Price Skewness and Competition in Multi-Sided Markets*

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March 2008

Abstract

The paper examines competition between multi-sided platforms and analyzes the profits and the pricing strategies that emerge. Focusing on a Stackelberg pricing game it shows how price discrimination helps a platform coordinating the choices of consumers. After providing general bonds on equilibrium profits, it characterizes equilibria assuming a particular resolution of the coordination game played by platforms' users, that favours the leader (favorable expectations). Even under the most advantageous conditions, a platform's may be adversely affected by the ability of competitors to exploit asymmetries in multi-sided externalities. Results are then used to characterize the equilibria of a game of perfect price-discrimination by competiting networks, where the largest network is shown to be too small from a welfare perspective, and of a game with two sides (with sequential and simultaneous prices). Various implication are then discussed, such the possibility of excessive entry or the incentives to interconnect.

1 Introduction

Platforms offer services used by different types of agents to interact with each other, thereby involving externalities. Platforms then rely on complex price schemes, involving discrimination and non-linear tariffs, to allocate the surplus between their members and provide adequate incentives. Following Caillaud and Jullien (2003), Rochet and Tirole (2003) and Armstrong

*The author is strongly indebted to Bernard Caillaud. The author thanks Jacques Crémer, Jean Tirole and Jean Charles Rochet for their detailed comments, Patrick Rey and Rossella Argenziano for useful discussions, as well as seminar participants at IDEI, University of Essex, University College London, University of East Anglia, Ente Einaudi, London Business School, ZEW, CESifo and the European Summer Symposium in Economic Theory. This work builds extensively on a research paper initiated while visiting the European University Institute, entitled "Competing in Network Industries: Divide and Conquer" (2000).

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(2006), the literature on two-sided markets has improved substantially our understanding of these strategies, and provided valuable insights on the functioning of several markets.[1] This includes for instance the payment cards market (Wright 2003, Rochet and Tirole (2002)), intermediation services (Caillaud and Jullien (2001, 2003)), mobile telephony termination charges (Wright (2003), Armstrong (2003)), or the media market (Anderson and Coates (2005), Gabszewicz, Laussel and Sonnac (2001a,b), Ferrando et al (2003)), video games (Hagiu (2006)).

As emphasized by the literature on two-sided markets, the design of the price structure must account for the need to coordinate the users. The price charged to a user reflects not only the cost of service but also the externalities associated with his/her participation. One intuition for this specificity of a platform's pricing strategies is that each individual is not only a user but also an "input", since his/her participation creates value for other user. As an input, the user is a scarce resource. Firms then compete both to sell the outputs (interactions) and to buy the inputs (participation). Competition may then lead to aggressive strategies offering very advantageous terms, even a subsidy, to some sides and exploiting the network externalities to generate profits on other sides: a divide-and-conquer strategy (Innes and Sexton (1993), Caillaud and Jullien (2001, 2003)). Such a strategy allows a platform overcoming the coordination problem by transferring part of the surplus to targeted customers and creating a bandwagon effect. Notice that this bears similarity with the dynamic mechanism that leads a network to subsidize early customers, as analyzed by Farell and Saloner (1986) and by Katz and Shapiro (1986,1992).[2]

This paper analyses "divide&conquer" pricing strategies in a model of platform competition encompassing both vertical and horizontal differentiation. The focus is on price skewness and on the extent of market power that one can expect in such markets. Multi-sided externalities are defined as externalities between distinct and well identified groups of agents, for which it is possible to charge different prices. Groups may be reduced to a single individual. The model focuses on participation externalities, assuming that the utility of an agent is affected by the number of participants of each other side.[3] The paper analyzes equilibrium prices and profits, providing new insights on the nature of business strategies used by platforms to conquer or

[1] See Rochet and Tirole (2006) or Jullien (2005) for general presentations.

[2] See Bensaid and Lesne (1996) or Cabral, Salant and Woroch (1999) for applications to dynamic monopoly pricing. Recent progress on the modeling of dynamic competition between networks with forward looking consumers has been made in Fudenberg and Tirole (2000) and Cabral (2008).

[3] A very enlightning analysis of price skewness and competition in two-sided markets with usage externalities is Weyl (2006).

preserve market shares. The analysis is then used to characterize equilibria and efficiency in two cases of interest: perfect price-discrimination in a one-sided market with network effects; and two-sided market.

The multi-sided market set-up encompasses not only multi-sided platforms where distinct types of users buy distinct services, but also situation where a network price-discriminates between users of the same service.[4] For instance, software suppliers discriminate between residential and professional users, as well as geographic areas. As it will be clear below, two groups of users that are subject to price-discrimination by a network can be treated as two distinct sides in a multi-sided market context. The analysis is thus of interest for understanding platform competition, and network competition under price discrimination. The set-up thus covers a wide range of activities, such as intermediation activities, financial market places, telecommunication services, postal services, medias, operating systems and software applications, research centers,....

The paper examines a situation where various types of users have different valuations for the goods and services available on the platforms, as well as for the participation of other sides. It is assumed that each side is homogeneous, that users register with only one platform - there is single-homing -, and that multi-sided externalities are not too large compared to the value attached by a user to the direct consumption of the goods offered by the platform. To this respect, this work is complementary to Caillaud and Jullien (2003), Ambrus and Argenziano (2003), or Damiano and Li (2005b) who consider pure intermediation services.

The core of the paper characterizes the profit that a platform can obtain in a sequential pricing game where it is a Stackelberg leader. This allows to analyze the ability of the leader to exert market power, and also provides an upper bound on the profit of a platform in the simultaneous move game. The analysis of the simultaneous pricing game is presented for the case of two sides only.

Due to multi-sided externalities, consumers face a coordination problem in their purchasing decision, that may generate multiple equilibria (Katz and Shapiro (1985)). The paper addresses this issue in two steps. First, some bounds are derived for the profits of the leading platform that are valid for any equilibrium. Then the analysis focuses on a particular equilibrium selection for the subgame where users choose between platforms, deriving

[4] Second degree price discrimination is not considered here. Damiano and Li (2005a) propose an analysis of endogenous sorting with a self-selection mechanism for a matching monopoly.

the equilibria and characterizing the conditions under which the bounds are attained. The selection criteria captures the idea that there is a "focal" platform that any user joins whenever this is his choice in at least one equilibrium of the subgame. This extends the concept of favorable expectations used by Hagiu (2006) and by Caillaud and Jullien (2003). To this extent, this paper is also complementary to contributions on two-sided markets that focus on selection criteria limiting to some extent the level of coordination failure (Ambrus and Argenziano (2003) for instance).

Divide&conquer strategies target some groups with subsidies and charge other groups, and the paper shows how these strategies can be designed in a multi-sided context by interpreting the price charged to one side as the combination of a subsidy and of a charge. With two sides, the side generating the highest externality is a natural candidate for a subsidy. However there is no obvious ranking with multiple sides and the paper also shows how to identify the sides targeted with low prices.

In a competitive context, divide-and-conquer strategies eliminate the ability of a platform to capture a positive share of the surplus generated by the multi-sided externalities. The effect is stronger when externalities are asymmetric: sides valuing less the participation of others become the object of an intense competition as they are more value-enhancing relatively to others. Competition for these sides then dissipates the profits obtained with bandwagon effects, to a point that may even prevent the existence of a pure strategy equilibrium in a simultaneous pricing game. This strong intensity of competition is one of the key difference with the more conventional analysis of network competition with uniform prices.

With multi-sided externalities, the market allocation is inefficient, and not surprisingly there may be excessive sales by the focal platform. More surprisingly, there may be excessive sales by the non focal platform. Even a focal platform offering a uniformly superior quality may fail to cover the market and may have to leave some profitable niche for the rival, as a mean to soften competition on other sides. Faced to this problem, one strategy for the platform is to focus more on horizontal differentiation at the expense of vertical differentiation. For instance it may be optimal to degrade the quality offered to some targeted sides, as a commitment to compete only for other sides, thereby inducing market sharing.

Another consequence of tough competition is that platforms may be better off when interconnecting their services, i.e. allowing their members to interact with the other platform's members. Without interconnection, platforms are very aggressive in building market shares, exploiting multi-sided externalities through cross-subsidies. Interconnection suppresses the motives for cross-subsidization and restores the benefits of differentiation.[5]

[5] This reasoning assumes away exclusionary motives and externalities within sides.

As mentioned above, there is a close connection between the economics of multi-sided markets, and the economics of price-discrimination in network industries. While there is already a substantive literature on competition with network effects under uniform prices (see Katz and Shapiro (1994) and Economides (1996)), little attention has been devoted to price-discrimination, and this despite the fact that the practice is widespread for network services. The paper illustrates this by applying the analysis to perfect price-discrimination by competing networks with size related network effects. Increasing the value of network externalities raises the equilibrium size of the largest network, but not as much as welfare maximization would require. Thus in equilibrium the largest network is too small. This result is similar to the result obtained by Argenziano (2005) for a model of network competition without price-discrimination.

The paper is organized as follows. Section 2 introduces the ideas with a very simple free vs pay platform example. Section 3 presents the model and the assumptions. Section 4 presents general results concerning the analysis of competitive strategies and equilibria. Section 5 then defines the concept of favorable expectations and analyses the model under this assumption. The remaining sections discuss specific cases. Section 6 presents the application to perfect price-discrimination with one-sided network effects, while section 7 focuses on the two-sided market case, discussing equilibria for the sequential and the simultanous pricing game. Section 8 discusses the implications of the results. Section 9 concludes.

2 Warm-up: divide&conquer and inefficient entry

Consider an incumbent, free-access platform A serving two homogeneous sides of mass 1, $j = 1, 2$. The utility of a side j member at the platform when n_ℓ^A agents participate on the other side is $U_j^A = u_j + \beta_j n_\ell^A$, where $u_j > 0$ and $\beta_j > 0$ is the value attached to the participation of the other side. By convention assume that $\beta_1 < \beta_2$. If the platform is the only platform on the market, all consumers participate.

Suppose now that a commercial platform B wishes to enter. The platform charges prices:

$$p^B = \{p_1^B, p_2^B\}.$$

Prices can be negative (see the discussion below). The platform is vertically differentiated by a factor δ, so that the utility of a side j member at the platform B is $U_j^B = u_j - \delta + \beta_j n_\ell^B - p_j^B$.

When designing its prices, platform B must account for potential coordination failure that may prevent consumers from exploiting profitable opportunities if this requires a joint move of both sides. Indeed, when the two prices are such that

$$u_j - \delta - p_j^B > u_j + \beta_j, \tag{1}$$

both sides coordinating on platform A is an equilibrium allocation of consumers: given that the other side joins platform A, a consumer should join platform A. We say that platform B "faces unfavorable expectations" whenever for all prices verifying condition (1), no consumer joins platform B. When this is the case one may conjecture that entry should be deterred to some extent, but we now show that this is not the case.

Faced to such unfavorable expectations platform B can use the following strategy:

- $p_1^B < -\delta - \beta_1$ (divide);
- $p_2^B < -\delta + \beta_2$ (conquer).

With this price p_1^B, a member of side 1 prefers joining platform B alone than A with side 2, since $u_1 + \beta_1 < u_1 - \delta - p_1^B$. Thus any equilibrium allocation of consumers is such that side 1 joins platform B. An immediate consequence is that side 2 also joins platform B. Indeed the choice for a member of side 2 is now between joining side 1 or staying with A. Under the above condition, the former dominates unambiguously, since its members obtain a utility u_2 at platform A and $u_2 - \delta + \beta_2 - p_2^B > u_2$. Thus "both sides at B" is the unique equilibrium allocation of consumers at these prices (formally this results from iterated elimination of dominated strategies for consumers).

As a result the maximal profit $p_1^B + p_2^B$ that platform B can secure with this strategy is

$$\pi^B = -2\delta + \beta_2 - \beta_1.$$

The remarkable feature is that, since $\beta_2 - \beta_1 > 0$, platform B is able to generate a positive profit out of network externalities even when $\delta = 0$ is positive, meaning that an equally efficient platform can enter faced to a free platform and obtain positive profit. Platform B subsidizes one side and chooses the target for subsidy. Here it is side 1 because $\beta_1 < \beta_2$: *the side generating higher cross-group externality* (lower β_j) *is targeted.* The reason is that targeting the low externality side requires a subsidy smaller than the recoupment β_2 obtained on the high externality side.

More generally, platform B obtains positive profits whenever $\delta < \frac{\beta_2 - \beta_1}{2}$, thus for positive values of δ. There is inefficient entry: *platform B may conquer the market even if it is less efficient.*

The remaining of the paper extends these intuitions to the case where platform A sets prices and there are more sides.

3 The model

Consider two competing platforms, denoted A and B, with a production cost normalized to 0. The platforms bundle a consumption good and an intermediation service subject to externalities. There are J different types of users, each represented by a "side" composed of a mass m_j of identical agents with a unit demand. The set of sides is denoted $\mathcal{J}$.

Denote u_j^k the utility (or intrinsic value) derived by a member of side j from the good of platform k, and let $\delta_j = u_j^A - u_j^B$ be the "stand-alone" quality differential for side j.

The intermediation service value depends on the composition of the population that the platform allows to reach, and each side values differently the participation of every other side to the platform. The valuation of a user on side j for the participation of n_l members of side $l \neq j$ is $\beta_{jl} n_l$, where the coefficients β_{jl} are nonnegative. The base model doesn't consider the possibility of within-side network effects, so $\beta_{jj} = 0$, but the extension to the case where users care also about their own side raises no difficulty and is presented at the end of the section. The choice of not considering it here is made first to focus on the multi-sided aspect, and second because welfare conclusions would be more ambiguous.

