reform and moves first while lobby 0 is pro-status quo and moves second. Votes are assumed to be observable. A strategy for each lobby is a profile of offers where the offer made to each legislator is assumed to be based on his/her vote and to be honored irrespective to the voting outcome. The net payoff of a lobby is its gross willingness to pay less the total amount of payments made to the legislators who ultimately vote for the policy advocated by this lobby. The legislators are assumed to care about how they cast their vote (independent of the income) and monetary offers. Therefore, voters do not truly act strategically as their voting behavior is simply a best response to the pair of offers made by the lobbies and is independent of the decisions of other legislators. We focus on the complete-information environment where the lobbies's and the legislators' preferences are known to the lobbies when they bid. We characterize the main features of the subgame perfect equilibrium of this game as a function of the following key parameters of the environment.

· The maximal willingness to pay of each lobby for winning[4] (i.e. to have their favorite policy selected). These two numbers represent the economic stakes under dispute and deter-

[1] We depart from voluminous literature based on the common agency setting in abandoning the assumption that policies are set by a single individual or by a cohesive, well-disciplined political party. In reality, most policy decisions, are made not by one person but by a group of elected representatives acting as a legislative body. Even when the legislature is controlled by a single party (as it is necessarily the case in a two-party system if the legislature consists of a unique chamber), the delegation members do not always follow the instructions of their party leaders.

[2] Hereafter, we will often refer to the two alternatives as being the status quo (alternative 0) versus the change or reform (alternative 1). While simplistic, many policy issues fit that formulation like for instance : to ratify or not a free-trade agreement, to forbid or not a free market for guns, to allow or not abortion.

[3] By legislators we mean here all individuals who have a constitutional role in the process of passing legislation. This may include individuals from what is usually referred to as being the executive branch like for instance the president or the vice-president.

[4] Or, under an alternative interpretation, their respective budgets.

mine the intensity and asymmetry of the competition.

· The voting rule describing the legislative process.

· The heterogeneity across legislator's preferences.

The binary setting considered in this paper is the simplest setting where we can tackle the joint influence of these three inputs on the final outputs. The first item consists of a single number per lobby: how much money this lobby is willing (able) to invest in this competition. The second item is also very simple. In this simplistic institutional setting, with no room for agenda setting or other sophisticated legislative action which would arise in the case of large multiplicity of issues[5], we only need to know what are the winning coalitions i.e. the coalitions of legislators in position to impose the reform if the coalition unanimously supports this choice. Despite its apparent simplicity, this combinatorial object is extremely rich to accommodate a wide diversity of legislatures. Banks and Groseclose and Snyder focus on the standard majority game while Diermeier and Myerson consider the general case as we do. The third item describes the differences between the legislators others than those already attached to the preceding item if these legislators are not equally powerful or influent in the voting process. This "second" heterogeneity dimension refers to the differences between their intrinsic preferences for the reform versus the status quo. This difference measured in monetary units can be large or small and negative or positive. Diermeier and Myerson disregard this dimension by assuming that legislators are indifferent between the two policies while Banks and Groseclose and Snyder consider the general situation but derive their results under some specific assumptions. We assume that legislators prefer unanimously the reform to the status quo but differ with respect to the intensity of their preference.

The first contribution consists in identifying the conditions under which the lobby moving first will make positive offers to some legislator . In this sequential game, the lobby moving last has an advantage as it can react optimally to the offers of its opponent without any further possibility or reaction. If the asymmetry is too weak , lobby 1 will abandon the prospect of influencing the legislature as it will be rationally anticipating its defeat; in fact, it will make offers only if it anticipates to win for sure. If it does not make any offer, it is enough for lobby 0 to compensate a minimal winning coalition of legislators for their intrinsic preferences towards reform. Lobby 1 will participate if its willingness to pay or budget is larger than the willingness to pay or budget of lobby 0. This minimal amount of asymmetry, that we call the *victory threshold*, defines by how much the stake of lobby 1 must overweight

[5]Many formal models of the legislative process have been developed by social scientists to deal with more complicated choice environments. We refer the reader to Grossman and Helpman (2001) for lobbying models with more than two alternatives.

the stake of lobby 0 to make sure that lobby 1 wins the game. Our first result states that the calculation of the victory threshold amounts to calculate the supremum of a linear form over a convex polytope which is closely related to the polytope of balanced families of coalitions introduced in cooperative game theory to study the core and other solutions. The practical value of this result relies on the fact that we can take advantage of the voluminous amount of work which has been done on the description of balanced collections. When heterogeneity across legislators' preferences is ignored, the victory threshold only depends upon the simple game describing the rules of the legislature. It corresponds to what has been called by Diermeier and Myerson, the hurdle factor of the legislature. Quite surprisingly, this single parameter acts a summary statistic as long as we want to predict the minimal budget that lobby 1 needs to invest to win the game.

Our second contribution consists in connecting the problem of the computation of the hurdle factor to the *covering problem*, which is one of the most famous, but also difficult, problem in the combinatorics of sets or hypergraphs. We provide a short overview of the state of the art of this literature, show the connection with another famous parameter of a simple game, and illustrate through a variety of simple games, how the *hurdle factor* looks like. Once again, once it is noted that the hurdle factor is the fractional covering number of a specific hypergraph, we can take advantage of the enormous body of knowledge in that area of combinatorics.

The third contribution consists in showing that the hurdle factor can alternatively be calculated, surprisingly, as the maximum of specific criteria of equity over the set of imputations of a cooperative game with transferable utility attached to the simple game of the legislature. The specific equity criterion is the minimum over all coalitions of what the members of the coalitions get in the imputation and what they could get on their own, i.e. the first component in the lexicographic order supporting the nucleolus. We use that result to show how to calculate the hurdle factor for the important class of weighted majority games. While there is a link between the weights of the legislators and the hurdle factor when the game is homogeneous, we show that the relation is more intricate in the general case.

The connection with the theory of cooperative games turns to be even more surprising as it allows to provide a complete characterization of the second dimension of the optimal offer strategy of lobby 1. From what precedes, we know that the size of the lobbying budget is the hurdle factor times the willingness to pay (or budget) of lobby 0. It remains to understand how this budget is going to be allocated across the legislators. This is of course an important question as we would like to understand what are the characteristics of a legislator which determine the willingness of lobby 1 to buy its support and the amount that he will receive for the selling his vote . As already discussed, legislators differ along two lines: their intensity of their preference for lobby 1 and their position/power in the legislature. Likely the price of the vote of a legislator will be a function of both parameters. We show that the set of equilibrium offers is the least core of the cooperative game used to calculate the hurdle factor. It may contain multiple solutions but the nucleolus[6] is always one of them. We illustrate the calculation of these prices in the case of some important real world simple games and we revisit the model proposed by Diermeier and Myerson of the optimal determination of the hurdle factor of a legislative chamber given the other components of the legislative environment. One important conclusion is that these prices have little to do with the power of a legislator as calculated through either the Banzhaf's index (Banzhaf (1962), (1968)) or the Shapley-Shubik's index (Shapley and Shubik (1954)). This suggests that the axiomatic theory of power measurement may not be fully relevant to predict the payoffs of the players in a game like this one[7].

The paper contains two more contributions. Some legislators will not receive any offer from lobby 1. We may wonder what will be the identity and the number of legislators who will receive a positive offer. It is difficult to answer this question without being specific on either the preferences of the legislators or the simple game. We provide a characterization of this set in the case where the simple game is the standard majority game i.e. we describe the conditions under which lobby 1 will target a minority, a minimal winning coalition or a supermajority and whether it will bribe in priority those who are the more or the less reluctant to support the reform[8]. The last contribution aims to show that the results of this paper are preserved, up to some slight modifications, when we assume that legislators pay attention to the outcome rather than to their individual vote.

Related Literature

The literature on lobbying is very dispersed and voluminous[9]. The closest papers to ours are Banks (2000), Dekel, Jackson and Wolinsky (2006a,b), Diermeier and Myerson (1999), Groseclose and Snyder (1996), Young (1978 a, b, c) and Shubik and Young (1978d). Like us, they all consider the binary setting and assume that legislators care about their vote and money rather than the outcome. As already mentioned, the two-round sequential vote buying model that we consider is from the fundamental contribution of Groseclose and Snyder.

[6]Another appearance of the nucleolus in a non cooperative setting is Montero (2006) in a bargaining framework a la Baron-Ferejohn.

[7]This echoes Snyder, Ting and Ansolabehere (2005).

[8]Should the lobby seek to solidify support among those legislators who would be inclined to support its positions anyway, or should it seek to win over those who might otherwise be hostile to its views?

[9]We refer the reader to Grossman and Helpman (2001) for a description of the state of the art.

Banks as well as Diermeier and Myerson also consider this game. Their specific assumptions and focus are however quite different from ours. Banks and Groseclose and Snyder are primarily interested in identifying the number and the identity of the legislators who will receive an offer in the case of the simple majority game. By considering this important but specific symmetric game, they eliminate the possibility of evaluating the impact of the legislative power on the outcome. However, they consider more general profiles of legislators's preferences: instead of our unanimity assumption in favor of a reform, Banks assumes that a majority of legislators has an intrinsic preference for the status quo. This implies that lobby 1 needs to bribe at least a majority to win; Banks provides conditions on the profile under which this majority will be minimal or maximal but does not determine the optimal size in the general case. Diermeier and Myerson assume instead that legislators do not have any intrinsic preference but consider an arbitrary simple game. Their main focus is on the architecture of multicameral legislatures and on the optimal behavior of each chamber under the presumption that it can select its own hurdle factor to maximize the aggregate offer made to its members. Our paper is very much related to the contributions of Young who has analyzed a similar game and derived independently proposition 4. He should receive credit for being the first one to point out the relevance of the least core and the nucleolus to predict some dimensions of the equilibrium strategies of the lobbyists[10].

Dekel, Jackson and Wolinsky examine an open-ended sequential game where lobbies alternate in increasing their offers to legislators. By allowing lobbies to keep responding to each other with counter-offers, their game eliminates the asymmetry and the resulting second mover advantage of the Groseclose and Snyder 's game. Several settings are considered depending upon the type of offers that lobbies can make to legislators (Up-front payments versus promises contingent upon the voting outcome) and upon the role played by budget constraints[11]. The difference in the budgets of the lobbies plays a critical role in determining which lobby is successful when lobbies are budget constrained, and the difference in their willingness to pay plays an important role when they are not budget constrained. When lobbies are budget constrained, their main result states that the winning lobby is the one

[10]While we were working on this project, Ron Holzman pointed out to us the relevance of the notion of least core for our problem. After completing our paper, we have discovered, while reading Montero (2006), that Young (1978a,b) did reach the same conclusion a long ago. In fact, he wrote four remarkable papers on this topic containing many more results and insights. In (1978c), he presents a model of lobbying without opposition where the legislators have the leadership (they "post" the price to which they are willing to sell their vote and, then, the lobby select coalition). Young and Shubik (1978) develops another version of the competitive model, that they call the session lobbying game, where the nucleolus is the equilibrium.

[11]These considerations which are irrelevant in the case of our two-round sequential game are important in their game.

whose budget plus half of the sum of the value that each legislator attaches to voting in favor of this lobby exceeds the corresponding magnitude calculated for the other lobby. In contrast, when lobbies are not budget constrained, what matter are the lobbies's valuations and the intensity of preferences of a particular "near-median" group of legislators. The lobby with a-priori minority support wins when its valuation exceeds the other lobby's valuation by more than a magnitude that depends on the preferences of that near-median group. With our terminology, we can say that their main results are motivated by the derivation of the victory threshold(s). Once the value of these threshold(s) are known, the identity of the winner as well as the lobbying expenditures and the identity of bribed legislators follow. Note however that they limit their analysis to the simple majority game and are not in position to evaluate the intrinsic role of the simple game and the legislative power of legislators.

Note finally that the version of our game where the two lobbies make their offers simultaneously instead of sequentially has the features of a Colonel Blotto game. These games are notoriously difficult to solve and very little is known in the case of asymmetric players.

2 The Model and the Game

In this section, we describe formally the main ingredients of the problem as well as the lobbying game which constitute our model of vote-buying by lobbyists.

The external forces that seek to influence the legislature are represented by two players, whom we call lobby 0 and lobby 1. Lobby 1 wants the legislature to pass a bill (change, proposal, reform) that would change some area of law. Lobby 0 is opposed to this bill and wants to maintain the status quo. Lobby 0 is willing to spend up to W_0 dollars to prevent passage of the bill while lobby 1 is willing to pay up to W_1 dollars to pass the bill. Sometimes, we refer to these two policies in competition as being policies 0 and 1. We assume that $\Delta W \equiv W_1 - W_0 > 0$. While this assumption may receive different interpretations[12], we will assume here that the two lobbies represent faithfully the two opposite sides of the society on this binary social agenda and therefore that policy 1 is the socially efficient policy. We could consider that the two lobbies represent more private or local interests and that W_1 and W_0 ignore the implications of these policies on the rest of the society: in that case the reference to social optimality should be abandoned. Finally, we could consider instead the budgets B_1 and B_0 of the two lobbies, and assume that they are budget constrained i.e.

[12] As explained forcefully in Dekel, Jackson and Wolinsky (2006), in general, the equilibrium predictions will be sensitive to the type of offers that can be made by the lobbies and whether they are budget constrained or not. As explained later, these considerations are not relevant in the case of our lobbying game.

that $B_1 \leq W_1$ and $B_0 \leq W_0$. Under that interpretation, the ratio $\frac{W_1}{W_0}$ should be replaced by the ratio $\frac{B_1}{B_0}$.This ratio which is (by assumption) larger than 1 will be a key parameter in our equilibrium analysis. Depending upon the interpretation, it could measure the intensity of the superiority of the reform as compared to the status quo or the ex ante advantage of lobby 1 over lobby 0 in terms of budgets.

