

# Equilibrium Asset Pricing And Portfolio Choice Under Asymmetric Information

Bruno Biais, Toulouse University (Gremaq–CNRS, IDEI)

Peter Bossaerts, California Institute of Technology and Ecole Polytechnique Fédérale Lausanne

Chester Spatt, Carnegie Mellon University

**Revised March 04, 2009**

*Acknowledgment:* We gratefully acknowledge financial support from the Institute for Quantitative Research in Finance, a Moore grant at the California Institute of Technology, the Swiss Finance Institute and the Financial Markets and Investment Banking Value Chain Chair, sponsored by the Fédération Bancaire Française. This is a substantial revision of a previous paper entitled: Equilibrium Asset Pricing Under Heterogeneous Information. Many thanks to the editor, Matt Spiegel, and an anonymous referee for very helpful comments. Many thanks also to participants at seminars at American University, Arizona State University, Babson College, Carnegie Mellon University, Columbia University, CRSP (University of Chicago), ESSEC, Federal Reserve Bank of New York, George Washington University, Harvard Business School, INSEAD, the Interdisciplinary Center (IDC), London Business School, London School of Economics, Oxford University, Securities and Exchange Commission, Toulouse University, UC Davis, the University of Connecticut, the University of Maryland, University of Toronto, University of Vienna, the NBER Spring General Equilibrium Theory Workshop, the Q-Group Spring Conference, the 2002 North American Summer Meetings of the Econometric Society, the European Finance Association meetings, the CIRANO Portfolio Choice Conference, the Conference on “Institutions, Liquidity and Asset Prices” at the Swedish Institute of Financial Research, the University of Utah Winter Finance Conference, the meetings of the American Finance Association, the University of Michigan Symposium on “Information in Trading,” and the Conference on Trading Strategies and Financial Market Inefficiency at Imperial College, especially Geert Bekaert, Michael Brennan, Jacinta Cestone, Bernard Dumas, Phil Dybvig, Wayne Ferson, Rick Green, John Heaton, Matt Prisker, Jean Marc Robin and Ian Tonks.

Corresponding Author:

Professor Chester Spatt

Tepper School of Business

Carnegie Mellon University

Pittsburgh, PA 15213

412-268-8834

email: [cspatt@andrew.cmu.edu](mailto:cspatt@andrew.cmu.edu)

# Equilibrium Asset Pricing And Portfolio Choice

## Under Asymmetric Information

### *Abstract*

We analyze theoretically and empirically the implications of information asymmetry for equilibrium asset pricing and portfolio choice. In our partially revealing dynamic rational expectations equilibrium, portfolio separation fails and indexing is not optimal. We show how uninformed investors should structure their portfolios, using the information contained in prices to cope with winner's curse problems. We implement empirically this price-contingent portfolio strategy. Consistent with our theory, the strategy outperforms economically and statistically the index. While momentum can arise in the model, in the data the momentum strategy does not outperform the price-contingent strategy, as predicted by the theory.

# 1 Introduction

The theory of financial markets under homogeneous information has generated a rich body of predictions, extensively used in the financial industry, such as the optimality of indexing, the restrictions imposed by absence of arbitrage and equilibrium-based pricing relations. In contrast, the theory of capital markets under asymmetric information has not been used much to guide asset pricing and portfolio allocation decisions.

The goal of the present paper is to derive some of the implications of partially revealing rational expectations equilibria for asset pricing and asset allocation, and to test their empirical relevance. We analyze an overlapping generations multi-asset economy, in the line of the seminal paper of Admati (1985) and its extension to the dynamic case by Watanabe (2005). Agents live for one period and trade in the market for  $N$  risky securities, generating cash flows at each period. Some investors have private information about the future cash flows, while others are uninformed. Revelation is only partial because the demand of informed investors reflects their random endowment shocks, along with their signals. Equilibrium prices are identical to those which would obtain in a representative agent economy where i) the market portfolio would be equal to the supply of securities augmented by the aggregate risky endowment shock and ii) the beliefs of the representative agent would be a weighted average of the informed and uninformed agents' beliefs. This pricing relation cannot be directly relied upon in the econometrics since the beliefs of the representative agent are not observable by the econometrician. Hence, to test our model, we instead focus on portfolio choice.

We show that portfolio separation does not obtain, as investors hold different portfolios, reflecting their different information sets. Compared to the portfolio of aggregate risks, uninformed agents invest more in assets about which they are more optimistic than the informed agents. To cope with this winner's curse problem, the uninformed agents optimally extract information from

prices. Thus they hold the optimal price-contingent portfolio, i.e., the portfolio which is mean-variance efficient conditional upon the information revealed by prices. This enables uninformed agents without endowment shocks to outperform the index. The agents with endowment shocks are willing to pay a premium to hedge their risk. Outperformance reflects the reward to providing insurance against this risk, while optimally extracting information from prices.

The information set of the econometrician is comparable to that of uninformed agents with no endowment shocks. To confront our model to the data, we empirically implement the optimal price-contingent strategy of the uninformed agents. We test the key implication from our theory that this portfolio outperforms the index. We use monthly U.S. stock data over the period 1927-2000. We extract the information contained in prices by projecting returns onto (relative) prices. We use the corresponding expected returns and variance-covariance matrix to construct the conditional mean-variance optimal portfolio of the uninformed agent. We then compare the performance of this portfolio, as measured by its Sharpe ratio, to that of the value-weighted CRSP index. We find that the optimal price-contingent portfolio outperforms the index, both economically and statistically.

The optimal price-contingent portfolio allocation strategy we analyze is entirely based on ex-ante information. Portfolio decisions made at the beginning of month  $t$  rely on price and return data prior to month  $t$ . Thus, we only use information available to market participants when they chose their portfolios. Hence, our result that the optimal price-contingent allocation strategy outperforms the index differs from the findings of Fama and French (1996). They show that, based on return means, variances and covariances estimated as empirical moments over a period including month  $t$  as well as later months, an optimal combination of their “factor portfolios” outperforms the index. However, Cooper, Gutierrez, and Marcum (2005) show that, if one estimates these empirical moments using only information prior to month  $t$ , the outperformance of the value or size strategy becomes insignificant.

To construct our price-contingent investment portfolio we use relative prices. The latter are

correlated with past returns, on which momentum strategies rely.<sup>1</sup> We show that momentum can arise as a feature of the rational expectations equilibrium price process. The performance of momentum strategies would lead to a rejection of this theory only if it exceeded the performance of our price-contingent strategy. Empirically, we find that the momentum strategy does not outperform the price-contingent strategy. In addition, the correlation structure on which our price-contingent strategy is based is more complex than the positive serial correlation corresponding to momentum. The correlations we empirically estimate and use in our price-contingent strategies have variable signs. This is consistent with our theoretical framework. Similarly, in the multi-asset rational expectations equilibrium analyzed by Admati (1985), the correlation between prices and subsequent returns can be positive, negative or insignificant.

To illustrate our theoretical findings we perform a numerical analysis and simulation of the equilibrium price dynamics. This exercise highlights the implementability of our noisy rational expectations analytic framework and illustrates how momentum effects can arise in equilibrium.

Our noisy rational expectations model is directly in the line of the insightful theoretical analysis of Watanabe (2005), who extends the overlapping-generations model of Spiegel (1998) to the asymmetric information case. While the structure of our model is similar to his, our focus differs. Watanabe (2005) analyzes the effect of asymmetric information and supply shocks on return volatility and trading volume. In particular, he shows that trading volume has a hump-shaped relation with information precision and is positively correlated with absolute price changes. He also shows how private information accuracy can increase volatility. We don't focus on trading volume or volatility, and instead tackle other issues, e.g., is there a CAPM return relation and is indexing optimal under information asymmetry? Moreover, while the analysis of Watanabe (2005) is theoretical, an important contribution of our paper is to confront to the data the empirical implications of the theory.

Our model is also related to the theoretical analysis of Brennan and Cao (1997). In their finite

horizon model they obtain that the optimal demand of each agent at each point in time is the same as if there were no further trading opportunities. Thus, given past information, the equilibrium at each point in time has the same structure as in the one-period case analyzed by Admati (1985). In contrast, in our infinite horizon model the equilibrium at each point in time differs from the equilibrium of the one-period model. Expectations about future prices influence the decisions of the traders and hence the current price.<sup>2</sup> Thus there are two rational expectations loops: On the one hand, agents have rational expectations about the link between the current signals and the current price, as in Grossman and Stiglitz (1980), Admati (1985) and Brennan and Cao (1997). On the other hand, there is also a link between the current price and the next period's price function, as in the seminal analysis of Lucas (1978).<sup>3</sup> These two rational expectations loops are also at play in Wang (1993). The main differences between our analysis and his are that we analyze the multi-asset case and we consider an overlapping generations model. Also, our model is designed to set the stage for our econometric analysis, while his analysis is purely theoretical. Several papers have analyzed empirical applications of the noisy rational expectations framework. Cho and Krishnan (2000) estimate the primitive parameters of the Hellwig (1980) single risky asset model. Brennan and Cao (1997) and Grundy and Kim (2002) study the implications of partially revealing rational expectations equilibria for international investment flows and volatility, respectively. Our analysis differs from these because we focus on the empirical implications of the theory for the performance of various portfolio strategies.

The next section presents our theoretical model. Our econometric approach is discussed in Section 3. The empirical results are in Section 4. Section 5 concludes. Proofs are in the Appendix.

## 2 Multi-Asset Dynamic Rational Expectations Equilibrium

### 2.1 Assumptions

#### 2.1.1 The sequence of events

Consider the following overlapping generations model. Agents participate in the market during two consecutive periods. They have CARA utilities with risk aversion parameter  $\gamma$ . There are  $N$  risky assets, each generating a stream of cash flows at time  $t = 0, 1, \dots, \infty$ .

**In period  $t$ :** Generation  $t$  enters the market at the beginning of the period. There is a mass one continuum of competitive agents  $a \in [0, 1]$ . Each agent receives cash  $K$  at this point in time. Each agent  $a$  will also receive a state-contingent endowment  $e_a^t$  at time  $t + 1$ , which includes labor income and revenue from other assets than stocks. Furthermore, agent  $a$  observes signal  $y_a^t$ , about next period's payoff. After privately observing signals, agents trade  $q_a^t$  at price  $p^t$ . Market clearing requires that the new generation ( $t$ ) buy the entire supply of tradeable assets (denoted  $x$ ) from the previous generation ( $t - 1$ ).

**In period  $t+1$ :** At the beginning of the period, the vector of asset payoffs ( $f^{t+1}$ ) and the state contingent endowments of the consumption good are realized and distributed. Cash flows and aggregate endowments are publicly observed. Then, generation  $t$  sells its holdings at price  $p^{t+1}$ . Before leaving the market, they consume their wealth:

$$W_a^t = q_a^{t'}(f^{t+1} + p^{t+1}) + (K - q_a^{t'}p_a^t)(1 + r) + e_a^t,$$

where  $r$  is the exogenous risk-free rate.

From the perspective of agent  $a$  at time  $t$ ,  $e_a^t$  and  $f^{t+1} + p^{t+1}$  are two random variables. To analyze the distribution of the final wealth of the agent at time  $t + 1$  we need to consider the joint distribution of these two random variables. To do this, it is convenient to project  $e_a^t$  onto



$f^{t+1} + p^{t+1}$ :

$$e_a^t = (z_a^t)'(f^{t+1} + p^{t+1}) + \eta_a^{t+1},$$

where  $E(\eta_a^{t+1}|f^{t+1} + p^{t+1}) = 0$ . This equation is simply a statistical representation, and does not rely on any economic assumption about prices, cash flows and endowments.<sup>4</sup> Yet, it can be given an economic interpretation. The regression coefficient,  $z_a^t$ , measures the sensitivity of the endowment shocks to the risky assets' returns.  $\eta_a^{t+1}$  corresponds to the component of the shock that is not spanned by the risky assets. When receiving her endowment at time  $t$ , agent  $a$  does not know what the future prices and cash flows or the noise term ( $\eta_a^{t+1}$ ) will be. On the other hand, she knows the probabilistic structure of her random endowment and observes  $z_a^t$  at time  $t$ . As will be apparent below, because we assume it is uncorrelated with the other random variables in the model,  $\eta_a^{t+1}$  does not affect the decisions of the agents. In contrast,  $z_a^t$  does. Hence, we will hereafter (somewhat loosely) refer to  $z_a^t$  as the endowment shock of the agent.<sup>5</sup> Summarizing the above discussions, the sequence of events is depicted in Figure 1.

### 2.1.2 Informed and uninformed agents

A fraction  $\lambda$  of the agents has observed a private signal about the payoff of the asset at the next period. The other agents are uninformed. In general, all agents could receive endowment shocks. To reduce the notational and computational burden, we consider a simpler case. We assume that only the informed agents have endowment shocks, i.e., for the uninformed  $z_a^t = 0$ . Random endowment shocks for the informed agents are needed to prevent prices from being fully revealing. To further simplify the analysis we assume the informed agents all observe the same signal:  $y^t = f^{t+1} + \epsilon^t$ . One should bear in mind that these restrictions are made only for the sake of simplicity. We checked that in a more complicated model, where uninformed agents could receive endowment shocks, qualitatively similar results obtain.<sup>6</sup> The econometric implications of our analysis that we examine in the next sections, are not affected by these simplifying assumptions.

One advantage of the theoretical framework we develop here is that it is in line with the empirical approach taken in the following sections. The uninformed agents in the theoretical model solve the same problem as the econometrician in the empirical analysis in the next section.

The aggregate endowment shock at time  $t$  is  $z^t$ . The endowment shock of the informed agent indexed by  $a$  is:  $z_a^t = \frac{z^t + \zeta_a^t}{\lambda}$ , where the  $\zeta_a^t$  are i.i.d. across agents. The law of large numbers writes as:

$$\int_{a \in S} z_a^t da = z^t,$$

where  $S$  is the set of informed agents. We assume  $z^t$  follows an autoregressive process:

$$z^t = \rho_z z^{t-1} + \epsilon_z^t, \tag{1}$$

where  $\epsilon_z$  is white noise, with variance  $\sigma_z^2$ , and  $\rho_z$  is a coefficient matrix with eigenvalues strictly less than one, so that  $z$  is stationary.

Since all the informed agents observe the same signal, they don't need to filter out information from the price. Hence they don't need to form beliefs about the shocks of the others. The uninformed agents, in contrast, need to infer information from prices. In doing this, however, they don't need to be concerned about the distributions of the individual shocks. In equilibrium, only the aggregate shock ( $z^t$ ) matters for them. Finally note that our assumption that uninformed agents have no endowment shock simplifies the analysis, since it rules out inferences about  $z^t$  from their own shock.

We assume the cash flows generated by the risky assets at each period follow an autoregressive process:

$$f^{t+1} = \mu + \rho_f f^t + \epsilon_f^t, \tag{2}$$

where  $\epsilon_f$  is white noise with variance  $\sigma_f^2$ ,  $\mu$  is a non-negative constant, and  $\rho_f$  is such that  $f$  is stationary.