As we will see in section 6, the model encompasses also the case of perfect price discrimination which obtains when $m_j = 1$ for all j and each side is interpreted as an individual.

If n_l^A and n_l^B members of side l join A and B respectively, a side j user joining platform k obtains a utility

$$U_j^k = u_j^k + \sum_{l \neq j} \beta_{jl} n_l^k - p_j^k.$$

To fix ideas, it is assumed that $\beta_{21} \geq \beta_{12}$. I focus on the case where the network externalities at the platform level are not too large compared to the intrinsic value of the good. More specifically it is assumed that:

Assumption 1: $\forall j : u_j^B \geq \sum_{l \neq j} \beta_{jl} m_l$.

To give an example, the benefits of using the same text editor are confined within small communities, and presumably smaller than the value of using a text editor. As another example, consider third generation mobile services. Base services include voice services or email access under universal connectivity. For these services, the penetration rate is very high, and assuming that operators don't engage into network based price discrimination, we can reasonably assume that there is no network effect. But the provision of more sophisticated services such as TV, and internet services have a multi-sided nature.

One can also reinterpret this property as the result of a sufficient level compatibility between services or standards in a context of imperfect compatibility (see Jullien (2006)). However the assumption may not hold for matching agencies for instance, for which network effects swamp the intrinsic value.

The core of the paper focuses on the Stackelberg game where A is a leader. The competitive game is thus composed of three stages:

Stage 1.1 : Platform A sets prices P^A.

Stage 1.2 : Platform B sets prices P^B.

Stage 2 : Users simultaneously decide which platform to join if any.

Equilibria for the simultaneous move equilibria, as well as the reverse sequential timing, are discussed afterward.

There are several reasons for focusing on this game. From a policy perspective, the Stackelberg game allows us to better understand the level of market power that a firm can expect by focusing on the reaction of its rivals and the competitive constraint it faces. In particular we will be able to derive bounds on the profit of platform A, that are also valid in a simultaneous pricing game. Second, it ensures tractability, while the simultaneous pricing game raises problems of multiplicity, as well as problems of existence.

Faced to prices $P^A = \{p_1^A, .., p_J^A\}$ and $P^B = \{p_1^B, ..., p_J^B\}$, users coordinate on a rational expectation equilibrium allocation (REA) defined as an allocation $\left\{n_j^k\right\}_{j=1,J}^{k=A,B}$ such that the choice of each individual is optimal given the prices and the choices of other consumers. To avoid technicalities and w.l.o.g. in the present context of homogeneous sides and positive network externalities, we restrict attention to REAs such that $n_j^k \in \{0, 1\}$. An allocation rule, denoted $\mathcal{A}(P^A, P^B)$, is a mapping from the set of prices to the set of users allocations, that assigns to every price vector a REA. An equilibrium consists in an allocation rule $\mathcal{A}$, and of equilibrium prices for the game where sides are allocated according to $\mathcal{A}$. I will discuss a specific allocation rule in section 5, but for the first part of the paper, I will only assume the following mild assumption:

Assumption 2: Let $\mathcal{S}$ be the set of sides joining platform B in $\mathcal{A}(P^A, P^B)$. For any price vector $\hat{P}^B$ such that $\hat{p}_j^B = p_j^B$ for $j \in \mathcal{S}$, all sides $j \in \mathcal{S}$ join platform B in $\mathcal{A}(P^A, \hat{P}^B)$.

The assumption is here to prevent pathological allocations, and it is a mild assumption compatible with the selection criteria used in the two-sided literature.[6] The idea is that if side l is not buying from platform B, then changing the price for side l can only be a "good news" for the clients of platform B, as this can either have no effect or attract this side which benefits all customers of platform B. It is substantial only when there is the possibility of coordination failure between customers of platform B, inducing multiple allocations. Then it could be the case that in situations where side l would never buy from platform B, customers of platform B use the price on side l as a coordination device, and join platform B only for particular values of p_l^B although this is irrelevant for them. It also rules out situations where a deviation from equilibrium by platform B that could attract side l is prevented by customers coordinating in such a manner that they would leave the platform B if side l were to join, despite the fact that their welfare would increase.

The assumption is compatible with the selection criteria used in the two-sided market literature. Indeed most criteria tend to limit the lack of coordination through various means, and assumption 2 is a very mild version of this. For instance, Ambrus and Argenziano (2003) use a different concept, coalitional rationalizability, with a similar property. The concept of favorable expectations that is developed later on verifies this condition.

4 General results

4.1 Preliminary considerations

Before we state the main results, a few points are worth mentioning. Assumptions 1 and 2 imply first that in equilibrium all sides join one platform. Indeed platform B could always propose a slightly positive price to members of side l if they don't buy and attract them, without loosing any customer. We denote by $\mathcal{K}$ the set of sides joining platform A in equilibrium, while $\mathcal{J}\backslash\mathcal{K}$ is the set of side joining platform B.

For the same reason, a member of a side $j \in \mathcal{K}$ cannot obtain a larger utility by joining platform B at a positive price. Otherwise platform B could just propose this price along with a zero price to other sides buying from A, without altering the prices for the sides on its platform: because of Assumption 2, the targeted side j and all the previous customers of platform B (sides in $\mathcal{J}\backslash\mathcal{K}$) would join platform B, and the profit of platform B would increase.

[6] The assumption is slightly stronger than what is required for the results that follow. Indeed we need that: **(2bis)** For any $\left\{\hat{p}_j^B\right\}_{j\notin\mathcal{S}}$ and $\varepsilon > 0$, there exists $\hat{P}^B$ such that: *i)* prices are $\hat{p}_j^B$ for $j \notin \mathcal{S}$, *ii)* $\left|\hat{p}_j^B - p_j^B\right| < \varepsilon$ for $j \in \mathcal{S}$, *iii)* sides $j \in \mathcal{S}$ join B in $\mathcal{A}(P^A, \hat{P}^B)$.

Assumption 2 is more intuitive and simplifies the presentation of the results.

Thus in equilibrium, it must be the case that the utility obtained with A by a user of a side $j \in \mathcal{K}$ is larger than the minimal utility that this user may obtain with platform B :

$$\forall j : u_j^A + \sum_{l \in \mathcal{K}} \beta_{jl} m_l - p_j^A \geq u_j^B + \sum_{l \notin \mathcal{K}} \beta_{jl} m_l.$$

Using $\delta_j = u_j^A - u_j^B$, selling to sides within the set $\mathcal{K}$ requires platform A to set prices such that

$$\forall j : p_j^A \leq \delta_j + \sum_{l \in \mathcal{K}} \beta_{jl} m_l - \sum_{l \notin \mathcal{K}} \beta_{jl} m_l \leq u_j^A, \tag{2}$$

where the latter inequality follows from Assumption 1. Condition (2) implies that the utility level that side j would obtain joining A alone is nonnegative: $u_j^A - p_j^A \geq 0$. This ensures that in any subgame following $A's$ pricing decision, side j joins A if not B. Thus when discussing potential deviation for platform B, we need not worry about the possibility that some users buying from A in equilibrium may decide to stay out of the market, which simplifies greatly the analysis.

4.2 Divide&conquer

The general idea of the paper is that even faced with the worst situation a follower platform B can use a "divide-and-conquer" strategy to overcome its disadvantage. To do so, platform B has to favor some sides: platform B can charge a very low price to one side, if selling to this side allows to sell to another side at a high price.[7] To follow insights from Innes and Sexton (1993), one may view the situation as one in which platform B faces users who have the possibility to form a coalition (to join A). To prevent the formation of the coalition, platform B may need to "bribe" some users. However it needs not bribe all users but only enough of them to ensure that the value of the sub-coalition composed of the remaining users is reduced to a point where it becomes unattractive. The characteristic of divide&conquer strategies is that they allow to capture demand for any allocation rule $\mathcal{A}$, because they are constructed in such a way that joining the platform is a dominant strategy for the targeted sides.[8] They thus solve the coordination problem and allow a platform capturing part of the value of network externalities. This subsection presents an intuitive characterization of these strategies and of their effects on profits.

[7] Notice that given that costs are normalized to 0, prices should be interpreted as margins and a negative price is a price below marginal cost. In the model, the real price may, but need not, be negative. One interpretation in this case is that the customer receives free access to the platform, and is subsidized through additional free goods and services (see Amelio and Jullien (2007)).

[8] See Segal 2003 for similar ideas applied to constracting with externalities.

As an introduction, suppose that there are only two sides and that platform A sets prices $p_j^A \leq u_j^A$. Reminding that $\beta_{12} \leq \beta_{21}$, let us build a strategy that enables platform B to attract both sides. For any prices such that $p_j^B \geq p_j^A - \delta_j - \beta_{jl} m_l$, assumption 2 implies that side j joins platform A if the other side does. Thus to attract some users, platform B needs to set one price p_j^B below $p_j^A - \left(\delta_j + \beta_{jl} m_l\right)$, say p_1^B. Then, as in the warm-up section, a member of side 1 joins platform B irrespective of what the other side does. Following the logic of the warm-up section, platform B can attract side 2 with a price such that $u_2^B + \beta_{21} m_1 - p_2^B \geq u_2^A - p_2^A$. The maximal price that platform B can set on side 2 is thus $p_2^A - \delta_2 + \beta_{21} m_1$. By attracting side 1, platform B has reduced the attractiveness of platform A for side 2 by an amount $\beta_{21} m_1$, and has increased its own attractiveness by the same amount. The overall profit from this strategy is

$$\pi^B = p_1^A m_1 + p_2^A m_2 - \sum_{j=1}^{2} \delta_j m_j + (\beta_{21} - \beta_{12}) m_1 m_2. \tag{3}$$

Since $\beta_{21} - \beta_{12} \geq 0$, platform B is able to generate a positive profit out of network externalities. Thus there is a strong value attached to being able to react to the opponent's prices by choosing who to subsidy.

The key part of the paper consists in extending this insight to J sides. For this purpose, suppose again that platform A sets prices such that $p_j^A \leq u_j^A$ for all j. Then, from the preliminary considerations, all consumers buy from one platform. This simplifies the exposition as it implies that a consumer always prefer joining platform A to staying out of the market.

Suppose that platform B tries to sell to the subset $\mathcal{L}$ and for the sake of exposition set $\mathcal{L} = \{1, 2, ..., L\}$. If platform B sets the prices at levels such that for all sides $u_j^A + \sum_{l \neq j} \beta_{jl} m_l - p_j^A > u_j^B - p_j^B$, then there is a REA where no consumer joins platform B. To avoid this platform B can set one price at a level that guarantees the willingness of a member of side j to join alone. Say that for side 1 :

$$u_1^B - p_1^B > u_1^A + \sum_{l>1} \beta_{1l} m_l - p_1^A. \tag{4}$$

The resulting price includes a subsidy equal to the maximal value of externalities $\sum_{l>1} \beta_{1l} m_l$. Then joining platform B is a dominant strategy for a member of side 1. Given that this is commonly known, any price p_2^B such that

$$u_2^B + \beta_{21} m_1 - p_2^B > u_2^A + \sum_{l>2} \beta_{2l} m_l - p_2^A \tag{5}$$

induces a member of side 2 to join as well.

It follows that if both (4) and (5) are verified, sides 1 and 2 join platform B in any market allocation. The process can then continue: a member of side 3 joins at a price that makes it more attractive to join sides 1 and 2, as opposed to staying with sides above 3 at platform A. More generally, side j is charged by platform B a price that ensures that its members join given that all sides $l < j$ join:

$$u_j^B + \sum_{l<j} \beta_{jl} m_l - p_j^B > u_j^A + \sum_{l>j} \beta_{jl} m_l - p_j^A \tag{6}$$

The pricing strategy is thus built in such a way that after j rounds of elimination of dominated strategies in stage 2, there are j sides for which the only remaining strategy is to join platform B. The condition 6 allow to design prices enabling platform B to sell to $\mathcal{L}$ irrespective of the allocation rule.