The legislature is described by a *simple game*[13] i.e. a pair $(N, \mathcal{W})$, where $N = \{1, 2, ..., n\}$ is the set of legislators and $\mathcal{W}$ the set of *winning* coalitions satisfies: $S \in \mathcal{W}$ and $S \subseteq T$ implies $T \in \mathcal{W}$. The interpretation is the following. A bill is adopted if and only if the subset of legislators who voted for the bill forms a winning coalition. From that perspective, the set of winning coalitions describes the rules operating in the legislature to make decisions. A coalition C is *blocking* if $N \backslash C$ is not winning: some legislators (at least one) from C are needed to form a winning coalition. We will denote by $\mathcal{B}$ the subset of blocking coalitions[14]; from the definition, the status quo is maintained as soon as the set of legislators who voted against the bill forms a blocking coalition. The simple game is called *proper* if $S \in \mathcal{W}$ implies $N \backslash S \notin \mathcal{W}$. The simple game is called *strong* if $S \notin \mathcal{W}$ implies $N \backslash S \in \mathcal{W}$ and *constant-sum* if it is both proper and strong i.e. equivalently if $\mathcal{B} = \mathcal{W}$[15]. The set of minimal (with respect to inclusion) winning (blocking) coalitions will be denoted $\mathcal{W}_m$ ($\mathcal{B}_m$). A legislator is a *dummy* if he is not part of any minimal winning coalition, while a legislator is a *vetoer* if he belongs to all blocking coalitions. A group of legislators forms an *oligarchy* if a coalition is winning iff it contains that group i.e. each member of the oligarchy is a vetoer and the oligarchy does not need any extra support to win i.e. legislators outside the oligarchy are dummies. When the oligarchy consists of a single legislator, the game is called dictatorial. In some cases it will be possible to order, partially or totally, the legislators according to *desirability* as defined by Maschler and Peleg (1966). Legislator $i \in N$ is at least as desirable as legislator $j \in N$ if $S \cup \{j\} \in \mathcal{W}$ implies $S \cup \{i\} \in \mathcal{W}$ for all $S \subset N \backslash \{i, j\}$. Legislators i and j are symmetric or interchangeable if $S \cup \{j\} \in \mathcal{W}$ iff $S \cup \{i\} \in \mathcal{W}$ for all $S \subset N \backslash \{i, j\}$.

In this paper, all legislators are assumed to be biased towards policy 1 i.e. all of them will vote for policy 1 against policy 0 if no other event interferes with the voting process. Under the interpretation privileged in this paper, this assumption simply means that legislators vote for the policy maximizing the aggregate social welfare. This assumption is of course

[13]In social sciences it is sometimes called a committee or a voting game. In computer science, it is called a quorum system (Holzman, Marcus and Peleg (1997)) while in mathematics, it is called a hypergraph (Berge (1989), Bollobas(1986)). An excellent reference is Taylor and Zwicker (1999).

[14]In game theory, $(N, \mathcal{B})$ is often called the dual game.

[15]When the simple game is constant-sum, the two competing alternatives are treated equally.

controversial[16] and becomes even more so when we abandon the interpretation in terms of social efficiency. It is introduced here for the sake of simplicity as, otherwise, we would have to consider an additional parameter of differences among the legislators that we prefer to ignore for the time being. Indeed, in contrast to Banks (2000) and Groseclose and Snyder (1996), our assumption on the preferences of legislators rule out the existence of horizontal heterogeneity. However, legislators also value money and we introduce instead some form of vertical heterogeneity. Precisely, we assume that legislators differ among themselves according to their willingness to depart from social welfare. The type of legislator i, denoted by α_i, is the minimal amount of dollars that he needs to receive in order to sacrify one dollar of social welfare. Therefore if the policy adopted generates a level of social welfare equal to W, the payoff of legislator i if he receives a transfer t_i is:

$$t_i + \alpha_i W.$$

This payoff formulation is compatible with two behavioral assumptions. Either, the component W appears as soon as the legislator has voted for a policy generating a level of social welfare W regardless of the fact that this policy has been ultimately selected or not: we will refer to this model, as *behavioral model P*, where P stands for procedural. Or, the component W appears whenever the policy ultimately selected generates a level of social welfare W regardless of the fact that the legislator has voter for or against this policy: we will refer to this model, as *behavioral model C*, where C stands for consequential. In this paper, we will focus exclusively on the behavioral model P and explain in the last section how to adjust the results in the case of behavioral model C.

To promote passage of the bill, lobby 1 can promise to pay money to individual legislators conditional on their supporting the bill. Similarly, lobby 0 can promise to pay money to individual legislators conditional on their opposing the bill. We denote by $t_{i0} \geq 0$ and $t_{i1} \geq 0$ the (conditional) offers made to legislator i by lobbies 0 and 1 respectively. The corresponding n-dimensional vectors will be denoted respectively by t_0 and t_1.

The timing of actions and events that we consider to describe the lobbying game is the following[17].

1. Nature draws the type of each legislator.

2. Lobby 1 make contingent monetary offers to individual legislators.

3. Lobby 0 observes the offers made by lobby 1 and makes contingent monetary offers to individual legislators

[16] It is however very common in the recent literature on lobbying (Grossman and Helpman (2001)).

[17] Specific details and assumptions will be provided in due time.

4. Legislators vote.
5. Payments (if any) are implemented.

This game has $n+2$ players. A strategy for a lobby is a vector in $\Re^n_+$. Each legislator can chose among two (pure) strategies: to oppose or to support the bill.

The important thing to note is that the two lobbies move in sequence[18]. Following Banks (2000), Diermeier and Myerson (1999) and Groseclose and Snyder (1996), we assume that lobby 1, the advocate for change, makes the first move and announces its offers first, and lobby1's offers are known to lobby 0 when lobby 0 makes its offers to induce legislators to oppose the bill. This sequential version of the lobbying game should be contrasted with the version where either the two lobbies move simultaneously or where lobbies make offers in an open-ended sequential and alternating bidding process. As pointed out by Dekel, Jackson and Wolinsky (2006), in such case the detailed specification of the type of offers as well as the budget constraints (if any) may matter. For instance, we can assume that lobbies' offers are either up-front payments or campaign promises honored only if the policy supported by the lobby is ultimately selected. In the case where the moves are simultaneous campaign promises with budget constraints, the lobbying game belongs to the family of Colonel Blotto games, a class of discontinuous two-player zero-sum games which are notoriously difficult to solve; existence and characterization of equilibria in mixed strategies has been proved in the symmetric case i.e. when $W_1 = W_0$ and for some very specific simple games, like the simple majority game. In our case, where each lobby moves only once and in sequence, these differences do not matter. The specificity of the sequential game which is considered here has been criticized by several authors including Dekel, Jackson and Wolinsky (2006) and Grossman and Helpman (2001). In particular, in this game, there is a strong second mover advantage. Note however, that we can see the results of this paper as answering alternatively the following questions: how much asymmetric must be the budgets or valuations of the two lobbies to ensure the existence of a pure strategy Nash equilibrium in the simultaneous version of the game i.e. in this generalized Colonel Blotto game? When it exists, how the offers made to legislators in such equilibrium look like and depend upon their personal characteristics.

To complete the description of the game, it remains to specify what are the informations

[18] when the two lobbies act simultaneously, that we call compactly (with a slight abuse in the terminology) Nash equilibria. We show that they are efficient but exist only under very stringent conditions. Then, we explore the set of subgame perfect Nash equilibria in the case where the two lobbies move in sequence, that we call Stackelberg equilibria and show the critical role played by the efficiency threshold. These results are derived without putting too much structure on the simple game. In our final part, we look specifically at the case of the majority game with three legislators and calculate the Nash equilibrium in mixed strategies.

held by the players when they act. In this paper, we have already implicitly assumed that the votes of the legislators are observable, i.e. open voting, and that the vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ of legislators's types is common knowledge and without loss of generality such that $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$.. We refer to this informational environment as *political certainty*. It has two implications: first, the lobbies know the types of the legislators when they make their offers and second, each legislator knows the type of any other legislator when voting. The environment where the type α_i of legislator i is a private information, to which we refer as *political uncertainty*, is analyzed in Le Breton and Zaporozhets (2007) in the case where the two lobbies move simultaneously.

3 The Victory Threshold

In this section, we begin our examination of the *subgame perfect Nash equilibria* of the lobbying game. Hereafter, we will refer to them simply as equilibria. Our first objective is to calculate a key parameter of the game, that we call the *victory threshold*. Once calculated, this parameter leads to the following preliminary description of the equilibrium. Either, the ratio $\frac{W_1}{W_0}$ is larger than or equal to the victory threshold and then lobby 1 makes an offer and wins the game, or $\frac{W_1}{W_0}$ is smaller than the victory threshold and then lobby1 does not make any offer and lobby 0 wins the game. The victory threshold depends both upon the vector of types α and the simple game $(N, \mathcal{W})$. Given the second mover advantage, the victory threshold is larger than or equal to 1. Therefore,while necessary, $W_1 > W_0$ is not sufficient in general to guarantee the victory of lobby 1. The victory threshold provides the smallest value of the relative differential leading to such victory.

The equilibrium of the lobbying game can be easily described. Let $t_1 = (t_{11}, t_{21}, ..., t_{n1}) \in \Re^n_+$ be lobby 1's offers. Lobby 0 will find profitable to make a counter offer if there exists a blocking coalition S such that:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i W_1 \right) < \sum_{i \in S} \alpha^i W_0 + W_0.$$

Indeed, in such case, there exists a vector $t_0 = (t_{10}, t_{20}, ..., t_{n0})$ of offers such that:

$$t_{i1} + \alpha^i W_1 < t_{i0} + \alpha^i W_0 \text{ for all } i \in S \text{ and } \sum_{i \in S} t_{i0} < W_0.$$

The first set of inequalities implies that legislators in S will vote against the bill while the last one simply says that the operation is beneficial from the perspective of lobby 0.

Therefore , if lobby 1 wants to make an offer that cannot be cancelled by lobby 0, it must satisfy the list of inequalities:

$$\sum_{i\in S}\left(t_{i1}+\alpha^{i}\Delta W\right)<W_0 \text{ for all } S\in\mathcal{B}.$$

The cheapest offers t_1 meeting these constraints are the solutions of the following linear program:

$$\begin{gathered}\underset{t_1}{Min}\sum_{i\in N}t_{i1}\\ \text{subject to the constraints}\\ \sum_{i\in S}\left(t_{i1}+\alpha^{i}\Delta W\right)\geq W_0 \text{ for all } S\in\mathcal{B}\\ \text{and } t_{i1}\geq 0 \text{ for all } i\in N.\end{gathered}\tag{1}$$

Lobby 1 will find profitable to offer the optimal solution t_1^* of problem (1) if the optimal value to this linear program is less than W_1. It is then important to be able to compute this optimal value. To do so, we first introduce the following definition from combinatorial theory.

Definition 1. A family of coalitions $\mathcal{C}$ is *(sub)balanced* if there exists a vector $\delta\in\Re^{\#\mathcal{C}}$, called *(sub)balancing coefficients*, such that:

$$\begin{aligned}\sum_{S\in\mathcal{C}_i}\delta(S) &\leq (=1) \text{ for all } i\in N\\ \text{and } \delta(S) &\geq 0 \text{ for all } S\in\mathcal{C}.\end{aligned}$$

The following result summarizes the equilibrium analysis of the sequential game.

Proposition 1. Either (i) $W_1\geq\sum_{S\in\mathcal{B}}\delta(S)\left[W_0-\sum_{i\in S}\alpha^{i}\Delta W\right]$ for all vectors of subbalancing coefficients δ attached to $\mathcal{B}$ and then lobby 1 offers an optimal solution t_1^* to problem (1) and lobby 0 offers nothing and so the bill is passed. Or (ii) $W_1<\sum_{S\in\mathcal{B}}\delta(S)\left[W_0-\sum_{i\in S}\alpha^{i}\Delta W\right]$ for at least one vector of subbalancing coefficients δ attached to $\mathcal{B}$ and then both lobbyists promise nothing and so the bill is not passed.

Proof: Let $v^*(\mathcal{B},\boldsymbol{\alpha})$ be the optimal value of problem (1). From the duality theorem of linear programming, $v^*(\mathcal{B}, \boldsymbol{\alpha})$ is the optimal value of the following linear program:

$$\underset{\gamma}{Max} \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right]$$

subject to the constraints

$$\sum_{S \in \mathcal{B}_i} \delta(S) \leq 1 \text{ for all } i \in N$$

and $\delta(S) \geq 0$ for all $S \in \mathcal{B}$.

The conclusion follows. □

This result[19] leads to several conclusions. If $W_0 - \sum_{i \in S} \alpha^i \Delta W \leq 0$ for all $S \in \mathcal{B}$, then $\delta = 0$ is a solution and therefore $v^*(\mathcal{B}, \boldsymbol{\alpha}) = 0$. We are in case (i) but lobby 1 promises nothing. If instead, $W_0 - \sum_{i \in S} \alpha^i \Delta W > 0$ for at least one $S \in \mathcal{B}$, then $v^*(\mathcal{B}, \boldsymbol{\alpha}) > 0$. Note further that for any vector of subbalancing coefficients δ attached to $\mathcal{B}$:

$$\begin{aligned} \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha_i \Delta W \right] &= W_0 \sum_{S \in \mathcal{B}} \delta(S) - \Delta W \sum_{S \in \mathcal{B}} \delta(S) \sum_{i \in S} \alpha^i \\ &= W_0 \sum_{S \in \mathcal{B}} \delta(S) - \Delta W \sum_{i \in N} \alpha^i \sum_{S \in \mathcal{B}_i} \delta(S) \\ &\geq W_0 \sum_{S \in \mathcal{B}} \delta(S) - \Delta W \sum_{i \in N} \alpha^i. \end{aligned}$$

We deduce:

$$\sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right] \geq W_0 \gamma^*(\mathcal{B}) - \Delta W \sum_{i \in N} \alpha^i,$$

and therefore

$$v^*(\mathcal{B}, \boldsymbol{\alpha}) + \Delta W \sum_{i \in N} \alpha^i \geq W_0 \gamma^*(\mathcal{B}), \tag{2}$$

where $\gamma^*(\mathcal{B}) \equiv v^*(\mathcal{B}, \mathbf{0})$, called hereafter the *hurdle factor*[20], is the value of the problem:

$$\underset{\gamma}{Max} \sum_{S \in \mathcal{B}} \delta(S)$$

subject to the constraints

$$\sum_{S \in \mathcal{B}_i} \delta(S) \leq 1 \text{ for all } i \in N$$

and $\delta(S) \geq 0$ for all $S \in \mathcal{B}$.

[19] Note that we could replace $\mathcal{B}$ by $\mathcal{B}_m$ in the statement of proposition 1.
[20] This terminology is due to Diermeier and Myerson (1999).

After simplifications, we deduce that if we are in case (i), then:

$$\frac{W_1}{W_0} \geq \frac{\gamma^*(\mathcal{B}) + \sum_{i \in N} \alpha^i}{1 + \sum_{i \in N} \alpha^i}. \tag{3}$$

Inequality (3) is simply a necessary condition for case (i) to prevail. It is also sufficient for any problem where it can be shown that all the coordinates of t_1^*, the solution to problem (1), are strictly positive. Indeed, in that case, we deduce from the complementary slackness condition, that:

$$\sum_{S \in \mathcal{B}_i} \delta(S) = 1 \text{ for all } i \in N,$$

and (2) becomes an equality.