The private signals are equal to the realization of the cash flow plus a noise term:

$$y^t = f^{t+1} + \epsilon^t.$$

The variance of  $\epsilon^t$  is denoted by  $\sigma^2$ . All the random variables are assumed to be jointly normal.

## 2.2 Analysis

### 2.2.1 Prices and demand

We look for a stationary linear rational expectations price function:

$$p^t = Af^t + By^t - Cz^t - Dz^{t-1} + G. \quad (3)$$

Imposing this stationary form, where the coefficients are time independent, rules out bubbles (which are non stationary). The price function in equation (3) is similar to the equilibrium price function in equation (3) in Admati (1985). The main difference is that in Admati (1985) the equilibrium price is a function of only two variables, the final cash flow and the current supply shock, while in the present model it is also a function of the prior cash flow and the prior shock. In our dynamic analysis, it is necessary to include these two additional variables to summarize the past of the process. Note however that, in the simple environment we consider, it is enough to include only the prior cash flow and the prior shock. Previous realizations of the variables don't enter the equilibrium price function. This differs from Brennan and Cao (1997) and Wang (1993) where the entire past history entered the price function. Our price equation is similar to that obtained by Watanabe (2005), up to some minor differences reflecting alternative modeling choices. For example, like Admati (1985), Watanabe (2005) assumes all agents observe private signals with independent errors. By the law of large numbers, their errors integrate out. Hence, in his analysis, the price is a function of the realized cash flow. This differs from our analysis where there is only one private signal.

The final wealth of agent  $a$  can be rewritten as:

$$W_a^t = (q_a^t + z_a^t)'(f^{t+1} + p^{t+1}) + (K - q_a^{t'} p_a^t)(1 + r) + \eta_a^{t+1}.$$

Because of linearity, conditionally on the price, all the random variables are jointly normal. Thus, with CARA utilities, the program of agent  $a$  at time  $t$  is:

$$\text{Max}_{q_a^t} E[W_a^t | I_a^t] - \frac{\gamma}{2} \text{Var}[W_a^t | I_a^t],$$

where  $I_a^t$  is the information set of agent  $a$  at time  $t$ . That is:

$$\text{Max}_{q_a^t} E[(q_a^t + z_a^t)'(f^{t+1} + p^{t+1}) - q_a^{t'} p^t(1+r) + \eta_a^{t+1} | I_a^t] - \frac{\gamma}{2} \{ \text{Var}[(q_a^t + z_a^t)'(f^{t+1} + p^{t+1}) | I_a^t] + \text{Var}(\eta_a^{t+1}) \}. \quad (4)$$

All expectations and variances are taken conditionally upon the information set of the agent at the beginning of period  $t$ , including in particular the publicly observed variables:  $f^t$  and  $z^{t-1}$ . For brevity, we hereafter omit to writing these variables explicitly in the conditioning set. The optimal demand of the informed agents is:

$$\frac{1}{\gamma} (\text{Var}[f^{t+1} + p^{t+1} | y^t])^{-1} \{ E[f^{t+1} + p^{t+1} | y^t] - p^t(1+r) \} - z_a^t.$$

$\text{Var}[f^{t+1} + p^{t+1} | y^t]$  is constant over time because of linearity and stationarity of the pricing function and joint normality of the variables. The inverse of this matrix is  $\tau_y$ , the precision of the information of the informed agents. The demand of the informed agents is written as:

$$\frac{\tau_y}{\gamma} \{ E[f^{t+1} + p^{t+1} | y^t] - p^t(1+r) \} - z_a^t.$$

This demand reflects the endowment shock  $z_a^t$ , as agents seek to trade away from their undiversified endowments, to hold more balanced portfolios. For example, consider an agent working for Exxon, whose income and wealth are exposed to the risk of this firm and, more generally, to the oil industry. This agent will form his optimal portfolio taking into account his exposure to this firm and industry.

The demand of the uninformed agent is:

$$(\gamma \text{Var}[f^{t+1} + p^{t+1}|p^t])^{-1} \{E[f^{t+1} + p^{t+1}|p^t] - p^t(1+r)\}.$$

That is:

$$\frac{\tau_p}{\gamma} \{E[f^{t+1} + p^{t+1}|p^t] - p^t(1+r)\}, \quad (5)$$

where  $\tau_p$  denotes the precision of the information set of the uninformed agents. Importantly, this information set includes the current price. Equation (5) shows that, as discussed in Grossman and Stiglitz (1980), the demand of the uninformed agent is contingent on prices, not only because of the standard effect of price on demand, but also because the expectation of future cash flows and their variance are conditional on prices. Hereafter, we refer to this demand as the “optimal price contingent strategy.”

### 2.2.2 Equilibrium

Market clearing implies:

$$\lambda \tau_y [E[f^{t+1} + p^{t+1}|y^t] - p^t(1+r)] + (1-\lambda) \tau_p [E[f^{t+1} + p^{t+1}|p^t] - p^t(1+r)] = \gamma(x + z^t).$$

This yields the market-clearing price:

$$p^t = \frac{\omega E[f^{t+1} + p^{t+1}|y^t] + (I - \omega) E[f^{t+1} + p^{t+1}|p^t] - \gamma(\lambda \tau_y + (1-\lambda) \tau_p)^{-1}(x + z^t)}{1+r}, \quad (6)$$

where the constant  $\omega$  is such that:

$$\omega = (\lambda \tau_y + (1-\lambda) \tau_p)^{-1} \lambda \tau_y,$$

and  $I$  is the identity matrix. Defining the average expectation as:

$$E^m[f^{t+1} + p^{t+1}] = \omega E[f^{t+1} + p^{t+1}|y^t] + (1-\omega) E[f^{t+1} + p^{t+1}|p^t],$$

and the average variance–covariance matrix as:

$$\Omega^m = (\lambda\tau_y + (1 - \lambda)\tau_p)^{-1},$$

one obtains the pricing relation stated in the following proposition.

**Proposition 1:** *If there exists a stationary linear rational expectations equilibrium, prices are such that:*

$$p^t = \frac{1}{1+r} \{E^m[f^{t+1} + p^{t+1}] - \gamma\Omega^m(x + z^t)\}. \quad (7)$$

Equation (7) essentially tells us that the portfolio of aggregate risk,  $x + z^t$ , is mean-variance optimal for the average expectations vector:  $E^m[f^{t+1} + p^{t+1}]$  and variance–covariance matrix:  $\Omega^m$ . Otherwise stated, equilibrium prices are identical to those which would obtain in a homogeneous information–representative agent economy, where the market portfolio would be equal to the supply of securities ( $x$ ) augmented by the aggregate risky endowment ( $z^t$ ) and the representative agent would have the average beliefs  $E^m$  and  $\Omega^m$ . A CAPM return-covariance relationship holds, from the perspective of the representative agent, relative to this augmented portfolio ( $x + z^t$ ), which differs from the index ( $x$ ). After standard manipulations, equation (7) yields the classic CAPM equilibrium return equation, for our representative agent economy:

$$E^m(r_i^{t+1}) - r = \beta_i^t (E^m(r_m^{t+1}) - r),$$

where  $r_i^{t+1} = (p_i^{t+1} + f_i^{t+1} - p_i^t)/p_i^t$  is the return on asset  $i$ ,  $r_m^{t+1} = \sum_{j=1}^N (x_j + z_j^t)r_j^{t+1}$  is the return on the portfolio of aggregate risks, expectations are taken from the point of view of the representative investor’s average belief, and  $\beta_i^t$  is defined as:

$$\beta_i^t = \frac{Cov(r_i^{t+1}, r_m^{t+1} | z^t)}{Var(r_m^{t+1} | z^t)}.$$

At time  $t$ , the representative agent knows  $z^t$ . But through time  $z^t$  follows a stochastic process. Hence betas are stochastically evolving.

This pricing relation cannot be directly relied upon in the econometrics to estimate a representative agent model, however. Indeed, the beliefs of the representative agent are not observable by the econometrician. Nor is the portfolio of aggregate risk ( $x + z$ ) observable, a point that is related to the Roll (1977) Critique.<sup>7</sup> Thus we take another route to confront our model to the data, as explained in the next section.

The aggregation result we obtain in our dynamic rational expectation equilibrium context is related to previous results obtained in static models, but it also differs from them. First, the pricing equation (7) differs from that stated in Corollary 3.5 in Admati (1985). Admati (1985) characterizes the ex-ante expected price, computed by averaging across all realizations of the random variables. She shows that an aggregate CAPM obtains on average across possible realizations of the random variables. This contrasts with the equilibrium relationship (7) obtained in the present paper, which holds in every possible state of the world. Second, as in Mayers (1974), investor's risk exposure and hence market pricing are affected by the non-tradeable endowment shocks. Third, our aggregation result with heterogeneous beliefs reflects in part our assumption that agents have exponential utility. This is in line with the Gorman aggregation results obtained by Wilson (1968) (see also Huang and Litzenberger, 1988, p. 146-148) and Lintner (1969). In these analyses, however, beliefs are exogenous, and inconsistent with rational expectations. In ours, aggregation obtains with endogenous beliefs and rational expectations. Fourth, DeMarzo and Skiadas (1998) also offer a theoretical analysis of a CAPM with heterogeneous information, but our model differs from theirs. On the one hand they allow for a more general class of utility functions than we do. On the other hand, a key ingredient in our model is that the aggregate portfolio of risks is unknown by the uninformed agents, which prevents prices from being fully informative. In contrast, the CAPM result obtained by DeMarzo and Skiadas (1998) reflects their assumption that the aggregate supply of each of the risky assets is common knowledge for all the agents.<sup>8</sup> Finally, O'Hara (2003)

also focuses on the impact of asymmetric information on portfolio choice and asset pricing in the Grossman and Stiglitz framework. Our dynamic analysis differs from the static model of O'Hara (2003).

Simple manipulations of the market-clearing condition (6) yield the following result:

**Proposition 2:** *If there exists a stationary linear rational expectations equilibrium, the demand of the uninformed agent is:*

$$(x + z_t) + \frac{\lambda}{\gamma} [\tau_p \{E[f^{t+1} + p^{t+1}|p^t] - p^t(1 + r)\} - \tau_y \{E[f^{t+1} + p^{t+1}|y^t] - p^t(1 + r)\}].$$

The demand of the uninformed agent is equal to the portfolio of aggregate risk,  $x + z^t$ , which she would hold if information was homogeneous, minus a correction term. The latter underscores the winner's curse problem faced by the uninformed agent. She invests more than the aggregate portfolio in asset  $i$  when she is more optimistic about this asset than the informed agent, while she invests less otherwise. Hence, portfolio separation does not obtain in our asymmetric information environment. Different agents hold different portfolios, to the extent that they have different information sets.<sup>9</sup>

In contrast with the homogeneous information case, the uninformed agent does not buy the portfolio of aggregate risk ( $x + z$ ). She does not do so intentionally, because she does not know this portfolio. However, she invests optimally given the information she observes, which includes in particular the current prices. Thus the portfolio she holds outperforms the index. One might wonder how uninformed agents can obtain such superior performance. It arises because the agents with endowment shocks are willing to pay a premium to hedge their risk. Thus, the performance obtained by the uninformed agents reflects the price other agents are willing to pay for insurance.

Since the uninformed agent does not know the structure of the portfolio of aggregate risk, her deviation from this portfolio can be interpreted as *estimation risk*. In the past (e.g., Kandel and Stambaugh [1996]), estimation risk has been studied under homogeneous information, in which



case it only adds to variance. In our asymmetric information setting, estimation risk also yields a winner's curse. Consequently, our analysis introduces a new dimension to the nature of estimation risk.

Computing explicitly the demand functions of the informed and uninformed agents and equating the price relation obtained from the market-clearing condition with the conjectured price function (3), we obtain the next proposition:

**Proposition 3:** *There exists a linear stationary rational expectations equilibrium, if there exists a solution  $B, C, D$  to the following system:*

$$B = \Phi^{-1}(\lambda\tau_y + (1 - \lambda)\tau_p)^{-1}\lambda\tau_y[I - (1 + r)^{-1}\rho_f]^{-1}\text{Var}(\epsilon_f^t)\text{Var}(\epsilon_f^t + \epsilon^t)^{-1},$$

$$C = \Phi^{-1}\gamma(\lambda\tau_y + (1 - \lambda)\tau_p)^{-1},$$

$$D = \Phi^{-1}(C\rho_z + D)\rho_z + [I - (1 + r)\Phi^{-1}][I - (1 + r)^{-1}\rho_f]^{-1}\text{Var}(\epsilon_f^t)B' + (C\rho_z + D)\text{Var}(\epsilon_z^t)C' \\ (B\text{Var}(\epsilon_f^t + \epsilon^t)B' + C\text{Var}(\epsilon_z^t)C')^{-1}(C\rho_z + D),$$

where  $\Phi, \tau_y$  and  $\tau_p$  are functions of  $B, C, D$  given by equations (20), (22), and (23) in the Appendix. The two other parameters of the equilibrium price function  $A$  and  $G$  are also functions of  $B, C$  and  $D$  given in equations (24) and (28) in the Appendix.

The system of nonlinear equations stated in Proposition 3 is rather complicated. This contrasts with the elegant closed-form solutions obtained by Admati (1985). The additional complexity we face here stems from the dynamic nature of the problem. The price at time  $t$  reflects the expectations of the agents about the cash flow *and the price* at time  $t + 1$ . Thus, the rational expectations loop is more complicated than in the one-period case. In the latter, agents must have rational expectations about the link between the current price and the current signals. In the dynamic case, there is also a link between the current price and the next period's price function. A similar complexity arises in the dynamic analysis of asset pricing under asymmetric information

by Wang (1993).<sup>10</sup> Watanabe (2005) obtain a slightly closer characterization of equilibrium than we do. This reflects alternative modeling choices. For example, he assumes dividends and supply shocks are random walks, which simplifies the computations. Our more complicated structure helps generating rich dynamics.

### 2.2.3 Special cases

In special cases, the existence and properties of equilibrium can be analyzed explicitly. First consider the simplest case, without endowment shocks and without private signals. In this case, market clearing yields:

$$p^t = (1 + r)^{-1} \{E(p^{t+1} + f^{t+1}) - \gamma \tau^{-1} x\},$$

where  $\tau^{-1} = Var(\epsilon_f)$ . The standard dynamic-CAPM obtains. All agents have the same beliefs and they all hold the market portfolio, i.e., indexing is optimal. In this case, there exists a unique stationary linear rational expectations equilibrium, spelled out in the next corollary.<sup>11</sup>

**Corollary 1:** *When there are no endowment shocks and no private signals, there exists a unique stationary, linear, rational expectation price function:  $p^t = Af^t + G$ , where:  $A = \rho_f[(1+r)I - \rho_f]^{-1}$  and  $G = r^{-1}[(I + A)\mu - \gamma \tau^{-1}x]$ .*

The empirical asset pricing literature has documented important stylized facts about the time-series of stock returns. It has been found that returns were predictable (see, e.g., Cochrane, 1999). In particular, stocks whose return over the past twelve months is low relative to that of others tend to underperform, while stocks with recent strong returns tend to outperform. Hence, shorting the losers and investing the proceeds in recent winners generates high expected returns. This strategy has become known as the momentum strategy. It has been analyzed in depth by Chan, Jegadeesh and Lakonishok (1996). These empirical findings have motivated theoretical analyses based on the assumptions that some investors are irrational. Our framework offers an opportunity to check

whether momentum and predictability are consistent with equilibrium in a dynamic CAPM where all agents are rational.<sup>12</sup>

The time  $t + 1$  return is:

$$R^{t+1} = p^{t+1} + f^{t+1} - p^t. \quad (8)$$

To implement the momentum strategy, we form a portfolio investing in assets proportionally to their excess return relative to the average across all assets. This leads indeed to buying winners and selling losers. More precisely, the strategy involves purchasing:

$$\frac{R_i^t - \bar{R}^t}{N}, \quad (9)$$

units of asset  $i$ , where  $R_i^t$  is the return on this asset in period  $t$ ,  $\bar{R}^t$  is the average return (computed across assets) for period  $t$ , and  $N$  is the number of risky assets.