More generally, platform B has the choice of the set $\mathcal{L}$ of sides targeted, and of the order in which sides are subsidized. Indeed platform B benefits from subsidizing the sides valuing less the network effects, since the required subsidy is smaller than the value of the network externalities that platform B can extract from the other sides. Denote $\sigma(.)$ a permutation on the set of sides , where $\sigma(l) > \sigma(j)$ means that j is "ranked" before l. The interpretation is that platform B sets prices such that a member of side j is willing to join provided that it is sure that all sides ranked below in the targeted set $\mathcal{L}$ join as well:

$$u_j^B + \sum_{\substack{l\in\mathcal{L} \\ \sigma(l)<\sigma(j)}} \beta_{jl} m_l - p_j^B > u_j^A + \sum_{\substack{l\in\mathcal{L} \\ \sigma(l)>\sigma(j)}} \beta_{jl} m_l + \sum_{l\notin\mathcal{L}} \beta_{jl} m_l - p_j^A.$$

The minimal profit that platform B can secure with the subset $\mathcal{L}$ of sides is obtained by summing the corresponding prices over the sides pondered by the size m_l of sides:

$$\sum_{j\in\mathcal{L}} \left(p_j^A m_j - \delta_j m_j + \sum_{\substack{l\in\mathcal{L} \\ \sigma(l)<\sigma(j)}} \beta_{jl} m_l - \sum_{\substack{l\in\mathcal{L} \\ \sigma(l)>\sigma(j)}} \beta_{jl} m_l - \sum_{l\notin\mathcal{L}} \beta_{jl} m_l \right) m_j > \sum_{j\in\mathcal{L}} p_j^B m_j.$$

Optimizing on the order σ and the subset $\mathcal{L}$ yields a lower bound on platform B's profit:

$$\pi^B \geq \max_{\mathcal{L}} \left\{ \sum_{j\in\mathcal{L}} p_j^A m_j - \sum_{j\in\mathcal{L}} \delta_j m_j + \Omega(\mathcal{L}) - \mathcal{B}(\mathcal{L}) \right\}. \tag{7}$$

where

$$\Omega(\mathcal{L}) = \max_{\sigma(.)} \left\{ \sum_{j \in \mathcal{L}} \sum_{\substack{l \in \mathcal{L} \\ \sigma(l) > \sigma(j)}} (\beta_{lj} - \beta_{jl}) m_l m_j \right\} \tag{8}$$

$$\mathcal{B}(\mathcal{L}) = \sum_{j \in \mathcal{L}} \sum_{l \in \mathcal{J} \backslash \mathcal{L}} \beta_{jl} m_l m_j; \ \mathcal{B}(\mathcal{J}) = 0 \tag{9}$$

$\Omega(\mathcal{L})$ captures the profit that platform B can generate with cross-subsidization, the maximization coming from the optimal choice of targeting. This term has a very simple interpretation: when side j is attracted "before" side l, platform B must give a subsidy to the members of side j equal to their opportunity cost $\beta_{jl} m_l$ of leaving side l, but it can charge an extra amount $\beta_{lj} m_j$ to the members of side l corresponding to the value of joining side j. The net effect is then $(\beta_{lj} - \beta_{jl}) m_l m_j$. Because platform B can choose the order of targeting, $\Omega(\mathcal{L})$ is non-negative

Lemma 1 *For all $\mathcal{L}$, $\Omega(\mathcal{L}) \geq 0$, with equality if and only if for all j, l within $\mathcal{L}$, $\beta_{jl} = \beta_{lj}$. Moreover $\Omega(\mathcal{L}) + \Omega(\mathcal{L}') \leq \Omega(\mathcal{L} \cup \mathcal{L}')$.*

Proof. See Appendix. ■

$\mathcal{B}(\mathcal{L})$ is the total value of the network effects between sides within $\mathcal{L}$ and sides outside $\mathcal{L}$. Since platform B sells only to $\mathcal{L}$, it must compensate its customers for the foregone value of network externalities.

Thus, when platform B reacts to the prices of platform A , it has the ability to exploit the presence of network effects within the group of targeted consumers, but he may be disadvantaged by the potential network externalities between its customers and the customers of platform A. The objective in this section is then to identify which effect dominates: the favorable expectations advantaging platform A, or the ability to react with cross-subsidies advantaging platform B. We will show that the latter dominates, and in particular that platform A cannot obtain a profit larger than $\sum_{j \in \mathcal{L}} \delta_j m_j$ when selling to a subset $\mathcal{L}$ of sides, which represents simply the quality differential associated with the good offered by the platform.

4.3 Maximal equilibrium profits for the leader

With the property 7 characterizing minimal profit for platform B, we can now analyze the strategic possibilities opened to the leader. Suppose that platform A decides to sell to sides within a subset $\mathcal{K}$. How much profit can it expect, given the possibility that platform B uses a divide&conquer strategy.

Platform A can sell in equilibrium to $\mathcal{K}$ only if platform B cannot deviate and attract profitably some subset of $\mathcal{K}$ with a divide&conquer strategy. A bound on profits can then be obtained by deriving the condition for the subset $\mathcal{K}$ itself. One difference with the above discussion of divide&conquer is that we have to consider a deviation from an equilibrium allocation were platform B sells if $\mathcal{K} \neq \mathcal{J}$, while above we were considering only whether platform B could sell or not. But Assumption 2 implies that restricting attention to strategies with constant prices on $\mathcal{J}\backslash\mathcal{K}$ ensures that these sides always buy from platform B. We can then ignore them for the analysis of the market allocation and apply the above reasoning on the subset $\mathcal{K}$. We need also to incorporate the fact that having a secured clientele raises the relative value of platform B compared to platform A. Then to obtain an upper bound on profit, it is sufficient to state a necessary condition for platform B not being able to attract the whole set $\mathcal{K}$ with a divide&conquer strategy.

Proposition 1 *Suppose that platform A sells to sides within $\mathcal{K}$ in equilibrium, then it's profit is smaller than or equal to* $\Pi^A(\mathcal{K}) = \sum_{j\in\mathcal{K}} \delta_j m_j - \Omega(\mathcal{K}) - \mathcal{B}(\mathcal{K})$.

Proof. Starting from equilibrium prices, consider a deviation by B that maintains constant all the prices outside $\mathcal{K}$. Then Assumption 2 ensures that in any REA, sides in $\mathcal{J}\backslash\mathcal{K}$ buy from B. We can thus treat them as passive and focus on the sides in $\mathcal{K}$. Given that it is common knowledge that sides in $\mathcal{J}\backslash\mathcal{K}$ buy from B, a consumer in $\mathcal{K}$ anticipates a utility $u_j^B + \sum_{l\notin\mathcal{K}} \beta_{jl} m_l$ if she is the only from $\mathcal{K}$ joining B. The resulting reduced game is thus the same game as above played on the sides $j \in \mathcal{K}$ only instead of $\mathcal{J}$, and where the intrinsic value u_j^B is replaced by $u_j^B + \sum_{l\notin\mathcal{K}} \beta_{jl} m_l$. Then for the set $\mathcal{L} = \mathcal{K}$, platform B can increase its profits by

$$\sum_{j\in\mathcal{K}} p_j^A m_j - \sum_{j\in\mathcal{K}} \left(u_j^A - u_J^B - \sum_{l\notin\mathcal{K}} \beta_{jl} m_l \right) m_j + \Omega(\mathcal{K}) = \sum_{j\in\mathcal{K}} p_j^A m_j - \sum_{j\in\mathcal{K}} \delta_j m_j + \Omega(\mathcal{K}) + \mathcal{B}(\mathcal{K}).$$

For this to be an equilibrium this must be non-positive

$$\sum_{j\in\mathcal{K}} p_j^A m_j - \sum_{j\in\mathcal{K}} \delta_j m_j + \Omega(\mathcal{K}) + \mathcal{B}(\mathcal{K}) \leq 0. \tag{10}$$

and thus

$$\sum_{j\in\mathcal{K}} p_j^A m_j \leq \sum_{j\in\mathcal{K}} \delta_j m_j - \Omega(\mathcal{K}) - \mathcal{B}(\mathcal{K}).$$

■

Condition (10) states that using a divide&conquer strategy to conquer the whole market is not profitable for platform B. It is the same as the above equation (7) but accounting for the fact that since sides outside $\mathcal{K}$ join platform B in any case, the value of platform B is augmented by the corresponding value $\mathcal{B}(\mathcal{K})$ of multi-sided effects that sides in $\mathcal{K}$ would be sure to obtain if they join platform B.

One point worth noticing when comparing (7) and (10) is that the difference between the two formulas is $2B(\mathcal{K})$, reflecting the fact that platform A sells to sides outside $\mathcal{K}$ in the former case, while platform B does in the latter case. This illustrates the fact that the opportunity cost for platform A of not serving side j is equal to twice the value of the externality that this side generates for its customers. The reason is that the value of the externality is transferred to platform B if it sells to the side instead of platform A, thus the total value of platform A is reduced by $\mathcal{B}(\mathcal{K})$ while the value platform B can offer to platform $A's$ increases by $\mathcal{B}(\mathcal{K})$.

A key implication of the results is that platform A can't benefit from the presence of multi-sided externalities, since its profit is bounded by the profit $\sum_{j\in\mathcal{K}} \delta_j m_j$ it would obtain if no users attached a value to the participation of other sides ($\beta_{jl} \equiv 0$, for all j and l). This contrasts with the case of networks effects between members of the same side, as S would benefit from the presence of positive network effects within sides (see the discussion section).

5 Favorable expectations and focal platform

Given the bounds derived above, the question is now to understand when these bounds can be reached and how. To generate maximal profits for platform A we need to exhibit the allocation rule that is the most favorable to platform A. This allocation should capture the idea that platform A benefits from an advantage because users' tend to coordinate on it. While this notion may be ambiguous in a general context, it takes the form of a simple selection criteria when there are positive network externalities, due to the following result:

Lemma 2 *Fix the prices P^A and P^B, and consider all REAs for these prices. Let $\mathcal{K}$ be the set of sides with a positive participation to platform k in* ***at least one*** *REA and S be the set of sides with full participation to platform s in* ***all*** *REAs. Then there exists a REA, denoted $\mathcal{D}^k(P^k, P^s)$, such that all sides in $\mathcal{K}$ join platform k, and only sides in S join platform s.*

Proof. See Appendix ■

The key feature is that the value of a platform uniformly increases when new users are added to its customer base. According to a bandwagon effect, moving sides from s to k raises the incentives to join k for all individuals and reduces the incentives to join s. Thus there exists a unique REA that at the same time maximizes the market share of platform k and minimizes the market share of platform s. This follows from the fact that the users allocation process exhibits strategic complementarity (see Topkis (1979), Vives (1990)).

The most favorable allocation for platform A is one where it dominates the coordination process in the sense that users coordinate on $\mathcal{D}^A(P^A, P^B)$. To avoid inexistence problems that may be created by discontinuities at indifference points of users, the allocation is allowed to differ from $\mathcal{D}^A$ at such points. Defining $\bar{\mathcal{D}}^A$ as the closure of $\mathcal{D}^A$:[9]

Definition 1 (Favorable expectations) *Platform k is focal if*

$$\forall (P^A, P^B),\ \mathcal{A}(P^A, P^B) \in \bar{\mathcal{D}}^k(P^A, P^B).$$

Admittedly a strong criteria, the assumption allows capturing in a simple way the role of users coordination in the emergence of market power in platform industries. The term "favorable expectations" is borrowed from Hagiu (2004) who, building on Caillaud and Jullien (2003), uses a similar concept to analyze the emergence of dominant platforms in a two-sided model where sides choose the platform sequentially. It refers to the fact that coordination failures in the market are always resolved in favor of platform A.

Favorable expectation verifies the version (2bis) of Assumption 2 and thus is compatible with it.

Which firm is focal at a particular point in time depends on history or some exogenous factor. I must stress that being a focal platform refers only to the consumers coordination process, and need not imply that the firm appears to be dominant. Indeed market power results from the combined effects of the coordination process, the quality of services and the timing of prices.

To see the difference with other criteria, consider the case of two networks competing with uniform prices for identical agents who care only about the size of the network. Then the Pareto criterion or coalitional rationalizability imply that all consumers join the network that generates the highest surplus, so that the most efficient platform serves the market. By contrast, under favorable expectations for platform A, all consumers coordinate on platform A as long that the surplus obtained on platform A including the value of network effects is larger that the surplus that an individual would obtain alone on platform B, thus excluding the value of network effects. Platform A can then sell even in some cases where it is less efficient than platform B.

[9] For almost all prices, $\bar{\mathcal{D}}^A$ coincides with $\mathcal{D}^A$. At points of discontinuity, agents of some side joining A are indifferent between A and B. A slight reduction of the price of B for this side would induce them to join B, and the bandwagon effect may induce other sides to follow. $\bar{\mathcal{D}}^A(P^A, P^B)$ thus includes also $\lim_{\varepsilon \to 0} \mathcal{D}^A(P^A, P^B - \varepsilon)$.

Notice that understanding the best-reply of one platform under favorable expectations is also useful for the more general case where $\mathcal{A}$ is not restricted. Indeed for given prices of platform A, the criteria minimizes the payoff of platform B. Thus the easiest way to support any equilibrium is to assume that, following a deviation from equilibrium strategies by platform k, users coordinate on $\mathcal{D}^{-k}(P^{-k}, P^k)$, which minimizes the deviation profits, without restricting the allocation of consumers on the equilibrium path. This methodology is used for instance in Caillaud and Jullien (2003) to characterize the set of equilibrium profits and allocations of a pure matching model under simultaneous pricing and no restriction except the monotonicity of the equilibrium selection.

Since they favor platform A, one can conjecture that a focal leader should be able to generate the maximal profits. We now investigate whether the bound in proposition 1 is the maximal profits.

The first case is when platform A sells to all sides. Indeed in this case it sets all prices below the intrinsic values: $p_j^A \leq u_j^A$. Faced to a focal platform, the divide&conquer strategies described in section 4.2 are the only strategies that allows platform B to sell. Then platform A covers the market if and only if the profit derived in equation (7) is non-positive for all $\mathcal{K}$. We show in appendix that there exists a pricing strategy that verifies these conditions and yields maximal profits.

Proposition 2 *A focal platform A can obtain the profit* $\Pi^A(\mathcal{J})$ *by selling to all sides.*

Proof. See Appendix ■

The profit that platform A generates when covering the market is then equal to the total quality differential on the good minus the profit that platform B could get by exploiting multi-sided externalities: $\Pi^A(\mathcal{J}) = \sum_{j=1}^{J} \delta_j m_j - \Omega(\mathcal{J})$. Notice that the result is very general and doesn't depends on assumption 2.

Whether the bound on profits can also be obtained when platform A doesn't cover the market and sell only to a subset $\mathcal{K}$ is a more delicate issue, because we did not consider all divide&conquer strategies. Indeed the bound were obtained by fixing the prices set by platform B for the sides it serves in equilibrium and varying the other prices only. It thus ignore the possibility that platform B uses more complex strategies involving all the prices.

These more complex strategies don't matter when multi-sided effect are pairwise symmetric, that is when for all $j, l : \beta_{jl} = \beta_{lj}$. Indeed, in this case the implicit ranking of sides is irrelevant so that proposition 1 gives the profit.

