Indeterminacy Produces Determinacy

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Abstract
This note discusses the observational equivalence between determinate and indeterminate equilibrium when the economy is driven purely – although arbitrary – by fundamental shocks.

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1 Introduction

Beyer and Farmer (2003, 2004) have pointed out in a series of examples that it is not possible to decide whether actual data is generated by a determinate or an indeterminate equilibrium. For example, they construct an indeterminate model driven purely by a non–fundamental sunspot variable that is observationally equivalent with a determinate model driven purely by a fundamental shock.[1] This note discusses this issue when the indeterminate model is arbitrary driven by fundamental shocks. We introduce a sunspot variable that is correlated with the fundamental shock and we determine some restrictions on the correlation structure that lead to observational equivalence. This note also completes Beyer and Farmer (2003, 2004) and provides a new example of observational equivalence in a sticky price model.

2 An introductory example

The model is a single linear equation that takes the form

$$y_t = a_i E_t y_{t+1} + b_i x_t \qquad i = 1, 2 \tag{1}$$

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[1] Kamihigashi (1996) and Cole and Ohanian (1999) have already pointed out this type of observational equivalence, but Beyer and Farmer (2003, 2004) extend this result to econometric issues.

We assume that x_t is AR(1)

$$x_t = \rho x_{t-1} + \varepsilon_t \tag{2}$$

where ε_t is $iid(0, \sigma_\varepsilon^2)$. The assumption that x_t is AR(1) is widely used in macroeconomic dynamics. It allows to simply represent the time series behavior of the forcing variable. Moreover, one must remark that x_t can be correlated with y_t. In this case, we have just to modify (1), taking into account for the feedback effect of x_t on y_t. The AR(1) specification then represents the pure exogenous component in x_t.

We consider two situations indexed by $i = 1, 2$. In the first case, the equilibrium is determinate, *i.e.* $|a_1| < 1$. Using (2), the stationary solution is given by:

$$y_t = \frac{b_1}{1 - a_1 \rho} x_t \Longleftrightarrow (1 - \rho L)\, y_t = \frac{b_1}{1 - a_1 \rho} \varepsilon_t \tag{3}$$

where L is the lag operator. In the second case, the equilibrium is indeterminate, *i.e.* $|a_2| > 1$. The solution now takes the form

$$y_{t+1} = \frac{1}{a_2} y_t - \frac{b_2}{a_2} x_t + \varepsilon_{t+1}^y$$

or equivalently

$$y_t = \frac{1}{a_2} y_{t-1} - \frac{b_2}{a_2} x_{t-1} + \varepsilon_t^y$$

where ε_t^y is a martingale difference sequence, $E_{t-1}\varepsilon_t^y = 0$, that can be correlated with the fundamental shock ε_t. Let us consider the following linear relation[2]

$$\varepsilon_t^y = \pi \varepsilon_t \tag{4}$$

The parameter π arbitrary rules the dependency of ε_t^y to fundamental. As Benhabib and Farmer (2000) and Matheny (1998) have pointed out, the value of π is critical for the dynamic properties of the economy. We now use the degree of freedom provided by the parameter π in order to construct an indeterminate equilibrium with the same likelihood as the determinate equilibrium. Using (4), the solution rewrites

$$\left(1 - \frac{1}{a_2} L\right) (1 - \rho L)\, y_t = \pi \left(1 - \left(\rho + \frac{b_2}{a_2 \pi}\right) L\right) \varepsilon_t \tag{5}$$

When $\pi = b_1/(1 - a_1\rho)$ and $b_2 = \pi(1 - a_2\rho)$, equations (3) and (5) are observationally equivalent. This result means that it is not possible for an econometrician to decide whether data y_t is generated by a determinate or an indeterminate equilibrium. Note that when $\rho = 0$, the restrictions becomes $\pi = b_1 = b_2$. This introductory example shows that we can easily construct equivalent economies when some restrictions are placed on the correlation structure of the sunspot shock with the fundamental.

[2]Note that we can introduce an independent non–fundamental shock. We do not consider this type of disturbance and the economy is only affected by the fundamental shock.