The following corollary provides information about the time series properties of returns and the profitability of this momentum strategy in the special case of our model where a dynamic CAPM obtains. For simplicity, we focus on the case where risky assets are ex-ante identical and independent and  $\rho_f = \bar{\rho}_f I$ , where  $\bar{\rho}_f \in [0, 1]$ .

**Corollary 2:** *When there are no endowment shocks and no private signals, under our assumptions individual return autocorrelations are positive, and the expected return on the momentum strategy is positive for large  $N$ .*

In our framework, cash flows  $f^t$  are positively serially correlated. The corollary states that equilibrium returns will be positively correlated too. Momentum and predictability arise, yet pricing is rational and indexing is optimal. As a result, momentum and predictability cannot be relied upon to outperform the passive strategy of holding the index. Nor do momentum and predictability suffice to reject CAPM pricing.

Another interesting special case is when there are endowment shocks but no private signal, i.e.

$\lambda = 0$ , as analyzed by Spiegel (1998).<sup>13</sup> In this case the standard CAPM no longer holds. The relevant measure of aggregate risk is not the index, but the index augmented by the aggregate endowment shock. All agents have the same beliefs and share risk equally. As shown by Spiegel (1998), when equilibrium exists it is generically not unique. Multiplicity arises because of the circularity involved by the dynamic rational expectations loop: the price function depends upon the expectation of the price function. To illustrate this assume for simplicity that  $\rho_z = 0$ . Denote  $Var(\epsilon_z) = \sigma_z^2 I$  and  $Var(\epsilon_f) = \sigma_f^2 I$ , where  $\sigma_z^2$  and  $\sigma_f^2$  are scalars. Manipulating the system in Proposition 3 yields the following result:<sup>14</sup>

**Corollary 3:** *Assume there are no private signals ( $\lambda = 0$ ), endowment shocks are white noise ( $\rho_z = 0$ ) and the risky assets are ex-ante identical and independent. If  $1 + r < \bar{\rho}_f + 2\gamma\sigma_f\sigma_z$ , there does not exist a linear rational expectation equilibrium. In contrast, if  $1 + r > \bar{\rho}_f + 2\gamma\sigma_f\sigma_z$ , then there exist  $2^N$  solutions to the quadratic matrix equation:*

$$\tau^{-1} = \left[1 + \frac{\bar{\rho}_f}{(1+r) - \bar{\rho}_f}\right]^2 \sigma_f^2 I + \left[\frac{\gamma}{1+r}\right]^2 \sigma_z^2 (\tau^{-1})' \tau^{-1}. \quad (10)$$

*Each of these solutions corresponds to a linear rational expectation equilibrium price function of the form:  $p^t = Af^t - Cz^t + G$ , where:  $A = \rho_f ((1+r)I - \rho_f)^{-1}$ ,  $C = \frac{\gamma}{1+r} \tau^{-1}$  and  $G = r^{-1}[(I + A)\mu - \gamma\tau^{-1}x]$ .*

In equilibrium a variety of return dynamics can arise. Since returns reflect cash flows ( $f^t$ ) and since the latter are persistent, there is some persistence in returns, which can generate momentum. On the other hand, returns also include a transient component ( $z^t$ ). This can give rise to mean reversion. Depending on parameter values, one effect or the other can dominate. Hence, momentum or reversals can arise in our dynamic rational expectations equilibrium. This is illustrated in the simulations presented next.

## 2.2.4 Numerical analysis

To illustrate quantitatively major aspects of our model, such as equilibrium momentum, the performance of the price-contingent strategy and the effect of variation in the proportion of informed agents, we now offer a numerical analysis of the dynamic rational expectations equilibrium. This analysis is cast in the context of the simple case characterized in the next corollary.<sup>15</sup>

**Corollary 4:** *Consider the case where  $\rho_z = \mu = 0$ ,  $N = 2$  and the cash-flows and endowments of the two risky assets are ex-ante identically and independently distributed, and denote:*

$$\text{Var}(\epsilon_f) = \begin{pmatrix} \sigma_f^2 & 0 \\ 0 & \sigma_f^2 \end{pmatrix}, \text{Var}(\epsilon) = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \text{ and } \text{Var}(\epsilon_z) = \begin{pmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_z^2 \end{pmatrix}.$$

*There exists a symmetric equilibrium where  $B, C, \Phi, \tau_y, \tau_p$  and  $\omega$  are symmetric 2 by 2 matrices, if there exists a solution  $b, c$  to the following system of scalar equations:*

$$b = \frac{\sigma_f^2 \lambda \iota_y (\lambda \iota_y + (1 - \lambda) \iota_p)^{-1}}{\phi (\sigma_f^2 + \sigma^2) (1 - \rho_f (1 + r)^{-1})}, \quad (11)$$

$$c = \frac{\gamma}{\phi (\lambda \iota_y + (1 - \lambda) \iota_p)}, \quad (12)$$

*where the scalars  $\phi, w, \iota_y, \iota_p$  are such that:*

$$\phi = (1 + r) - \frac{\sigma_f^2 [1 - \lambda \iota_y (\lambda \iota_y + (1 - \lambda) \iota_p)^{-1}] b}{((\sigma_f^2 + \sigma^2) b^2 + \sigma_z^2 c^2) (1 - \rho_f (1 + r)^{-1})}, \quad (13)$$

$$\iota_y^{-1} = \frac{\sigma_f^2 \sigma^2}{(\sigma_f^2 + \sigma^2) (1 - \rho_f (1 + r)^{-1})^2} + ((\sigma_f^2 + \sigma^2) b^2 + \sigma_z^2 c^2), \quad (14)$$

$$\iota_p^{-1} = \frac{\sigma_f^2 [1 - b^2 \sigma_f^2 \{(\sigma_f^2 + \sigma^2) b^2 + \sigma_z^2 c^2\}]}{(1 - \rho_f (1 + r)^{-1})^2} + ((\sigma_f^2 + \sigma^2) b^2 + \sigma_z^2 c^2). \quad (15)$$

*The solutions to this system:  $b$  and  $c$  and the corresponding values of  $\phi, \iota_y$ , and  $\iota_p$  are the eigenvalues of the symmetric matrices  $B, C, \tau_y, \tau_p$ , and  $\Phi$ .  $D = 0$  and  $A$  is obtained from  $B$  and  $C$  as in Proposition 3.*

To conduct our numerical analysis, we rely on Corollary 4. Thus, we focus on symmetric equilibria.<sup>16</sup> We discuss below the instances where results qualitatively differ across equilibria. Also, we focus on a set of parameters such that  $1 + r > \bar{\rho}_f + 2\gamma\sigma_f\sigma_z$ , so that linear equilibria exist in the symmetric information case, and we select parameters such that the momentum strategy (as defined in the description leading to Corollary 2) can generate positive average returns. Thus, we set  $\rho_f = 0.990$ , since momentum arises when cash flow are persistent. As discussed by Spiegel (1998, page 436), it is plausible that the variance of supply shocks is relatively small compared to that of cash flows. Correspondingly, we set  $\sigma_f^2 = 0.100$  and  $\sigma_z^2 = 0.010$ . Furthermore we focus on the case where informed signals are quite precise and set:  $\sigma^2 = 0.001$ . Note also that we set  $r = 0.10$ ;  $\rho_z = 0$ ;  $\gamma = 1.735$  and we normalize  $x$  to 1.<sup>17</sup>

Figure 2 plots the equilibrium parameters as a function of  $\lambda$ , for one of the two symmetric equilibria. For simplicity, we focus on the diagonal terms. First note that, when  $\lambda = 0$ ,  $B = 0$ , which is a sanity check for our analysis: when no agent in the market is privately informed, the private signal is not revealed in prices at all. Second, note that  $A$  decreases and  $B$  increases with  $\lambda$ . As  $\lambda$  increases, more private information gets impounded in prices and  $B$  increases.<sup>18</sup> Correspondingly, less weight is given to the prior expectation of cash flows (reflecting  $f^t$ ) and  $A$  decreases.

For the equilibrium on which we focus, Figure 3 displays average returns for the momentum and price-contingent portfolios, as defined in the theoretical analysis above (see equations (9) and (5) respectively). Estimates are based on 60-period average returns and 500 random replications. Results are shown as a function of  $\lambda$ , for  $\lambda \leq 8\%$ . The left panel shows average momentum returns in excess of average index returns. The right panel displays the average excess returns earned by the price-contingent strategy. 95% confidence intervals are depicted.

On average, the price contingent strategy generates larger returns than the market portfolio, thus illustrating our theoretical analysis. Of course, this also obtains in the three other equilibria.

The average return on the price contingent strategy goes down with  $\lambda$ . The source of performance for this strategy is the supply of liquidity to agents with endowment shocks. The cost of this strategy stems from trading with superiorly informed agents. When  $\lambda$  goes up, the former effect remains constant, while the latter is strengthened.

In the equilibrium on which we focus, the momentum portfolio also beats the index for  $\lambda$  above 5%. This shows that asymmetric information can give rise to momentum in a dynamic noisy rational expectations equilibrium. Otherwise stated, outperformance of the index by the momentum portfolio should not be viewed as evidence of deviation from rational expectations. It should be noted, however, that in the other equilibria we simulated the momentum portfolio did not outperform the index. Thus the excess performance of the momentum portfolio is not a robust feature of our equilibria, in contrast with the excess performance of the price contingent portfolio.

### **3 Econometric approach**

#### **3.1 Testable restriction implied by theory**

Hansen and Singleton (1982) test whether the representative agent invests optimally. This is not feasible in our setting, where the econometrician does not observe the endowment shocks and the signals necessary to construct the representative investor. On the other hand, the information set of the econometrician is similar to that of the uninformed agent in our model. The uninformed agent's demand is the solution to a simple mean–variance portfolio choice problem, where the information content of prices is used to estimate expected returns and variances. This is the price–contingent strategy defined in equation (5). Our theoretical analysis implies that, in the partially revealing rational expectations equilibrium, this investment strategy fares better, in mean–variance terms, than indexing. In contrast, if the CAPM holds, then indexing is optimal. Hence, to test the partially revealing rational expectations equilibrium against the CAPM, we compare the performance of the

index to that of the price–contingent portfolio strategy of the uninformed agent.

We focus upon what is likely to be the most robust implication of the theory, i.e., that prices contain information, rather than on more parametric aspects of the model. Spiegel (1998) and Watanabe (2005) show that equilibrium multiplicity arises in dynamic rational expectations equilibria. Yet, irrespective of which equilibrium the agents in the economy select, equation (5) still describes the optimal investment strategy of a rational uninformed investor. This empirical implication is also robust to alternative parametrizations of our model, e.g., in which some uninformed agents would be exposed to endowment shocks or informed agents would observe various signals. While, in equation (5), the optimal strategy involves conditioning on price *levels*, in our econometric analysis, for statistical reasons, we will use *relative* prices instead.

Our comparison of the performance of the price–contingent strategy to that of the index is in the line of empirical investigations of asset pricing models, testing the efficiency of a particular portfolio, such as a market proxy. However, our approach differs from previous empirical results on the inefficiency of market proxies.

First, our theory implies that indexing should be inferior to a portfolio strategy constructed using public information available to the agent when he makes his investment decision. Hence our empirical analysis will be *ex-ante*, i.e., to form portfolios at date  $t$ , we will use only data observed prior to  $t$ . In contrast, Fama and French [1996] and Davis, Fama and French [2000] rely on *ex-post* information. They find that proxies of the market portfolio are mean-variance suboptimal relative to some *ex-post* determined combination of three specific factor portfolios: the market proxy itself, a portfolio long in small firms and short in large firms, and a portfolio long in value stock and short in growth stock. Cooper, Gutierrez and Marcum [2005] show that if one uses only information in prior returns to determine optimal combinations, the improvement from investing in Fama and French’s factor portfolios is insignificant.

Second, we compare indexing against a portfolio strategy based on specific information suggested



by theory, namely, relative prices, as opposed to information obtained through an exhaustive search over all available information. Without the discipline that theory imposes, such an exercise runs into the danger of data snooping. Information in relative prices has never been explicitly conditioned on before. This information may have been *implicitly* conditioned on in other studies, such as those evaluating momentum investment. We discuss the difference between our price-contingent strategy and momentum investment in the next section.

### 3.2 The Data

We focus on monthly returns on U.S. common stock listed on the NYSE, AMEX and NASDAQ, as recorded by CRSP for the period from July 1927 until December 2000. The null hypothesis is that the value-weighted CRSP index is optimal. This index has been used as the market proxy in previous empirical studies. Against the null hypothesis, we test the hypothesis that the index is outperformed by the price-contingent portfolio. In principle, one can construct those portfolios by combining individual stocks. This requires, however, that one handle thousands of different stocks, correlating their returns to their prices, a computationally challenging exercise. A more parsimonious approach is to use groups of stocks as building blocks for our portfolios.

A natural choice for these groups of stocks is to focus on the six portfolios which have been used extensively in the empirical asset pricing literature. These are specific portfolios constructed from a double sort of the securities based on the size of the issuing firms as well as the ratio of book value to market value. We will refer to them as the six FF benchmark portfolios. Portfolio 1 selects stocks of large companies with low ratio of book to market value. Portfolio 2 also selects large companies, but with medium book to market value. Portfolio 3 is comprised of large value companies. Portfolios 4 to 6 are analogous to portfolios 1 to 3, but for small firms only. All portfolios are value-weighted. Details can be found on Ken French's website.

Monthly returns are taken from Ken French's web site.<sup>19</sup> We use the returns that are adjusted

for the substantial transaction costs caused by flows of individual assets in and out of the portfolios. Such flows are the result of changes in firm size, book and market values.

The choice of these assets for studying the information content of prices is suitable in light of the well-known dispersion in their relative prices over time, enhancing the potential power of our statistical analysis. However, the size or value strategies do not significantly outperform indexing on an ex ante basis, as shown by Cooper, Gutierrez and Markum [2005]. Hence, the performance of price-contingent strategies using only ex ante information to form portfolios of these assets cannot be due to data mining.