Proposition 3 *When multi-sided effect are pairwise symmetric* ($\forall j, l : \beta_{jl} = \beta_{lj}$), *a focal platform A can obtain the profit* $\Pi^A(\mathcal{K})$ *by selling to sides within* $\mathcal{K}$.

Proof. See Appendix ■

When multi-sided externalities are symmetric ($\beta_{jl} = \beta_{lj}$), their effects in the divide-and-conquer strategy cancel out and $\Omega(\mathcal{K})$ vanishes and $\Pi^A(\mathcal{K}) = \sum_{j\in\mathcal{K}} \delta_j m_j - \mathcal{B}(\mathcal{K})$. Multi-sided externalities with a platform are neutral for platform A in this case, and what matters is the value of externalities not realized between platforms. In this case platform A chooses the set $\mathcal{K}$ so as to maximize $\Pi^A(\mathcal{K})$. In particular it covers the market if $B(\mathcal{K}) > -\sum_{j\notin\mathcal{K}} \delta_j m_j$ for all $\mathcal{K}$.

Compared to this case, introducing some asymmetry can only worsen the case for platform A. In particular if $\delta_j \equiv 0$ for all j, the Stackelberg leader leaves the whole market to its rival. The results also shows that

Corollary 1 *If platform A is focal, $\Pi^A(\mathcal{J}) \geq 0$ and for all subsets $\mathcal{K}$*

$$\sum_{j\notin\mathcal{K}} \delta_j m_j + \mathcal{B}(\mathcal{K}) \geq \Omega(\mathcal{J}) - \Omega(\mathcal{K}).$$

then platform A covers the market at equilibrium.

Proof. The result follows from the comparison of the value of the profit when platform A covers the market (proposition 2) and the bounds on market sharing profits (proposition 1). ■

Focal platform B

To complete this section let us just mention what occurs if instead of platform A, platform B is the focal platform. The difference with before is that platform B can capture all the surplus from network effects without relying on cross-subsidization. By attracting a new side, platform B could then capture the value of the externalities between this side and its own clients, but also the value created to its clients when the new side joins the platform. This generates a total potential value differential in favor of platform B equal to the sum of the quality differential and the total increase in the value of network effects on platform A when it attracts the whole population instead of sides within the equilibrium set $\mathcal{K}$ only.

Proposition 4 *Suppose that platform B is focal. The profit that platform A obtains when it sells to sides within $\mathcal{K}$ is $\pi^A = \sum_{j\in\mathcal{K}} \delta_j m_j - \mathcal{B}(\mathcal{J}\backslash\mathcal{K}) - \mathcal{B}(\mathcal{K})$.*

Proof. See Appendix ■

In particular, if platform A covers the market it obtains profits

$$\pi^A = \sum_{j\in\mathcal{J}} \delta_j m_j - \sum_{(j,l)\in\mathcal{J}\times\mathcal{J}} \beta_{jl} m_j m_l.$$

6 Perfect price-discrimination with uniform network effects

As pointed above, the analysis is the same if a side is constituted by a single individual. Setting $m_j = 1$, then J is the number of individuals and p_j is interpreted as an individualized price. Thus one application of the multi-sided framework developed is the case of perfect price-discrimination with network effects. Let us apply this framework to the conventional case where consumers care only about the size of the network, by assuming in this section:

Property 1: $\forall j, l,\ \beta_{jl} = \beta,\ m_j = 1,\ \delta_j \geq \delta_{j+1};$

where individuals are ranked by decreasing order of preference for network A.

Assume for the moment that all individuals have identical taste for the goods, so that for all $j:\ \delta_j = \delta$. From proposition (3), when able to set different prices to different sides, platform A covers the whole market with profit δ if it is positive, while platform A cannot profitably attract a set of consumers if $\delta < 0$.

Thus, with uniform preferences, the equilibrium allocation is efficient. The non focal platform is able to overcome the coordination problem and to pass the full value of the surplus to its customers, which eliminates inefficiencies.[10]

Allowing the intrinsic utility to differ across consumers, the profit for the network A serving individuals in $\mathcal{K}$ is $\sum_{j \in \mathcal{K}} \delta_j - \beta K(J-K)$ where K is the cardinal of $\mathcal{K}$, which implies that network A serves a set $\mathcal{K} = \{1, ..., K\}$ and choose the size

$$K^A(\beta) = \arg\max_K \sum_{j=1}^{K} \delta_j - \beta K(J-K)$$

We see that the benefits of adding one more side is

$$\delta_{K+1} - \beta(K+1)(J-K-1) + \beta K(J-K) = \delta_{K+1} - \beta(J-2K),$$

larger than δ_{K+1} if $K > \frac{J}{2}$. When it serves more than half of the market, the leader would serve all sides for which it has a larger quality u_j^k and more, but if it is smaller than its rival, it may refrain from including some side for which it is more efficient. The reason is that the leader benefits from increasing asymmetries between the size of the network as this minimizes the value of the externalities between users of different platform.

[10] One consequence of this is that if we drop assumption 2, and consider a simultaneous pricing game, there is a unique efficient equilibrium.

Proposition 5 *For $\beta' > \beta$: $\left|K^A\left(\beta'\right) - \frac{J}{2}\right| \geq \left|K^A(\beta) - \frac{J}{2}\right|$. If the profit is quasi-concave in K, then either $K^A\left(\beta'\right) \geq K^A(\beta) \geq J/2$ or $K^A\left(\beta'\right) \leq K^A(\beta) \leq J/2$.*

Proof. See Appendix. ■

Under quasi-concavity, increasing β increases the size of the largest network. Quasi-concavity is required as for large β, some tipping may occur where a small network A decides to cover the market. For instance suppose that $\sum_{j=1}^{K}\delta_j$ is maximal at $K^A(0) < 1/2$, but that $\sum_{j=1}^{J}\delta_j > 0$. For a low but positive β, network A reduces its market share to $K^A < K^A(0)$. But at some point, its profit will fall below $\sum_{j=1}^{J}\delta_j$, so that it will prefer to serve the whole population and $K^A(\beta)$ may jump to J.

Thus competition with perfect price discrimination tends to generate less balanced allocations than competition based solely on intrinsic values. However the allocation is too balanced from a social welfare perspective. Indeed welfare writes as

$$\begin{aligned} & \sum_{j=K+1}^{J} u_j^B + \sum_{j=1}^{K} u_j^A + \beta K^2 + \beta(J-K)^2 \\ = & \left(\sum_{j=1}^{J} u_j^B + \beta J^2\right) + \left(\sum_{j=1}^{K}\delta_j - 2\beta K(J-K)\right) \end{aligned}$$

Comparing with the profit of A, we see that, up to the first constant term, the difference between $A's$ profit and welfare is $\beta K(J-K)$, the value of the externalities between members of different platforms, maximal at $K = J/2$, when the platforms are of equal sizes. Divide&conquer strategies by network B prevents network A from capturing all the value of the externalities generated when attracting more sides. Here network A obtains only half of them. This implies that the allocation is biased toward the allocation generating the least value of network effects, namely the balanced allocation.

Proposition 6 *Let total welfare be maximal at K^*. Then either $K^* \leq \min\left\{1/2, K^A(\beta)\right\}$ or $K^* \geq \max\left\{1/2, K^A(\beta)\right\}$. If in addition welfare is strictly quasi-concave in K, then either $K^* \leq K^A(\beta) \leq 1/2$, or $K^* \geq K^A(\beta) \geq 1/2$.*

Proof. The equilibrium size of A maximizes $w(K) + \beta K(J-K)$, where $w(K) = \sum_{j=1}^{K}\delta_j - 2\beta K(J-K)$ denote the welfare gain. Suppose that $K^* \geq 1/2$. Then for $K > K^*$, $\beta K^*(J-K^*) > \beta K(J-K)$ so that $K^A(\beta) \leq K^*$. If in addition $w(K)$ is quasi-concave ■

Thus the market share of the network that should be the largest from a welfare perspective is too small in equilibrium. If we add the requirement that welfare is quasi-concave in K, still the largest network is the right one. Therefore welfare maximization would require increasing the size of the largest network. The conclusion is similar to the result of Argenziano (2005) on one-sided externalities. She analyzes a model of competition between networks with uniform prices and concludes that the largest network size is suboptimal in equilibrium. The same result holds under perfect price-discrimination but for different reasons. In the case of uniform prices, the result follows from the relation between mark-ups and network effects. In the present case, this is due to an attempt of the leader to protect its market share, faced to "excessive" competition due to price-discrimination.

7 Two-sided markets with a focal platform

In order to obtain more specific results on market shares and profits, let us focus on a two-sided market ($J = 2$) with a focal platform A. I start with the sequential timing but will extend the results to a simultaneous pricing.

7.1 Sequential timing

If platform A is focal and covers the market, its profit is (remind that $\beta_{21} \geq \beta_{12}$):

$$\bar{\pi}^A = \delta_1 m_2 + \delta_2 m_2 - (\beta_{21} - \beta_{12}) m_1 m_2.$$

The question here is whether platform A prefers to cover the market, or to let platform B sell to some side.

To fix ideas, suppose that platform A gives up on side 2 and serves only side 1. From proposition (1), platform A cannot expect more than $\delta_1 m_1 - \beta_{12} m_1 m_2$. But this bound may not be attainable. The reason is that proposition (1) didn't consider all possible strategies that platform B can use. Indeed, to sell to side 1, platform B could just undercut (by an amount $\delta_1 - \beta_{12} m_2$) the price p_1^A charged by platform A to this side keeping the same price for side 2, which yields the above bound on platform A's profit. But an alternative strategy for platform B is to exploit the bandwagon effect associated to the success of attracting side 1. To do so, platform B increases the price it charges to side 2 by an amount equal to the value ($\beta_{21} m_1$) attached by side 2 to the participation of side 1. Obviously this raises again the issue of coordination failure, as at the new price side 2 could prefer to join platform A. Platform B must then secure side 1 by undercutting the price set by platform A by a much larger amount than before, indeed by an amount $\delta_1 + \beta_{12} m_2$. If side 2 values the externality much more than side 1, this strategy is the most profitable for platform B, and prevent it requires that platform A reduced its price below the level of proposition 1. As a result we obtain:

Lemma 3 *When $J = 2$ and platform A sells to side j only, its profit is* $\pi_j^A = \delta_j m_j - \max\{\beta_{jl}, \beta_{lj} - \beta_{jl}\} m_l m_j$.

Proof. See Appendix. ■

Thus, if $\beta_{21} > 2\beta_{12}$, irrespective of whether platform A covers the whole market or sell to 1 only, the main limitation on its strategy is always the indirect profit that platform B can obtain on side 2 by attracting side 1 with a subsidy.

Intuition suggests that, when sharing the market and selling only to side j, platform A should leave to platform B the opportunity to gain a high profit on side l alone and thus set a high price p_l^A. The proof of the lemma shows that the optimal price on side l is then $p_l^A = u_l^A + \beta_{lj} m_j$: it lets platform B capturing its maximal profit $u_l^B m_l$, with a price $p_l^B = u_l^B$, while maintaining the willingness of side 2 to join platform A along with side 1. An interpretation is that platform A induces cooperation by platform B with some type of "stick and carrot" strategy which can be stated as: "you can serve one side with high profit, but don't try to be aggressive on the other side, or I will take it back".

Let us now turn to the equilibrium analysis, which amounts to the comparison of the three levels of profit, π^A, π_1^A and π_2^A. The equilibrium is then as follows:

Both sides join platform A if

$$\delta_1 m_1 + \delta_2 m_2 \geq (\beta_{21} - \beta_{12}) m_1 m_2,$$
$$\delta_1 \geq -\beta_{12} m_2 \text{ and } \delta_2 \geq \inf\{0, \beta_{21} - 2\beta_{12}\} m_1;$$

Side 1 joins platform A and side 2 joins platform B if:

$$\delta_1 \geq \max\{\beta_{12}, \beta_{21} - \beta_{12}\} m_2 \text{ and } \delta_2 < \inf\{0, \beta_{21} - 2\beta_{12}\} m_1;$$

Side 2 joins platform A and side 1 joins platform B if:

$$\delta_1 < -\beta_{12} m_2 \text{ and } \delta_2 \geq \beta_{21} m_1.$$

In the remaining cases, all sides joins platform B.

The market share of platform A is represented in the space (δ_1, δ_2), for the case $\beta_{12} < \beta_{21} < 2\beta_{12}$ in Figure 1. Straight lines delineate the ranges where platform A covers the market, or sells only to one side, or to none. The dotted lines delineate the same range but for the allocation that maximizes total surplus.

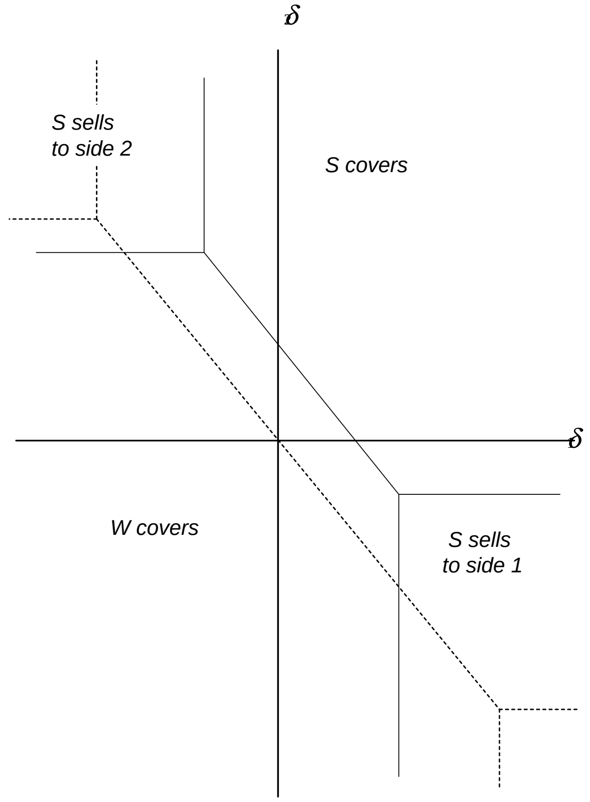

Figure 1

The first point is that platform A may be unable to sell, even in cases where both δ_1 and δ_2 are positive. Even a platform that offer better qualities and dominates the coordination process in stage 2 may not be able to generate a positive profit on the market. While this is exacerbated by the sequential timing and the second mover advantage of platform B, we will see that this is deeper and the same conclusion holds under a simultaneous timing.