The force of proposition 1 is to reduces the derivation of the victory threshold to the exploration of the geometry of a convex polytope: the polytope of vector of subbalancing coefficients. To use it efficiently, it may be appropriate to consider an arbitrary family of balanced coalitions i.e. with edges not necessarily in $\mathcal{B}$. If we define the function Φ over coalitions of N as follows:

$$\Phi(S) = \left\{ \begin{array}{c} W_0 - \sum_{i \in S} \alpha^i \Delta W \text{ if } S \in \mathcal{B}, \\ 0 \text{ otherwise.} \end{array} \right.$$

then the duality argument used in the proof of proposition 1 shows that in the statement we can trivially replaced "$\sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right]$ for all vectors of subbalancing coefficients δ attached to $\mathcal{B}$" by "$\sum_{S \subset N} \delta(S) \Phi(S)$ for all vectors of balancing coefficients δ". The first formulation is useful as soon as we are in position to characterize the vector of subbalancing coefficients attached to the family of coalitions $\mathcal{B}$ i.e. to the simple game[21]. This amounts first to explore the combinatorics of simple games. A classification of simple games was first provided by Morgenstern and von Neumann (1944) and further explored by Isbell (1956)(1959). The second formulation takes advantage of the tremendous volume of research accomplished in cooperative game theory. Indeed, is well know since Bondareva (1963) and Shapley (1967) that a game with transferable utility has a nonempty core iff it is balanced. As pointed out by Shapley, this amounts to check the balancedness inequalities for the extreme points of the polytope of balanced collections of coalitions. He demonstrated that vector δ is an extreme point of the polytope of balanced collections iff the collection of coalitions $\{S\}_{S \subseteq N : \delta(S) > 0}$ is minimal in terms of inclusion within the set of balanced collections of coalitions. A minimal balanced collection has at most n sets[22]. Peleg (1965) has

[21] Holzman, Marcus and Peleg (1997) contains results on the polytope of balancing coefficients for an arbitrary proper and strong simple game.

[22] We refer the reader to Owen (2001) for a complete and nice exposition of this material.

given an algorithm for constructing the minimal balanced sets inductively. We illustrate the mechanical use of proposition 1 through a sequence of simple examples .

Example 1. Consider the simple majority game with 3 legislators where $S \in \mathcal{B}_m$ iff $\#S = 2$ i.e. $S = \{1,2\}$, $\{1,3\}$ and $\{2,3\}$. The set of vectors of subbalancing coefficients is the polytope described by the set of extreme points

$$(0,0,0)\,,(1,0,0)\,,(0,1,0)\,,(0,0,1)\,,\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},0,\frac{1}{2}\right),\left(0,\frac{1}{2},\frac{1}{2}\right) \text{ and } \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

From the ordering of the α_i and proposition 1, we deduce that

$$v^*(\mathcal{B},\alpha) = \ Sup\left(W_0 - (\alpha_1+\alpha_2)\,\Delta W,\ \frac{3W_0 - 2\,(\alpha_1+\alpha_2+\alpha_3)\,\Delta W}{2},\ 0\right),$$

$$\gamma^*(\mathcal{B}) = \frac{3}{2}W_0.$$

The first (respectively second) term is the largest whenever $(\alpha_1+\alpha_2)\,\Delta W \leq W_0 \leq 2\alpha_3\Delta W$ (respectively $W_0 \geq 2\alpha_3\Delta W$) and $v^*(\mathcal{B},\alpha) = 0$ whenever $(\alpha_1+\alpha_2)\,\Delta W \geq W_0$.

We will examine later how to derive the optimal (offers) of lobby 1 and in particular the personal characteristics of the legislators who are offered some positive amount. This will depend obviously on two main features: α_i i.e. his/her personal propensity to vote against social welfare and also its position in the family of coalitions. If legislator i is a dummy then, obviously, $t_{i1} = 0$. But if he is not a dummy, then in principle all situations are conceivable: he may receive something in all optimal offers, in some of them or in none of them. It will be important to know the status of a legislator according to this classification in three groups. In example 1, no legislator is a dummy. However, if $(\alpha_1+\alpha_2)\,\Delta W \leq W_0 \leq 2\alpha_3\Delta W$, then the relevant extreme point is $(1,0,0)$. Since then $\sum_{S\in\mathcal{B}_3}\gamma(S) < 1$, we deduce from complementary slackness that $t_{31} = 0$.

Example 2. Consider the simple game with 4 legislators[23] where $S \in \mathcal{B}_m$ iff $S = \{1,2\}$, $\{1,3\}$, $\{1,4\}$ or $\{2,3,4\}$. According to Shapley (1967), the minimal balanced families of coalitions are (up to permutations):

$$\{\{1,2,3\}\,,\{1,2,4\}\,,\{1,3,4\}\,,\{2,3,4\}\}\,,\{\{1,2\}\,\{1,3\}\,\{1,4\}\,\{2,3,4\}\}\,,$$
$$\{\{1,2\}\,\{1,3\}\,\{2,3\}\,\{4\}\}\,,\{\{1,2\}\,\{1,3,4\}\,\{2,3,4\}\}$$

with the following respective vectors of balancing coefficients$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}\right),\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1\right)$

[23] As demonstrated by Von Neumann and Morgenstern ((1944), 52C), this is the unique strong simple four-person game without dummies.

and $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.We deduce from proposition 1 that:

$$\begin{aligned} v^*(\mathcal{B},\alpha) &= Sup\left(\begin{array}{c} W_0-(\alpha_1+\alpha_2)\Delta W,\ \frac{3W_0-(2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)\Delta W}{2}, \\ \frac{6W_0-(3\alpha_1+4\alpha_2+4\alpha_3+4\alpha_4)\Delta W}{4},\ \frac{5W_0-3(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Delta W}{3}, 0 \end{array}\right), \\ \gamma^*(\mathcal{B}) &= \frac{5}{3}W_0. \end{aligned}$$

Example 3. Consider the following simple game with 3 legislators and $S \in \mathcal{B}_m$ iff $S = \{1,2\}$or $\{1,3\}$. The set of vectors of subbalancing coefficients is the polytope described by the set of extreme points $(0,0), (1,0), (0,1)$. We deduce from proposition 1 that:

$$v^*(\mathcal{B},\alpha) = Sup\left(W_0-(\alpha_1+\alpha_2)\Delta W,\ 0\right),$$

$$\gamma^*(\mathcal{B}) = W_0.$$

Example 4. Consider the simple game with 5 legislators where $S \in \mathcal{B}_m$ iff $S = \{1,2\}$, $\{1,3\}$, $\{1,4,5\}$, $\{2,3,4\}$ or $\{2,3,5\}$. The geometry of the polytope becomes more intricate. We will demonstrate later, through a different technique, that when $\alpha = \mathbf{0}$, the relevant extreme point is the vector $\left(\frac{1}{5},\frac{1}{5},\frac{3}{5},\frac{2}{5},\frac{2}{5}\right)$ i.e.

$$\gamma^*(\mathcal{B}) = \frac{9}{5}W_0.$$

4 Complements and Extensions

Proposition 1 constitutes an important element of the toolkit to determine the victory threshold. In this second section, we continue this exploration of the problem having in mind to add more elements in the toolkit. In the first subsection, we show that in the special case where $\alpha = 0$, our problem is strongly connected to one of the most famous problem in the combinatorics of sets. We elaborate on the relationship with this branch of applied mathematics and show how to take advantage of this body of knowledge to get a better understanding of our own questions, on top of which is the determination of the hurdle factors attached to a simple game. In the second subsection, we show, quite surprisingly, that the set of equilibrium offers to the legislators made by the first mover lobby coincides with the least core (and always contains the nucleolus) of the simple game. We then explore within the important class of weighted majority games, how the personal positions of the legislators in the simple game translate into some personal prices: we show that more desirable legislators get more and evaluate through real world examples the differences between their respective prices. In the third subsection, we characterize the size and the composition of the coalitions of legislators receiving an offer in the case of the simple majority game and for an arbitrary α. Finally, in the last subsection, we show how to adjust the current formulation and results when instead of assuming that legislators are procedural, we assume that they are consequential.

4.1 Fractional Matchings and Coverings

The main purpose of this section is to connect our problem to the covering problem which is considered to be one of the most famous problems in the combinatorics of sets. As pointed out by Füredi (1988), "the great importance of the covering problem is supported by the fact that apparently all combinatorial problems can be reformulated as the determination of the covering number of a certain hypergraph". A *hypergraph* is an ordered pair $H = (N, \mathcal{H})$ where N is a finite set of n vertices and $\mathcal{H}$ is a collection of subsets of N called edges. The *rank* of H is the integer $r(H) \equiv Max\{\#E : E \in \mathcal{H}\}$. If every member of $\mathcal{H}$ has r elements, we call it $r-$uniform. An $r-$uniform hypergraph H is called $r-$partite if there exists a partition $\{N_k\}_{1\leq k\leq K}$ of N such that $\#(N_k \cap E) = 1$ holds for all $E \in \mathcal{H}$ and all $k = 1, ...K$. The maximum *degree* of the hypergraph H, denoted $D(H)$, is the number $Max_{i\in N} Deg_H(i)$ where $Deg_H(i) \equiv \#\{E \in \mathcal{H} : i \in E\}$. A hypergraph is $D-$regular if $Deg_H(i) = D(H)$ for all $i \in N$. Given an integer k, a hypergraph is $k-$*wise intersecting* if any k edges of it have a non-empty intersection; intersecting is used in place of 2-intersecting. An $(r, \lambda)-$design is a hypergraph $(N, \mathcal{H})$ such that for all $i \in N$, $Deg_H(i) = r$ and for all $\{i, j\} \subseteq N$, $\#\{S \in \mathcal{H} : \{i, j\} \subseteq S\} = \lambda$. It is called symmetric if $n = \#\mathcal{H}$. Then, $n = \frac{r^2-r+\lambda}{\lambda}$. A projective plane of order n, denoted by $PG(2, n)$ is a symmetric $(n + 1, 1)$ design. More generally, a $t-$dimensional finite projective space of order q, denoted by $PG(t, q)$, where q is a primepower, is an $(r, \lambda)-$design with $r = q^t + q^{t-1} + ... + 1$ and $\lambda = q^{t-1} + ... + q + 1$. A hypergraph is an $r-$clique (or a maximal intersecting family of rank r) if any two edges intersect in at least one point and it cannot be extended to another intersecting family by adding a new $r-$set.

Given an integer k, a k-cover of H is a vector $t \in \{0, 1, ..., k\}^n$ such that:

$$\sum_{i\in S} t_i \geq k \text{ for all } S \in \mathcal{H}. \tag{4}$$

A $k-$matching of H is a collection $\{E_1, ..., E_s\}$(repetitions are possible) such that $E_j \in \mathcal{H}$ for all $j = 1, ...s$ every $i \in N$ is contained in at most k of E_j. A $1-$cover ($1-$matching) is simply called a cover (matching) of H. Note that a cover is simply a set T intersecting every edge of H i.e. $T \cap E \neq \varnothing$ for all $E \in \mathcal{H}$ while a matching is a collection of pairwise disjoint members of $\mathcal{H}$. A $k-$cover t^* minimizing $\sum_{i\in N} t_i$ subject to the constraints (4) is called an optimal $k-$cover and $\gamma_k^*(H) \equiv \sum_{i \in N} t_i^*$ is called the $k-$covering number. A $k-$matching δ^*maximizing $\sum_{S \subseteq N} \delta(S)$ is called an optimal $k-$matching and $\mu_k^*(H) \equiv \sum_{S \subseteq N} \delta^*(S)$ is called the $k-$matching number. When $k = 1$, $\gamma_1^*(H)$ is the minimum cardinality of the covers and is called the *covering number* of H while $\mu_1^*(H)$ is the maximum cardinality of a matching and is called the *matching number* of H. A hypergraph H is $\gamma-$*critical* if each of its subfamilies has a smaller covering number i.e. $\gamma_1^*((N, \mathcal{H} - \{E\})) < \gamma_1^*(H)$ for all $E \in \mathcal{H}$.

A *fractional cover* of H is a vector $t \in \Re^n$ such that:

$$\begin{aligned} \sum_{i \in S} t_i &\geq 1 \text{ for all } S \in \mathcal{H} \\ \text{and } t_i &\geq 0 \text{ for all } i \in N. \end{aligned} \tag{5}$$

A fractional matching of H is a vector $\delta \in \Re^{\#\mathcal{H}}$ such that:

$$\begin{aligned} \sum_{S \in \mathcal{H}_i} \delta(S) &\leq 1 \text{ for all } i \in N \\ \text{and } \delta(S) &\geq 0 \text{ for all } S \in \mathcal{H}. \end{aligned} \tag{6}$$

A fractional cover t^* minimizing $\sum_{i \in N} t_i$ subject to the constraints (5) is called an optimal fractional cover and $\gamma^*(H) \equiv \sum_{i \in N} t_i^*$ is called the *fractional covering number.* A fractional matching δ^*maximizing $\sum_{S \subseteq N} \delta(S)$ subject to the constraint (6) is called an optimal fractional matching and $\mu^*(H) \equiv \sum_{S \subseteq N} \delta^*(S)$ is called the *fractional matching number.*

It follows immediately from these definitions that the hurdle factor of the simple game $(N, \mathcal{W})$ is the fractional covering number of $H = (N, \mathcal{B})$. If , in contrast to what has been assumed in the preceding section, money is available in indivisible units, then the appropriate parameter becomes $\gamma_{W_0}^*(H)$ where the integer W_0 is the value of policy 0 for lobby 0 (when $\mathcal{C} = \mathcal{B}$ i.e. when lobby 0 is the follower) expressed in monetary units. The case where $W_0 = 1$ is of particular interest as it describes the situation where lobby 0 has a single unit of money to spend in the process. The problem is now purely combinatorial: whom should be the legislators on which lobby 1 should spend one unit to prevent lobby 0 from a targeting a unique pivotal legislator[24]. Hereafter, the integer $\gamma_1^*(H)$ will be called the *integral hurdle factor.* While we will focus mostly on the divisible case, it is interesting to note the implications of indivisibilities on the equilibrium outcome of the lobbying game. Note finally that if we invert the order of moves between the two lobbies, then the relevant simple game is the *dual game* $(N, \mathcal{B})$ and the corresponding hurdle factor that we will call the *dual hurdle*

[24] To support that interpretation, we need however to assume that a legislator who is indifferent breaks the tie in direction of lobby 0.

factor, is the fractional covering number of $H = (N, \mathcal{W})$. The following developments apply equally to both hurdle factors and we will often use the symbol $\mathcal{H}$ without specifying whether $\mathcal{H} = \mathcal{B}$ or $\mathcal{H} = \mathcal{W}$. For an arbitrary hypergraph H, we have the inequalities:

$$\mu_1^*(H) \leq \frac{\mu_k^*(H)}{k} \leq \mu^*(H) = \gamma^*(H) \leq \frac{\gamma_k^*(H)}{k} \leq \gamma_1^*(H). \tag{7}$$

We deduce immediately from these inequalities that the value of the hurdle factor increases with the "degree" of indivisibilities; indivisibilities act as additional integer constraints in the linear program describing the determination of the optimal fractional matchings and coverings. We spend the rest of this section in the calculation of the hurdle and integral hurdle factors[25] for some important hypergraphs and a short exposition of some results from the theory of hypergraphs providing bounds and estimates of these numbers

The calculation of the covering number of an arbitrary hypergraph is an NP-hard problem in contrast to the determination of the fractional covering number which amounts to solve a linear program. The examples presented below arise from the theory of simple games. In some cases, the hypergraph $\mathcal{H}$ describes the family of minimal winning coalitions while in some others it represents the family of minimal blocking coalitions.