It is not obvious how to measure the relative prices on which our portfolio allocation strategy will be based. We opted to use the weights in a buy-and-hold portfolio of the six FF benchmark portfolios, reinvesting returns (including dividends) into the FF portfolio that generates them. More specifically, let  $r_i^t$  denote the rate of return on FF benchmark portfolio  $i$  ( $i = 1, \dots, 6$ ) over month  $t$ . ( $t = 1$  corresponds to 7/1927). Let  $p_i^t$  denote our measure of the relative price of portfolio  $i$  at the beginning of month  $t$ . It is computed as follows:

$$p_i^t = \frac{p_i^{t-1}(1 + r_i^t)}{\sum_{j=1}^6 p_j^{t-1}(1 + r_j^t)}, \quad (16)$$

$t > 0$ . Notice that  $\sum_{i=1}^6 p_i^t = 1$ , so our prices are effectively portfolio weights in a portfolio of the six FF benchmark portfolios. This portfolio starts with \$1 at the end of 6/1927. The initial weights used to initialize this procedure are arbitrarily chosen and we checked that they do not affect our empirical results. Our proxies for relative prices are thus weights in a value-weighted portfolio.

One could be concerned about persistence in the prices. Our portfolio allocation strategy will be based on projections of a month's returns onto the vector of relative prices at the beginning of the month. The properties of estimated projection coefficients are known to be unusual when the explanatory variables exhibit persistence. In particular, the significance of the projection coefficients may be spurious, and the out-performance of the resulting price-contingent strategy may be a

statistical artifact. To be sure that this is not the case, it is imperative that we perform an ex-ante portfolio performance evaluation. As mentioned before, this is exactly what we do. If the significance of the projection coefficients is not spurious, persistence in the regressors (relative prices) is actually a virtue. Standard least squares projection coefficients converge faster.

### 3.3 Portfolio Allocation Strategy

The allocation strategy we implement is in line with the mean variance optimization described in our theoretical analysis in equation (4), which gave rise to the optimal price-contingent strategy given in equation (5). For each month in the sample, referred to as the *target month*, we determine the composition of the portfolio that promises the highest conditional expected return for a volatility equal to that of the benchmark CRSP index. Thus, to determine the portfolio  $q^t$ , we solve:<sup>20</sup>

$$\text{Max}_{q^t} E(q^t r^{t+1} | p^t), \text{ s.t.}, \text{Var}(q^t r^{t+1} | p^t) = V_{index}^t,$$

where  $V_{index}^t$  is the conditional variance of the index as of  $t$ . Determining this portfolio requires estimating expectations ( $E(q^t r^{t+1} | p^t)$ ) and variances ( $\text{Var}(q^t r^{t+1} | p^t); V_{index}^t$ ).

In line with our theory, we estimate conditional expectations by projecting returns onto relative prices. Variances and covariances are estimated from the errors of these projections. To determine the optimal portfolio for any target month, we use observations from the sixty-month period *prior* to the target month. That is, our analysis is entirely ex-ante, i.e., only based on information that investors had available at the beginning of the target month. Generalized Least Squares (GLS) are used to estimate the coefficients in projections of returns onto prices, to adjust for the substantial autocorrelation in the error (it sufficed to adjust for first-order autocorrelation). No further adjustments were made, although one obviously could think of many potential improvements (Iterated Least Squares, higher-order autocorrelation in the error term, autoregressive heteroscedasticity, etc.).<sup>21</sup>

To estimate the volatility of the index  $V_{index}^t$ , we perform GLS projections of index returns onto the prices of the 6 FF benchmark portfolios over the 60 months prior to the target months, accounting for first-order autocorrelation in the error term. The ex-ante volatility of the index return is then obtained from the standard deviation of the prediction error of these GLS projections. Estimating the volatility of the price-contingent portfolio is more complicated, as discussed below.

### 3.4 Accounting For Errors In Estimating Optimal Portfolio Weights

Because optimal weights for our price-contingent strategy are based on estimated expected returns, variances and covariances, we inevitably introduce estimation error. When we base ex-ante volatility estimates on the covariance matrix of the prediction errors from GLS projections of returns onto prices, we fail to properly account for estimation error. As a result, the ex-post volatilities may be higher. The ex-post volatilities can readily be estimated as mean squared differences between returns actually recorded over the target months and ex-ante expected returns (from the GLS projections). In contrast, since no estimation of optimal portfolio weights is involved, the ex-ante volatility of the market indexing strategy is likely to be a good estimate of its ex-post volatility. Consequently, we suspected that the ex-post volatility of our price-contingent strategy may be much higher than that of the market portfolio, even if volatilities matched ex-ante. The data confirmed our suspicion.

To accommodate estimation error, one could directly adjust estimates of the ex-ante volatilities of the price-contingent strategy. The necessary adjustments are rather involved, and unfortunately, they require additional assumptions on the data generating process that reduce the robustness of the inference (e.g., the projection errors are jointly normal and independent over time).<sup>22</sup>

Instead, we correct for estimation error by matching ex-post volatilities, as is often done in finance.<sup>23</sup> In particular, we determine the right combination of our price-contingent strategy with investment in the market portfolio that generates the same ex-post volatility as the index.<sup>24</sup> With

our return history, a 50-50 combination ensures that variances are matched. As a result, in the performance evaluation to follow, we will compare the returns of two strategies: (i) a strategy that constantly re-invests 50% in our optimal price-contingent portfolio and 50% in the market portfolio; (ii) 100% market indexing. These two strategies generated approximately the same ex-post volatility over the period 7/1927 to 12/2000.

For brevity, we will refer in the sequel to the 50-50 combination of the optimal price-contingent portfolio and the market as our price-contingent strategy. But bear in mind that it in fact mixes indexing with optimal price-contingent investing.

### 3.5 Performance Evaluation

To evaluate the performance of the optimal price-contingent strategy, we compare its Sharpe ratio to that of the index. For each month  $t$ , we compute the average return of the price-contingent portfolio as a 60-month moving average centered on  $t$ . We proceed similarly for the index. We estimate the volatility of the index as the mean squared difference between its return and that predicted by the GLS regression. The difference between the Sharpe ratios of the two portfolios is estimated as the difference between the two average returns, divided by the volatility of the index. We use the same denominator for the two Sharpe ratios, since the price-contingent strategy is constructed to have the same volatility as the index.

To complement this comparison of Sharpe ratios and evaluate the statistical significance of the outperformance, we use a  $z$ -test. Its intuition can be summarized as follows: We compute the average difference between the returns on the price-contingent portfolio and the index; we divide it by the standard deviation of this difference; we scale up this ratio by the square root of the number of observations. Relying on the functional central limit theorem we compute a confidence interval.<sup>25</sup> More precisely, we compute partial  $z$ -statistics as follows. Let  $r_C^t$  denote the return on the CRSP over month  $t$ . Let  $r_P^t$  denote the month- $t$  return on our price-contingent portfolio. For

a sample that starts at  $T_1$  and ends at  $T_2$ , the partial  $z$ -statistics are computed from the partial sums of the difference between the return on the price contingent portfolio and that on the index:

$$z_{T_1, T_2, t} = \frac{1}{\sqrt{T_2 - T_1}} \sum_{\tau=T_1+1}^t \frac{r_P^\tau - r_C^\tau}{\sigma}.$$

We estimate  $\sigma$  as

$$\hat{\sigma} = \sqrt{\frac{1}{T} \sum_{\tau=1}^T \{r_P^\tau - r_C^\tau\}^2}.$$

This is a (heteroscedasticity-consistent) estimate of the standard deviation of the return differences, under the null that the expected return differences equal zero.<sup>26</sup>

The partial  $z$ -statistics form a stochastic process on  $[T_1, T_2]$ , so they are easy to visualize. The functional central limit theorem predicts that, in large samples (meaning  $T_2 - T_1 \rightarrow \infty$ ),

$$z_{T_1, T_2, t} \sim W\left(\frac{t - T_1}{T_2 - T_1}\right),$$

where  $W$  denotes a standard Brownian motion on  $[0, 1]$ . Note that the usual  $z$ -statistic over  $[T_1, T_2]$  has  $t = T_2$  and hence:  $z_{T_1, T_2, T_2} \sim W(1)$ , i.e., its asymptotic distribution is standard normal, in accordance with the usual central limit theorem. Confidence bands of 95% can readily be computed as:

$$\pm 1.97 \sqrt{\frac{t - T_1}{T_2 - T_1}}.$$

We provide plots of the partial  $z$ -statistics for  $T_1 = 0$  (before the start of our sampling period, i.e., 6/1927), and  $T_2 = T$  (the end of our sampling period, namely, 12/2000). That is, we report  $z_{0, T, t}$ . In that case, the 95% confidence intervals are given by:

$$\pm 1.97 \sqrt{\frac{t}{T}}.$$

One can compute confidence intervals starting at any  $T_1 > 0$  and conditional on the partial  $z$ -statistic at that point,  $z_{0, T, T_1}$ . These derive from the fact that<sup>27</sup>

$$z_{0, T, t} - z_{0, T, T_1} = z_{T_1, T, t} \sqrt{\frac{T - T_1}{T}} \sim W\left(\frac{t - T_1}{T - T_1}\right) \sqrt{\frac{T - T_1}{T}}.$$

$(T_1 < t \leq T)$ . Hence, the confidence interval starting  $T_1$  and conditional on  $z_{0,T,T_1}$  equals

$$z_{0,T,T_1} \pm 1.97 \sqrt{\frac{t - T_1}{T}}.$$

We plot such conditional confidence intervals at ten-year intervals.

## 4 Empirical Results

### 4.1 Main Results

The main results are displayed in Figures 4 and 5. The average return on our price-contingent strategy is 1.4% per month (18% on an annual basis); that of the market portfolio equals 1.2% (15% on an annual basis). The ex-post standard deviation on both portfolios is 21% per year. Figure 4 shows the evolution of the difference in Sharpe ratio between the optimal price-contingent portfolio and the CRSP index. The average difference in Sharpe ratios is 0.048. Our price-contingent strategy thus adds substantially to the achievable return. Figure 4 demonstrates that, with the exception of a few subperiods, our price-contingent optimal strategy outperforms the CRSP index consistently since the beginning of the sampling period.

Figure 5 displays the evolution of the partial  $z$ -statistic. It confirms that the outperformance was significant. Consider the evolution of the  $z$ -statistic from the beginning of the sample period (1927) to its end (2000). The square-root function depicts the confidence bounds. The  $z$ -statistic crosses the confidence bound, indicating significant outperformance, as soon as the 1930's. The final value of the statistic, at the end of the sample, reaches a highly significant value of 2.7. The gradual increase in the  $z$ -statistic indicates that the outperformance of the price-contingent strategy is not the effect of a few outliers. Figure 5 also enables the reader to check the significance of the outperformance of the price-contingent strategy for any of the decades in our sample: 1932–1942, 1942–1952, ..., 1982–1992, 1992–2000. The  $z$ -statistic is positive at the end of the decade in all but one ten-year subperiod; the corresponding  $p$ -level is 0.06.<sup>28</sup> The performance is significant at the

5% level in 2 out of 7 ten-year subperiods; the corresponding  $p$  level is about 0.05.<sup>29</sup> That is, there is little doubt about the significance of the outperformance.

Thus we find that our price-contingent allocation strategy significantly outperforms the index. This is consistent with our Noisy Rational Expectations model, where prices reflect economically relevant information, while at the same time not fully revealing all of it.

## 4.2 Perspective

To build perspective, we now compare the performance of our price-contingent strategy to that of alternative strategies, which have been claimed to improve upon indexing: the size, value and momentum strategies (see e.g., Cochrane [1999], Cooper, Gutierrez and Marcum [2005], Davis, Fama and French [2000], Fama and French [1996], Chan, Jegadeesh and Lakonishok [1996] and Lewellen [2002]).

The size strategy exploits the difference between the returns of small firms and those of large firms. The standard portfolio that implements this strategy is a zero-investment strategy, taking a long position in small firms and a short position in large firms. It is referred to as the Fama-French SMB (“small minus big”) portfolio.<sup>30</sup> Analogously, the value strategy exploits the difference between the returns of firms with high book value of equity relative to market value and firms with low book-to-market ratio. It is a zero-investment portfolio taking a long position in high-value firms, while shorting low-value firms. It is referred to as the Fama-French HML (“high minus low”) portfolio. The momentum strategy exploits persistence in stock returns. It establishes a long position in recent winners, while shorting recent losers. Usually, a twelve-month window is considered to determine if stocks have been winners or losers. Portfolio weights are proportional to the difference between the return over the previous 12 months relative to the average performance.

Note that these are zero-investment strategies, not portfolios. Therefore, returns are not defined for these strategies (computing returns would involve dividing by 0) and it is impossible to position



them directly in a mean–return/variance space.<sup>31</sup> To be able to measure returns and analyze mean–variance efficiency, Cochrane (1999) combines these zero–investment strategies with the risk–free asset. This gives rise to portfolios, with well defined returns, which can be cumulated over time. Furthermore, to control for risk, Cochrane (1999) designs these portfolios to match their volatility with that of the index. We follow the approach in Cochrane (1999), but we combine the SMB, HML and momentum strategy with the index, instead of the riskfree asset.<sup>32</sup> Since SMB, HML and the momentum portfolio are zero-investment portfolios, combining them with the index amounts to adjusting standard market portfolio weights. For instance, investing in the market plus SMB translates into over-weighting small firms and under-weighting large firms (relative to the index).

Of these strategies, the combination of the index and a momentum portfolio comes closest in spirit to our price-contingent strategy, because both exploit information in past returns. As shown in Section II, the momentum effect is perfectly consistent with our theoretical framework. It is, however, a secondary effect, which means that the momentum strategy should not out-perform our price-contingent strategy when evaluated in mean-variance space. The relationship between our price-contingent strategy and size and value investing is less clear, because there is no role for firm size or book value of assets in our theoretical framework. To the extent that size and value are secondary effects, enhancement of indexing by skewing weights towards small firms or high book-to-market value ratios should not lead to out-performance relative to our price-contingent strategy.

Figure 6 compares the performance of the price–contingent, size, value and momentum portfolios. We plot the evolution of the wealth of an investor starting on 6/30/1932 with one dollar and investing according to one of five possible strategies. The five strategies were constructed to have the same ex–post monthly return volatility over the period 7/1932–12/2000. Hence, the ordering in mean-variance space can be readily inferred from the relative wealth levels that the strategies generate. The five strategies are as follows.

1. *Indexing*: The investor holds the CRSP value-weighted index throughout the 68 1/2-year period.
2. *Indexing With Size Enhancement*: The investor invests 95% in the index and 5% in three-month Treasury bills, adding 0.2 units of the (zero-investment) Fama-French SMB portfolio for every dollar invested in the index, and re-balancing at the end of each month.
3. *Indexing With Value Enhancement*: The investor invests 95% in the index and 5% in three-month Treasury bills, adding 0.25 units of the (zero-investment) Fama-French HML portfolio for every dollar invested in the index, and re-balancing at the end of each month.
4. *Indexing With Momentum Enhancement*: The investor invests 95% in the index and 5% in three-month Treasury bills, adding 0.2 units of a momentum portfolio for every dollar invested in the index, and re-balancing at the end of each month. The (zero-investment) momentum portfolio invests in each of the 6 FF benchmark portfolios in proportion to the return they generated over the previous 12 months relative to the average return.
5. *Price-Contingent Strategy*: The investor puts 50% of wealth in the index, and 50% in the optimal price-contingent strategy with the same ex-ante volatility as the index; the portfolio is re-balanced monthly.