The second point is that the profit loss due to multi-sided externalities is always higher when platform A serves only one side than when it serves two sides. As a consequence, if platform A sells, it will not give up on a side for which it offers a higher intrinsic utility:

$$\text{for } \ l \neq j, \ \delta_l \geq 0 \Rightarrow \bar{\pi}^A \geq \pi_j^A.$$

The motive for abandoning one side is thus grounded into a quality advantage of platform B. In particular market sharing can only occur when there is a sufficient degree of horizontal differentiation ($\delta_j > 0 > \delta_l$).

Compared with the case where there is no externality (where platform A serves side j if $\delta_j \geq 0$), the range of parameters for which the market is shared between the two platforms is reduced. Without surprise, network effects tends to generate some tipping, extending the range of market covering by one platform. But this can translate into an increase or a decrease of the focal platform's market share. When platform A has a large quality differential on some side (δ_j high) and a rather small disadvantage on the other (δ_l negative but small in absolute value), platform A will cover the market, in order to secure side j. On the other hand, with strong horizontal differentiation, platform A is not able to obtain positive profits and leaves the whole market to platform B.

It is also fairly easy to see in the two-sided market case that when the value attached to the externalities by one side moves closer to the value attached by the other side, the focal leader's profit raises (weakly) and platform A is more likely to serve both sides. Thus, network effects are the more detrimental to platform A's profits and market shares, the more asymmetric they are.

7.2 Simultaneous pricing with a focal platform

The Stackelberg profits derived above are upper bounds on the profits that the platforms can achieve in the simultaneous pricing game. With network effects however, these profits may not be reached in a simultaneous pricing game. This section discusses equilibria for the case where platforms A and B set they prices simultaneously. Formal derivation is in appendix.

Consider an equilibrium where platform A covers the market. From the analysis of platform $B's$ best response in the sequential game, platform $A's$ prices must verify:

$$p_j^A \leq \delta_j + \beta_{jl} m_l, \tag{11}$$

$$0 \leq p_1^A m_1 + p_2^A m_2 \leq \bar{\pi}^A; \tag{12}$$

along with the condition for the market allocation:

$$p_j^B \geq p_j^A - \delta_j - \beta_{jl} m_l. \tag{13}$$

The question is now whether one can set prices for platform B in such a way that A doesn't deviate. This requires binding condition 13 on both sides. When platform A's prices are nonnegative, this is sufficient to obtain an equilibrium. However, some price may need to below cost, and in this case platform A may be tempted to serve only the other side. One way to look at this issue is to analyze the opportunity cost of not selling to side j served at a loss. platform A looses the (negative) income $p_j m_j = (\delta_j + \beta_{jl} m_l) m_j$, and since side j joins platform B instead of platform A, the valuation of platform A by the other side decreases by an amount equal to the value of the externality $\beta_{lj} m_j$, while the valuation of platform B increases by the same amount. This forces platform A to decrease the price for side l by twice this amount. The net opportunity cost for platform A of not selling to side j is thus $\left(\delta_j + \beta_{jl} m_l + 2\beta_{lj} m_l\right) m_j$. Platform A prefers to cover the market rather than to sell to l only when this opportunity cost is positive.

Indeed an equilibrium with platform A selling to both sides exists if $\bar{\pi}^A > 0$ and $\left(\delta_j + \beta_{jl} m_l + 2\beta_{lj} m_l\right) m_j$ for both sides.

A similar reasoning allows to define for the case where platform B covers the market.[11] From proposition 4, the maximal profit that platform B can expect is

$$\bar{\pi}^B = -\delta_1 m_2 - \delta_2 m_2 - (\beta_{21} + \beta_{12})\, m_1 m_2.$$

An equilibrium where platform B covers the market exists if $\bar{\pi}^B \geq 0$ and the opportunity cost for platform B of not selling to side j is positive which yields (differences arise because A is focal).

$$\delta_1 - (\beta_{12} + 2\beta_{21})\, m_2 \leq 0 \textit{ and } \delta_2 - \beta_{21} m_1 \leq 0.$$

The analysis of market sharing situations follows the same lines. Suppose that platform A serves side j and platform B serves side l. Then it must be profitable for platform A not to sell to side l. The relevant criteria is again the opportunity cost for platform A of not selling to side l which must be nonpositive. There exists an equilibrium with simultaneous pricing where platform A sells to side j and platform B sells to side $l \neq j$ if and only if:

$$if\ j = 1 : \delta_1 + (\beta_{12} - 2\beta_{21})\, m_2 \geq 0 \textit{ and } -\delta_2 - (\beta_{21} + 2\beta_{12})\, m_1 \geq 0\ ;$$
$$if\ j = 2 : -\delta_1 - (\beta_{12} + 2\beta_{21})\, m_2 \geq 0 \textit{ and } \delta_2 - \beta_{21} m_1 \geq 0.$$

The global equilibrium configuration is depicted on Figure 2. Dotted lines delineate the range of market sharing in the sequential game.

[11] With $\beta_{12} = \beta_{21}$, the results applies by labeling side 1 the side that maximizes $\delta_j m_j$.

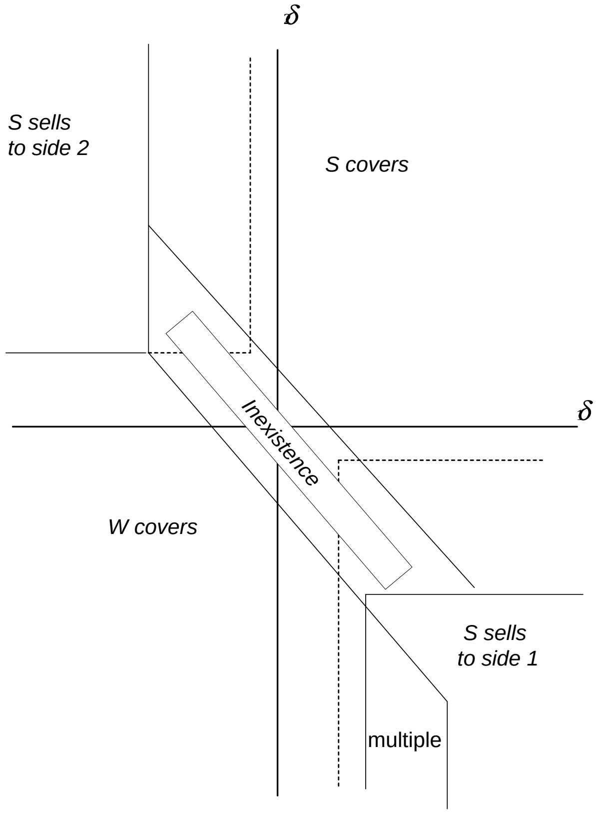

Figure 2

Compared to the sequential game, there is less market sharing. In particular platform A covers more often. Notice also that platform A obtains less than its Stackelberg profit when it sells to side 1 only, and when it covers the market while it were sharing the market as a leader.

7.3 Welfare

Let us compare the equilibrium configuration with the efficient allocation. Market sharing maximizes total surplus whenever $\delta_1\delta_2 < 0$ and $(\beta_{12} + \beta_{21})m_1m_2 < \min\{|\delta_1|\, m_1, |\delta_2|\, m_2\}$, which corresponds to a pattern of strong horizontal differentiation with small network effects. Otherwise platform A should cover the market if $\delta_1 m_1 + \delta_2 m_2 > 0$, while platform 2 should cover the market if $\delta_1 m_1 + \delta_2 m_2 < 0$. The dotted lines in Figures 1 show the delimitations of the various range of quality differentials. There can be excessive sales by platform A which can take the form of inefficient market covering by platform A or inefficient market sharing.

More surprisingly, given the fact that platform A benefits from favorable expectations, there is also the possibility of excessive sales by the non focal platform B both in the sequential game and in the simultaneous game. It occurs when it would be optimal that platform A covers the market, and nevertheless it cannot sell at all or sell only to one side. Price discrimination may allow platform B to generate a profit that outweigh the quality differentials in favor of an efficient leader.

In the sequential game, platform A may prefer to let its competitor sell to one side, although it would be more efficient that it cover the market. The motive for platform A is that this weakens competition on the other side.

To illustrate this effect, set $m_1 = m_2 = 1$, and suppose that side 1 values only platform A but not network effects or platform B: $u_1^B = 0$, $\beta_{12} = 0$. Suppose that side 2 has no value for platform A: $u_2^A = 0$. Assuming that $u_1^A > \beta_{21} > u_2^B$, the efficient allocation has both sides at platform A. However the optimal strategy for A is to give up on side 2 by setting a price β_{21} on side 2. This allows to charge $u_1^A - \beta_{21}$ to side 1, while platform B charges u_2^B to side 2. This conclusion holds despite the fact that side 2 would be willing to pay a larger amount (β_{21}) to join platform A than the price charged by platform B.

To see why this is the case, consider what occurs if platform A lower its price p_2^A to zero. Platform B can not sell to side 2 alone at a positive price and has no other option than to attract both sides or none. But with a divide-and-conquer strategy subsidizing side 1, platform B can generate profit: $\left(p_1^A - u_1^A\right) + \left(u_2^B + \beta_{21}\right)$. To avoid that platform A would need to reduce p_1^A below $u_1^A - \beta_{21} - u_2^B$. This means reducing the price on side 1 by an amount u_2^B. Thus even at a price that would generate almost no profit on side 2, platform A would have to reduce its price p_1^A by a finite amount compared to the level it can charge with market sharing.

In both cases, market sharing or market covering by platform A, selling to side 1 allows platform B to create and capture some extra value on side 2. Thus, despite $u_1^B = 0$, side 1 has some value for platform B, who is willing to compete for it. But in the market sharing case, this value is just the externality for side 2, while when platform A covers the market, it is the sum of the externality and the intrinsic value, since both come together. By giving away side 2, platform A allows its competitor to extract the intrinsic value without having to fight for the other side, reducing the value attached by platform B to the participation of side 1 to its platform. Thus platform A weakens competition.

8 Discussions and extensions

8.1 Platform interconnection

Compare the result with a situation without the multi-sided effects. This occurs when platforms interconnect and allow clients to interact with members of the both platforms. An example of this strategy is provided by the alliances between European financial market places.

Suppose that users of interconnected platforms benefit from externalities with all users, irrespective of their choice, while the components u_j^k are preserved. For any individual, the comparison between the two platforms reduces to the comparison between the goods', thus of $u_j^A - p_j^A$ and $u_j^B - p_j^B$. The competition game reduces to a standard Bertrand type game, where each side constitutes a specific market. Platform A sells to side j if and only if $u_j^A \geq u_j^B$. The profit that the interconnected platform A is then $\sum_{j\in\mathcal{J}} \max\{\delta_j, 0\}m_j$. It is then immediate that:

Corollary 2 *The profit of platform A is larger when platforms are interconnected.*

Proof. A obtains less than $\max_{\mathcal{K}} \left\{\sum_{j\in\mathcal{K}} \delta_j m_j\right\} = \sum_{j\in\mathcal{J}} \max\{\delta_j, 0\} m_j$. ■

While externalities raises the incentive to reach a large population, platform A is not able to appropriate the efficiency gains associated with these externalities. Moreover, it may have to sell to some sides despite the fact that $\delta_j < 0$, because there is an opportunity cost of letting them joining the competitor. Choosing to interconnect may then be one way to avoid these problems.

Notice that, by using assumption 2, we derived bounds for platform $A's$ profits that hold for any selection of the market allocation of users. Moreover, it is fairly easy to see from the proof that these bounds are valid for the case where platforms set prices simultaneously. It follows that when platforms set prices simultaneously, the profit of each platform in a pure strategy equilibrium is smaller than the profit it obtains when platforms are interconnected. This conclusion doesn't rely on assumption 2.

These conclusions rely however on the absence of within-sides network effects. Indeed positive network effects between members of the same side may refrain platform A from interconnecting with platform B, as it can obtain an extra profit on each side served, that vanishes under interconnection (see section 8.3). The profit that platform A can obtain by cornering the market is now augmented by the total value of within-side network effects (see below). Thus platform $A's$ profit may be above the interconnected profit.

8.2 Strategic degradation of quality for targeted customers

One of the general principle that emerges is that head-to-head competition dissipates profit. One way to escape from such situation of intense competition is to achieve enough horizontal differentiation. In the present model, this means shifting from a market covering equilibrium to a more peaceful market sharing situation. When a platform controls the quality of the good at the individual level, it can reach this objective by degrading quality for some customers.[12]

With simultaneous pricing, there are cases where platform A would prefer to share the market, but cannot because at any price that allows platform B to sell, platform A has the ability to attract the whole market. Degrading the quality on one side is one way to induce the more profitable market sharing situation.