3 Observational equivalence in the sticky price model

We now investigate the observational equivalence between the determinate and indeterminate equilibrium with respect to money supply shock in a sticky price model. The relative deviations of output $\widehat{y}_t$ and money growth $\widehat{\gamma}_t$ from their steady state values are given by:

$$\begin{aligned} \widehat{y}_t &= \frac{a}{a-(1-a)(1+\chi)} E_{t-1}\widehat{y}_{t+1} + \varepsilon_t + \rho^2 \frac{1-a}{a-(1-a)(1+\chi)} \widehat{\gamma}_{t-1} \qquad (6) \\ \widehat{\gamma}_t &= \rho\widehat{\gamma}_{t-1} + \varepsilon_t \qquad (7) \end{aligned}$$

The parameter of habit persistence $a \in [0, 1)$ rules the effect of the lagged aggregate consumption on individual decisions. The parameter $\chi \geq 0$ is the inverse of the labor supply elasticity. The parameter $\rho \in [0, 1)$ is associated to the exogenous money growth rule and ε_t is an innovation. The model is presented in more details in appendix.

As in introductory example, we consider two versions of the sticky price model. Model 1 corresponds to the determinate case. We set $a = 0$ and $\chi = 0$. From (6), we deduce the reduced form:

$$\widehat{y}_t = \varepsilon_t - \rho^2\widehat{\gamma}_{t-1}$$

Using (7), output follows an $ARMA(1, 1)$ process

$$(1 - \rho L)\,\widehat{y}_t = (1 - \rho(1 + \rho)L)\,\varepsilon_t \qquad (8)$$

Model 2 corresponds to the indeterminate case. Indeterminacy occurs if a is sufficiently large (see Auray, Collard and Fève (2004)), *i.e.* when $a \in (a^\star, 1)$ where $a^\star \equiv (1+\chi)/(3+\chi) \geq 1/3$. In this case, output dynamics takes the form:

$$E_{t-1}\widehat{y}_{t+1} = \varphi\widehat{y}_t - \varphi\varepsilon_t - \rho^2\left(\frac{1-a}{a}\right)\widehat{\gamma}_{t-1}$$

or equivalently

$$\widehat{y}_t = \varphi\widehat{y}_{t-1} - \varphi\varepsilon_{t-1} - \rho^2\left(\frac{1-a}{a}\right)\widehat{\gamma}_{t-2} + \varepsilon^y_t$$

where $\varphi = 1 - (1 - a)(1 + \chi)/a$ and ε^y_t satisfies $E_{t-2}\varepsilon^y_t = 0$. Let us now introduce the following correlation of the sunspot shock with the fundamental

$$\varepsilon^y_t = \pi_1\varepsilon_t + \pi_2\varepsilon_{t-1} \qquad (9)$$

This function is consistent with the rational expectations equilibrium as $E_{t-2}\varepsilon_{t-2+\tau} = 0$ for $\tau \geq 1$. Using (7) and (9), output follows an $ARMA(2, 2)$ process

$$(1 - \varphi L)\,(1 - \rho L)\,\widehat{y}_t = \pi_1\left(1 - \left(\rho + \frac{\varphi - \pi_2}{\pi_1}\right)L + \left(\frac{\varphi\rho - \pi_2\rho - \rho^2(1-a)/a}{\pi_1}\right)L^2\right)\varepsilon_t \qquad (10)$$

We now investigate the observational equivalence between (8) and (10).

Proposition 1 *For any $\chi > (1-\rho)/(1+\rho)$, equations (8) and (10) are observationally equivalent when $\pi_1 = 1$, $\pi_2 = -\rho^2$ and $a = \chi/(1+\chi-\rho) > a^\star$*

Proposition 1 shows that we can construct an indeterminate equilibrium that produces exactly the same likelihood as the determinate equilibrium. Note that the equivalence holds if (9) introduces particular weights on each innovation. The sunspot shock is positively correlated with current money injections but negatively correlated with the lagged innovation. An additional restriction is imposed on the habit persistence parameter, provided some conditions on χ in order to ensure that model 2 is always indeterminate. When money supply is *iid*, there is no restriction on a and/or χ, as (9) allows to match exactly the determinate equilibrium. In this case, equations (8) and (10) are observationally equivalent when $\pi_1 = 1$ and $\pi_2 = 0$.