The portfolio structures in strategies 2 to 5 are designed to ensure that the (ex-post) variance of the portfolio matches that of the index. While the price contingent portfolio is simply combined with the index, the size, value and momentum portfolios are also combined with Treasury bills (for 5% of the portfolio.) This was necessary to match their volatility with that of the index. The SMB, HML and momentum portfolios have virtually zero correlation with the market (as also found by Cochrane, 1999). Hence, simply adding these zero-investment portfolios to the market would have increased the volatility of the portfolio above that of the index. To offset this increase, 5% of the

wealth is invested in Treasury bills.<sup>33</sup> In contrast, it was not necessary to include the risk-free asset in the price-contingent strategy to match its volatility with that of the index.

Figure 6 confirms the performance of the value and momentum strategies. Combining the market portfolio with the Fama-French HML portfolio improves upon indexing; a combination with the momentum portfolio does even better. The standard t-statistics for the outperformance of the size, value, momentum and price-contingent portfolios relative to the market are -0.45, 1.46, 2.14, and 2.66, respectively. Figure 6 also illustrates that the price-contingent strategy outperforms the four alternatives. Even compared to the best alternative (i.e., the momentum strategy) the outperformance of the price contingent strategy is marginally significant (the t-statistic for the difference between the performance of these two strategies is 1.88.)

The reader may wonder why we did not rely on an approach that has become standard in finance: Fama and French regressions. The two approaches are analogous. We compare the performance of identically risky portfolios, relying on Sharpe ratios and cumulated returns. Fama and French regressions test whether some combination of the index and the value or size strategies outperforms the index. But our test is more specific since we ask whether a particular portfolio rather than “some combination of FF factors” outperforms the index. Furthermore, the specific portfolio we consider is suggested by theory and is constructed using only ex-ante information.

### **4.3 The Nature Of The Return-Price Relationship**

In principle, the return-price relationships that are at the heart of the success of our price-contingent strategy can be rather counter-intuitive. For instance, as shown in Admati (1985, Section 4, pages 641 to 646), it is possible that a relatively high price for a given asset be associated with a relatively low rate of return. All depends on the correlation structure between the payoffs in the multi-asset economy. The pattern of correlations thus arising is much more complex and richer than the simple pattern of continuations upon which the momentum strategy relies.

To document this point we estimated the partial correlation between a portfolio’s return and its own price. Below is a list of the average slope coefficients in the GLS projections of the returns of the 6 FF benchmark portfolios onto prices.<sup>34</sup> We only report the slope coefficient corresponding to a portfolio’s own price. Each sixty-month estimation period prior to a target month generates one estimate. The sample standard deviation of the estimated slope coefficients is reported in parentheses. We only report results from non-overlapping sixty-month periods.<sup>35</sup> The FF benchmark portfolios are identified as holding stock in big firms (B), small firms (S), high-value (H), medium-value (M) or low-value firms (L).

BL	BM	BH	SL	SM	SH
-0.51	0.84	-1.53	-2.01	-0.47	-5.47
(0.25)	(0.60)	(0.77)	(1.27)	(1.00)	(1.69)

The (partial) correlation between a portfolio’s return and its own price tends to negative, especially for firms with extremely high or low book-to-market value ratios. Again, this points to the difference between the price–contingent strategy and the momentum strategy.

#### 4.4 The structure of the price–contingent portfolio

To better document the nature of the optimal price contingent strategy, we also computed the average weight it places on the six Fama and French portfolios. To obtain these numbers, we took the weights placed by the optimal price contingent strategy on each of the 6 portfolios for each month in our sample. We then averaged these weights across months. The average weights are reported in the next table. For the sake of comparison, the table also reports the structure of the portfolio of the six Fama and French portfolios which best replicated the index during our sample period. When considering these numbers, one should bear in mind that they are averages, around which actual weights fluctuate significantly throughout the sample period.

	BL	BM	BH	SL	SM	SH
Price contingent strategy	.525	.435	.035	-.490	.555	-.055
Index replication	.580	.270	.070	.050	.030	.000

The table shows that the optimal price contingent strategy places a lot of weight on big capitalization stocks with a low ratio of book to market value. This is approximately in line with the weight placed on those stocks by the portfolio replicating the index. In contrast, the optimal price contingent strategy involves shorting small capitalization stocks with a low ratio of book to market value. This departs markedly from the portfolio replicating the index. The price contingent strategy also places a relatively large weight on the medium value stocks, be they small or big capitalization stocks. Finally, like the portfolio replicating the index, it places negligible weight on high value small stocks. In line with the results presented in Subsections 4.2 and 4.3, these figures illustrate that the optimal price contingent strategy differs from the size and value strategies.

## 5 Conclusion

This paper studies the implications of information asymmetry for equilibrium asset pricing and portfolio choice. In our dynamic multi-asset rational expectations model, prices are only partially revealing because the demand of informed investors reflects their random endowment shocks, along with their signals. Equilibrium prices are set as in a representative agent economy where the market portfolio would include the aggregate risky endowment shock and the beliefs of the representative agent would average those of the informed and the uninformed. This pricing relation cannot be directly tested since the beliefs of the representative agent are not observable to the econometrician. On the other hand, the information set of the econometrician is comparable to that of some agents in the model: the uninformed agents with no endowment shocks. We implement empirically the optimal portfolio strategy implied by our theory for these agents. Thus we construct their condi-

tional mean–variance optimal portfolio. Consistent with our theory, we find that it outperforms the index both economically and statistically.

There is still ample scope for improving the performance of price–contingent strategies. Our results are based on rather crude groupings of stocks. Less aggregate groupings should be contemplated, as well as other groupings (e.g., industry-based portfolios).<sup>36</sup> Our estimation of the correlation between returns and prices is based on simple linear generalized least squares. We did not investigate more sophisticated specifications or estimation strategies, such as nonlinear least squares or conditional heteroscedasticity. No attempt was made to estimate the optimal window size on which to estimate the correlation between prices and returns. Refining the statistical analysis along those and other lines may yield more powerful information extraction and consequently superior performance.

The significant outperformance we uncover suggests that the price–contingent investment approach is a valuable complement to fundamental and quantitative investment analysis. It should be emphasized that our results are out of sample, so that the outperformance we obtain is based on information that was available to the investors at the time portfolio allocation decisions had to be made. Our results suggest that value can be created not only in traditional ways, by designing optimal portfolios (quantitative investment analysis) or estimating cash flows (fundamental investment analysis), but also by studying price formation in the marketplace and using the results to infer information about future returns that only competitors observe directly. Our setting provides a reconciliation between the philosophies of active and passive portfolio management as investors tilt their portfolios in favor of the assets for which they are particularly optimistic and in that sense follow active strategies.

## Figure Legends

**Figure 1:** Sequence of events in the theoretical model.

**Figure 2:** Estimates of price equation coefficients as a function of the proportion lambda ( $\lambda$ ) of informed agents. Parameters: Two identical risky securities with uncorrelated payoffs and signals, both in unit supply;  $r = 0.100$ ;  $\rho_f = 0.990$ ,  $\rho_z = 0$ ;  $\sigma_f^2 = 0.100$ ,  $\sigma_z^2 = 0.010$ ,  $\sigma^2 = 0.001$ ;  $\gamma = 1.735$ .

**Figure 3:** Estimates of 60-period average return (in excess of the market) of the momentum portfolio (left panel) and the optimal price contingent portfolio of an uninformed agent without endowment shocks (right panel) as a function of the proportion ( $\lambda$ ) of informed agents. Vertical bars denote 95% confidence interval. Parameters: Two identical risky securities with uncorrelated payoffs and signals, both in unit supply;  $r = 0.100$ ;  $\rho_f = 0.990$ ,  $\rho_z = 0$ ;  $\sigma_f^2 = 0.100$ ,  $\sigma_z^2 = 0.010$ ,  $\sigma^2 = 0.001$ ;  $\gamma = 1.735$ .

**Figure 4:** Evolution of the difference between the Sharpe ratios of: (i) a strategy with 50% in the CRSP value-weighted index (the market index) and 50% in the optimal price-contingent portfolio whereby the return-prices relationship is estimated from the sixty months prior to the target month [weights change as a function of (a) expected returns based on relative prices and the estimated price-return relationship and (b) corresponding prediction error variances; the ex-ante volatility of this portfolio is the same as that of the market index], and (ii) the market index, 7/1927-12/2000. The two strategies generate the same ex-post volatility. The difference in Sharpe ratios is estimated on the basis of a moving, fixed-length window of sixty months centered around the target month.

**Figure 5:** Evolution of the partial  $z$ -statistic of the difference in return between: (i) a strategy with 50% in the CRSP value-weighted index (the market index) and 50% in the optimal price-contingent portfolio whereby the return-prices relationship is estimated from the sixty months prior to the target month [weights change as a function of (a) expected returns based on relative prices and the estimated price-return relationship and (b) corresponding prediction error variances; the ex-ante

volatility of this portfolio is the same as that of the market index], and (ii) the market index, 7/1927-12/2000. The two strategies generate the same ex-post volatility. Strategy (i) outperforms strategy (ii) when the partial  $z$ -statistic is positive; the performance is significantly different from zero in a given ten-year period if the partial  $z$ -statistic moves outside the 95% confidence region bounded by the square-root function anchored at the beginning of the ten-year period.

**Figure 6:** Wealth evolution from investing one dollar on 6/30/1932 and monthly re-investing returns, various strategies (natural log scale). All strategies generated the same (ex-post) volatility over the period 7/1932-12/2002. The strategies are: (i) Indexing (solid line; 100% investment in the CRSP value-weighted index [the market portfolio]); (ii) Indexing with Size Enhancement (dash-dotted line; 95% investment in the market portfolio, plus 20% invested in the Fama-French SMB zero-investment portfolio and 5% in three-month Treasury bills; the SMB portfolio is long small firms and short large firms); (iii) Indexing with Value Enhancement (dashed line; 95% investment in the market portfolio, plus 25% invested in the Fama-French HML zero-investment portfolio and 5% in three-month Treasury bills; the HML portfolio is long firms with high book-to-market value ratio and short firms with low book-to-market value ratio); (iv) Indexing with Momentum Enhancement (dotted line; 95% investment in the market portfolio, plus 20% invested in the standard zero-investment momentum portfolio and 5% in three-month Treasury bills; the momentum portfolio is long recent winners and short recent losers among the 6 FF benchmark portfolios; winners and losers are determined by the return over the previous 12 months relative to the average); (v) Our price-contingent strategy (heavy solid line; 50% in the market portfolio and 50% in an optimal price-contingent portfolio, as explained in the captions of Figures 4 and 5).



## Appendix: Proofs

**Proof of Proposition 2:** The market-clearing condition is:

$$\gamma(x + z^t) = \lambda\tau_y\{E[f^{t+1} + p^{t+1}|y^t] - (1+r)p^t\} + (1-\lambda)\tau_p\{E[f^{t+1} + p^{t+1}|p^t] - (1+r)p^t\}.$$

Thus:

$$\begin{aligned} & \tau_p \frac{E[f^{t+1} + p^{t+1}|p^t] - (1+r)p^t}{\gamma} \\ = & (x + z^t) - \frac{\lambda}{\gamma} [\tau_y\{E[f^{t+1} + p^{t+1}|y^t] - (1+r)p^t\} - \tau_p\{E[f^{t+1} + p^{t+1}|p^t] - (1+r)p^t\}]. \end{aligned}$$

Noting that the left-hand-side of this inequality is equal to the demand of the uninformed agent, we obtain the proposition.

QED

**Proof of Proposition 3:** By the projection theorem:

$$E[f^{t+1} + p^{t+1}|p^t] = E(f^{t+1} + p^{t+1}) + Cov(f^{t+1} + p^{t+1}, p^t)(Var(p^t))^{-1}(p^t - E(p^t)).$$

Substituting the conjectured price function (3) and the recursive definition of  $f^{t+1}$  and  $z^{t+1}$  ((2) and (1) respectively) into this conditional expectation:

$$\begin{aligned} & E[f^{t+1} + p^{t+1}|p^t] \\ = & \mu + \rho_f f^t + E[Af^{t+1} + By^{t+1} - Cz^{t+1} - Dz^t + G] \\ & + Cov(\rho_f f^t + \epsilon_f^t + Af^{t+1} + By^{t+1} - Cz^{t+1} - Dz^t, Af^t + By^t - Cz^t - Dz^{t-1}) \\ & \times (Var(Af^t + By^t - Cz^t - Dz^{t-1}))^{-1} \\ & \times [p^t - E(p^t)]. \end{aligned}$$

Denote:  $\psi = I + A + B\rho_f$ . The intercept simplifies to:

$$\begin{aligned} & (I + A)\mu + (I + A)\rho_f f^t + E[By^{t+1} - (C\rho_z + D)z^t] + G \\ &= (I + A)\mu + \psi\rho_f f^t - (C\rho_z + D)\rho_z z^{t-1} + G + B(I + \rho_f)\mu. \end{aligned}$$

The slope becomes:

$$\begin{aligned} & Cov(\rho_f f^t + \epsilon_f^t + A(\rho_f f^t + \epsilon_f^t) + By^{t+1} - C(\rho_z z^t + \epsilon_z^{t+1}) - Dz^t), \\ & Af^t + By^t - C(\rho_z z^{t-1} + \epsilon_z^t) - Dz^{t-1}) \\ & \times (Var(Af^t + By^t - Cz^t - Dz^{t-1}))^{-1} \end{aligned}$$

Now, at time  $t$ ,  $f^t$  and  $z^{t-1}$  are known, hence the slope simplifies to:

$$Cov((I + A)\epsilon_f^t + By^{t+1} - C(\rho_z z^t + \epsilon_z^{t+1}) - Dz^t, By^t - C\epsilon_z^t) \times (Var(By^t - Cz^t))^{-1}.$$

Substituting the recursive equation for  $z$  and  $y$ , after some simplifications we get that:

$$\begin{aligned} & Cov(f^{t+1} + p^{t+1}, p^t)(Var(p^t))^{-1} = \\ & (\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C') (BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C')^{-1}. \end{aligned} \quad (17)$$

Finally,

$$E[p^t] = E(Af^t + By^t - Cz^t - Dz^{t-1}) + G = (\psi - I)f^t - (C\rho_z + D)z^{t-1} + G + B\mu.$$

Putting all this together, the conditional expectation of the uninformed agents is:

$$\begin{aligned} & E[f^{t+1} + p^{t+1}|p^t] \\ &= (\psi + B)\mu + \psi\rho_f f^t - (C\rho_z + D)\rho_z z^{t-1} + G \\ &+ \{\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'\} \times [p^t - (\psi - I)f^t + (C\rho_z + D)z^{t-1} - G - B\mu] \\ &\times (BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C')^{-1}. \end{aligned}$$