Example 5 (Qualified Majorities/minorities). Consider the case of an arbitrary symmetric simple game i.e. $S \in \mathcal{H}$ iff $\#S = q$ where q is a fixed integer. In that case, it is easy to show that $\gamma^*(\mathcal{H}) = \frac{n}{q}$. For instance, in the case of the winning coalitions of the majority game ($q = \frac{n+1}{2}$ is n is odd and $q = \frac{n+2}{2}$ is n is even), we obtain:

$$\gamma^*(\mathcal{H}) = \begin{cases} \frac{2n}{n+1} \text{ if n is odd,} \\ \frac{2n}{n+2} \text{ if n is even.} \end{cases}$$

which tends to 2 when n tends to infinity. In contrast:

$$\gamma_1^*(\mathcal{H}) = \begin{cases} \frac{n+1}{2} \text{ if n is odd,} \\ \frac{n+2}{2} \text{ if n is even.} \end{cases}$$

When n is odd the family of blocking coalitions of the majority game coincides with the family of winning coalitions. Instead, when n is even, the family of minimal blocking coalitions $\mathcal{H}$ is the family of subsets of cardinality $\frac{n}{2}$ and then $\gamma^*(\mathcal{H}) = 2$ while $\gamma_1^*(\mathcal{H}) = \frac{n+2}{2}$.

Example 6 (Symmetric Simple Games). The games considered in example 5 display a total symmetry[26] in the sense that the group of automorphisms of the simple game is the all group of permutations. We can consider simple games exhibiting some regularity without

[25]It has been demonstrated by Chung, Furedi, Garey and Graham (1988) that for any rational number x, there exists a, hypergraph $H = (N, \mathcal{H})$ such that $\mu^*(\mathcal{H}) = x$.

[26]This hypergraph is often called the complete $q-$graph.

displaying such level of symmetry[27]. This is the case of the (r, λ) $-$design and in particular the projective planes of order n $PG(2, n)$ which have been defined earlier. Simple calculations show that $\gamma^* (r, \lambda) = \frac{r-1}{\lambda} + \frac{1}{r}$ and therefore $\gamma^* (PG(2, n)) = n + \frac{1}{n+1}$.

Another example of hypergraph displaying some symmetry due to Erdös and Lovasz (1975) is the following[28]. Consider a set S with $2\gamma - 2$ elements where γ is a given integer. For each partition $\pi = (P, P')$ of S where $P \cup P' = S$ and $\#S = \#S' = \gamma - 1$, take a new element i_π. Let $N \equiv S \cup_\pi \{i_\pi\}$ and let $\mathcal{H}$ be the collection of all $\gamma-$tuples of the form $P \cup \{i_\pi\}$ where $\pi = (P, P')$ is a partition. Then, it is easy to verify that $\gamma^*(H) = 2$ and $\gamma_1^*(H) = \gamma$.

Example 7 (Compound Simple Games). Another class of hypergraphs describing an important class of voting procedures is the following. Let $(N_r, \mathcal{W}_r)_{1 \leq r \leq R}$ be a family of R hypergraphs with $N_r \cap N_t = \varnothing$ for all $r, t = 1, ..., R$ with $r \neq t$. Let $(N, \mathcal{W})$ be such that $N = \cup_{r=1}^R N_r$ and $S \in \mathcal{W}$ iff $S \cap N_r \in \mathcal{W}_r$ for all $r = 1, ..., R$. This is the definition of a multicameral legislature as defined by Diermeier and Myerson (1999): a reform is approved if it is approved in all the different R chambers according to the rules (possibly different) in use in the chambers. It is easy to show that:

$$\gamma^*(\mathcal{B}) = \sum_{r=1}^{R} \gamma^*(\mathcal{B}_r).$$

This multicameral system is a special case of a compound simple game as defined first by Shapley (1962). Let $\left(\{1, ..., R\}, \widetilde{\mathcal{H}}\right)$ be a hypergraph on the set of chambers: $\widetilde{\mathcal{H}}$ describes the power of coalitions of chambers (Diermeier and Myerson (1999)'s definition corresponds to the case where $\widetilde{\mathcal{H}} = \{\{1, ..., R\}\}$ i.e. each chamber has a veto power). In general, $S \in \mathcal{H}$ iff:

$$\{r \in \{1, ..., R\} : S \cap N_r \in \mathcal{W}_r\} \in \widetilde{\mathcal{W}}.$$

The computation of $\gamma^*(\mathcal{W})$ is now more intricate. If $\left(\{1, ..., R\}, \widetilde{\mathcal{W}}\right)$ is uniform as well as $(N_r, \mathcal{W}_r)$ for all $r = 1, ..., R$, then $(N, \mathcal{W})$ is also uniform. Füredi (1981)'s inequality gives an upper bound on $\gamma^*(\mathcal{W})$.

Consider the case where $R = 2K + 1$ and $\#N_r = 2n_r + 1$ for all $r = 1, ..., R$ where K, $n_1, ...n_R$ are integers and assume that $\left(\{1, ..., R\}, \widetilde{\mathcal{W}}\right)$ and $(N_r, \mathcal{W}_r)$ for all $r = 1, ..., R$

[27] Von Neumann and Morgenstern (1944) offers a nice definition of symmetry based on the group of automorphisms of the game i.e. the group of permutations leaving invariant the winning coalitions. We may for instance require this group to be $k-$transitive for some integer k. With such definition, the symmetry of the game increases with the value of k.

[28] The nucleus coterie constructed by Holzman, Marcus and Peleg (1997) and the symmetric hypergraph considered by Le Breton (1989) bear similarities with that hypergraph.

are the simple majority games. Exploiting the symmetry of the game, the determination of an optimal fractional cover is equivalent to the determination of a vector $(t_1, ..., t_R) \in \Re_+^R$ minimizing $\sum_{r=1}^{R}(2n_r + 1)t_r$ subject to the constraints:

$$\sum_{r \in S} (n_r + 1)\, t_r \geq 1 \text{ for all } S \subset \{1, ..., R\} \text{ such that } \#S = K + 1.$$

With the change of variables $T_r = (n_r + 1)\, t_r$, the problem is equivalent to the minimization of $2\sum_{r=1}^{R}(\frac{n_r+\frac{1}{2}}{n_r+1})T_r$ subject to the constraints:

$$\sum_{r \in S} T_r \geq 1 \text{ for all } S \subset \{1, ..., R\} \text{ such that } \#S = K + 1.$$

This problem is almost identical to the covering problem for the majority game considered in example 1. The only difference lies in the fact that the weights on the variables do not need to be the same if the population in the chambers differ in size. When they are identical, using the calculation in example 1, we deduce that:

$$\gamma^*(\mathcal{W}) = \frac{(2K + 1)\,(2n + 1)}{(n + 1)\,(K + 1)},$$

which tends to 4 when n becomes large.

It is difficult in general to derive the exact value of $\gamma^*(\mathcal{H})$ when $\mathcal{H}$ is the family of minimal blocking or winning coalitions describing the decision making process of the legislature. Then, it becomes valuable to get some estimates of these numbers and the results established in the theory of hypergraphs can be useful in that respect.

Example 8 (Simple Games with Restrictions). The simple game describes the rules of the legislature but may also encompass some information about characteristics of the legislators suggesting that some coalitions should be declared as unfeasible. The population of legislators may be partitioned according to several types like for instance gender, geography, ethnicity or ideology and some coalitions, corresponding to some particular mixing of the types, may simply then be considered as irrelevant.

A first important classical example is the case of a one dimensional left-right ideological axis. In such setting, we may assume that disconnected coalitions are unlikely to form. So even if to pass a law, say a majority of legislators suffices, some of these hypothetical majorities can be safely disregarded. When $n = 2k + 1$ and the legislators are ordered from left to right, all minimal winning coalitions contain the legislator with index $k + 1$ i.e. the median legislator. The median legislator is a veto player and the fractional number of this "truncated" majority game is equal to 1 instead of 2.

This situation could be generalized by considering that the legislators are the vertices of a tree and that only connected coalitions with a majority of legislators can form[29]; the preceding example corresponds to the specific case where the tree is a line segment

If however, instead of being ordered on a line or a tree, legislators are ordered on a circle, then it is easy to see that the hurdle factor of the majority remains unchanged as the corresponding uniform hypergraph is $D-$regular.

Finally, legislators may be initially partitioned into K types i.e. $N = \cup_{k=1}^{K} N_k$ and in some legislatures, it is conceivable that to block a proposal, you need at least a legislator of each type. In this setting, the dual game is not the symmetric game according to which a coalition is blocking if it contains at least K legislators; instead, it is required that it contains at least K players of different types. Of course, some of these coalitions may be irrelevant for the reasons discussed earlier. The resulting set of minimal blocking coalitions define a $K-$partite hypergraph. Suppose for instance that $K = 2$ (legislators are either male or female) that $\#N_1 = \#N_2$ and that a proposal is blocked if the coalition contains at least one female and one male. If all such coalitions are likely to form, the set of minimal blocking coalitions consists of all pairs composed with a male and a female. If on the other hand, legislators are also differentiated according to left and right, then it is reasonable to assume that only pairs of legislators with the same ideology form. Let p_M and p_F to denote respectively the proportions of left legislators in the male and female populations and assume, for the sake of simplicity, that $p_M < \frac{1}{2}$ and $p_M + p_F = 1$. Using Konig's theorem on bipartite graphs, it is easy to show that:

$$\mu_1^*(H) = \mu^*(H) = \gamma^*(H) = \gamma_1^*(H) = np_M.$$

This last example is peculiar as the long chain of inequalities (7) degenerates into a perfect equality: the integral and fractional hurdle factors are equal and coincide themselves with the integral and fractional matching numbers. Calculating the matching number is quite easy as it amounts to find a partition of the set of legislators into the largest possible number of blocking coalitions. Therefore, when the above equalities hold, the calculation of the hurdle factor becomes very easy. Interestingly enough, the same chain of equalities hold in the case where the players are ordered on a line or a tree and only connected coalitions can form. A hypergraph for which this is true is called *normal* and a nice characterization has been obtained by Lovasz (1972)[30]. $r-$partite hypergraphs are typically not normal when $r \geq 3$.

[29] Given a tree, Alon (2001) has proved that if is a collection of subgraphs of the tree, each having at most d components, then $\gamma_1^*(H) \leq 2d^2 \mu_1^*(H)$.

[30] See also Lovasz (1975b).

Example 9 (Vetoers and Oligarchies). When we consider the blocking hypergraph attached to a simple game, the fractional and integral covering numbers are likely to be large numbers when its set of edges contains many small coalitions. This will happen as soon as in the simple game, a coalition is winning if it contains most of the players. The extreme case of such situation is unanimity according to which a coalition is winning if it contains all the legislators. In such case, any singleton is a blocking coalition and then $\mu_1^*(H) = \gamma^*(H) = \gamma_1^*(H) = n$. The closest situation to unanimity is the case where a coalition is winning if it contains at least $n-1$ legislators. This case has been extensively studied by several authors including Lucas (1966), Maschler (1963) and Owen (1968). In such a case the set of blocking coalitions consists of the set of pairs of legislators. In such a case, we obtain $\mu_1^*(H) = \gamma^*(H) = \gamma_1^*(H) = \frac{n}{2}$. However, in many circumstances some pairs can be simply ignored by lobby 1 as they are unlikely to form for multiple reasons like for instance ideological disagreement. Consider the case where the legislators are located in a multidimensional Euclidean ideological space and let d_{ij} denote the distance between $i$and j. It seems reasonable to assume that two legislators will act together iff their distance does not exceed some exogenous threshold ρ. The set of relevant minimal blocking coalitions is then the set of pairs $\{i, j\}$ such that $d_{ij} \leq \rho$. Another nice illustration is the case of a bicameral system where to be winning a coalition must contain all the members or at least one of the two chambers. In such case, the blocking hypergraph consist of all pairs with one member in each chamber. Under these conditions, the blocking hypergraph is set a subset of the set of pair of vertices i.e. what is usually called a *graph;* in the bicameral illustration, the graph is simply the complete bipartite graph. In the case of a graph, the largest possible value of $\gamma^*(H)$ is $\frac{n}{2}$ which is (as we have just seen) realized when the graph is complete. From the point of view of matchings, it correspond to what is called in graph theory as a *perfect matching*. If there is a perfect matching, we deduce from (7) that $\gamma^*(H) = \frac{n}{2}$. If the graph is bipartite, Hall's theorem[31] provides necessary and sufficient condition for the existence such a perfect matching. For an arbitrary graph, Tutte's beautiful theorem[32] also provides necessary and sufficient condition for the existence such a perfect matching.

When there is no perfect matching, we can still explore the set of maximum matchings and obtain a lower bound on $\gamma^*(H)$ through inequality (7). The celebrated Edmonds-Gallai Structure theorem offers deep insights on the structure of any maximum matching. In order to make the best possible use of proposition 1 in such case, let us see how the family of balanced collections looks like here. It has been demonstrated by Balinski (1972) that δ is

[31]We refer the reader to Furedi (1988) for the theorems cited in the treatment of example 5.

[32]It also follows from that theorem that $\mu_2^*(H) = 2\gamma^*(H) = \gamma_2^*(H)$.

an extreme point of the polytope of fractional matchings iff there exists a collection Q of node-disjoint edges and odd cycles such that:

$$\delta(\{i,j\}) = \begin{cases} 1 \text{ if } \{i,j\} \in Q, \\ \frac{1}{2} \text{ if } \{i,j\} \text{ belongs to an odd cycle of } Q, \\ 0 \text{ otherwise.} \end{cases}$$

In the above examples, we have examined the covering numbers of either the family of winning or the family of blocking coalitions. In the case of the hypergraph of winning coalitions, i.e. when we want to calculate the dual hurdle factor, it traditionally assumed that it is intersecting i.e. that the simple game $(N, \mathcal{W})$ is proper. In such case, it should be clear from what precedes that the pattern of intersection of winning coalitions plays some role in the determination of the integral and fractional hurdle factors. A cover is a set which intersects every edge. When the simple game is proper, the set of minimal winning coalitions is an intersecting family. Any set in $\mathcal{W}$ is therefore a cover. This implies that the integral covering number is then smaller than $Min_{E\in\mathcal{W}}\#E$. The knowledge of the integral hurdle factor provides a useful information of the smallest size of a group of legislators in position to control collectively the legislative process. When it is equal to 1, we have the familiar notion of a vetoer. When the number is equal to k, this means that there is a subset of k legislators which is represented in any winning coalition and that no subset of smaller size has this property. When the game is strong, the optimal cover is itself a winning coalition: a vetoer is then a dictator.

Along these lines, the following proposition relates the integral hurdle factor to another key parameter of a simple game known as the Nakamura's number (Nakamura (1978)). The Nakamura's number provides the exact largest possible value of the cardinality of the set of alternatives such that the core of the voting game resulting from the list of winning coalitions is non-empty for every conceivable profile of preferences. This parameter has attracted a lot of attention in the theory of voting and committees.

Definition . Let $G = (N, \mathcal{W})$ be a simple game. The Nakamura's number of G, is the integer:

$$\nu(G) = \begin{cases} Min_{\mathcal{W}'\subseteq\mathcal{W}}\ \#\mathcal{W}' \text{ such that: } \cap_{S\in\mathcal{W}'} S = \varnothing, \\ +\infty \text{ if } \cap_{S\in\mathcal{W}} S \neq \varnothing. \end{cases}$$

Proposition 2. For any simple game

$$\gamma^*(\mathcal{H}) \leq \gamma_1^*(\mathcal{H}) \leq 1 + \frac{(Min\ \#S : S \in \mathcal{H}) - 1}{\nu(G) - 2} \text{ if } \nu(G) \neq \infty$$

and

$$\gamma^*(\mathcal{H}) = \gamma_1^*(\mathcal{H}) = 1 \text{ if } \nu(G) = \infty.$$

Proof: If $\nu(G) < \infty$, it follows from the definition of the Nakamura number prove that the collection $\mathcal{H}$ of minimal winning coalitions is a $(\nu(G) - 1)$-intersecting family. The conclusion follows from an inequality established in Lovasz (1979). If $\nu(G) = \infty$, then any $\{i\}$ with $i \in T \equiv \cap_{S \in \mathcal{W}} S$ is obviously an optimal cover.$\square$

There is an obvious trade-off between the number of minimal winning coalitions and the magnitude of the hurdle factors[33]. Suppose that to reflect equity among the legislators, all minimal winning coalitions are of the same size r i.e. $\mathcal{W}_m$ is $r-$uniform. If we have many coalitions in $\mathcal{W}_m$, the hurdle factors are more likely to be large numbers. The hurdle factors are often very sensitive to the addition or the deletion of a coalition from $\mathcal{W}_m$. These patterns correspond to what has been defined above as $\gamma-$critical hypergraphs. For instance, it is easy to check that the hypergraphs attached to the simple games in example 1, the Erdos-Lovasz's simple game in example 2 and the cyclic majority game are $\gamma-$critical. How small can be a $r-$uniform hypergraph if we want the covering number to be at least equal to r?

To answer this question, let s be an integer. A set T is an $s-$multicover of H is either $\#(E \cap T) \geq s$ or $T \supseteq E$ for all $E \in \mathcal{H}$. A $1-$multicover of H is simply a cover of H. The $s-$multicover number of H is the smallest cardinality of an $s-$multicover of H. If H is $s-$intersecting, then T is called a nontrivial $s-$multicover if $1 \leq \#(T \cap E) < \#E$ for all $E \in \mathcal{H}$. As defined earlier, an $r-$clique is an intersecting $r-$uniform hypergraph which does not have a non-trivial cover of size at most r. We can also consider the family $H_{(r,\gamma)}$ of $r-$uniform hypergraphs H which are intersecting and such that $\gamma_1^*(\mathcal{H}) = \gamma$ where γ is a given integer less than or equal to r. If we denote by $m(r)$ the minimum number of edges in an $r-$clique and by $n(r)$ the minimum number of edges for a hypergraph in $H_{(r,r)}$, it can be shown that:

$$3r \leq m(r) \leq r^5 \text{ for all } r \geq 4$$

and

$$\frac{8}{3}r - 3 \leq n(r) \text{ for all } r \text{ and } n(q+1) \leq 4q\sqrt{q}Log\ q \text{ if } q \text{ is a primepower.}$$

This means that for all r there exist small $r-$cliques and small $r-$uniform hypergraphs with an integral hurdle factor equal to r but also that some minimal critical number of edges are necessary for the properties to be fulfilled. Füredi (1988) reports many results describing some of the properties of critical hypergraphs and specific families of hypergraphs. For instance, he presents an upper bound on the cardinality of $s-$multicover when the

[33] See Idzik, Katona and Vohra (2001) for an exploration of the intersecting balanced families of sets.

hypergraph is a symmetric (r, λ) −design[34], an upper bound on the number of vertices of an arbitrary $\gamma-$critical $r-$uniform hypergraph and an upper bound on the number of edges of an arbitrary $\gamma-$critical hypergraph of rank r.

As already pointed out, the derivation of the integral or fractional hurdle factors is often intricate. The chain of inequalities provides bounds which are sometimes very rough as illustrated for instance by example 1. It is of interest to explore if the calculation of the integral and fractional matching numbers helps in providing decent bounds on the integral and fractional hurdle factors and if, besides the inequality, the relationships between the integral and fractional hurdle factors can be sharpened in some cases.

We conclude this section by reporting on a sample of useful results answering these questions. Lovasz (1974)(1975b) has demonstrated that if H is an $r-$partite hypergraph, then $\gamma_1^*(\mathcal{H}) \leq \frac{r}{2}\gamma^*(\mathcal{H})$ and if H is a hypergraph of rank r, then:

$$\frac{\gamma_2^*(\mathcal{H})}{2\gamma^*(\mathcal{H})} \leq \frac{r}{2}, \frac{\gamma_r^*(\mathcal{H})}{r\gamma^*(\mathcal{H})} \leq 2 - \frac{1}{r-1}, \frac{\gamma_k^*(\mathcal{H})}{k\gamma^*(\mathcal{H})} \leq 1 + \frac{r-1}{k} \text{ for all integer } k.$$

Frankl and Füredi (1986) have demonstrated that if H is $2-$wise $s-$intersecting hypergraph of rank r different from a symmetric (r, λ) −design then:

$$\gamma^*(\mathcal{H}) \leq \frac{r-1}{s} + \frac{1}{r} - \frac{r-s}{r(r-1)s}.$$

Lovasz (1975a) has derived the following lower bound on the ratio of the hurdle factors:

$$\frac{\gamma^*(\mathcal{H})}{\gamma_1^*(\mathcal{H})} \geq \frac{1}{1 + LogD(H)}.$$

When we apply this inequality to the simple majority games in example 1, we obtain

$$\frac{\gamma^*(\mathcal{H})}{\gamma_1^*(\mathcal{H})} \geq \frac{1}{1 + C^{n-1}_{\frac{n-1}{2}}} \simeq \frac{1}{1 + nLog\ 2}.$$

Finally, some results have been established between the covering and matching numbers. For instance, Aharoni, Erdos and Linial (1985) have demonstrated that for any hypergraph H:

$$\mu_1^*(\mathcal{H}) \geq \frac{(\gamma^*(\mathcal{H}))^2}{n - \frac{m-1}{\#\mathcal{H}}(\gamma^*(\mathcal{H}))^2} \geq \frac{(\gamma^*(\mathcal{H}))^2}{n}.$$

and

$$\gamma_1^*(\mathcal{H}) \leq \begin{cases} Min\left\{n, \sqrt[3]{\mu_1^*(\mathcal{H})nLog\frac{\#\mathcal{H}}{\sqrt{\mu_1^*(\mathcal{H})n}}}\right\} \text{ if } \#\mathcal{H} > e\sqrt{\mu_1^*(\mathcal{H})n}, \\ \#\mathcal{H} \text{ if } \#\mathcal{H} \leq e\sqrt{\mu_1^*(\mathcal{H})n}, \end{cases}$$

[34]Boros, Caro, Füredi and Yuster (2001) demonstrate that for non-uniform hypergraphs, the maximum cardinality of an optimal cover is less than $1.98\sqrt{n}(1 + O(1))$.

where $e \simeq 2.718$ as usual. To conclude, let us mention that the Ryser's conjecture claims that if H is $r-$partite then:

$$\gamma_1^*(\mathcal{H}) \leq (r-1)\,\mu_1^*(\mathcal{H}).$$

It holds true for $r = 2$ and $r = 3$ (Aharoni (2001)) but the problem is still wide open for $r \geq 4$.

4.2 Weighted majority Games

In this section, we focus on the important class of weighted majority games. A simple game is a weighted majority game if there exists a vector $\omega = (\omega_1, ..., \omega_n, q)$ of $(n+1)$)non negative numbers such that a coalition S is in $\mathcal{W}$ iff $\sum_{i \in S} \omega_i \geq q$; $\omega_i > 0$ is the weight attached to legislator[35] i. The vector $\omega \equiv (\omega_1, ..., \omega_n)$ is called a *representation* of the simple game. It is important to note that the same game may admit several representations. A simple game is *homogeneous* if there exists a representation ω such that $\sum_{i \in S} \omega_i = \sum_{i \in T} \omega_i$ for all $S, T \in \mathcal{W}_m$. This representation is called *the* homogeneous representation of the simple game as Isbell (1956)[36] has demonstrated that an homogeneous simple game admits a unique (up to multiplication by a constant) . The homogeneous representation ω for which $\sum_{i \in N} \omega_i = 1$ is called the homogeneous normalized representation and a homogeneous representation ω for which ω_i is an integer for all $i \in N$ is called an integral representation.

Consider an arbitrary cooperative game with transferable utility (N, V) and let $x \in X_n \equiv \left\{ y \in \Re_+^n : \sum_{i=1}^n y_i = V(N) \right\}$. Let $\theta(x)$ be the 2^n dimensional vector[37] whose components are the numbers $V(S) - \sum_{i \in S} x_i$ arranged according to their magnitude i.e. $\theta(x) \geq \theta(x)$ for $1 \leq i \leq j \leq 2^n$. The *nucleolus* of (N, V) is the unique vector $x^* \in X_n$ such that $\theta(x^*)$ is the minimum, in the sense of the lexicographic order, of the set $\{\theta(y) : y \in X_n\}$. The *least core*[38] is the subset of X_n consisting of the vectors x such that $\theta_1(x) = \theta_1(x^*)$. It will be denoted $LC(V, N)$; by construction $x^* \in LC(V, N)$

To any simple game, we attach the cooperative game with transferable utility (N, V) defined as follows:

$$V(S) = \begin{cases} 1 \text{ if } S \in \mathcal{W}, \\ 0 \text{ if } S \notin \mathcal{W}. \end{cases}$$

[35] In many applications, it is more relevant (if party discipline is strong) to assume that the players in the legislature are the different parties to which the legislators belong rather than the legislators themselves ; in such case, ω_i denotes the number of legislators affiliated to party i.

[36] See also the generalization by Ostmann (1987).

[37] This vector is called the vector of excesses attached to x.

[38] This notion was first introduced by Maschler, Peleg and Shapley (1979).

In such case, only minimal winning coalitions matter in "minimizing" the vector of excesses. The least core consists in the subset of vectors x such that

$$x \in \underset{y \in S_n}{ArgMax} \underset{S \in \mathcal{W}_m}{Min} \sum_{i \in S} y_i,$$

where $S_n \equiv \left\{ y \in \Re^n_+ : \sum_{i=1}^n y_i = 1 \right\}$. Let:

$$C^* \equiv \underset{y \in S_n}{Max} \underset{S \in \mathcal{W}_m}{Min} \sum_{i \in S} y_i.$$

The following simple assertion holds true for any simple game.

Proposition 4. Let $(N, \mathcal{W})$ be a simple game. Then, $\gamma^*(\mathcal{W}) = \frac{1}{C^*}$.

Proof: By definition of C^*, there exists $y \in \Re^n_+$ such that:

$$\sum_{i=1}^n y_i = 1 \text{ and } \sum_{i \in S} y_i \geq C^* \text{ for all } S \in \mathcal{W}_m.$$

Therefore the vector z such that $z_i \equiv \frac{y_i}{C^*}$ for all $i = 1, ..., n$ verifies:

$$\sum_{i=1}^n z_i = \frac{1}{C^*} \text{ and } \sum_{i \in S} z_i \geq 1 \text{ for all } S \in \mathcal{W}_m$$

implying that $\gamma^*(\mathcal{W}) \leq \frac{1}{C^*}$.

Assume that $\gamma^*(\mathcal{W}) < \frac{1}{C^*}$. This means that there exist a vector $z \in \Re^n_+$ such that:

$$\sum_{i=1}^n z_i = \gamma^*(\mathcal{W}) \text{ and } \sum_{i \in S} z_i \geq 1 \text{ for all } S \in \mathcal{W}_m.$$

Therefore the vector y such that $y_i \equiv \frac{z_i}{\gamma^*(\mathcal{W})}$ for all $i = 1, ..., n$ verifies:

$$\sum_{i=1}^n y_i = 1 \text{ and } \sum_{i \in S} y_i \geq \frac{1}{\gamma^*(\mathcal{W})} \text{ for all } S \in \mathcal{W}_m.$$

Since $\frac{1}{\gamma^*(\mathcal{W})} > C^*$, this contradicts our definition of C^*.□

The proof is also quite instructive by itself as it also demonstrates that the set of optimal fractional covers of $(N, \mathcal{W})$ is, up to a division by C^*, the least core of the game induced by the simple game. Since the set of optimal fractional covers is, up to the multiplication by W_0, the set of offers to legislators made by lobby 1 at equilibrium, the least core provides a complete characterization of the equilibrium behavior of lobby 1.

Proposition 4 raises a number of questions. First, is it simple to calculate the quantity for some particular families of simple games? Second, how the least core looks like i.e. how are treated the different legislators? We answer these questions when $(N, \mathcal{W})$ is a weighted majority game

Peleg (1968) has demonstrated[39] that the normalized homogeneous representation of an homogeneous strong weighted majority game $(N, \mathcal{W})$ coincides with the nucleolus x of (N, V). Similarly, the integral representation of the nucleolus (which is well defined) is the minimum representation of the game i.e. the unique minimal integral representation of the game. Since the nucleolus is an element of the least core, proposition 4, combined with Peleg's result, provides a nice and simple way to calculate $\mu^*(\mathcal{W})$ for strong homogeneous weighted majority games. The task amounts to discover the weight of each minimal winning coalition in the normalized homogeneous representation. For instance, the weighted majority game resulting from a legislature with 4 parties where the number of representatives of each party is described by the vector $\omega = (49, 17, 17, 17)$ is exactly the apex game considered in example 2. It is easy to see that the normalized homogeneous representation is here $\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$. It follows that $\gamma^*(\mathcal{W}) = \frac{5}{3}$.

The task is however more intricate when the simple game is not homogeneous[40]. Peleg has also proved that the minimal integral representation of the nucleolus is a minimal integral representation of the game if some condition is fulfilled, and has disproved by means of a counterexample of size 12 that the assertion holds true in general. He asks whether this assertion holds true when the simple game has a *minimum* integral representation. This conjecture has been disproved by Isbell (1969) by means of a counterexample of size 19. Therefore, within the class of non homogeneous weighted majority games[41], the relationship between the nucleolus (and then covering) and the set of minimal representations is less transparent. In such a case, the computation of $\gamma^*(\mathcal{W})$ can exploit the general algorithms which have been developed to calculate the nucleolus.

As already pointed out, Besides the knowledge of $\gamma^*(\mathcal{W})$, it is of interest to know how the amount of money $\gamma^*(\mathcal{W})W_0$ is allocated across legislators or parties. This question is of course very important as we would like to know what are the characteristics of legislator i (parties) besides α_i which determine the "price" of that legislator from the perspective of

[39] See also, Peleg and Rosenmüller (1992).

[40] Several authors including among others (Ostmann (1987), Peleg and Rosenmuller (1992), Rosenmuller (1987) and Sudhölter (1996)) have investigated the class of homogeneous weighted majority games which are not necessarily strong. Sudhölter has introduced a notion of nucleolus (called the modified nucleolus) which is a representation of the game when it is homogeneous.

[41] The question becomes even more complicated when we move outside the world of weighted simple games. as exemplified by the calculation of the nucleolus of compound simple games (Meggido (1974)).

lobby 1 (in fact, on the market for votes where the two lobbies compete). This will have to do with the position of i in the set of minimal winning coalitions[42]. In a weighted majority game, we expect intuitively this price to be positively correlated to the weight of the legislator, if not even exactly proportional to that weight. We have just seen that this intuition is correct in the case of homogeneous weighted majority game (for an appropriate vector of weights) but that the exact relationship between weights and price is less clear otherwise. At least qualitatively, for any offer x in the least core, the price is a non decreasing function of the weight. Indeed, assume on the contrary that legislators i and j are such that $\omega_i \leq \omega_j$, there exists at least a minimal coalition S containing i but not j[43] and $x_i > x_j$. Observe that for any minimal coalition S containing i but not j, there exists a minimal winning coalition T containing j but not i and such that $T\setminus\{i\}$ is contained in S. Consider $R = (S\setminus\{i\}) \cup \{j\}$ Either R is minimal and then set $T = R$. Or R is not minimal. Since S was winning, R is winning and since S was minimal, any minimal winning coalition contained in R must contain j. Let $\mathcal{S}$ be the family of coalitions S such that $\underset{R\in\mathcal{W}_m}{Min} \sum_{k\in R} x_k = \sum_{k\in S} x_k$. We claim[44] that there exist at least one coalition $S \in \mathcal{S}$ containing i. Indeed assume on the contrary that $i \notin \widehat{N} \equiv \cup_{S\in\mathcal{S}} S$ and let $y \in S_n$ be the vector such that $y_k = x_k + \frac{\varepsilon}{\widehat{n}}$ for all $k \in \widehat{N}$ and $y_i = x_i - \varepsilon$ where ε is a small positive number (which is well defined since $x_i > 0$). We deduce from this construction that $\underset{R\in\mathcal{W}_m}{Min} \sum_{k\in R} y_k > \underset{R\in\mathcal{W}_m}{Min} \sum_{k\in R} x_k$ contradicting our assumption that x an element of the least core of the simple game. Consider $S \in \mathcal{S}$ containing i and $T \in \mathcal{W}_m$ containing j but not i and such that $T\setminus\{i\}$ is contained in S . Since $x_i > x_j$ we deduce note $\sum_{k\in T} x_k < \sum_{k\in S} x_k$ contradicting our assumption that $S \in \mathcal{S}$.

Proposition 4 establishes an "unexpected" connection between the least core which has been developed in cooperative game theory and the set of equilibrium monetary offers in this non cooperative game. The exploration of the least core of the simple game reveals all what we expect to discover about the payoffs of the legislators in the lobbying game. We know that the nucleolus is one of those but as we will see the least core may also contain

[42]The pattern of these positions defines, in some sense, the power of legislator i. There is an extensive literature on the measurement of the power of players in simple games with a prominent place occupied by the Banzhaf 's index (1965)(1968) and the Shapley-Shubik 's index (1954). The view that any of this one dimensional measure of power helps in predicting the payoffs of legislators in strategic environments where their votes can be bought has been disputed by several authors (see for instance Snyder, Ting and Ansolabehere (2005) in the context of a legislative bargaining model).

[43]If the game is such that any minimal coalition containing i contains j, then the claim is false. The two legislators are equally desirable and there are simple games where some imputations of the least core (obviously not all) treat differently such players. Maschler and Peleg (1966) have shown that if two players are equally desirable, then there exists a representation where they have the same weight while Lapidot (1972) has demonstrated that the counting vector determines the game uniquely.

[44]An important result due to Kohlberg (1971) provides more precise informations on the family $\mathcal{S}$.

some other vectors. It is obvious that legislators who are dummies will never receive any offer from the lobbies. However, as illustrated through some of the following examples, there are situations where legislators who are not dummies do not receive any offer.

Example 10. The simple game considered in example 3 is a weighted majority game; $\omega = (2, 1, 1)$ is a representation. It is easy to see that the core (and therefore the least core and the nucleolus) is equal to the vector $(1, 0, 0)$. The first legislator gets all the money despite the fact neither legislator 2, nor legislator 3 is a dummy. Legislator 1 needs one of them to pass or to block (depending upon the interpretation of the hypergraph) the proposal. Legislators 2 and 3 are however perfect substitutes and in excess supply on this market. Their internal competition drives down their price to 0. Note otherwise that this game is not strong and therefore Peleg's theorem does not apply. In fact the nucleolus is not a representation of the game while the modified nucleolus $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ is so.

Example 11 (Vector Weighted Majority Games). In many situations, the type of a legislator, defined by a vector of traits or attributes, is an important parameter in the explicit description of winning or blocking coalitions: for instance, the type of a legislator may consist of the chamber to which he belongs (in a multicameral system), its gender, its geographic area (east, west, north, south), and so on. Let us assume that there are K mutually exclusive possible types and denote by n_k the number of legislators of type k. A coalition is a $K-$tuple of integers $m \equiv (m_1, ..., m_K)$ where $m_k \leq n_k$ for all $k = 1, ..., K$. We consider the setting where there exist J vectors $\left(a_1^j, ..., a_K^j, b^j\right) \in \Re_+^{K+1}$ such that a coalition m is winning iff:

$$\sum_{k=1}^{K} a_k^j m_k \geq b^j \text{ for } j = 1, ..., J.$$

This framework generalizes[45] the concept of weighted majority game: in the case of a strong weighted majority game, the set of different weights is the set of types, $J = 1$, $a = (1, 1, ..., 1)$ and $b = \left\lfloor \frac{\sum_{1 \leq k \leq K} n_k \omega_k}{2} + 1 \right\rfloor$[46]. The minimal winning coalitions are the lower vertices of the polyhedron described by the above inequalities and the hypercube $\prod_{k=1}^{K} \{0, 1, ...n_k\}$. We illustrate the calculation of the hurdle and dual hurdle factors and the least core in the following specific case which describes the U.S. federal legislative system[47]. Let $K = 4$,

[45] Taylor and Zwicker (1999) call vector weighted games such simple games. Among the real world voting systems which are vector-weighed, we can cite the system to amend the Canadian constitution and the different decision rules for the council of ministers of the EU like those prescribed by the treaty of Nice.

[46] $\lfloor x \rfloor$ denotes the integer part of x.

[47] This representation of the U.S. federal legislative system appears in Taylor and Zwicker (1999).

$J = 2$, $a^1 = \left(0, 1, \frac{1}{2}, \frac{33}{2}\right)$, $a^2 = (1, 0, 0, 72)$, $b^1 = 67$, $b^2 = 290$, $n_1 = 435$, $n_2 = 100$, $n_3 = 1$ and $n_4 = 1$. This simple game represents a bicameral system (the house of representatives and the senate) with two additional players: the vice president and the president. A coalition is winning if either it contains more than half the house and more than half the senate (with the vice president playing the role of tie-breaker in the senate), together with the support of the president or two-thirds of both the house and the senate to override a veto by the president). The problem of determination of the least core reduces to the minimization of $435x_1 + 100x_2 + x_3 + x_4$ with respect to $(x_1, x_2, x_3, x_4) \in \Re^4_+$ under the constraints:

$$\begin{aligned} 218x_1 + 50x_2 + x_3 + x_4 &\geq 1 \\ 218x_1 + 51x_2 + x_4 &\geq 1 \\ 290x_1 + 67x_2 &\geq 1. \end{aligned}$$

Note that if $x_3 + x_4 = 0$, then the relevant inequality is $218x_1 + 50x_2 \geq 1$. Since $\frac{435}{218} \simeq 1.9954 < \frac{100}{50} = 2$, the minimum is obtained in $\left(\frac{1}{218}, 0\right)$ leading to the value 1.9954. If instead $x_3 + x_4 > 0$, then the inequalities $290x_1 + 67x_2 \geq 1$ and $218x_1 + 50x_2 + x_3 + x_4 \geq 1$ are the relevant inequalities for the two remaining variables, as the slope of the objective is -4.35 and the slopes attached to these constraints are respectively -4.36 and -4.33 while the slope of the second one is -4.27, in the two-dimensional space corresponding to the variables x_1 and x_2. If $x_3 = 0$, then $x_4 = 72x_1 + 17x_2$ and the objective to be minimized is $507x_1 + 117x_2$. The minimum is obtained for $x_1 = 0$ and $x_2 = \frac{1}{67}$ leading to $x_4 = 1 - \frac{50}{67} = \frac{17}{67}$. If $x_3 > 0$, then proceeding as above, we obtain that $x_1 = 0$, $x_2 = \frac{1}{67}$ and $x_3 + x_4 = \frac{17}{67}$. Since, the lower bound on x_4 is $1 - \frac{51}{67} = \frac{16}{67}$, we obtain that the upper bound on x_3 is $\frac{1}{67}$. For any such solution, the value of the program is $\frac{117}{67} \simeq 1.746 < 1.9954$. We have just proved that the dual hurdle factor of the US federal legislative game is 1.746 and that the least core is a one dimensional convex set namely the convex hull of the vectors $\left(0, \frac{1}{67}, \frac{1}{67}, \frac{16}{67}\right)$ and $\left(0, \frac{1}{67}, 0, \frac{17}{67}\right)$. Interestingly, the members of the house do not get any offer despite the fact that there are not dummies and the offer made to the president is around 16 times larger than the offer made to any single senator or to the vice president (if any).

Let us now look at the U.S. federal legislative game from the point of view of the blocking coalitions i.e. at the hurdle factor. It is easy to see that the minimal blocking coalitions

(m_1, m_2, m_3, m_4) are the following:

$$\begin{aligned} m_1 &= 146,\ m_2 = m_3 = 0,\ m_4 = 1 \\ m_1 &= 0,\ m_2 = 34,\ m_3 = 0,\ m_4 = 1 \\ m_1 &= 218,\ m_2 = m_3 = m_4 = 0 \\ m_2 &= 51,\ m_1 = m_3 = m_4 = 0 \\ m_2 &= 50,\ m_3 = 1,\ m_1 = m_4 = 0. \end{aligned}$$

In such case, we obtain that the least core consists of the unique vector $\left(\frac{1}{218}, \frac{1}{51}, \frac{1}{51}, \frac{17}{51}\right)$ (which is the nucleolus) and that the hurdle factor is approximately 4.31[48].

Example 12 (The United Nations Security Council). The voters are the 15 countries that make up the security council, 5 of which are called permanent members whereas the other 10 are called nonpermanent members. Passage requires to total of at least 9 votes, subject to approval from any one of the 5 permanent members. It is easy to show that this simple game is a weighted majority game: assigning a weight of 7 to each permanent member, a weight of 1 to any nonpermanent member and a quota equal to 39 provides a representation. If lobby 1 acts to pass a reform (here a resolution), the problem of determination of the least core reduces to the minimization of $5x_1 + 10x_2$ with respect to $(x_1, x_2) \in \Re^2_+$ under the constraints:

$$x_1 \geq 1 \text{ and } 7x_2 \geq 1.$$

We deduce that the least core consists of the unique vector $\left(1, \frac{1}{7}\right)$ (which is the nucleolus) and that the hurdle factor $5 + \frac{10}{7}$ is approximately equal to 6.43.

If instead lobby 0 acts to block a reform, the problem of determination of the least core reduces to the minimization of $5x_1 + 10x_2$ with respect to $(x_1, x_2) \in \Re^2_+$ under the constraint:

$$5x_1 + 4x_2 \geq 1.$$

Now we obtain that the least core consists of the unique vector $\left(\frac{1}{5}, 0\right)$ (which is the nucleolus) and that the dual hurdle factor is equal to 1. Here, only the permanent members

[48] This result should be contrasted with the claims formulated by Diermeier and Myerson (1999) in their footnote 9. They write "....The veto-override provision is not significant. The $\frac{2}{3}$ veto override option allows that lobby 1 can get a bill passed by paying $3W_0$ in the house and $3W_0$ in the senate, rather than paying W_0 to the president plus $2W_0$ in the house and $2W_0$ to the senate. So the alternative legislative path that is allowed by the $\frac{2}{3}$ veto override has a hurdle factor of 6, which is higher than the hurdle factor of 5 that is available without it. *Thus our analysis predicts that lobbyists for change should generally ignore the more expensive option of overriding a presidential veto*, and should lobby just as they would if the congress were a purely serial bicameral legislature with a presidential veto...". This prediction differs sharply from ours as for instance they predict that the president will receive 20% of the "cake" while we predict that he will receive only 7.74% and that the bribe offered to the president will be 50 times larger than the bribe offered to any single senator while we predict that it will be 17 times larger.

receive an offer and with a hurdle factor equal to 1, lobbying expenditures by lobby 1 remain moderate. We could wonder what would be the consequences of limiting somehow the veto power of the permanent members and/or changing the level of the qualified majority to pass a reform. For instance, suppose that passage requires to total of at least 9 votes, subject to approval of at least 3 permanent members. The constraints now becomes:

$$3x_1 + 6x_2 \geq 1 \text{ and } 5x_1 + 4x_2 \geq 1.$$

In that case, both permanent and nonpermanent members are likely to receive offers as the least core consists of the convex hull of the vectors $\left(\frac{1}{3}, 0\right)$ and $\left(\frac{1}{9}, \frac{1}{9}\right)$ and the dual hurdle factor $\frac{5}{3}$ is approximately equal to 1.66. Consider finally the case where passage requires to total of at least 10 votes, subject to approval of at least 3 permanent members. The constraints now becomes:

$$3x_1 + 7x_2 \geq \text{ and } 5x_1 + 5x_2 \geq 1.$$

It is straightforward to show that the least core consists of the unique vector $\left(\frac{1}{10}, \frac{1}{10}\right)$ (which is the nucleolus) and that the dual hurdle factor is equal to 1.50.