To illustrate this phenomenon, suppose that platform A covers the market in equilibrium with prices $p_1^A = \delta_1 + \beta_{12} m_2 < 0$ and $p_2^A = \delta_2 - \beta_{21} m_1 m_2 > 0$ (which is compatible with equilibrium conditions): platform A sells to side 1 at a loss to protect its market. Consider what happens if platform A can at no cost and publicly reduce u_1^A before the price game (holding u_2^A constant) up to a point where the new quality differential falls short of $-(\beta_{12} + 2\beta_{21}) m_2$. Then the new equilibrium involves market sharing: platform A sells to side 2 only with profit $(\delta_2 - \beta_{21} m_1) m_2 > \bar{\pi}^A$. It is thus profitable to do so.

The point here is that platform A would like to commit not to compete on side 1, as an alternative to being 'forced' to include side 1 in its platform despite a competitive hedge in favor of platform B on this side. A targeted degradation of quality is one way to achieve such a commitment (a "puppy dog" strategy). The same phenomenon may hold for platform B as well.

More generally when a platform can choose the technology and affect perceived qualities, and when it can't gain a large quality advantage on both sides, it will have incentives to shift its technological choices toward the preferred technology of one side and the least preferred technology of the other side. This may generate inefficiencies in technological choices and even result in the choice of a dominated technology.

8.3 Within-side cross-effects

In the paper, it is assumed that there is no externality between members of the same side. One can allow for such an externality by setting $\beta_{jj} \geq 0$. It is straightforward to see that lemma 2 still holds, so that the concept of favorable expectations is well defined. Then the value of the within-side externalities β_{jj} can be attributed to the platform benefitting from favorable expectations :

Lemma 4 *Consider a market with $\beta_{jj} \geq 0$ for all j, and alternative market that differs only by the facts that for all j, $\beta_{jj} = 0$ while the value of platform*

[12] Both the motive for quality reduction and the way it is achieved differ substantially from models of damaged goods and screening as Denekere and McAfee (1996) and Hahn (2000).

$A's$ good is $u_j^A + \beta_{jj} m_j$. Then the two models generate the same maximal allocation rule $\mathcal{D}^A(P^A, P^B)$.

Proof. See Appendix. ■

Thus the market allocation at given prices can be derived "as if" the intrinsic value differential were $\hat{\delta}_j = \delta_j + \beta_{jj} m_j$ with no network effect within sides. The reason is that the favorable expectation selection criteria implies that a side can be treated as an homogenous entity (all its members join platform A or all its members join platform B) that accounts for the within-side externality only when considering joining platform A. Apart for the welfare analysis, all the results of the paper extend to this case replacing δ_j by $\hat{\delta}_j$.

8.4 Inexistence of a pure strategy equilibrium

With simultaneous pricing, a pure strategy equilibrium fails to exist when the platforms offer goods of similar characteristics. The natural question is whether this is due to the assumption 2 (favorable expectations).

Clearly, relaxing assumption 2 allows to extend the set of equilibria. But, although the range of existence increases when all market allocations are considered, the inexistence problem remains if platforms are not differentiated enough and there is some asymmetry in externalities. To see that, notice that in any pure strategy equilibrium of a simultaneous pricing game, the profit of a platform is bounded above by the Stackelberg profit it can derive when it is "focal". Otherwise there would be a strategy of the rival that would raises its profits by extending it market share. But if the differences between the intrinsic values of the goods are small, no platform can profitably sell as a Stackelberg leader. In particular, with two sides, a pure strategy equilibrium fails to exist if $\sum_j \left| u_j^A - u_j^B \right| m_j < \min\{\beta_{12}, \beta_{21} - \beta_{12}\} m_1 m_2$. The inexistence issue thus appears to be a robust phenomenon for network that are not differentiated.

9 Conclusion

By focusing on the effect of cross-group externalities the paper derives striking results that apply to multi-sided market and network competition with price-discrimination. Probably the most striking feature is that price-discrimination may reduce considerably the level of barriers to entry in network industries, and even be the source of inefficient entry. The analysis deserves to be extended in several directions.

For one thing, sides were supposed to be homogeneous, which favors tipping. Extension of multi-sided markets to heterogenous sides may help to understand market sharing configurations and differentiation strategies.

Most importantly, two dimensions are missing in the model: time and risk.

Divide&conquer strategies may require selling below cost, which may be very risky in environments with demand uncertainty, since the platform may fail to recover the subsidy on others sides. Accounting for risk should reduces the effectiveness as competitive tools.

The analysis should clearly be extended to account for dynamic considerations. Dynamics may allow to identify more precisely the pattern of cross-subsidy since sides may have to join a platform at different dates. A dynamic divide&conquer strategy may require to run negative cash-flows for sometime, and thus need financial resources. With an imperfect capital market, this means that platforms having access to a deep-pocket should have a strong advantage over financially constrained platform. The analysis of platform competition should then devote special care to financial aspects.

The strategies may also require negative prices and these may be hard to target. They may attract clients who just grab the subsidy and are not interested by the platform's services. Moreover these clients may generate negative externalities on other members of the platform. The strategy would then be to use in-kind subsidies targeted at attracting only active users of the platform, but these are usually costly while the model assumes no social cost associated with subsidies.

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A Appendix

Proof of lemma 1. Denote $\Omega^{\sigma}(\mathcal{J}) = \sum_{j\in\mathcal{J}}\left(\sum_{\sigma(l)>\sigma(j)}(\beta_{lj}-\beta_{jl})m_l m_j\right)$. For any order σ on $\mathcal{J}$ there is an exact reverse ordering $\hat{\sigma}$ and it verifies $\Omega^{\sigma}(\mathcal{J}) = -\Omega^{\hat{\sigma}}(\mathcal{J})$. Hence the maximum over all permutations is non-negative. The maximum is zero iff $\Omega^{\sigma}(\mathcal{J}) = 0$ for all σ. Suppose this is the case. Fix j, l. Consider a permutation σ such that $\sigma(j) = J-1$ and $\sigma(l) = J$, and $\bar{\sigma}$ that coincides with σ except that $\bar{\sigma}(j) = J$ and $\bar{\sigma}(l) = J-1$. Then $0 = \Omega^{\sigma}(\mathcal{J}) = \Omega^{\hat{\sigma}}(\mathcal{J}) + 2(\beta_{lj}-\beta_{jl})m_j m_l$ which implies that $\beta_{jl} = \beta_{lj}$. The same proof holds for $\mathcal{L} \subset \mathcal{J}$.

Moreover suppose σ and σ' are the argument in the maximum for $\mathcal{L}$ and $\mathcal{L}'$. Denote $\sigma\sigma'$ the order that ranks 2 sides of $\mathcal{L}$ according to σ, 2 sides of $\mathcal{L}'$ according to σ', and all sides $\mathcal{L}$ before all sides of $\mathcal{L}'$.

$$\Omega^{\sigma\sigma'}(\mathcal{L}\cup\mathcal{L}') = \Omega(\mathcal{L}) + \Omega(\mathcal{L}') + \sum_{j\in\mathcal{L}, l\in\mathcal{L}'}(\beta_{lj}-\beta_{jl})m_l m_j$$

Define similarly the order $\sigma'\sigma$ where $\mathcal{L}'$ is ranked before $\mathcal{L}$.

$$\Omega^{\sigma'\sigma}(\mathcal{L}\cup\mathcal{L}') = \Omega(\mathcal{L}) + \Omega(\mathcal{L}') + \sum_{j\in\mathcal{L}', l\in\mathcal{L}}(\beta_{lj}-\beta_{jl})m_l m_j$$

Since $\sum_{j\in\mathcal{L}', l\in\mathcal{L}}(\beta_{lj}-\beta_{jl})m_l m_j = -\sum_{j\in\mathcal{L}, l\in\mathcal{L}'}(\beta_{lj}-\beta_{jl})m_l m_j$, one of the two is non-negative. Thus $\Omega(\mathcal{L}\cup\mathcal{L}') \geq \Omega(\mathcal{L}) + \Omega(\mathcal{L}')$ ■

Proof of lemma 2. Consider two $REA : i = 1, 2$, where sides allocate according to ${}^i n_j^k \in [0, m_j]$ and let $\mathcal{K}_i$ be the set of side with ${}^i n_j^k > 0$, and A_i the set of sides with ${}^i n_j^s = m_j$. Then

$$\begin{aligned}
j &\in \mathcal{K}_i \Rightarrow u_j^k + \sum_{l\in\mathcal{K}_i}\beta_{jl}\left({}^i n_l^k\right) - p_j^k \geq \max\{u_j^s + \sum_{l\notin\mathcal{A}_i}\beta_{jl}\left({}^i n_l^k\right) + \sum_{l\in\mathcal{A}_i}\beta_{jl}m_l - p_j^s, 0\} \\
&\Rightarrow u_j^k + \sum_{l\in\mathcal{K}_i}\beta_{jl}m_l - p_j^k \geq \max\{u_j^s + \sum_{l\in\mathcal{A}_i}\beta_{jl}m_l - p_j^s, 0\} \\
j &\in \mathcal{J}\setminus(\mathcal{A}_i\cup\mathcal{K}_i) \Rightarrow 0 \geq \max\left\{u_j^k + \sum_{l\in\mathcal{K}_i}\beta_{jl}\left({}^i n_l^k\right) - p_j^k, u_j^s + \sum_{l\notin\mathcal{A}_i}\beta_{jl}\left({}^i n_l^k\right) + \sum_{l\in\mathcal{A}_i}\beta_{jl}m_l - p_j^s\right\} \\
&\Rightarrow \max\left\{u_j^k + \sum_{l\in\mathcal{K}_i}\beta_{jl}^i m_l - p_j^k, 0\right\} \geq u_j^s + \sum_{l\in\mathcal{A}_i}\beta_{jl}m_l - p_j^s
\end{aligned}$$

Now suppose that we impose that no member of in $\mathcal{J}\setminus(\mathcal{A}_1\cap\mathcal{A}_2)$ join s, and that all members of sides in $\mathcal{K}_1\cup\mathcal{K}_2$ join k. Since for the sides in $\mathcal{J}\setminus(\mathcal{A}_1\cap\mathcal{A}_2)$:

$$\max\{u_j^k + \sum_{l\in\mathcal{K}_1\cup\mathcal{K}_2}\beta_{jl}m_l - p_j^k, 0\} \geq u_j^s + \sum_{l\in\mathcal{A}_1\cap\mathcal{A}_2}\beta_{jl}m_l - p_j^s,$$

the minimal benefit that a member of a side in $\mathcal{J}\backslash \mathrm{A}_1 \cap \mathrm{A}_2$ can obtain when not joining s is larger than the maximal benefit it can gain when joining s. Therefore the optimal strategy is either to join k or not to join at all. Moreover for a user in $\mathcal{K}_1 \cup \mathcal{K}_2$, $u_j^k + \sum_{l \in \mathcal{K}_1 \cup \mathcal{K}_2} \beta_{jl} m_l - p_j^k \geq 0$, implying that the optimal strategy for this user is indeed to join k. Now take an equilibrium of the game with strategy spaces restricted as described. This is an equilibrium of the allocation game with the property that $n_l^k \in \{0,1\}$, $\mathcal{K}_1 \cup \mathcal{K}_2 \subset \mathcal{K}$ and $\mathrm{A} \subset \mathrm{A}_1 \cap \mathrm{A}_2$.

Taking a maximal element completes the proof. ■

Proof of propositions 2 and 3. Suppose that A wants to sell to $\mathcal{K}$ and A sets $p_j^A \leq u_j^A$ for $j \in \mathcal{K}$ and $p_j^A > u_j^A + \sum_l \beta_{jl} m_l$ for $j \notin \mathcal{K}$. Suppose that wants to sell to $\mathcal{L} = \{1, 2, ..., L\}$. Since A is focal, B cannot sell unless one side is willing to join alone since otherwise there would exists an allocation where no side joins B. Let let j for this side and set $\sigma(j) = 1$: it is such that

$$u_j^B - p_j^B \geq \max \left\{ 0, u_j^A + \sum_{l \in \mathcal{K}} \beta_{jl} m_l - p_j^A \right\}.$$

By the same reasoning, there must be some allocation willing to join if only side $\sigma^{-1}(1)$ joins. For this side we set $\sigma(j) = 2$. Recursively, if the strategy is such that all sides l ranked below some level, $l \in \sigma^{-1}(\{1, 2.., r\})$, joins even if other sides don't, attracting an additional side j requires that B sets a price such that

$$u_j^B + \sum_{\sigma(l) < \sigma(j)} \beta_{jl} m_l - p_j^B \geq \max \left\{ 0, u_j^A + \sum_{\substack{\sigma(l) > \sigma(j) \\ l \in \mathcal{K} \cap \mathcal{L}}} \beta_{jl} m_l + \sum_{l \in \mathcal{K} \backslash \mathcal{L}} \beta_{jl} m_l - p_j^A \right\}.$$

Notice that prices of A have been chosen so that the maximum obtains at zero if and only if $j \notin \mathcal{K}$. Using this and computing the sum of inequalities pondered by mass over the set $\mathcal{L}$ we obtain the maximal profit of B :

$$\begin{aligned} \sum_{j \in \mathcal{L}} p_j^B m_j \quad \leq \quad & \sum_{j \in \mathcal{K} \cap \mathcal{L}} p_j^A m_j - \sum_{j \in \mathcal{K} \cap \mathcal{L}} \delta_j m_j + \sum_{j \in \mathcal{L} \backslash \mathcal{K}} u_j^B m_j \\ & + \max_{\sigma} \left\{ \sum_{\substack{\sigma(l) < \sigma(j) \\ j, l \in \mathcal{L}}} \beta_{jl} m_j m_l - \sum_{\substack{\sigma(l) > \sigma(j) \\ j, l \in \mathcal{K} \cap \mathcal{L}}} \beta_{jl} m_j m_l \right\} - \sum_{j \in \mathcal{K} \cap \mathcal{L}} \sum_{l \in \mathcal{K} \backslash \mathcal{L}} \beta_{jl} m_j m_l \end{aligned}$$