For the informed agent, by the projection theorem:

$$E[f^{t+1} + p^{t+1}|y^t] = E[f^{t+1} + p^{t+1}] + Cov(f^{t+1} + p^{t+1}, y^t) (Var(y^t))^{-1} [y^t - E(y^t)].$$

The intercept is the same as for the uninformed agent. Now:

$$Cov(f^{t+1} + p^{t+1}, y^t) = \psi, \quad (18)$$

and:

$$Var(y^t)^{-1} = Var(\epsilon_f^t)Var(\epsilon_f^t + \epsilon^t)^{-1}. \quad (19)$$

After some manipulation, the slope simplifies to:  $(I + A + B\rho_f)Var(\epsilon_f^t) (Var(\epsilon_f^t + \epsilon^t))^{-1}$ . Also,  $E(y^t) = \mu + \rho_f f^t$ . Thus the conditional expectation for the informed agents is:

$$\begin{aligned} E[f^{t+1} + p^{t+1}|y^t] &= (I + A)\mu + \psi\rho_f f^t - (C\rho_z + D)\rho_z z^{t-1} + G + B(I + \rho_f)\mu \\ &\quad + \psi Var(\epsilon_f^t) (Var(\epsilon_f^t + \epsilon^t))^{-1} [y^t - \mu - \rho_f f^t]. \end{aligned}$$

Putting these results together, the price equation becomes:

$$\begin{aligned} p^t(1 + r) &= (I + A)\mu + \psi\rho_f f^t - (C\rho_z + D)\rho_z z^{t-1} + G + B(I + \rho_f)\mu \\ &\quad + \omega\psi Var(\epsilon_f^t) (Var(\epsilon_f^t + \epsilon^t))^{-1} [y^t - \mu - \rho_f f^t] \\ &\quad + (I - \omega)[\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'] \\ &\quad \times (BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C')^{-1} \\ &\quad \times [p^t - (\psi - I)f^t + (C\rho_z + D)z^{t-1} - G - B\mu] \\ &\quad - \gamma(\lambda\tau_y + (1 - \lambda)\tau_p)^{-1}(x + z^t). \end{aligned}$$

Denote:

$$\Phi = (1 + r)I - [I - (\lambda\tau_y + (1 - \lambda)\tau_p)^{-1}\lambda\tau_y]$$

$$[\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'] [BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C']^{-1}. \quad (20)$$

The price equation can be rewritten as:

$$\begin{aligned} \Phi p^t &= \{[(I + A) + B\rho_f - \omega\psi Var(\epsilon_f^t)(Var(\epsilon_f^t + \epsilon^t))^{-1}]\rho_f \\ &\quad - (I - \omega)[\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'] \\ &\quad \times (BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C')^{-1} (\psi - I)\} f^t \\ &\quad + \{\omega\psi Var(\epsilon_f^t)(Var(\epsilon_f^t + \epsilon^t))^{-1}\} y^t \\ &\quad - \{\gamma(\lambda\tau_y + (1 - \lambda)\tau_p)^{-1}\} z^t \\ &\quad - \{(C\rho_z + D)\rho_z \\ &\quad - (I - \omega)[\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'] \\ &\quad \times (BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C')^{-1} (C\rho_z + D)\} z^{t-1} \\ &\quad - (I - \omega)[\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'] \\ &\quad \times (BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C')^{-1} (G + B\mu) \\ &\quad + G + (\psi + B)\mu - \omega(I + A + B\rho_f)Var(\epsilon_f^t)(Var(\epsilon_f^t + \epsilon^t))^{-1}\mu \\ &\quad - \gamma(\lambda\tau_y + (1 - \lambda)\tau_p)^{-1}x. \end{aligned}$$

The conditional precision of the information of the informed agent is:

$$(\tau_y)^{-1} = Var(f^{t+1} + p^{t+1}) - Cov(f^{t+1} + p^{t+1}, y^t) (Var(y^t))^{-1} Cov(f^{t+1} + p^{t+1}, y^t)'.$$

The first term is:

$$Var(f^{t+1} + p^{t+1}) = \psi Var(\epsilon_f^t)\psi' + B[Var(\epsilon_f^t) + Var(\epsilon^{t+1})]B' + [C\rho_z + D]Var(\epsilon_z^t)[C\rho_z + D]' + CVar(\epsilon_z^t)C'. \quad (21)$$

Substituting from equations (18), (19), and (21):

$$(\tau_y)^{-1} = \psi Var(\epsilon_f^t)\psi' + BVar[(\epsilon_f^t) + Var(\epsilon^{t+1})]B$$

$$+[C\rho_z + D]Var(\epsilon_z^t)[C\rho_z + D]' + CVar(\epsilon_z^t)C' - \psi Var(\epsilon_f^t) (Var(\epsilon_f^t + \epsilon^t))^{-1} Var(\epsilon_f^t)' \psi'. \quad (22)$$

The conditional precision of the information of the uninformed agent is:

$$(\tau_p)^{-1} = Var(f^{t+1} + p^{t+1}) - Cov(f^{t+1} + p^{t+1}, p^t) (Var(p^t))^{-1} Cov(f^{t+1} + p^{t+1}, p^t)'.$$

Substituting from equations (17) and (21), we get:

$$\begin{aligned} (\tau_p)^{-1} &= \psi Var(\epsilon_f^t) \psi' + B[Var(\epsilon_f^t) + Var(\epsilon^{t+1})]B' + [C\rho_z + D]Var(\epsilon_z^t)[C\rho_z + D]' + CVar(\epsilon_z^t)C' \\ &\quad - \{\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'\} \times \{BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C'\} \\ &\quad \times \{\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'\}'. \end{aligned} \quad (23)$$

Identifying the price function resulting from the market-clearing condition with the conjectured price function, one gets the following system of equations:

$$\begin{aligned} \Phi A &= [(\psi - \omega\psi Var(\epsilon_f^t)(Var(\epsilon_f^t + \epsilon^t))^{-1})\rho_f \\ &\quad - (I - \omega)[\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'] (BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C')^{-1} (A + B\rho_f)]. \end{aligned} \quad (24)$$

$$\Phi B = \omega\psi Var(\epsilon_f^t)(Var(\epsilon_f^t + \epsilon^t))^{-1}. \quad (25)$$

$$\Phi C = \gamma(\lambda\tau_y + (1 - \lambda)\tau_p)^{-1}. \quad (26)$$

$$\Phi D = (C\rho_z + D)\rho_z$$

$$-(I - \omega)[\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'] (BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C')^{-1} (C\rho_z + D) \quad (27)$$

$$-\Phi G = (I - \omega)[\psi Var(\epsilon_f^t)B' + (C\rho_z + D)Var(\epsilon_z^t)C'] (BVar(\epsilon_f^t + \epsilon^t)B' + CVar(\epsilon_z^t)C')^{-1} (G + B\mu)$$

$$-(G + (\psi + B)\mu) + \omega\psi Var(\epsilon_f^t)(Var(\epsilon_f^t + \epsilon^t))^{-1}\mu + \gamma(\lambda\tau_y + (1 - \lambda)\tau_p)^{-1}x. \quad (28)$$

Note that equation (26) directly yields the expression for  $C$  in Proposition 3. To simplify this system of equations, we rely on the following lemma:

**Lemma 1:**  $\psi = [I - (1 + r)^{-1}\rho_f]^{-1}$ .<sup>37</sup>

**Proof of Lemma 1:** From (24) and the definition of  $\Phi$ , we have that:  $A = \Phi^{-1}\{[\psi - \omega\psi Var(\epsilon_f^t)(Var(\epsilon_f^t + \epsilon^t))^{-1}]\rho_f + [\Phi - (1 + r)I](\psi - I)\}$ . Combining this with (25), we obtain that:  $I + A + B\rho_f = I + \Phi^{-1}\{\psi\rho_f + [\Phi - (1 + r)I](\psi - I)\}$ , which, after some simplifications, yields the result.

QED

Substituting  $\psi$  from Lemma 1 and  $\omega$  in equation (25) we obtain the expression for  $B$  in Proposition 3. Noting that  $-\Phi^{-1}(I - \omega) = I - (1 + r)^{-1}\Phi^{-1}$  and substituting  $\psi$  and  $\omega$  into equation (27) we obtain the expression for  $D$  in Proposition 3. Substituting  $\psi$  from Lemma 1 in  $\tau_p$  and  $\tau_y$  from (22) and (23), we get  $\omega$ ,  $\tau_p$  and  $\tau_y$  as functions of the exogenous parameters and  $B, C$  and  $D$ , as stated in Proposition 3. Similarly, substituting  $\Phi$ ,  $\psi$ ,  $\omega$ ,  $\tau_p$  and  $\tau_y$  into (24) and (28) we get  $A$  and  $G$  as functions of  $B, C$ , and  $D$ , as stated in Proposition 3.

QED

**Proof of Corollary 1:** To establish that the price function stated in the corollary is an equilibrium price, we substitute the linear price function in the market-clearing condition and check that the equality holds for the coefficients stated in Corollary 1. Substituting:  $p^t = Af^t + G$ , into:  $p^t = (1 + r)^{-1}\{E(p^{t+1} + f^{t+1}) - \gamma\tau^{-1}x\}$ , yields:  $Af^t + G = (1 + r)^{-1}\{E(Af^{t+1} + G + f^{t+1}) - \gamma\tau^{-1}x\}$ . That is:  $Af^t + G = (1 + r)^{-1}\{(I + A)(\mu + \rho_f f^t) + G - \gamma\tau^{-1}x\}$ . Or:  $[(1 + r)A - (I + A)\rho_f]f^t = -rG + (I + A)\mu - \gamma\tau^{-1}x$ . For this to hold for each realization of  $f^t$ , we need that:  $(1 + r)A - (I + A)\rho_f = 0$  and:  $rG = +(I + A)\mu - \gamma\tau^{-1}x$ . This directly yields the coefficients stated

in Corollary 1.

QED

**Proof of Corollary 2:** Substituting the price function (3) and the recursive definition of  $f^{t+1}$  (2) into the return equation (8) yields:  $R^{t+1} = (I + A)\mu + ((A + I)\rho_f - A)f^t + (A + I)\epsilon_f^t$ . With ex-ante identical and independent dividend processes, the coefficient matrix  $A$  is diagonal with common diagonal element  $\bar{a}(> 0)$ . For the  $n$ th asset, the (unconditional) return autocovariance  $Cov(R_n^{t+1}, R_n^t)$  equals:

$$Cov(((\bar{a} + 1)\bar{\rho}_f - \bar{a})(\rho_f f_n^{t-1} + \epsilon_{f,n}^{t-1}) + (\bar{a} + 1)\epsilon_{f,n}^t, ((\bar{a} + 1)\bar{\rho}_f - \bar{a})f_n^{t-1} + (\bar{a} + 1)\epsilon_{f,n}^{t-1}).$$

Now, in this simple case we have:  $\bar{a} = \bar{\rho}_f(1 + r - \bar{\rho}_f)^{-1}$ , so:

$$(\bar{a} + 1)\bar{\rho}_f - \bar{a} = \frac{1 + r}{1 + r - \bar{\rho}_f}\bar{\rho}_f - \frac{\bar{\rho}_f}{1 + r - \bar{\rho}_f} = \frac{r\bar{\rho}_f}{1 + r - \bar{\rho}_f}.$$

Hence the covariance  $Cov(R_n^{t+1}, R_n^t)$  is :

$$\left(\frac{r\bar{\rho}_f}{1 + r - \bar{\rho}_f}\right)^2 \bar{\rho}_f Var(f_n^{t-1}) + \left(\frac{r\bar{\rho}_f}{1 + r - \bar{\rho}_f}\right)\left(\frac{1 + r}{1 + r - \bar{\rho}_f}\right) Var(\epsilon_{f,n}^{t-1}) > 0.$$

Now turn to the momentum strategy. Let  $M$  denote the (unconditional) expected return on asset  $n$ . Letting  $\bar{\mu}$  denote the  $n$ th element of  $\mu$  (they are all the same, so we drop the subscript  $n$ ), and denote  $M = E[R_n^{t+1}]$ . The position in asset  $n$  equals  $(1/N)(R_n^t - \bar{R}^t)$ . The return on the momentum portfolio equals:

$$\frac{1}{N}(R^t - \bar{R}^t \mathbf{1})' R^{t+1} = \frac{1}{N}(R^t - M \mathbf{1})' R^{t+1} - \frac{1}{N}(\bar{R}^t - M) \mathbf{1}' R^{t+1}.$$

As  $N \rightarrow \infty$ , the second term on the right-hand-side converges to zero (based on the principle that if a sequence of random variables  $X_N$  converges in probability to 0 and another one  $Y_N$  converges in probability to  $Y$ , then  $X_N Y_N$  converges in probability to zero as well). So, we only need to

investigate the first term:

$$E\left[\frac{1}{N}(R^t - M\mathbf{1})'R^{t+1}\right] = \frac{1}{N} \sum_{n=1}^N Cov(R_n^{t+1}, R_n^t) > 0.$$

QED

**Proof of Corollary 3:** With random endowment shocks and no private information, the linear price function is:  $p^t = Af^t - Cz^t + G$ , and Proposition 3 yields:  $A = \rho_f[(1+r)I - \rho_f]^{-1}$ ,  $C = \frac{\gamma}{1+r}\tau^{-1}$ ,  $G = r^{-1}[(I+A)\mu - \gamma\tau^{-1}x]$ , and:  $\tau^{-1} = [I+A]'Var(\epsilon_f^t)[I+A] + CVar(\epsilon_z^t)C'$ . Substituting for  $A$  and  $C$  in the latter equation, we obtain:

$$\tau^{-1} = [I + \rho_f[(1+r)I - \rho_f]^{-1}]'Var(\epsilon_f^t)[I + \rho_f[(1+r)I - \rho_f]^{-1}] + \frac{\gamma}{1+r}\tau^{-1}Var(\epsilon_z^t)\left(\frac{\gamma}{1+r}\tau^{-1}\right)'$$

Substituting  $\rho_f = \bar{\rho}_f I$ ,  $Var(\epsilon_z) = \sigma_z^2 I$  and  $Var(\epsilon_f) = \sigma_f^2 I$ , this rewrites as:

$$\tau^{-1} = (I + \bar{\rho}_f I ((1+r)I - \bar{\rho}_f I)^{-1})' \sigma_f^2 I (I + \bar{\rho}_f I ((1+r)I - \bar{\rho}_f I)^{-1}) + \left(\frac{\gamma}{1+r}\tau^{-1}\right)' \sigma_z^2 I \left(\frac{\gamma}{1+r}\tau^{-1}\right). \quad (29)$$

That is:

$$\tau^{-1} = \left[1 + \frac{\bar{\rho}_f}{(1+r) - \bar{\rho}_f}\right]^2 \sigma_f^2 I + \left[\frac{\gamma}{1+r}\right]^2 \sigma_z^2 (\tau^{-1})' \tau^{-1}. \quad (30)$$