4 Conclusions

This note provides an example of observational equivalence between determinate and indeterminate equilibrium. This equivalence property comes from a particular correlation structure of the sunspot shock with the fundamental. As in Farmer and Guo (1996), this can be the basis of a formal quantitative evaluation.

Appendix

The sticky price model

We present the main ingredients of the model. A representative infinitely lived household enters period t with nominal balances M_t brought from the previous period. The household supplies labor at the real wage W_t/P_t. During the period, the household also receives a lump–sum transfer from the monetary authorities in the form of cash N_t and profit from the firm Π_t. These revenues are then used to purchase a consumption bundle and money balances for the next period. Therefore, the budget constraint records $M_{t+1} + \int_0^1 P_{i,t}C_{i,t}di = W_t h_t + M_t + N_t + \Pi_t$. Money is held because the household must carry cash in order to purchase goods. She therefore faces a cash–in–advance constraint of the form $\int_0^1 P_{i,t}C_{i,t}di \leqslant M_t + N_t$. Each household has preferences over consumption and leisure represented by the following intertemporal utility function ,

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[\log(C_\tau - a\bar{C}_{\tau-1}) - \frac{\chi_0}{1+\chi} h_\tau^{1+\chi} \right],$$

where $\beta \in (0,1)$ is the discount factor, χ_0 a positive constant and $\chi \geq 0$ is the inverse of the labor supply elasticity. h_t denotes the number of hours supplied by the household. E_t is the expectation operator conditional on the information set in period t. C_t is a composite consumption index $C_t = \left(\int_0^1 C_{i,t}^{(\varepsilon-1)/\varepsilon} di \right)^{\varepsilon/(\varepsilon-1)}$, where $C_{i,t}$ is the quantity of good $i \in [0,1]$ consumed in period t and $\varepsilon > 1$ is the elasticity of substitution among consumption goods. The price of good i is $P_{i,t}$, with aggregate price given by $P_t = \left(\int_0^1 P_{i,t}^{1-\varepsilon} di \right)^{1/(1-\varepsilon)}$. We consider external habit specified in difference with one lag in aggregate consumption $\bar{C}_{t-1}$ which is unaffected by any one agent's decision, therefore joining the catching up with the Joneses literature. The parameter of habit persistence $a \in [0,1)$ rules the effect of the aggregate consumption.

The household determines her optimal consumption/saving choice, labor supply and money holding plan by maximizing utility subject to the budget and cash–in–advance constraints.

In this economy, there is a continuum of firms distributed uniformly on the unit interval. Each firm is indexed by $i \in [0,1]$ and produces a differentiated good with a linear technology $Y_{i,t} = Ah_{i,t}$, where $A > 0$. At the end of period $t-1$, *i.e.* before the observation of the realization of the money supply shocks in period t, firm i sets the price $P_{i,t}$ at which it will be selling good i during period t, for a given aggregate price P_t. The firm i will seek to maximize for a given wage W_t

$$\max_{P_{i,t}} E_{t-1} \left[\Phi_{t+1} \left(P_{i,t} Y_{i,t} - W_t h_{i,t} \right) \right],$$

subject to $Y_{i,t} = \left(P_{i,t}/P_t \right)^{-\varepsilon} C_t$. Φ_{t+1} is an appropriate dsicount factor.

Money is exogenously supplied according to the money growth rule $M_{t+1} = \gamma_t M_t$, where the gross rate of money growth γ_t follows a stationary stochastic process:

$$\log(\gamma_t) = \rho \log(\gamma_{t-1}) + (1-\rho)\log(\bar{\gamma}) + \varepsilon_t.$$

ε_t is a white noise with a variance σ^2 and $|\rho| < 1$.

An equilibrium is a sequence of prices and allocations, such that given prices, allocations maximize profits (when taking technological choice into account) and maximize utility (subject to the savings behavior), and all markets clear. In a symmetric equilibrium, all firms will set the same price P_t and choose identical output and hours. Moreover, all households have the same consumption and $C_t = \bar{C}_t \;\; \forall t$. Goods market clearing require $C_t = C_{i,t} = Y_t = Y_{i,t}$ for all $i \in [0,1]$ and all t. The equilibrium conditions are approximated by log–linearization arround the deterministic steady state.

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