The first immediate point is that side $j \notin \mathcal{K}$ does not constrain the profit as it appears only in the term $\sum_{j \in \mathcal{L} \backslash \mathcal{K}} u_j^B m_j + \sum_{\substack{\sigma(l) < \sigma(j) \\ j \in \mathcal{L}}} \beta_{jl} m_j m_l$ which has positive coefficient. Thus platform B sells to all sides outside $\mathcal{K} : \mathcal{J}\backslash\mathcal{K} \subset \mathcal{L}$; in particular $\sum_{j\in\mathcal{L}\backslash\mathcal{K}} u_j^B m_j = \sum_{j\notin\mathcal{K}} u_j^B m_j$

Now B will choose to sell only to $\mathcal{J}\backslash\mathcal{K}$ if adding sides can only reduce its profits compared to

$$\pi = \sum_{j\notin\mathcal{K}} u_j^B m_j + \max_{\sigma} \sum_{\substack{\sigma(l)<\sigma(j)\\ j,l\notin\mathcal{K}}} \beta_{jl} m_j m_l$$

This holds if for all $\mathcal{L}$ such that $\mathcal{J}\backslash\mathcal{K} \subset \mathcal{L}$:

$$\sum_{j\in\mathcal{K}\cap\mathcal{L}} p_j^A m_j \leq \sum_{j\in\mathcal{K}\cap\mathcal{L}} \delta_j m_j - \max_{\sigma}\left(\sum_{\substack{\sigma(l)<\sigma(j)\\ j,l\in\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{\substack{\sigma(l)>\sigma(j)\\ j,l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_j m_l\right) \tag{14}$$

$$+\max_{\sigma}\left(\sum_{\substack{\sigma(l)<\sigma(j)\\ j,l\notin\mathcal{K}}} \beta_{jl} m_j m_l\right) + \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{l\in\mathcal{K}\backslash\mathcal{L}} \beta_{jl} m_j m_l \tag{15}$$

For the case $\mathcal{L} = \mathcal{J}$, we have

$$\sum_{j\in\mathcal{K}} p_j^A m_j \leq \sum_{j\in\mathcal{K}} \delta_j m_j - \max_{\sigma}\left(\sum_{\substack{\sigma(l)<\sigma(j)\\ j,l\in\mathcal{J}}} \beta_{jl} m_j m_l - \sum_{\substack{\sigma(l)>\sigma(j)\\ j,l\in\mathcal{K}}} \beta_{jl} m_j m_l\right) \tag{16}$$

$$+\max_{\sigma} \sum_{\substack{\sigma(l)<\sigma(j)\\ j,l\notin\mathcal{K}}} \beta_{jl} m_j m_l \tag{17}$$

Let $s(.)$ be the order that maximizes $\sum_{\sigma(l)<\sigma(j);\ j,l\in\mathcal{J}} \beta_{jl} m_j m_l - \sum_{\sigma(l)>\sigma(j);\ j,l\in\mathcal{K}} \beta_{jl} m_j m_l$. Let A then set price

$$p_j^A = \delta_j - \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{J}}} \beta_{jl} m_l + \sum_{\substack{s(l)>s(j)\\ l\in\mathcal{K}}} \beta_{jl} m_l - \sum_{\substack{s(l)>s(j)\\ l\notin\mathcal{K}}} \beta_{lj} m_l + x_j$$

where $x_j > 0$ and

$$\sum_{j\in\mathcal{K}} x_j m_j = \max_{\sigma} \sum_{\substack{\sigma(l)<\sigma(j)\\ j,l\notin\mathcal{K}}} \beta_{jl} m_j m_l - \sum_{\substack{s(l)<s(j)\\ j,l\notin\mathcal{K}}} \beta_{jl} m_j m_l. \tag{18}$$

I claim that at these prices A sells to sides in K. First summing the equalities pondered by m_j and using $\sum_{j\in\mathcal{K}} \sum_{\substack{s(l)>s(j)\\ l\notin\mathcal{K}}} \beta_{lj} m = \sum_{j\notin\mathcal{K}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{K}}} \beta_{jl} m_l$, one can easily show that equation (16) holds with equality.

Second for $\mathcal{L}$ such that $\mathcal{J}\backslash\mathcal{K}\subset\mathcal{L}$:

$$\begin{aligned}\sum_{j\in\mathcal{K}\cap\mathcal{L}} p_j^A m_j &= \sum_{j\in\mathcal{K}\cap\mathcal{L}} \delta_j m_j - \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{J}}} \beta_{jl} m_j m_l \\ &+ \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{\substack{s(l)>s(j)\\ l\in\mathcal{K}}} \beta_{jl} m_j m_l - \sum_{j\notin\mathcal{K}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_l + \sum_{j\in\mathcal{K}\cap\mathcal{L}} x_j m_j\end{aligned}$$

Suppose equation (14) is violated, than one can find an order $\hat{\sigma}$ such that

$$\begin{aligned}&\sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{J}}} \beta_{jl} m_j m_l - \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{\substack{s(l)>s(j)\\ l\in\mathcal{K}}} \beta_{jl} m_j m_l + \sum_{j\notin\mathcal{K}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_l - \sum_{j\in\mathcal{K}\cap\mathcal{L}} x_j m_j \\ < & \sum_{\substack{\hat{\sigma}(l)<\hat{\sigma}(j)\\ j,l\in\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{\substack{\hat{\sigma}(l)>\hat{\sigma}(j)\\ j,l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_j m_l - \max_{\sigma}\left(\sum_{\substack{\sigma(l)<\sigma(j)\\ j,l\notin\mathcal{K}}} \beta_{jl} m_j m_l\right) - \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{l\in\mathcal{K}\backslash\mathcal{L}} \beta_{jl} m_j m\end{aligned}$$

Using (18), we have:

$$\begin{aligned}& \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{J}}} \beta_{jl} m_j m_l - \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{\substack{s(l)>s(j)\\ l\in\mathcal{K}}} \beta_{jl} m_j m_l + \sum_{j\notin\mathcal{K}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_l \\ & + \sum_{\substack{s(l)<s(j)\\ j,l\notin\mathcal{K}}} \beta_{jl} m_j m_l + \sum_{j\in\mathcal{K}\backslash\mathcal{L}} x_j m_j \\ < & \sum_{\substack{\hat{\sigma}(l)<\hat{\sigma}(j)\\ j,l\in\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{\substack{\hat{\sigma}(l)>\hat{\sigma}(j)\\ j,l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{l\in\mathcal{K}\backslash\mathcal{L}} \beta_{jl} m_j m\end{aligned}$$

which we can write:

$$\begin{aligned}&\sum_{\substack{s(l)<s(j)\\ j,l\in\mathcal{J}}} \beta_{jl} m_j m_l - \sum_{\substack{s(l)>s(j)\\ j,l\in\mathcal{K}}} \beta_{jl} m_j m_l < \sum_{\substack{\hat{\sigma}(l)<\hat{\sigma}(j)\\ j,l\in\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{\substack{\hat{\sigma}(l)>\hat{\sigma}(j)\\ j,l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{l\in\mathcal{K}\backslash\mathcal{L}} \beta_{jl} m_j m_l \\ &+ \sum_{j\notin\mathcal{K}\cap\mathcal{L}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{J}}} \beta_{jl} m_j m_l - \sum_{j\in\mathcal{K}\backslash\mathcal{L}} \sum_{\substack{s(l)>s(j)\\ l\in\mathcal{K}}} \beta_{jl} m_j m_l - \sum_{j\notin\mathcal{K}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_l - \sum_{\substack{s(l)<s(j)\\ j,l\notin\mathcal{K}}} \beta_{jl} m_j m_l - \sum_{j\in\mathcal{K}\backslash\mathcal{L}} x_j m_j\end{aligned}$$

Now define the order $\bar{\sigma}$ by $\bar{\sigma}(l)<\bar{\sigma}(j)$ if

$$\begin{aligned}\bar{\sigma}(l) &< \bar{\sigma}(j)\ if\ l\in\mathcal{L}\text{ and }j\notin\mathcal{L}, \\ \bar{\sigma}(l) &< \bar{\sigma}(j) \Longleftrightarrow \hat{\sigma}(l)<\hat{\sigma}(j)\ if\ j\in\mathcal{L}\ and\ l\in\mathcal{L}, \\ \bar{\sigma}(l) &< \bar{\sigma}(j) \Longleftrightarrow s(l)<s(j)\ \ if\ j\notin\mathcal{L}\ and\ l\notin\mathcal{L}.\end{aligned}$$

Then using $\mathcal{J} = \mathcal{L} \cup \mathcal{K} \backslash \mathcal{L}$

$$\sum_{\substack{\bar{\sigma}(l)<\bar{\sigma}(j)\\ j,l\in\mathcal{J}}} \beta_{jl} m_j m_l - \sum_{\substack{\bar{\sigma}(l)>\bar{\sigma}(j)\\ j,l\in\mathcal{K}}} \beta_{jl} m_j m_l =$$

$$\sum_{\substack{\hat{\sigma}(l)<\hat{\sigma}(j)\\ j,l\in\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{\substack{\hat{\sigma}(l)>\hat{\sigma}(j)\\ j,l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{j\in\mathcal{K}\cap\mathcal{L}} \sum_{l\in\mathcal{K}\backslash\mathcal{L}} \beta_{jl} m_j m_l$$

$$+ \sum_{j\in\mathcal{K}\backslash\mathcal{L}} \sum_{l\in\mathcal{L}} \beta_{jl} m_j m_l + \sum_{\substack{s(l)<s(j)\\ j,l\in\mathcal{K}\backslash\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{\substack{s(l)>s(j)\\ j,l\in\mathcal{K}\backslash\mathcal{L}}} \beta_{jl} m_j m_l$$

Given that

$$\sum_{\substack{s(l)<s(j)\\ j,l\in\mathcal{J}}} \beta_{jl} m_j m_l - \sum_{\substack{s(l)>s(j)\\ j,l\in\mathcal{K}}} \beta_{jl} m_j m_l \geq \sum_{\substack{\bar{\sigma}(l)<\bar{\sigma}(j)\\ j,l\in\mathcal{J}}} \beta_{jl} m_j m_l - \sum_{\substack{\bar{\sigma}(l)>\bar{\sigma}(j)\\ j,l\in\mathcal{K}}} \beta_{jl} m_j m_l,$$

we obtain

$$\sum_{j\notin\mathcal{K}\cap\mathcal{L}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{J}}} \beta_{jl} m_j m_l - \sum_{j\in\mathcal{K}\backslash\mathcal{L}} \sum_{\substack{s(l)>s(j)\\ l\in\mathcal{K}}} \beta_{jl} m_j m_l$$

$$-\sum_{j\notin\mathcal{K}} \sum_{\substack{s(l)<s(j)\\ l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_l - \sum_{\substack{s(l)<s(j)\\ j,l\notin\mathcal{K}}} \beta_{jl} m_j m_l - \sum_{j\in\mathcal{K}\backslash\mathcal{L}} x_j m_j$$

$$> \sum_{j\in\mathcal{K}\backslash\mathcal{L}} \sum_{l\in\mathcal{L}} \beta_{jl} m_j m_l + \sum_{\substack{s(l)<s(j)\\ j,l\in\mathcal{K}\backslash\mathcal{L}}} \beta_{jl} m_j m_l - \sum_{\substack{s(l)>s(j)\\ j,l\in\mathcal{K}\backslash\mathcal{L}}} \beta_{jl} m_j m_l$$

This implies

$$0 > 2 \sum_{j\in\mathcal{K}\backslash\mathcal{L}} \sum_{\substack{s(l)>s(j)\\ l\in\mathcal{K}\cap\mathcal{L}}} \beta_{jl} m_j m_l + \sum_{j\in\mathcal{K}\backslash\mathcal{L}} \sum_{\substack{s(l)>s(j)\\ l\notin\mathcal{K}}} \left(\beta_{jl} - \beta_{lj}\right) m_j m_l + \sum_{j\in\mathcal{K}\backslash\mathcal{L}} x_j m_j \tag{19}$$

Now suppose that $\mathcal{K} = \mathcal{J}$, the RHS of (19) is positive which yields a contradiction. Thus all conditions (14) are verified. The profit is than $\sum_j \delta_j m_j - \Omega\left(\mathcal{J}\right)$.

Suppose that $\beta_{jl} = \beta_{lj}$ for $j \in \mathcal{K}$ and $l \notin \mathcal{K}$, again the RHS of (19) is positive. All conditions (14) are verified and A obtains profits given by

(16) . Moreover

$$\begin{aligned}\sum_{\substack{\sigma(l)<\sigma(j)\\ j,l\in\mathcal{J}}}\beta_{jl}m_jm_l-\sum_{\substack{\sigma(l)>\sigma(j)\\ j,l\in\mathcal{K}}}\beta_{jl}m_jm_l &= \frac{1}{2}\sum_{j,l\in\mathcal{J}}\beta_{jl}m_jm_l-\frac{1}{2}\sum_{j,l\in\mathcal{K}}\beta_{jl}m_jm_l\\ &= \frac{1}{2}\sum_{j,l\notin\mathcal{K}}\beta_{jl}m_jm_l+B(\mathcal{K})\\ \text{and}\quad \sum_{\substack{\sigma(l)<\sigma(j)\\ j,l\notin\mathcal{K}}}\beta_{jl}m_jm_l &= \frac{1}{2}\sum_{j,l\notin\mathcal{K}}\beta_{jl}m_jm_l,\end{aligned}$$

so that the profit is $\sum_{j\in\mathcal{K}}\delta_jm_j-B(\mathcal{K})$. ■

Proof of proposition 4. In equilibrium A sets the price for $j\in\mathcal{K}$ at

$$p_j^A=u_j^A-\max\{u_j^B+\sum_{l\in\mathcal{L}}\beta_{jl}m_l-p_j^B,0\}+\sum_{l\in\mathcal{K}}\beta_{jl}m_l$$

which is the maximal price at which a member of side j is willing to join A when sides $l\in\mathcal{K}$ do. If A decides to attract sides in $\mathcal{H}\subset\mathcal{L}$, it can do so by setting prices for all sides in $\mathcal{K}\cup\mathcal{H}$, including those sides it already serves:

$$\hat{p}_j^A=u_j^A-\max\{u_j^B+\sum_{l\in\mathcal{L}\setminus\mathcal{H}}\beta_{jl}m_l-p_j^B,0\}+\sum_{l\in\mathcal{K}\cup\mathcal{H}}\beta_{jl}m_l$$

The gain in profit is then $\sum_{j\in\mathcal{H}}\hat{p}_j^Am_j+\sum_{j\in\mathcal{K}}(\hat{p}_j^A-p_j^A)m_j$. Notice first that for $j\in\mathcal{K}$, the price differential $\hat{p}_j^A-p_j^A$ is larger than $\sum_{l\in\mathcal{H}}\beta_{jl}m_lm_j$, with equality when $p_j^B\geq u_j^B+\sum_{l\in\mathcal{L}}\beta_{jl}m_l$.. The gain in $A's$ profit is thus minimal when B sets very high prices for these sides, with a lower bound $\sum_{j\in\mathcal{H}}\hat{p}_j^Am_j+\sum_{j\in\mathcal{K}}\sum_{l\in\mathcal{H}}\beta_{jl}m_lm_j$.

If B sells to all sides in $\mathcal{L}$, it sets prices such that for all subset $\mathcal{H}$:

$$\sum_{j\in\mathcal{H}}p_j^Bm_j\leq\sum_{j\in\mathcal{H}}(u_j^B-u_j^A)m_j+\sum_{j\in\mathcal{H}}\sum_{l\in\mathcal{L}\setminus\mathcal{H}}\beta_{jl}m_lm_j-\sum_{j\in\mathcal{H}}\sum_{l\in\mathcal{K}\cup\mathcal{H}}\beta_{jl}m_lm_j-\sum_{j\in\mathcal{K}}\sum_{l\in\mathcal{H}}\beta_{jl}m_lm_j$$

An upper bound on the profit is then obtained for $\mathcal{H}=\mathcal{L}$:

$$\sum_{j\in\mathcal{L}}p_j^Bm_j\leq\sum_{j\in\mathcal{L}}(u_j^B-u_j^A)m_j-\mathcal{B}(\mathcal{L})-\mathcal{B}(\mathcal{K}).$$

If it is a Stackelberg leader, B can obtain this profit by setting a price $p_j^B=u_j^B-u_j^A-\left(\sum_{l=1}^J\beta_{jl}m_l-\sum_{l\in\mathcal{K}}\beta_{lj}m_l\right)$ for all sides within $\mathcal{L}$. ■

Proof of proposition 5.

$$\begin{aligned}
& \sum_{j=1}^{K}\delta_j-\beta_1K(J-K)-\left(\sum_{j=1}^{K^A(\beta_2)}\delta_j-\beta_2K^A(\beta_2)(J-K^A(\beta_2))\right)\\
= & \sum_{j=1}^{K}\delta_j-\beta_2K(J-K)-\left(\sum_{j=1}^{K^A(\beta_2)}\delta_j-\beta_2K^A(\beta_2)(J-K^A(\beta_2))\right)\\
& +(\beta_2-\beta_1)\left(K(J-K)-K^A(\beta_1)(J-K^A(\beta_1))\right)\\
< & \ 0 \ if\ K^A(\beta_1)(J-K^A(\beta_1))<K(J-K).
\end{aligned}$$

Thus $K^A(\beta_1)(J-K^A(\beta_1))\geq K^A(\beta_2)(J-K^A(\beta_2))$ which reduces to

$$\left|K^A(\beta_2)-\frac{J}{2}\right|\geq\left|K^A(\beta_1)-\frac{J}{2}\right|.$$

Suppose that in addition the profit is quasi-concave in K at β_2. Then if $K^A(\beta_2)\geq\frac{1}{2}$, the profit at $\beta=\beta_1$ is increasing on $K<1/2$ and thus $K^A(\beta_1)\geq\frac{J}{2}$ which implies $K^A(\beta_2)\geq K^A(\beta_1)$. The symmetric argument holds for $K^A(\beta_2)\leq\frac{1}{2}$. ■

Proof of lemma 3 . Suppose that A sells only to side 1 (the same proof holds for side 2). Imposing w.l.o.g. $p_2^A\leq u_2^A+\beta_{21}m_1$, the equilibrium price of B is $p_2^B=p_2^A-\delta_2-\beta_{21}m_1$.

B will conform to selling to side 2 only if

$$\begin{aligned}
p_2^Bm_2 &\geq p_1^Am_1+p_2^Am_2-\delta_1m_1-\delta_2m_2+(\beta_{12}-\beta_{21})\,m_1m_2\\
p_2^Bm_2 &\geq p_1^Am_1-\delta_1m_1+u_2^Bm_2-\max\{u_2^A-p_2^A,0\}m_2+(\beta_{21}-\beta_{12})\,m_1m_2
\end{aligned}$$

This reduces to

$$\begin{aligned}
\delta_1m_1-\beta_{12}m_1m_2 &\geq p_1^Am_1\\
\delta_1m_1+\max\{u_2^A-p_2^A,0\}m_2-u_2^Am_2+p_2^Am_2-\beta_{21}m_1-(\beta_{21}-\beta_{12})\,m_1m_2 &\geq p_1^Am_1
\end{aligned}$$

The LHS of the second condition is maximal at the price $p_2^A=u_2^A+\beta_{21}m_1$, which yields:

$$\begin{aligned}
\pi_1^A &= \min\{\delta_1m_1-\beta_{12}m_1m_2,\delta_1m_1-(\beta_{21}-\beta_{12})\,m_1m_2\}\geq p_1^Am_1.\\
p_2^B &= u_2^B.
\end{aligned}$$

This completes the proof the lemma. ■

Equilibrium with $S=2$ and simultanous pricing. First a platform can't cover the market in equilibrium unless $\bar{\pi}^k\geq 0$.

Suppose that $\bar{\pi}^A\geq 0$. Set the prices p_j^A verifying conditions 11 and 12. Then $B's$ best response is not to sell at all provided that $\bar{\pi}^A\geq\pi^A$.

Equilibrium prices of B must verify $p_j^B = p_j^A - \delta_j - \beta_{jl} m_l$, since otherwise either B would sell or A could raise its price. Thus this is an equilibrium if A prefers to cover the market than to sell to one side only. To sell to side j alone, A must set a price $\hat{p}_j^A$ such that $u_j^A + \beta_{jj} m_j - \hat{p}_j^A \geq u_j^B + \beta_{jl} m_l - p_j^B$, which amounts to $\hat{p}_j^A = p_j^A - 2\beta_{jl} m_l$. Thus an equilibrium must verify $p_1^A m_1 + p_2^A m_2 \geq p_j^A m_j - 2\beta_{jl} m_l m_j$ or : $p_l^A \geq -2\beta_{jl} m_j$. Equilibrium prices then exist if $\delta_j + \beta_{jl} m_l \geq -2\beta_{lj} m_l$ for both sides .

Suppose now that $\bar{\pi}^B \geq 0$. Choose prices $p_j^B \leq -\delta_j + \beta_{jl} m_l$ verifying $p_1^W m_1 + p_2^W m_2 \leq \bar{\pi}^B$. A can't obtain a positive profit because its profits is the maximum between $\sum p_j^B m_j - \bar{\pi}^B$, $\left(p_1^B + \delta_1 - \beta_{12} m_2\right) m_1$ and $\left(p_2^B + \delta_2 - \beta_{21} m_1\right) m_2$, and they are all nonpositive. It must then be the case that for one side $p_j^B \leq p_j^A - \delta_j - \beta_{jl} m_l$ and for the other side $p_l^B \leq p_l^A - \delta_l + \beta_{lj} m_j$. Moreover this must hold at equality, since otherwise B could raise its prices. B can't obtain more than π^B by covering the market if $\sum p_j^A m_j - \bar{\pi}^A \leq \pi^B$. The unique prices that are compatible with the fact that B covers the market and obtains at most π^B are thus $p_1^A = p_1^B + \delta_1 + \beta_{12} m_2$, $p_2^A = p_2^B + \delta_2 - \beta_{21} m_1$. Notice that at price $p_1^A - \varepsilon$, side 1 joins B even alone. So $B's$ options if it doesn't cover the market are to sell to side 1 at p_1^B or to side 2 at $p_2^A - \delta_2 - \beta_{21} m_1 = p_2^B - 2\beta_{21} m_1$. It thus chooses to cover only if

$$\pi^B \geq \max\left\{p_1^B m_1, p_2^B m_2 - 2\beta_{21} m_1 m_2\right\}$$

or

$$\begin{aligned} p_2^B &\geq 0 \\ p_1^B &\geq -2\beta_{21} m_2 \end{aligned}$$

We can find such prices whenever $-2\beta_{21} m_2 \leq -\delta_1 + \beta_{12} m_2$, $0 \leq -\delta_2 + \beta_{21} m_1$.

Now assume that A sells to side 1 only. Each users should prefer to stay with its side rather than to join the other side. This yields

$$\begin{aligned} u_1^A + \beta_{11} m_1 - \max\left\{u_1^B + \beta_{12} m_2 - p_1^B, 0\right\} &\geq p_1^A \\ u_2^B - \max\left\{u_2^A + \beta_{22} m_2 + \beta_{21} m_1 - p_2^A, 0\right\} &\geq p_2^B \end{aligned}$$

Clearly to sustain the equilibrium, the best is to put prices p_1^B and p_2^A at their minimal value given profits, which yields:

$$p_1^A = p_1^B + \delta_1 - \beta_{12} m_2 \text{ and } p_2^B = p_2^A - \delta_2 - \beta_{21} m_1$$

Equilibrium conditions then write for B:

$$\begin{aligned} p_2^B m_2 &\geq p_1^A m_1 - \delta_1 m_1 - \beta_{12} m_1 m_2 \\ &\quad + u_2^B m_2 - \max\left\{u_2^A m_2 + \beta_{22} m_2^2 - p_2^A m_2, 0\right\} + \beta_{21} m_1 m_2 \\ p_2^B m_2 &\geq p_1^A m_1 + p_2^A m_2 - \delta_1 m_1 - \delta_2 m_2 + \beta_{12} m_1 m_2 - \beta_{21} m_1 m_2 \end{aligned}$$

For A we then get

$$p_1^A m_1 \geq p_2^B m_2 + p_1^B m_1 + \delta_1 m_1 + \delta_2 m_2 + (\beta_{12} + \beta_{21})\, m_1 m_2.$$

Overall this yields equilibrium conditions

$$\begin{array}{rcl} \delta_1 + (\beta_{12} - 2\beta_{21})m_2 & \geq & p_1^A \\ \delta_1 - \beta_{12} m_2 & \geq & p_1^A \\ -\delta_2 - (2\beta_{12} + \beta_{21})\, m_1 & \geq & p_2^B \end{array}$$

Given that $\beta_{12} \leq \beta_{21}$, the second constraint is implied by the first which gives the bounds on profits.

If A sells to side 2 the three conditions are the same, reverting the role of the two sides. Given that $\beta_{12} \leq \beta_{21}$, the relevant constraints are now $\delta_2 - \beta_{21} m_1 \geq p_2^A$ and $-\delta_1 - (\beta_{12} + 2\beta_{21})\, m_2 \geq p_1^B$. ■

Proof of lemma 4. Notice that lemma 1 applies and it implies that no side split between the two platforms. $\mathcal{D}^A(P^A, P^B)$ is then characterized by the following conditions

$$\begin{array}{rcl} j & \in & \mathcal{K}\; iff: \delta_j + \beta_{jj} m_j + \sum_{l \in \mathcal{K} \backslash j} \beta_{jl} m_l - p_j^A \geq \max\left\{ \sum_{l \in \mathcal{A} \backslash j} \beta_{jl} m_l - p_j^B, 0 \right\}, \\ j & \in & \mathcal{A}\; iff: \sum_{l \in \mathcal{A} \backslash j} \beta_{jl} m_l - p_j^B > \max\left\{ \delta_j + \beta_{jj} m_j + \sum_{l \in \mathcal{K} \backslash j} \beta_{jl} m_l - p_j^A, 0 \right\}, \\ j & \in & \mathcal{J} \backslash (\mathcal{A}_i \cup \mathcal{K}_i) \; otherwise. \end{array}$$

where the second inequality comes from the fact that joining platform j cannot generate more profit for an individual than when the whole side joins A or no platform. Notice that any allocation that verifies these conditions is a REA. We can thus define $\mathcal{D}^A(P^A, P^B)$ by taking a maximal element in the set of $REAs$ that verifies these conditions. ■