To solve the equilibrium, we must study if equation (30) admits a solution, and if, it does, find its roots. Denote:

$$\alpha = \left[\frac{\gamma}{1+r}\right]^2 \sigma_z^2 \quad \text{and} \quad \beta = \left[1 + \frac{\bar{\rho}_f}{(1+r) - \bar{\rho}_f}\right]^2 \sigma_f^2 = \left[\frac{1+r}{1+r - \bar{\rho}_f}\right]^2 \sigma_f^2,$$

and  $\tau^{-1} = Q$ . Equation (30) writes as:

$$aQ'Q - Q + bI = 0. \quad (31)$$

Since  $Q$  is a symmetric matrix, it can be written as:  $Q = V\Lambda V'$ , where  $V$  is the matrix of eigenvectors and  $\Lambda$  the diagonal matrix of eigenvalues. So we can rewrite (31) as:

$$\alpha(V\Lambda V')'V\Lambda V' - V\Lambda V' + \beta I = 0. \quad (32)$$



Note that  $V$  is also the eigenvector matrix of  $I$ , i.e.:  $VIV' = VV' = I$ . Note also that  $V = V'$  since  $V$  is symmetric. Thus, (32) is equivalent to:

$$\alpha V\Lambda'V'V\Lambda V' - V\Lambda V' + \beta I = V(\alpha\Lambda'\Lambda)V' - V\Lambda V' + V\beta IV' = 0. \quad (33)$$

Thus we can rewrite this equation as:

$$V(a\Lambda'\Lambda - \Lambda + bI)V' = 0. \quad (34)$$

(34) can be rewritten as:

$$VPV' = 0, \quad (35)$$

where  $P$  is the diagonal matrix, whose diagonal elements are roots of the quadratic scalar equation:

$$\alpha l^2 - l + \beta = 0. \quad (36)$$

Existence and multiplicity of equilibria will therefore depend on the sign of the discriminant of this equation:

$$\Delta = 1 - 4\alpha\beta = 1 - 4\left[\frac{\gamma}{1+r-\bar{\rho}_f}\right]^2\sigma_f^2\sigma_z^2.$$

If  $\Delta$  is negative then there is no solution to (34) and thus there does not exist a linear rational expectation equilibrium.  $\Delta$  is positive iff:

$$1+r > \bar{\rho}_f - 2\gamma\sigma_f\sigma_z. \quad (37)$$

If this inequality holds, then there are two roots to (34). In that case, there exists  $2^N$  possible matrices  $P$ , solving equation (35). Correspondingly there are  $2^N$  linear rational expectations equilibria.

QED

**Proof of Corollary 4:** Since endowment shocks are not serially correlated,  $D = 0$ . In this simple 2 assets case,  $A, B$  and  $C$  are two by two matrices and  $G$  is a two by one vector. In line with the symmetry of the distributions of the two assets, we focus on symmetric equilibria where

the coefficients are the same for the two assets in the parameter matrices and vector:  $A, B, C$  and  $G$ . We know from Proposition 3 how  $A$  and  $G$  can be obtained from  $B$  and  $C$ , so we focus only on the latter. Since  $B$  is a symmetric matrix, it can be expressed as:  $B = V\Lambda_B V'$ , where  $V$  is the matrix of eigenvectors and  $\Lambda_B$  the diagonal matrix of eigenvalues:

$$\Lambda_B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}. \quad (38)$$

In the symmetric equilibrium case, the matrices share common eigenvectors. Thus we can write:

$C = V\Lambda_C V'$ , where:

$$\Lambda_C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad (39)$$

is the matrix of eigenvalues of  $C$ . Similarly,  $\omega = V\Lambda_\omega V'$  and  $\Phi = V\Lambda_\Phi V'$ , where:

$$\Lambda_\omega = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \quad (40)$$

and,

$$\Lambda_\Phi = \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}. \quad (41)$$

Under our distributional assumptions,  $\Phi$  simplifies to:

$$\Phi = (1+r)I - \sigma_f^2(I - \omega)\psi B'((\sigma_f^2 + \sigma^2)BB' + \sigma_z^2 CC')^{-1}.$$

Using the decomposition of the matrices in eigenvectors and eigenvalues and Lemma 1, and noting that  $VV' = I$ , this yields:

$$V\Lambda_\Phi V' = (1+r)VV' - \sigma_f^2(VV' - V\Lambda_\omega V')[I - (1+r)^{-1}\rho_f]^{-1}V\Lambda_B V'((\sigma_f^2 + \sigma^2)BB' + \sigma_z^2 CC')^{-1}. \quad (42)$$

Note that  $BB' = V\Lambda_B\Lambda_B V'$  and  $CC' = V\Lambda_C\Lambda_C V'$ . Substituting (38) and (39), we get that:

$$((\sigma_f^2 + \sigma^2)BB' + \sigma_z^2 CC')^{-1} = V \begin{pmatrix} ((\sigma_f^2 + \sigma^2)b^2 + \sigma_z^2 c^2)^{-1} & 0 \\ 0 & ((\sigma_f^2 + \sigma^2)b^2 + \sigma_z^2 c^2)^{-1} \end{pmatrix} V'.$$

Also,

$$[I - (1+r)^{-1}\rho_f]^{-1} = V \begin{pmatrix} (1 - \rho_f(1+r)^{-1})^{-1} & 0 \\ 0 & (1 - \rho_f(1+r)^{-1})^{-1} \end{pmatrix} V',$$

and:

$$VV' - V\Lambda_\omega V' = V \begin{pmatrix} \sigma_f^2(1-w) & 0 \\ 0 & \sigma_f^2(1-w) \end{pmatrix} V'.$$

Substituting into (42), we get that:

$$V\Lambda_\Phi V' = V \begin{pmatrix} (1+r) - \frac{\sigma_f^2(1-w)b}{((\sigma_f^2 + \sigma^2)b^2 + \sigma_z^2 c^2)(1 - \rho_f(1+r)^{-1})} & 0 \\ 0 & (1+r) - \frac{\sigma_f^2(1-w)b}{((\sigma_f^2 + \sigma^2)b^2 + \sigma_z^2 c^2)(1 - \rho_f(1+r)^{-1})} \end{pmatrix} V'.$$

Hence:

$$\phi = (1+r) - \frac{\sigma_f^2(1-w)b}{((\sigma_f^2 + \sigma^2)b^2 + \sigma_z^2 c^2)(1 - \rho_f(1+r)^{-1})}.$$

Proceeding similarly, we get the eigenvalues of  $B, C, w, \tau_y,$  and  $\tau_p$  stated in Corollary 4.

QED

## References

Admati, A., 1985, "A Noisy Rational Expectations Equilibrium for Multiple Asset Securities Markets," *Econometrica*, 53, 629–657.

Biais, B., P. Bossaerts and C. Spatt, 2007, "Equilibrium asset pricing and portfolio choice under asymmetric information," IDEI Working paper n°474.

Bossaerts, P., 1995, "The Econometrics of Learning in Financial Markets," *Econometric Theory*, 11, 151-189.

Brennan, M., and H. Cao, 1997, "International Portfolio Investment Flows," *Journal of Finance*, 52, 1851-1880.

Chan, L., N. Jegadeesh, and J. Lakonishok, 1996, "Momentum Strategies," *Journal of Finance*, 51, 1681-1713.

Cho, J.-W., and M. Krishnan, 2000, "Prices as Aggregators of Private Information: Evidence from S&P 500 Futures Data," *Journal of Financial and Quantitative Analysis*, 35, 111-126.

Cochrane, J.H., 1999, "New Facts in Finance," *Economic Perspectives*, 23, 36–58.

Cooper, M., R. Gutierrez, and W. Marcum, 2005, "On the Predictability of Stock Returns in Real Time," *Journal of Business*, 78, 469–500.

Davis, J., E. Fama, and K. French, 2000, "Characteristics, Covariances, and Average Returns: 1929 to 1997," *Journal of Finance*, 55, 389–406.

DeMarzo, P., and C. Skiadas, 1998, "Aggregation, Determinacy, and Informational Efficiency for a Class of Economies with Asymmetric Information," *Journal of Economic Theory*, 80, 123-152.

Fama, E., and K. French, 1996, "Multifactor Explanations of Asset Pricing Anomalies," *Journal of Finance*, 51, 55–84.

Fama, E., and J. MacBeth, 1973, "Risk, Return, and Equilibrium: Empirical Tests," *Journal of Political Economy*, 81, 607–636.

Grossman, S., and J. Stiglitz, 1980, "On the Impossibility of Informationally Efficient Markets," *American Economic Review*, 70, 393–408.

Grundy, B., and Y. Kim, 2002, "Stock Market Volatility in a Heterogeneous Information Economy," *Journal of Financial and Quantitative Analysis*, 37, 1-27.

Hansen, L., and K. Singleton, 1982, "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models," *Econometrica*, 50, 1269-86.

Hellwig, M., 1980, "On the Aggregation of Information in Competitive Markets," *Journal of Economic Theory*, 22, 477-498.

Huang, C., and R. Litzenberger, 1988, *Foundations for Financial Economics*. North-Holland, New York.

Jimenez Garcia, S., 2004, "Information privee sur les marches financiers: Une etude de la prime de risque dans un cadre general." Ph.D. Dissertation, Grenoble University.

Kandel, E., and R. Stambaugh, 1996, "On The Predictability of Stock Returns: An Asset-Allocation Perspective," *Journal of Finance*, 51, 385-424.

Lewellen, J., 2002, "Temporary Movements in Stock Prices," MIT Sloan School working paper.

Lintner, J., 1969, "The Aggregation of Investors' Diverse Judgments and Preferences in Purely Competitive Security Markets," *Journal of Financial and Quantitative Analysis*, 4, 347-400.

Lucas, R., 1978, "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1445.

Mayers, D., 1974, "Portfolio Theory, Job Choice and the Equilibrium Structure of Expected Wages," *Journal of Financial Economics*, 1, 23-42.

O'Hara, M., 2003, "Presidential Address: Liquidity and Price Discovery," *Journal of Finance*, 58, 1335-1354.

Roll, R., 1977, "A Critique of the Asset Pricing Theory's Tests. Part I: On Past and Potential Testability of the Theory," *Journal of Financial Economics*, 4, 129–176.

Spiegel, M., 1998, "Stock Price Volatility in a Multiple Security Overlapping Generations

Model,” *Review of Financial Studies*, 11, 419–447.

Stapleton, R., and M. Subrahmanyam, 1978, “A Multiperiod Equilibrium Asset Pricing Model,” *Econometrica*, 46, 1077-1096.

Van Nieuwerburgh, S., and L. Veldkamp, 2005, “Information Acquisition and Portfolio Under-Diversification,” New York University Stern School of Business working paper.

Wang, J., 1993, “A Model of Intertemporal Asset Prices Under Asymmetric Information,” *Review of Economic Studies*, 60, 249–282.

Watanabe, M., 2005, “Rational Trend Followers and Contrarians in Excessively Volatile, Correlated Markets,” Rice University working paper.

Wilson, R., 1968, “The Theory of Syndicates,” *Econometrica*, 36,119-132.

## Notes

<sup>1</sup>See, e.g., Chan, Jegadeesh and Lakonishok (1996) and Lewellen (2002).

<sup>2</sup>Another difference is that finite horizon models are non-stationary, while we analyze stationary price equilibrium functions.

<sup>3</sup>There also is a similar rational expectations loop in Spiegel (1998). In that overlapping generations model, prices change because of supply shocks and public information, but there are no private signals.

<sup>4</sup>Investors can perform the projection because they know the parameters of the cash flows, endowment processes as well as the pricing equation.

<sup>5</sup>Our modeling and interpretation of state contingent endowment shocks differs slightly from that of Watanabe (2005). In our analysis,  $z_a^t$  is the sensitivity of personal income to risky asset returns. Watanabe (2005) models shocks in the supply of risky assets. In terms of theoretical modeling, there is no significant mathematical difference between the two approaches. But our interpretation emphasizes the difference between tradeable risky assets and aggregate risk exposure. This matters for the interpretation of our theoretical and empirical results.

<sup>6</sup>Van Nieuwerburgh and Veldkamp (2005) extends our analysis. They study portfolio implications of particular types of information heterogeneity. Their approach is static, however, as in Admati's original model.

<sup>7</sup>In the Roll Critique, the econometrician cannot observe the market portfolio. In our analysis, the econometrician and the uninformed agents cannot observe the portfolio of aggregate risk. While in the original Roll Critique, lack of observability has statistical consequences only, in our analysis it also has economic consequences, as it precludes full revelation of the private signals. That is, the Roll Critique states that the CAPM can be tested in principle, but not in practice; if our model of partial revelation is true, then the CAPM-like pricing equation cannot even be tested in principle.

<sup>8</sup>Proposition 6 in DeMarzo and Skiadas (1998) establishes that a CAPM holds in equilibrium. They assume that the endowment of agent  $i$  is  $e_i = a_i + b_i V$ , where  $V$  is the value of the asset and  $a_i$  and  $b_i$  are coefficients such that:  $a = \sum_i a_i$  and  $b = \sum_i b_i$  are common knowledge to all the agents (see Definition 4, pages 138 and 139). Hence, in this economy, the aggregate endowment of the risky assets is common knowledge.

<sup>9</sup>This contrasts with two-fund separation result obtained by the DeMarzo and Skiadas (1998) under the assumption that the aggregate supply of risky assets is common knowledge.

<sup>10</sup>There is an additional complexity arising in Wang (1993), but not in the present analysis. We assume the informed agents observe a signal on the next cash flow, which is then publicly observed. Wang (1993) assumes informed agents observe the stochastically evolving mean of the cash-flow process. Thus, the uninformed agents must continuously

learn about this state variable. Hence, to write the equilibrium price as a function of observable variables, one would have to include the entire history of the process.

<sup>11</sup>This is similar to the dynamic CAPM studied by Stapleton and Subrahmanyam (1978).

<sup>12</sup>The observation in Lucas (1978) that dynamic general equilibrium may give rise to a variety of types of predictability already foreshadowed that it would be possible to generate momentum in specific cases.

<sup>13</sup>As  $\lambda$  goes to 0,  $\int_{a \in S} z_a^t da$  is scaled such that in the limit  $z^t$  does not go to 0.

<sup>14</sup>We are grateful to the referee for suggesting to us this result and its proof.

<sup>15</sup>We are grateful to the referee for suggesting to us this result and its proof.

<sup>16</sup>In a previous draft of the paper, we also considered asymmetric equilibria. For these we could not rely on simple equations similar to those stated in Corollary 4. To analyze this more complicated case we relied on a numerical method known as perturbation analysis. See Biais, Bossaerts and Spatt (2007). Note also that the symmetric case gives rise to multiplicity to the extent that there are multiple eigenvectors. This would no longer arise with non-diagonal exogenous parameter matrices.

<sup>17</sup>It's not clear what an appropriate order of magnitude should be for the coefficient of absolute risk aversion in a CARA model. Indeed, one must bear in mind that cash does not affect choices in such a context, so that total wealth does not show up in the equilibrium restrictions. Wang (1993) normalizes the coefficient of absolute risk aversion to 1. We normalize the total endowment of risky assets to one, and arbitrarily set  $\gamma$  to 1.735. We checked that, when one varies the value of this coefficient, equilibrium parameters change (reflecting a change in the risk-return tradeoff), but not in a dramatic way, i.e., there is no local instability.