From 1954 to 1965, the simple game $(N, \mathcal{W})$ describing the council had 5 permanent members, 6 nonpermanent members and the qualified majority was equal to 7. Proceeding as above, we obtain that the hurdle factor $\gamma^* (\mathcal{B})$ was equal to 6.02 while $\gamma^* (\mathcal{W}) = 1$. The 1965 system is less vulnerable to lobbying than the 1954's one. It would be interesting to use this apparatus to evaluate some of the proposals to reform membership and voting rules of the United Nations Security Council. Many countries criticize the lack of representativeness of the current council. Among the proposals, we can find:

· The G4 proposal which ask the addition of 6 new permanent members without veto power and 4 new nonpermanent members.

· The African proposal which is similar to the G4 proposal except for the fact that it asks that the new permanent members also had a veto power and 5 new permanent members instead of 4.

· The "United for Consensus" proposal which simply asks for the addition of 10 new permanent members.

These proposals propose to increase the current size of 15 members to 25 or 26 members. In our setting being permanent or nonpermanent are equivalent. No specification of the required qualified majority is provided but given the historical attachment to a supermajority requirement of $60 - 63\%$, we could expect a quota equal to 15. The first, second and third proposals lead respectively to hurdle factors $\gamma (\mathcal{B})$ equal to 6.82, 12 and 6.82. A way to compromise between the first and second proposal could consist in offering to each pair (or triple) of new permanent members a veto power. To compromise with the third, we could increase the quota from 15 to 18. In general, a council composed of n_1 permanent members with regular veto voter, n_2 permanent members with veto voter offered to pairs, n_3 nonpermanent members and a quota equal to q leads to a hurdle factor equal to:

$$n_1 + \frac{n_2}{2} + \frac{n_3}{n_1 + n_2 + n_3 + 1 - q}.$$

Example 13 (Diermeier and Myerson's Multicameral Systems). These are the simple games considered by Diermeier and Myerson (1999). Their main objective in the paper is to determine the optimal hurdle factor of one of the chamber (say the house) given the hurdle factors of the other chambers where optimal means maximizing the expected aggregate amount of bribes received by the members of the house. The details will depend of course upon the beliefs concerning the bivariate random variable (W_0, W_1). Instead of the uncertainty framework considered in our paper, they assume that the two variables are independent and identically distributed random and they offer detailed calculations in the case where the marginals are either lognormal or uniform. Let t be the sum of the hurdle factors of the other chambers and s be the hurdle factor of the house and $F(s,t)$ be the corresponding expected income. Their central result asserts that the best response $s^*(t)$ of the house, which can be implemented through the choice of an appropriate simple game $(N, \mathcal{W})$, increases as the external hurdle factor t increases. At that stage, it is important to remind that they conduct their analysis under the assumption that there is no uncertainty about whom will be the lobby moving first. However, we can conceivably defend the view that in some circumstances, the lobby which wants the status quo to be preserved is acting first. If that is the case, the relevant simple game is the dual game and the relevant hurdle factor is the dual hurdle factor. If we assume the two situations to occur with probabilities[49] p and $1-p$, then in the case where there is no other chamber (unicameral legislature), the expected income is now:

$$pF(s,0) + (1-p)F(\widehat{s},0).$$

where $\widehat{s} = \gamma^*(\mathcal{W})$ is the dual hurdle factor. We have seen, through the above examples, that unless the simple game is constant-sum (in which case $\widehat{s} = s$), the two factors behave quite differently. A new trade-off appears as increasing s has now two effects: a direct effect as before (through a decrease of the probability of the event of active lobbying) and an indirect effect through a decrease of $\widehat{s}$. The following table provides the value of the optimal hurdle factor for different values of the parameters p and σ.

[49] Diermeier and Myerson assume $p = 1$.

Table 1: **Optimal Hurdle Factor in Lognormal Model.**

	σ									
p	0.6	0.8	1.0	1.2	1.3	1.5	1.6	1.7	2.0	3.0
1	1.0	1.0	1.2414	1.8516	2.3392	3.9872	5.3765	7.4056	21.9487	3122.3942
0.75	1.0	1.5486	1.6915	1.9293	2.189	3.5283	4.8525	6.8422	21.3253	3121.9111
0.5	2.0	2.0	2.0	2.0	2.0	2.0	1.4534	5.4101	20.01	3120.9446

The first line of table 1 is of course similar to the first line of table 3 in Diermeier and Myerson. An interesting observation is that moving from $p = 1$ to the more balanced assumption $p = \frac{1}{2}$ leads to the optimality of the standard majority game for a large range of values of σ (approximately when σ is less than 1.57). Interestingly enough, it is larger than the Diermeier-Myerson's optimal hurdle factor for small enough values of σ and smaller then. When σ gets larger than 1.57, the optimal[50] hurdle factor increases but stays smaller than Diermeier-Myerson's one.

In a truly multicameral legislature as defined by Diermeier and Myerson (1999), things get even more tricky. As defined in example 7, let $(N_r, \mathcal{W}_r)_{1\leq r\leq R}$ be a family of R hypergraphs with $N_r \cap N_t = \varnothing$ for all $r, t = 1, ..., R$ with $r \neq t$. Let $(N, \mathcal{W})$ be such that $N = \cup_{r=1}^{R} N_r$ and $S \in \mathcal{W}$ iff $S \cap N_r \in \mathcal{W}_r$ for all $r = 1, ..., R$ i.e. a reform is approved if it is approved in all the different R chambers according to the rules (possibly different) in use in the chambers. Given the hurdle factors $\gamma^* (\mathcal{W}_r)$ of each chamber $r = 1, ..., R$, let us calculate $\gamma^* (\mathcal{W})$. It is the value of the linear program:

$$Min_{t\in\Re_+^n} \sum_{r=1}^{R} \sum_{i\in N_r} t_{ir}$$

under the constraints

$$\sum_{r=1}^{R} \sum_{i\in S_r} t_{ir} \geq 1 \text{ for all } R-\text{tuple } (S_r)_{1\leq r\leq R} \text{ such that } S_r \in \mathcal{W}_r \text{ for all } r = 1, ..., R.$$

Let $\theta \in \Re_+^R$ be such that $\sum_{r=1}^{R} \theta_r = 1$. The value of the above program is less than the value of the program:

$$Min_{t\in\Re_+^n} \sum_{r=1}^{R} \sum_{i\in N_r} t_{ir}$$

[50] The function which is maximized displays interesting nonconvexities.

under the constraints

$$\sum_{i\in S_r} t_{ir} \geq \theta_r \text{ for all } R-\text{tuple } (S_r)_{1\leq r\leq R} \text{ such that } S_r \in \mathcal{W}_r \text{ for all } r = 1, ..., R.$$

But this new problem is decomposable into R disjoint minimization programs. We deduce from that argument that:

$$\gamma^*(\mathcal{W}) \leq \sum_{r=1}^{R} \theta_r \gamma^*(\mathcal{W}_r) \text{ for all } \theta \in \Re_+^R \text{ such that} \sum_{r=1}^{R} \theta_r = 1.$$

Since the above inequality is an inequality for any vector θ attached to a solution of the initial problem, we deduce:

$$\gamma^*(\mathcal{W}) = Min_{\theta\in\Re_+^R} \sum_{r=1}^{R} \theta_r \gamma^*(\mathcal{W}_r) \text{ under the constraint } \sum_{r=1}^{R} \theta_r = 1,$$

and therefore:

$$\gamma^*(\mathcal{W}) = Min_{1\leq r\leq R}\ \gamma^*(\mathcal{W}_r).$$

This result has important implications on the determination of the optimal dual hurdle factor by the house. Indeed, in the case where the first mover lobby is the lobby which wants to block the passage of the reform, the amount of money received by the house will depend critically upon how large is its dual hurdle factor compared to the dual hurdle factors of the other chambers. If it is larger than the smallest one, then the house will not be approached by the lobby in that case. The game describing the interaction between the chambers displays discontinuous payoff functions. In the case of two chambers and $p = \frac{1}{2}$, we obtain that the payoff of chamber 1[51] is equal to:

$$\begin{cases} \frac{F(\gamma^*(\mathcal{B}_1),\gamma^*(\mathcal{B}_2))}{2} + \frac{F(\gamma^*(\mathcal{W}_1),\gamma^*(\mathcal{W}_2))}{2} & \text{if } \gamma^*(\mathcal{W}_1) < \gamma^*(\mathcal{W}_2), \\ \frac{F(\gamma^*(\mathcal{B}_1),\gamma^*(\mathcal{B}_2))}{2} + \frac{F(\gamma^*(\mathcal{W}_1),\gamma^*(\mathcal{W}_2))}{4} & \text{if } \gamma^*(\mathcal{W}_1) = \gamma^*(\mathcal{W}_2), \\ \frac{F(\gamma^*(\mathcal{B}_1),\gamma^*(\mathcal{B}_2))}{2} & \text{if } \gamma^*(\mathcal{W}_1) > \gamma^*(\mathcal{W}_2), \end{cases}$$

if we assume that ties are broken equally. Of course, in the above expressions, there is a one to one relationship between the two hurdle factors $\gamma^*(\mathcal{W})$ and $\gamma^*(\mathcal{B})$. If we limit the implementation to symmetric quota games i.e. $S \in W$ iff $\#S \geq q$, we deduce from example 5 that $\gamma^*(\mathcal{W}) = \frac{n}{q}$ and $s_1 = \gamma^*(\mathcal{B}) = \frac{n}{n-q+1}$. If n is large, we deduce that:

$$\frac{q}{n}\gamma^*(\mathcal{W}) = \left(1 - \frac{q}{n}\right)\gamma^*(\mathcal{B}).$$

[51] The payoff of chamber 2 is obtained similarly.

Interestingly enough, if both chambers were acting under the presumption that the lobby which will move first is the pro-status quo lobby, then the game becomes a Bertrand game[52] where behavioral responses converge to the Nash equilibrium $(1,1)$. It would be interesting to know what we obtain in the general case. When it is taken for granted that the pro-reform lobby moves first, Diermeier and Myerson found convergence towards the Nash equilibrium $(2.20, 2.20)$ in the case of a bicameral legislature implemented by a quota of 54.5%; note that then $\gamma^* (\mathcal{W}) \simeq 1.835$.

4.3 Buying Supermajorities

In the two preceding subsections, we have ignored the impact of the vector α on the equilibrium outcomes of the lobbying game by assuming $\alpha = 0$. This was done in order to focus exclusively on the implications of the rules governing the decision process within the legislature and the "power" of the legislators resulting from that simple game. Following that path, we were able to cover a very large class of simple games describing many alternative institutional legislative settings and to isolate that component of the price of a legislator.

In this subsection and the following one, we reintroduce the vector but we focus our attention on a very special (while important) simple game: the classical majority game. In that respect, the analysis of this section is aligned with the framework of Banks (2000) and Groseclose and Snyder (1996)(2000). Given the symmetry of the simple game the legislators are all alike in terms of their power in the legislature. This means that if two legislators i and j receive different offers from the lobby, the rationale for this differential should be based on differences between α_i and α_j. We have seen, in the previous subsections, that some legislators endowed with a limited power within the legislature were, sometimes, totally ignored by the lobby. Here, a legislator i with a large α_i will be cheap for lobby 1 and expensive for lobby 0. Finally, we have also observed that most of the time the lobby was bribing a coalition strictly larger than a minimal winning coalition.

These considerations lead to a number of questions:

· What will be the size of the coalition of legislators receiving offers from the lobby? Since the α_i are nonnegative numbers, it can be the case that lobby 1 bribes a submajority coalition (at the extreme, nobody at all), a minimal majority or a supermajority (at the extreme everybody) depending upon the profile α. Which legislators will be part of that coalition?: when is it the case that the cheapest strategy of lobby 1 consists in bribing the whole legislature?

[52] The game arising from the assumption considered by Diermeier and Myerson (1999) displays the features of a Cournot game.

· Which legislators will be part of the bribed coalition? Shall we observe a flooded coalition as in Banks (2000) or a nonflooded coalition as in Groseclose and Snyder (1996)(2000) where flooded refers to the fact that lobby 1 bribes in priority the legislators more willing to support the reform.

· What are the differences between the offers received by the legislators who are in the coalition?

The following proposition answers these three questions in the case where $(N, \mathcal{W})$ is the classical majority game and n is odd[53] i.e. $n = 2k - 1$ for some integer $k \geq 2$.

Proposition 5. Let $t_1^* = (t_{11}^*, t_{21}^*, ..., t_{n1}^*)$ be an optimal offer by lobby 1. Then, there exists an integer m^* such that $t_{i1}^* > 0$ and $t_{i1}^* + \alpha^i \Delta W = t_{j1}^* + \alpha^j \Delta W$ for all $i, j = 1, ..., m^*$. Further, either $\frac{W_0}{k} > \alpha^k \Delta W$ and m^* is determined as the unique smallest integer m such that $\frac{W_0}{k} \leq \Delta W \alpha^m$ if any and $m^* = n$ otherwise. Or $\frac{W_0}{k} \leq \alpha^k \Delta W$ and m^* is the smallest value of $m \leq k - 1$ such that: $W_0 < \Delta W \left[\sum_{l=m+1}^{k} \alpha^l + m\alpha^{m+1}\right]$.

Proof: Assume without loss of generality that $\alpha^1 \leq \alpha^2 \leq ...... \leq \alpha^n$. Let $t_1^* = (t_{11}^*, t_{21}^*, ..., t_{n1}^*)$ be an optimal solution to problem (1) and $N^* \equiv \{i \in N : t_{i1}^* > 0\}$.

Claim 1: $t_{i1}^* + \alpha^i \Delta W = t_{j1}^* + \alpha^j \Delta W$ for all $i, j \in N^*$.

Assume on the contrary that $t_{i1}^* + \alpha^i \Delta W < t_{j1}^* + \alpha^j \Delta W$ for some $i, j \in N^*$. Then:

$$\sum_{l \in S} \left(t_{l1} + \alpha^l \Delta W\right) > W_0 \text{ for all } S \subseteq N \text{ such that } \#S = k \text{ and } i \in S. \tag{8}$$

Indeed if:

$$\sum_{l \in \widehat{S}} \left(t_{l1} + \alpha^l \Delta W\right) = W_0 \text{ for some } \widehat{S} \subseteq N \text{ such that } \#S = k \text{ and } i \in S,$$

then, we would obtain:

$$\sum_{l \in \left(\widehat{S} \setminus \{i\}\right) \cup \{j\}} \left(t_{l1} + \alpha^l \Delta W\right) < W_0$$

contradicting our assumption that t_1^* is a solution to problem (1). Let $t_1^{**} = (t_{11}^{**}, t_{21}^{**}, ..., t_{n1}^{**})$ be such that $t_{l1}^{**} = t_{l1}^*$ for all $l \neq i$ and $t_{i1}^{**} = t_{i1}^{**} - \varepsilon$ for some $\varepsilon > 0$. If ε is selected small enough, it follows from inequalities (8) that t_1^{**} meets the constraints of problem (1). Since further $\sum_{i \in N} t_{i1}^{**} < \sum_{i \in N} t_{i1}^*$, we contradict our assumption that t_1^* is a solution to problem (1).

Claim 2: $\alpha^i < \alpha^j$ for all $i \in N^*$ and $j \notin N^*$.

[53] Therefore this game is constant-sum.

Assume on the contrary that $\alpha^i \geq \alpha^j$ for some $i \in N^*$ and $j \notin N^*$. Then as in claim 1:

$$\sum_{l \in S} \left(t_{l1} + \alpha^l \Delta W \right) > W_0 \text{ for all } S \subseteq N \text{ such that } \#S = k \text{ and } i \in S.$$

Indeed, if:

$$\sum_{l \in \widehat{S}} \left(t_{l1} + \alpha^l \Delta W \right) = W_0 \text{ for some } \widehat{S} \subseteq N \text{ such that } \#S = k \text{ and } i \in S,$$

then, we would obtain:

$$\sum_{l \in \widehat{S} \setminus \{i\}} \left(t_{l1} + \alpha^l \Delta W \right) + \alpha^j \Delta W < W_0$$

contradicting our assumption that t_1^* is a solution to problem (1). The conclusion proceeds as in claim 1.

From claims 1 and 2, we deduce than an optimal strategy t_1^* is described by an integer $m^* \equiv \#N^*$ and a real number $t^* \equiv t_{l1}^* + \alpha^l \Delta W$ for all $l \in N^*$ such that $t^* < \alpha^j \Delta W$ for all $j = m^* + 1, ..., n$ and $t^* - \alpha^j \Delta W > 0$ for all $j = 1, ..., m^*$.

Consider first the case where $m^* \geq k$. The most severe constraint is attached to the coalition $S = \{1, ..., k\}$ and it takes the form:

$$t^* k \geq W_0.$$

Solving this for equality gives:

$$t^* = \frac{W_0}{k}$$

and a total cost for lobby 1 equal to:

$$\left(\frac{W_0}{k} \right) m^* - \Delta W \sum_{l=1}^{m^*} \alpha^l.$$

Since $\alpha^1 \leq \alpha^2 \leq .... \leq \alpha^n$, there is a unique value m^* of m such that $t^* = \frac{W_0}{k} \leq \Delta W \alpha^{m^*+1}$ and $t_{l1}^* = t^* - \alpha^l \Delta W > 0$ for all $l = 1, ..., m^*$.

Consider now the case where $m^* < k$. The most severe constraint is still attached to the coalition $S = \{1, ..., k\}$ and it takes now the form:

$$t^* m^* + \Delta W \sum_{l=m^*+1}^{k} \alpha^l \geq W_0.$$

Solving this for equality leads to:

$$t^* = \frac{W_0 - \Delta W \sum_{l=m^*+1}^{k} \alpha^l}{m^*}$$

and a total cost for lobby 1 equal to:

$$W_0 - \Delta W \sum_{l=1}^{k} \alpha^l.$$

This solution is valid iff:

$$t^* \leq \Delta W \alpha^{m^*+1},$$

i.e.

$$W_0 < \Delta W \sum_{l=m^*+1}^{k} \alpha^l + \Delta W m^* \alpha^{m^*+1}.$$

Since the function $\Delta W \sum_{l=m+1}^{k} \alpha^l + \Delta W m \alpha^{m+1}$ is increasing in m and takes the value $k \Delta W \alpha^k$ when $m+1=k$, we are left with two cases.

Either $\frac{W_0}{k} > \Delta W \alpha^k$ and m^* is determined as the unique smallest integer m such that $\frac{W_0}{k} \leq \Delta W \alpha^m$ if any and $m^* = n$ otherwise. Or $\frac{W_0}{k} \leq \Delta W \alpha^k$. Then let $\underline{m}$ be the smallest value of $m \leq k-1$ such that:

$$W_0 < \Delta W \left[\sum_{l=m+1}^{k} \alpha^l + m \alpha^{m+1} \right].$$

Since $\frac{W_0}{k} \leq \alpha^k \Delta W$, $\underline{m}$ is well defined. On the other hand, since:

$$t^* - \alpha^j \Delta W > 0 \text{ for all } j = 1, ..., m^*$$

we must have:

$$\frac{W_0 - \Delta W \sum_{l=m^*+1}^{k} \alpha^l}{m^*} \geq \Delta W \alpha^{m^*},$$

and therefore $m^* = \underline{m}$ □

Let us examine in turn, how proposition 5 answers the three questions formulated at the beginning of the subsection. Note first that if:

$$\frac{W_0}{k} > \Delta W \alpha_n \text{ i.e. } \frac{W_1}{W_0} < 1 + \frac{1}{k \alpha^n},$$

then the lobby 1 cheapest offer would consist in bribing all the legislators. The corresponding cost is $\frac{nW_0}{k}-\ \Delta W\sum_{l=1}^{n}\alpha^l$ and lobby 1 will therefore find profitable to do so iff:

$$W_1 \geq \frac{nW_0}{k} - \Delta W \sum_{l=1}^{n} \alpha^l \text{ i.e. } \frac{W_1}{W_0} \geq \frac{\left(\frac{2k-1}{k}\right) + \sum_{i\in N}\alpha^i}{1+\sum_{i\in N}\alpha^i},$$

i.e. inequality (3) since $\mu^*(\mathcal{B}) = 2 - \frac{1}{k}$. For lobby 1 to bribe at least a majority of legislators, it is necessary and sufficient that:

$$\frac{W_0}{k} > \Delta W \alpha^k \text{ i.e. } \frac{W_1}{W_0} < 1 + \frac{1}{k\alpha^k}.$$

It will bribe a minimal majority if:

$$1 + \frac{1}{k\alpha^{k+1}} \leq \frac{W_1}{W_0} < 1 + \frac{1}{k\alpha^k}.$$

The corresponding cost is $W_0-\ \Delta W\sum_{l=1}^{k}\alpha^l$and lobby 1 will therefore always find profitable to do so. At the other extreme, if:

$$W_0 < \Delta W \sum_{l=1}^{k} \alpha^l,$$

then, lobby 1 does not offer any bribe.

The proposition answers the two other questions. The legislators with the lowest α are approached first. These are the legislators who are the more expensive from the perspective of lobby 1 and so, even if any legislator has an intrinsic preference for the reform, there is a sense in which we can say that lobby 1 bribes a nonflooded coalition. Finally, the offers which are made to two distinct bribed legislators, say i and j compensate exactly for the differences in the parameters α_i and α_j: their gross benefits to vote for the reform instead of the status quo are ultimately identical. Groseclose and Snyder (1996) and Banks (2000) refer to such vector of offers as a levelling schedule.

While the assumptions are different, proposition 5 shares some common features with Banks's main result[54]. He assumes that there is a majority of legislators who have an intrinsic preference for the status quo: without lobbying, the reform is rejected by the legislature. We assume instead that the intrinsic preferences of the legislators are unanimously oriented towards the reform side. Under his assumption, lobby 0 has a double advantage, to be second mover in the game and to have a majority of partisans, while, in our case, the second advantage is entirely eliminated. Both Banks and ours prove the optimality of levelling schedules

[54]The comparison with Groseclose and Snyder is more difficult as most of the analysis in their 1996's paper assumes a continuum of voters.

but his coalition is nonflooded while it is in our case. We provide a complete characterization of the optimal size m^* while Banks provide necessary and sufficient conditions for this coalition to be minimal winning on one hand and universalistic on the other hand[55].

4.4 Procedural versus Consequential

The results of this paper have been derived under the assumption that the legislators were *procedural* i.e. their preferences are defined by how they vote rather than by the alternative that prevails. It is natural whether the bulk of the analysis is preserved if we assume instead that the legislators are *consequential* i.e. only concerned by the outcome of the vote. Under this new behavioral assumption, it is important for any legislator to predict accurately the voting behavior of the others in order to determine if his vote is pivotal or not while in the procedural case, it could decide optimally upon his course of action irrespective of the action of others. Surprisingly, we will see that the results derived in the consequential framework are quite closed to those obtained in the procedural framework. We will show how to adjust proposition 1 to accommodate this new setting for an arbitrary profile α[56].

Let $(N, \mathcal{W})$ be an arbitrary simple game. A coalition $S \subseteq N$ will be called minimal blocking + if there exists $T \in \mathcal{B}_m$ and $i \notin T$ such that $S = T \cup \{i\}$. Let us denote by $\mathcal{B}_{m+}$ the family of minimal blocking + coalitions. To prepare for proposition 6, let us examine intuitively the reaction of the different legislators when the pair of vectors of standing offers is (t_0, t_1). If $t_{0i} = 0$, then legislator i do not have to form expectations about the behavior of others as he will vote for the reform. If instead, $t_{0i} > 0$ then legislator must wonder if he is pivotal or not. If he feels like being pivotal, he will vote for the reform iff $t_{1i} + \alpha^i \Delta W \geq t_{0i}$ while if he feels not to be pivotal, he will vote for the reform iff $t_{1i} \geq t_{0i}$. For any such legislator, the question is therefore to determine whether he will be pivotal or not. He will be pivotal if he anticipates that the coalition of legislators who have received an offer from lobby 0 and such that $t_{1i} + \alpha^i \Delta W < t_{0i}$ is in $\mathcal{B}_m$. If the coalition who has received an offer from lobby 0 is a coalition S larger than $T \in \mathcal{B}_m$, then not everybody in this coalition can be pivotal. We can however consider a profile of votes within S where a subset $T \in \mathcal{B}_m$ vote for the status quo while the others in $S \backslash T$ vote for the reform but this would not be a Nash equilibrium as the best response for those in $S \backslash T$ is also to vote for the status quo. Since they all vote for the status quo, and S is not minimal, none of these legislators is pivotal and they all vote for the status quo. From the perspective of lobby 0, the cheapest coalition of

[55] Under his assumptions, lobby 1 cannot consider a submajority coalition. Further, in his framework, a full characterization of m^* looks difficult.

[56] Of course, if $\alpha = 0$, the two settings coincide.

that sort is a coalition in $\mathcal{B}_{m+}$. This reasoning leads to the following two relevant strategic options for lobby 0.

· Either, lobby 0 bribes exclusively the members of a minimal blocking coalition S and offers to any legislator i in S a bribe t_{0i} such that $t_{1i} + \alpha^i \Delta W < t_{0i}$.

· Or, lobby 0 bribes exclusively the members of a minimal blocking + coalition S and offers to any legislator i in S a bribe t_{0i} such that $t_{1i} < t_{0i}$.

Any other response to the offer of lobby 1 is dominated. The trade-off is pretty transparent: a minimal blocking coalition contains less legislators than a minimal blocking + coalition but they cost more individually. With minimal blocking + coalitions, everything is as if we were in the situation where $\alpha = 0$.

The subgame-perfect equilibrium of this sequential version of the lobbying game can be easily described. Let $t_1 = (t_{11}, t_{21}, ..., t_{n1}) \in \Re^n_+$ be lobby 1's offers. Lobby 0 will find profitable to make a counter offer if either there exists a minimal blocking coalition S such that:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i W_1\right) < \sum_{i \in S} \alpha^i W_0 + W_0$$

or a minimal blocking + coalition S such that:

$$\sum_{i \in S} t_{i1} < W_0.$$

Indeed, in both cases, there exists a vector $t_0 = (t_{10}, t_{20}, ..., t_{n0})$ of offers such that:

$$\text{either } t_{i1} + \alpha^i W_1 < t_{i0} + \alpha^i W_0 \text{ or } t_{i1} < t_{i0} \text{ for all } i \in S \text{ and } \sum_{i \in S} t_{i0} < W_0.$$

Therefore , if lobby 1 wants to make an offer that cannot be cancelled by lobby 0, it must satisfy the list of inequalities:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i \Delta W\right) < W_0 \text{ for all } S \in \mathcal{B}_m$$

and

$$\sum_{i \in S} t_{i1} < W_0 \text{ for all } S \in \mathcal{B}_{m+}.$$

The cheapest offer t_1 meeting these constraints is solution of the following linear program:

$$\underset{t_1 \in \Re_+^n}{Min} \sum_{i \in N} t_{i1}$$

subject to the constraints

$$\sum_{i \in S} \left(t_{i1} + \alpha^i \Delta W \right) \geq W_0 \text{ for all } S \in \mathcal{B}_m$$

$$\text{and } t \sum_{i \in S} t_{i1} \geq W_0 \text{ for all } S \in \mathcal{B}_{m+}.$$

Lobby 1 will find profitable to offer the optimal solution t_1^* of problem (1) if the optimal value to this linear program is less than W_1. It is then important to be able to compute this optimal value.

The following proposition, which is the consequential counterpart of proposition 1 describes the equilibrium outcome of the lobbying game[57].

Proposition 6. Either (i) $W_1 \geq \sum_{S \in \mathcal{B}_m} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right] + \sum_{S \in \mathcal{B}_{m+}} \delta(S) W_0$ for all vectors of subbalancing coefficients δ attached to $\mathcal{B}_m \cup \mathcal{B}_{m+}$ and then lobby 1 offers an optimal solution t_1^* to problem (1) and lobby 0 offers nothing and so the bill is passed. Or (ii) $W_1 < \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right]$ for at least one vector of subbalancing coefficients δ attached to $\mathcal{B}_m \cup \mathcal{B}_{m+}$ and then both lobbyists promise nothing and so the bill is not passed.

As in the behavioral case, the determination of the equilibrium outcome has been reduced to the examination of the value of a linear program which is slightly more intricate than the program obtained under behavioral model P. Both intuition and the calculations above suggest that the normalized value $v_C^*(\alpha, \mathcal{W})$ of this program lies somewhere between $\frac{\gamma^*(\mathcal{B}) + \sum_{i \in N} \alpha^i}{1 + \sum_{i \in N} \alpha^i}$ and $\gamma^*(\mathcal{B})$. As before, the comparison of $\frac{W_1}{W_0}$ and v_C^* leads to the determination of the optimal strategy of lobby 1. Since this new program has more constraints, we deduce that $v_C^*(\alpha, \mathcal{W})$ is at least equal to the victory threshold derived in the procedural case. The exact calculation of this parameter will be the subject of further research.

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