<sup>18</sup>We checked that in the other symmetric equilibria  $B$  also increases with  $\lambda$ . But, while  $C$  decreases with  $\lambda$  in the equilibrium upon which we focus, it increases in the other symmetric equilibrium.

<sup>19</sup>[http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data\\_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html)

<sup>20</sup>In accordance with our theory and extant empirical studies, short-sale constraints are not imposed.

<sup>21</sup>Since the error terms in the return-price projections will be correlated, one may want to use Seemingly Unrelated Regressions (SUR). Because the regressors are the same for each of the six projections, however, SUR boils down to ordinary Least Squares.

<sup>22</sup>For an in-depth analysis, see, e.g., Kandel and Stambaugh [1996].

<sup>23</sup>See, e.g., Cochrane [1999], Figure 6.

<sup>24</sup>Cochrane [1999] compares the performance of indexing against that of alternative strategies by combining the latter with investment in Treasury bills. In contrast, we combine our price-contingent strategy with investment in



the market portfolio.

<sup>25</sup>See Bossaerts (1995) for an earlier illustration of the use of partial  $z$ -statistics.

<sup>26</sup>Results do not change qualitatively if  $\hat{\sigma}$  is computed without the assumption that the expected return differences equal zero.

<sup>27</sup>The functional central limit theorem predicts that the asymptotic behavior of the partial  $z$ -statistic is that of the standard Brownian motion. This not only means that partial  $z$ -statistics are normally distributed (one could have derived that from a standard central limit theorem), but, more importantly, that increments are independent. It is precisely this independence that allows us to “re-start” the confidence intervals at any  $T_1 > 0$  as if a new, independent sample was produced.

<sup>28</sup>This  $p$ -level is based on a simple binomial test evaluating the probability of at least  $x$  positive outcomes (performance) in  $n$  independent trials (periods) when the probability of a positive outcome is 0.5. In the above,  $x = 6$  and  $n = 7$ .

<sup>29</sup>This  $p$ -level is based on a simple binomial test evaluating the probability of at least  $x$  rejections in  $n$  independent trials (periods) when the probability of a rejection is 0.05. In the above,  $x = 2$  and  $n = 7$ .

<sup>30</sup>Historical monthly returns of SMB and HML (referred to later) can be retrieved from Ken French’s web site; these were used in the present study.

<sup>31</sup>This problem did not arise in our numerical analysis above, where, in line with our Cara–Normal model, we considered dollar returns, as opposed to percentage returns.

<sup>32</sup>When combining the zero–investment strategies with the risk-free asset only, the combinations generated inferior performance. Thus, our choice to combine with the market index effectively stacks the deck against finding evidence that the price contingent strategy outperforms strategies based on size, value or momentum.

<sup>33</sup>Treasury bill return data are from CRSP.

<sup>34</sup>These GLS projections were the inputs of our optimal price-contingent portfolio strategy.

<sup>35</sup>The results are not sensitive to the choice of series of non-overlapping sixty-month periods.

<sup>36</sup>Jimenez Garcia (2004) offers an interesting empirical analysis of asset pricing under private information, relying on industry groupings.

<sup>37</sup>We thank the referee for suggesting to us this result.

Figure 1:

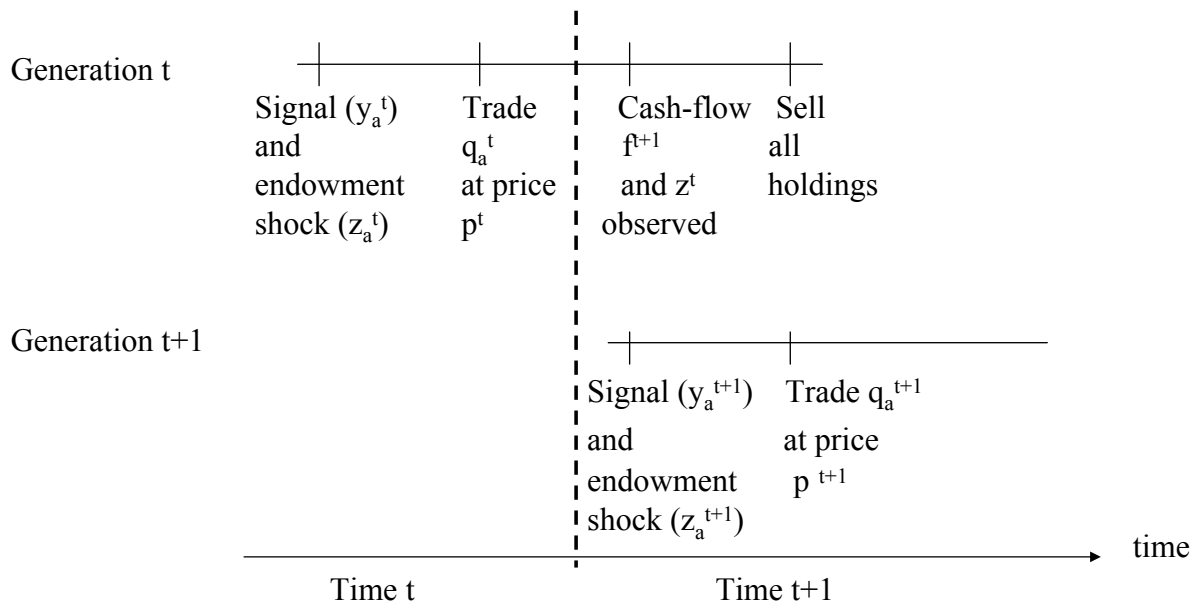


Figure 2:

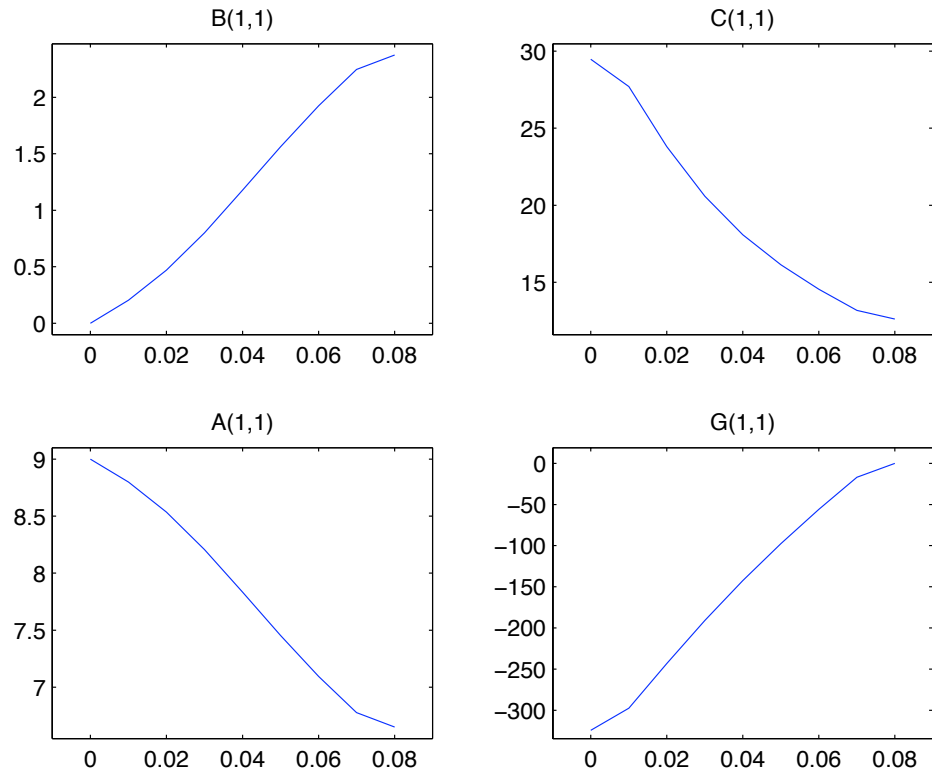


Figure 3:

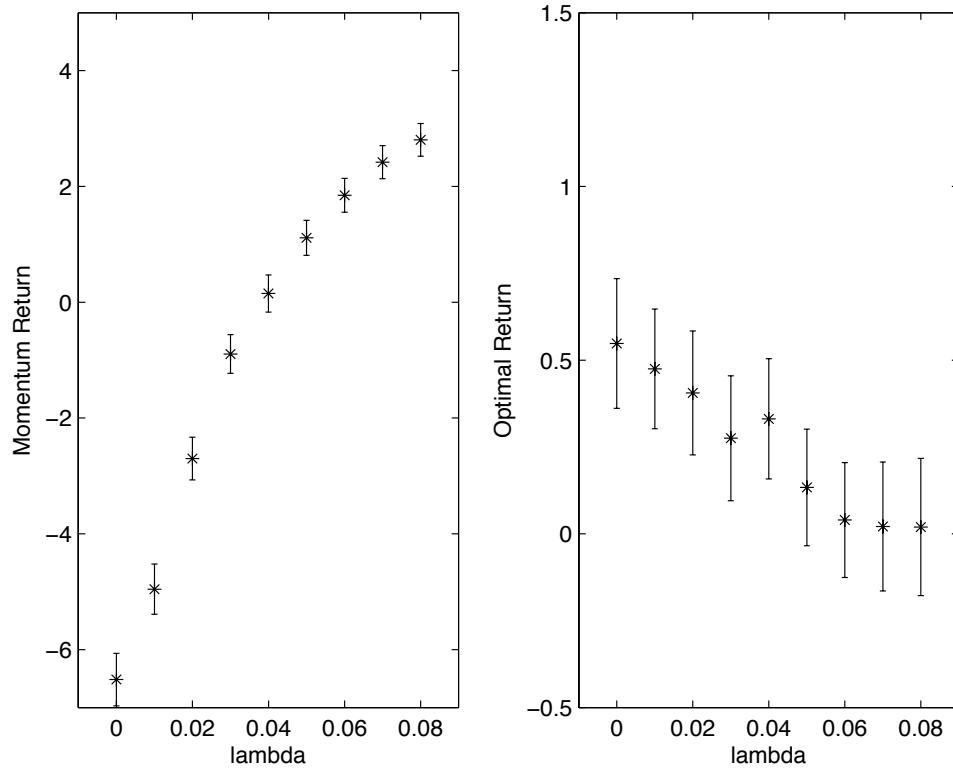


Figure 4:

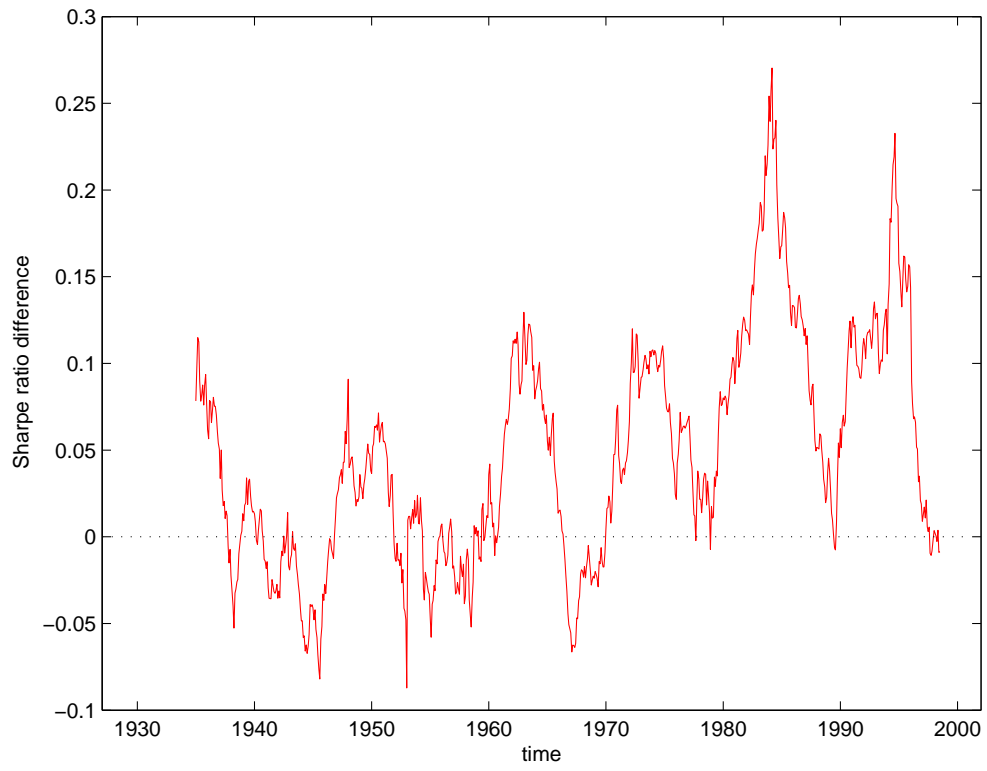


Figure 5:

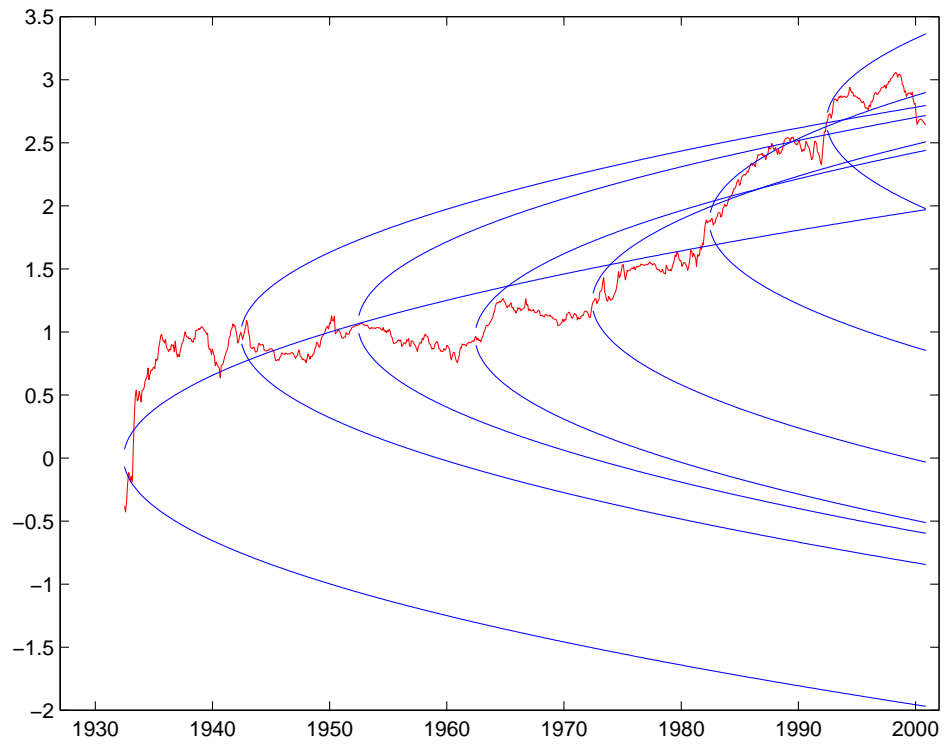


Figure 6:

