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Progressive and merging-proof taxation

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Keywords: taxation, progressivity, merging-proofness, consistency, operators.

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Abstract

We investigate the implications and logical relations between *progressivity* (a principle of distributive justice) and *merging-proofness* (a strategic principle) in taxation. By means of two characterization results, we show that these two principles are intimately related, despite their different nature. In particular, we show that, in the presence of continuity and consistency (a widely accepted framework for taxation) *progressivity* implies *merging-proofness* and that the converse implication holds if we add an additional strategic principle extending the scope of *merging-proofness* to a multilateral setting. By considering operators on the space of taxation rules, we also show that *progressivity* is slightly more robust than *merging-proofness*.

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1 Introduction

In modern welfare states, income tax is a major source of state funds and an essential policy measure for the enhancement of distributive justice. In the framework introduced by O'Neill (1982) and Young (1988), we focus on a specific principle of distributive justice, known as *progressivity*, which says that, for any pair of agents, the one with higher income should face a tax rate at least as high as the rate the other faces.[1] We investigate the implications of this principle as well as its relation to another principle that prevents

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[1] We refer readers to Young (1994), Moulin (2002) and Thomson (2003, 2006) for extensive treatments of diverse problems (such as taxation, bankruptcy, cost sharing or surplus sharing) fitting this framework.

any gain (tax cut) from strategic merging among taxpayers. This second principle, called *merging-proofness*, has been studied in this same context by de Frutos (1999), Ju (2003), Ju et al. (2007) and Moreno-Ternero (2007), among others, and in similar contexts by Sprumont (2005) and Moulin (2008), among others.

Merging-proofness is an axiom introduced from a strategic consideration independent of any principle of distributive justice. However, we find that it is in fact related to the progressivity principle. Based on two characterization results imposing either *progressivity* or *merging-proofness*, as well as some standard axioms in the literature, we show that (essentially) any *progressive* tax rule is *merging-proof*. This is an extra advantage of imposing *progressivity*.[2] Conversely, we show that a suitable "multilateral" extension of *merging-proofness* allows to ensure *progressive* taxation too.

A recent study by Thomson and Yeh (2008) gives a novel classification of tax rules and axioms based on *operators* on the space of tax rules. The robustness of an axiom can be tested by studying its preservation through the application of operators.[3] There is a natural operator, known as "minimal burden operator", preserving *progressivity* but not *merging-proofness*. We find an additional, yet mild axiom that helps *merging-proofness* be preserved. The other known operators preserve either both or none of the two principles. Thus, our analysis shows that *progressivity* is slightly more robust than *merging-proofness*.

The rest of the paper is organized as follows. In Section 2, we present the model and basic concepts. In Section 3, we define the axioms. In Section 4, we state and prove the characterization results and the logical relations between the axioms. In Section 5, we state and prove our results on operators. We conclude in Section 6 with some further insights. For a smooth passage, we defer some proofs and provide them in the appendix.

2 Model and basic concepts

We study taxation problems in a variable population model. The set of potential taxpayers, or *agents*, is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element N. We denote by $\mathbb{R}^N_+$ the cross-product of $|N|$ copies of $\mathbb{R}_+$ indexed by the members of N.[4] For each $i \in N$, let $y_i \in \mathbb{R}_+$ be i's (taxable) *income* and $y \equiv (y_i)_{i\in N}$ the income profile. A (taxation) *problem* is a triple consisting of a population $N \in \mathcal{N}$, an income profile $y \in \mathbb{R}^N_+$, and a tax revenue $T \in \mathbb{R}_+$ such that $\sum_{i\in N} y_i \geq T$. Let $Y \equiv \sum_{i\in N} y_i$. To avoid unnecessary complication, we assume $Y = \sum_{i\in N} y_i > 0$. Let $\mathcal{D}^N$ be the set of taxation problems with population N and $\mathcal{D} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, y, T) \in \mathcal{D}$, a *tax profile* is a vector $x \in \mathbb{R}^N$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq y_i$ and (ii) $\sum_{i\in N} x_i = T$. We refer to (i) as *boundedness* and (ii) as *balance*.[5] A (taxation) *rule* on $\mathcal{D}$, $R\colon \mathcal{D} \to \bigcup_{N\in\mathcal{N}} \mathbb{R}^N$, associates

[2] The most widely referred advantage of *progressive* taxation, and perhaps the main reason for its ubiquity in income tax schedules worldwide, is that it typically reduces income inequality. The connection between *progressive* taxation and *inequality reduction* that had long been perceived by a number of authors in the literature on tax functions (e.g., Musgrave and Thin, 1948; Fellman, 1976; Jakobsson, 1976; Eichhorn et al., 1984), has also been recently scrutinized and properly established for the framework we analyze here in a companion paper (Ju and Moreno-Ternero, 2008).

[3] Preservation of an axiom means that if a rule satisfies an axiom so does the rule obtained by applying the operator.

[4] Alternatively, the superscript N may refer to a set pertaining to the agents in N. Which interpretation is intended should be unambiguous from the context.

[5] Note that *boundedness* implies that each agent with zero income pays zero tax.

with each problem $(N,y,T) \in \mathscr{D}$ a tax profile $R(N,y,T)$ for the problem. For each $N \in \mathscr{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^N$, let $z_M \equiv (z_i)_{i \in M}$.

We now provide some examples of rules. The *head tax* distributes the tax burden equally subject to no agent paying more than her income. The *leveling tax* equalizes post-tax income across agents subject to no agent being subsidized. The *flat tax* equalizes tax rates across agents. These three rules are examples of rules in the following family introduced by Young (1987).

A rule R is *parametric* if there is a function $f : [a,b] \times \mathbb{R}_+ \to \mathbb{R}$, where $a,b \in \mathbb{R} \cup \{\pm\infty\}$, such that (i) f is nondecreasing and continuous in the first variable; (ii) for each $x \in \mathbb{R}_+$, $f(a,x) = 0$ and $f(b,x) = x$; (iii) for each $(N,y,T) \in \mathscr{D}$, there is $\lambda \in [a,b]$ such that, for each $i \in N$, $R_i(N,y,T) = f(\lambda, y_i)$. We call f a *parametric representation of* R.

Parametric representations (also parametric rules) are continuous in revenue but not necessarily in income. When a parametric representation is jointly continuous in both arguments, it is called a *continuous parametric representation* and the rule represented by it is called a *continuous parametric rule*. The three rules mentioned earlier are such rules. They have the following parametric representations:

- Head tax: $f^H(\lambda,y) = \min\{-\frac{1}{\lambda}, y\}$, for each $\lambda \in \mathbb{R}_-$ and each $y \in \mathbb{R}_+$.
- Leveling tax: $f^L(\lambda,y) = \max\{y - \frac{1}{\lambda}, 0\}$, for each $\lambda \in \mathbb{R}_+$ and each $y \in \mathbb{R}_+$.
- Flat tax: $f^F(\lambda,y) = \lambda \cdot y$, for each $\lambda \in [0,1]$ and each $y \in \mathbb{R}_+$.

3 Axioms

We now define our two main axioms.

Progressivity postulates that, for any pair of agents, the one with higher income should face a tax rate at least as high as the rate the other faces.

Progressivity. For each $(N,y,T) \in \mathscr{D}$ and each $i,j \in N$, if $0 < y_i \leq y_j$, $R_i(N,y,T)/y_i \leq R_j(N,y,T)/y_j$.

Our second axiom prevents a rule from being manipulated by a pair of agents through merging their incomes.

Merging-proofness. For each $(N,y,T) \in \mathscr{D}$ and each pair $i,j \in N$ with $i \neq j$, if $y' \in \mathbb{R}_+^{N \setminus \{j\}}$ is such that $y'_i = y_i + y_j$ and $y'_{N\setminus\{i,j\}} = y_{N\setminus\{i,j\}}$, $R_i(N,y,T) + R_j(N,y,T) \leq R_i(N\setminus\{j\}, y', T)$.[6]

We will investigate logical relations between the two axioms. We will invoke some of the following standard axioms in the process.[7]

First, any two agents with equal incomes should pay equal taxes.

Equal treatment of equals. For each $(N,y,T) \in \mathscr{D}$ and each pair $i,j \in N$ with $y_i = y_j$, $R_i(N,y,T) = R_j(N,y,T)$.

[6] We consider only pairwise merging. This restriction is for simplicity and without loss of generality. The same results are obtained considering merging incomes of more than two agents into the total income of a single representative agent. We consider later multiple representative agents and formulate a stronger axiom. This axiom is much stronger as we show in Lemma 2 and Proposition 2.

[7] We refer readers to Thomson (2003, 2006) for a detailed discussion of these axioms.

Next is the stronger requirement that tax payments should not depend on the names of taxpayers.

Anonymity. For each $(N,y,T) \in \mathscr{D}$, each $N' \in \mathscr{N}$, each bijection $\pi : N \to N'$, and each $i \in N$, $R_{\pi(i)}(N',(y_{\pi(i)})_{i\in N},T) = R_i(N,y,T)$.

The next axiom expresses the robustness of a rule under the departure of some agents with their contributions. Apply the rule to a problem and imagine some agents paying their tax contributions and leaving. The axiom says that if we revise the situation and apply the rule to it, it should assign to each of the remaining agents the same contribution as it had originally.

Consistency. For each $(N,y,T) \in \mathscr{D}$, and each $M \subset N$, if $x = R(N,y,T)$ then $x_M = R(M,y_M,\sum_{i\in M} x_i)$.

Finally, the next axiom says that small changes in incomes or revenue do not produce a jump in tax payments.

Continuity. For each $N \in \mathscr{N}$, each $(N,y,T) \in \mathscr{D}^N$ and each sequence $\{(N,y^n,T^n) : n \in \mathbb{N}\}$ in $\mathscr{D}^N$, if (y^n,T^n) converges to (y,T) then $R(N,y^n,T^n)$ converges to $R(N,y,T)$.

4 Characterizations and logical relations among axioms

We will show that among rules satisfying *consistency* and *continuity*, *progressivity* implies *merging-proofness*. This follows from the characterizations of two families of rules imposing either *progressivity* or *merging-proofness*. The following lemma is useful for the characterizations, as well as for establishing the converse implication that *merging-proofness*, together with an additional axiom extending its scope to a multilateral setting, also implies *progressivity*.

Lemma 1. *Merging-proofness and consistency together imply anonymity.*

The proof is provided in the appendix.

The next two properties of parametric representations are crucial in our characterization. A parametric representation $f\colon [a,b]\times\mathbb{R}_+ \to \mathbb{R}$ is *superhomogeneous in income* if for each $\lambda \in [a,b]$, each $y_0 \in \mathbb{R}_+$, and each $\alpha \geq 1$, $f(\lambda,\alpha y_0) \geq \alpha f(\lambda,y_0)$.[8] A parametric representation $f\colon [a,b]\times\mathbb{R}_+ \to \mathbb{R}$ is *superadditive in income* if for each $\lambda \in [a,b]$ and each pair $y_0, y_0' \in \mathbb{R}_+$, $f\left(\lambda, y_0 + y_0'\right) \geq f(\lambda,y_0) + f\left(\lambda,y_0'\right)$.[9]

Proposition 1. *Let R be a rule satisfying consistency and continuity. Then*:
1. *R is progressive if and only if it has a continuous parametric representation that is superhomogeneous in income.*
2. *R is merging-proof if and only if it has a continuous parametric representation that is superadditive in income.*[10]

[8]It is worth noting that, although there might be different representations of a parametric rule, superhomogeneity in income is invariant; that is, either every representation is superhomogeneous in income or none of them is.

[9]Like superhomogeneity, superadditivity in income is also invariant with respect to the choice of the representation.

[10]This strengthens Theorem 2 in Ju (2003) by dropping *equal treatment of equals*.

Proof. Statement 1. Note that *progressivity* implies *equal treatment of equals*. Young (1987, Theorem 1) shows that the *continuous* parametric rules are the only rules satisfying *consistency*, *equal treatment of equals*, and *continuity*. Thus, using Young's result, we just need to show that a parametric rule is *progressive* if and only if it has a parametric representation that is superhomogeneous in income. Let R be a parametric rule and $f\colon [a,b]\times\mathbb{R}_+\to\mathbb{R}$ a parametric representation of R. Assume that R is *progressive*. Let $\lambda\in[a,b]$, $y_0>0$ and $\alpha\geq 1$. Let $T^\lambda\equiv f(\lambda,y_0)+f(\lambda,\alpha y_0)$ and $N\equiv\{1,2\}$. Then, $R(N,(y_0,\alpha y_0),T^\lambda)=(f(\lambda,y_0),f(\lambda,\alpha y_0))$. By *progressivity*, $f(\lambda,y_0)/y_0\leq f(\lambda,\alpha y_0)/(\alpha y_0)$. Thus $\alpha f(\lambda,y_0)\leq f(\lambda,\alpha y_0)$, which shows that f is superhomogeneous in income. Conversely, assume that f is superhomogeneous in income. Let $(N,y,T)\in\mathscr{D}$ and $i,j\in N$ be such that $0<y_i\leq y_j$. Let $\lambda\in[a,b]$ be such that $R(N,y,T)=(f(\lambda,y_i))_{i\in N}$. Then, by superhomogeneity, $f\left(\lambda,y_j\right)=f(\lambda,\frac{y_j}{y_i}\cdot y_i)\geq\frac{y_j}{y_i}\cdot f(\lambda,y_i)$. Thus

$$\frac{R_j(N,y,T)}{y_j}=\frac{f(\lambda,y_j)}{y_j}\geq\frac{f(\lambda,y_i)}{y_i}=\frac{R_i(N,y,T)}{y_i},$$

which shows the *progressivity* of R. $\Diamond$

Statement 2. Ju (2003) shows that a parametric rule is *merging-proof* if and only if it has a parametric representation that is superadditive in income. From here, Young's result and Lemma 1 conclude the proof. $\Diamond$ $\square$

As a consequence of Proposition 1, a logical relation between *progressivity* and *merging-proofness* can be established.

Corollary 1. *Let R be a rule satisfying consistency and continuity. If R is progressive then R is merging-proof.*

Proof. It suffices to show that superhomogeneity in income implies superadditivity in income. To do so, let y_0 and y_0' be such that $0<y_0\leq y_0'$. Let $\alpha\equiv(y_0+y_0')/y_0'$. Then, by *superhomogeneity*, $f\left(\lambda,\alpha y_0'\right)\geq\alpha f\left(\lambda,y_0'\right)$, that is, $f\left(\lambda,y_0+y_0'\right)/\left(y_0+y_0'\right)\geq f(\lambda,y_0')/y_0'$. Thus, $f(\lambda,y_0+y_0')\geq f(\lambda,y_0')+\frac{y_0}{y_0'}f(\lambda,y_0')$. By *superhomogeneity*, $f\left(\lambda,\frac{y_0'}{y_0}y_0\right)\geq\frac{y_0'}{y_0}f(\lambda,y_0)$, thus $\frac{y_0}{y_0'}f(\lambda,y_0')\geq f(\lambda,y_0)$. Hence $f(\lambda,y_0+y_0')\geq f(\lambda,y_0')+f(\lambda,y_0)$, which shows that f is *superadditive* in income. $\square$

Remark 1. Without *consistency* and *continuity*, the logical relation between *progressivity* and *merging-proofness* in Corollary 1 does not hold, as shown by Example 1 in Section 5.

Note that any convex function $f\colon\mathbb{R}\to\mathbb{R}$, whose graph passes through the origin, is superhomogeneous. Hence by Proposition 1 and Corollary 1, any rule with a continuous parametric representation that is convex in income is *progressive* and *merging-proof*. As mentioned above, both the leveling tax and the flat tax have such representations. Thus, they are both *progressive* and *merging-proof*. The same argument applies to two other classical tax rules proposed by Cohen-Stuart and Cassel (formulated as rules in our model by Young, 1988).

Corollary 1 shows that, among rules satisfying *consistency* and *continuity*, *progressivity* implies *merging-proofness*. However, the converse does not hold.[11] To fully understand the

[11] An example of a rule satisfying *merging-proofness* but violating *progressivity* can be provided upon request.

logical relation and, in particular, how much weaker than progressivity merging-proofness is, we search for a condition that fills the gap between the two axioms. We accomplish this aim through the next investigation of "multilateral merging" and, more precisely, the non-manipulability axiom that arises when we focus on the special case in which agents merging have uniform incomes.

Merging-proofness pertains to merging the incomes of two agents, or any set of agents in its standard formulation, into the total income of a "single" representative. One may well consider the possibility of "multiple" representatives instead. That is, merging incomes of a set of agents into incomes of a proper "subset" of agents with the total income unchanged. Such merging is said to be *multilateral*. The next axiom says that no group of agents can reduce their total tax payment through multilateral merging.

Multilateral Merging-Proofness. For each $(N,y,T) \in \mathscr{D}$, each $S \subseteq N$ and each non-empty $S' \subseteq N \backslash S$, if $(\alpha_i)_{i \in S} \in \Delta^{|S|}$ and $y' \in \mathbb{R}_+^{N \backslash S'}$ are such that for each $i \in S$, $y'_i = y_i + \alpha_i \sum_{j \in S'} y_j$, and $y'_{N \backslash [S \cup S']} = y_{N \backslash [S \cup S']}$, then $\sum_{i \in S} R_i(N,y,T) + \sum_{j \in S'} R_j(N,y,T) \leq \sum_{i \in S} R_i(N \backslash S', y', T)$.

Evidently, this axiom implies *merging-proofness*. It turns out to be much stronger as we show that *multilateral merging-proofness*, together with *consistency*, implies the following dual axiom of *merging-proofness* that has also been frequently considered in the literature (e.g., de Frutos (1999), Ju (2003), Ju et al. (2007) and Moreno-Ternero (2007)):

Splitting-Proofness. For each $(N,y,T) \in \mathscr{D}$ and each pair $i,j \in N$ with $i \neq j$, if $y' \in \mathbb{R}_+^{N \backslash \{j\}}$ is such that $y'_i = y_i + y_j$ and $y'_{N \backslash \{i,j\}} = y_{N \backslash \{i,j\}}$, $R_i(N,y,T) + R_j(N,y,T) \geq R_i(N \backslash \{j\}, y', T)$.

Lemma 2. *Multilateral merging-proofness and consistency together imply splitting-proofness.*

Proof. Let R be a rule satisfying *multilateral merging-proofness* and *consistency*. Let $(N,y,T) \in \mathscr{D}$. For ease of notation, assume, without loss of generality, that $N \equiv \{1,2,\ldots,n\}$ and consider the problem in which agents 1 and 2 merge their incomes into 1's income, i.e., $(N \backslash \{2\}, (y_1+y_2, y_3, \ldots, y_n), T) \in \mathscr{D}$. Note that (N,y,T) is obtained after agent 1 in $(N \backslash \{2\}, (y_1 + y_2, y_3, \ldots, y_n), T)$ splits her income $y_1 + y_2$ into y_1 and y_2. We aim to show that $R_1(N,y,T) + R_2(N,y,T) \geq R_1(N \backslash \{2\}, (y_1+y_2, y_3, \ldots, y_n), T)$. To do so, let us consider the auxiliary problem in which agents 1 and 2 transfer their incomes to a new agent $n+1$, but do not leave, i.e., $(N \cup \{n+1\}, (0,0,y_3,\ldots,y_n,y_1+y_2), T) \in \mathscr{D}$. The multilateral merger of the incomes of 1, 2, and $n+1$ into the incomes y_1 and y_2 of 1 and 2 yields (N,y,T). Thus, applying *multilateral merging-proofness* to this auxiliary problem and the original one, as well as invoking *boundedness*, we obtain

$$R_1(N,y,T) + R_2(N,y,T) \geq R_{n+1}(N \cup \{n+1\}, (0,0,y_3,\ldots,y_n,y_1+y_2), T). \tag{1}$$

On the other hand, by *consistency*,

$$R_{n+1}(N \cup \{n+1\}, (0,0,y_3,\ldots,y_n,y_1+y_2), T) = R_{n+1}(\{3,\ldots,n+1\}, (y_3,\ldots,y_n,y_1+y_2), T). \tag{2}$$

Because *multilateral merging-proofness* implies *merging-proofness*, Lemma 1 guarantees that R satisfies *anonymity*, which implies that

$$R_{n+1}(\{3,\ldots,n+1\}, (y_3,\ldots,y_n,y_1+y_2), T) = R_1(N \backslash \{2\}, (y_1+y_2,y_3,\ldots,y_n), T). \tag{3}$$

Combining (1)-(3), we get $R_1(N,y,T)+R_2(N,y,T) \geq R_1(N\backslash\{2\},(y_1+y_2,y_3,\ldots,y_n),T)$, as desired. Since the argument would have also worked for any other pair of agents, instead of 1 and 2, we have just shown that R satisfies *splitting-proofness*. □

As a result, we have the following:

Proposition 2. *A rule satisfies multilateral merging-proofness and consistency if and only if it is the flat tax.*

Proof. It is straightforward to show that the flat tax satisfies *multilateral merging-proofness* and *consistency*. Conversely, let R be a *consistent* rule satisfying *multilateral merging-proofness*. In particular, R satisfies *merging-proofness*. Now, by Lemma 2, R also satisfies *splitting-proofness*. It follows from Ju et al., (2007, Theorem 9) that R is the flat tax. □

We have therefore shown that among the *consistent* rules, *multilateral merging-proofness* is much stronger than *progressivity*. A technical relaxation of it, however, focusing on a special type of multilateral merging, will turn out to fill the gap between *progressivity* and *merging-proofness*. For each $y \in \mathbb{R}_+^N$ and each $S \subseteq N$, y_S is *uniform* if for each pair $i,j \in S$, $y_i = y_j$. For each $(N,y,T) \in \mathscr{D}$, each $S \subseteq N$ and each non-empty $S' \subseteq N\backslash S$, we say that $y' \in \mathbb{R}_+^{N\backslash S'}$ is obtained from y through a *uniformity preserving* (multilateral) *merge* in $S \cup S'$ if the two profiles have the same incomes on $N\backslash[S\cup S']$, and both $y_{S\cup S'}$ and y'_S are uniform with the same total income ($\sum_{i\in S\cup S'} y_i = \sum_{i\in S} y'_i$). That is, $y'_{N\backslash[S\cup S']} = y_{N\backslash[S\cup S']}$ and there is $y_0 \in \mathbb{R}_+$ such that, for each $i \in S\cup S'$, $y_i = y_0$ and, for each $i \in S$, $y'_i = y_i + |S'|y_0/|S| = y_0\left(\frac{|S'|+|S|}{|S|}\right)$.

Focusing on uniformity preserving merges, we define:

Uniformity Preserving Multilateral Merging-Proofness. For each $(N,y,T) \in \mathscr{D}$, each $S \subseteq N$ and each non-empty $S' \subseteq N\backslash S$, if y' is obtained from y through a uniformity preserving merge of agents in $S\cup S'$, then $\sum_{i\in S} R_i(N,y,T) + \sum_{j\in S'} R_j(N,y,T) \leq \sum_{i\in S} R_i(N\backslash S', y', T)$.

An implicit assumption made in our formulation of *multilateral merging-proofness* (also present in the formulation of "reallocation-proofness" studied widely in numerous models; see Ju et al., 2007) is that a group of agents who successfully manipulate tax allocations, and reduce their total tax payment, can always share the tax cut so that they all benefit. The validity of this assumption depends on how capable agents are of reaching a cooperative agreement to share the tax cut. Uniformity preserving merges are a special type of merges where this assumption is arguably most valid. The reason being that in this case all agents involved in a merge are equal ex-ante and ex-post, and so the equal division of the total benefit from the merge constitutes a focal agreement to be reached, provided they accept the basic equal treatment principle. Thus, *uniformity preserving multilateral merging-proofness* can be considered as a natural weakening of *multilateral merging-proofness*, when guaranteeing a post-manipulation agreement is important.

Now we are ready to state our next result:

Proposition 3. *Let R be a rule satisfying consistency and continuity. Then, the following statements are equivalent*:
1. *R satisfies progressivity*;
2. *R satisfies merging-proofness and uniformity preserving multilateral merging-proofness*;
3. *R has a continuous parametric representation that is superhomogeneous in income.*

Proof. By Proposition 1, it only remains to show that statements 2 and 3 are equivalent. First, let R be a *consistent* and *continuous* rule that satisfies *merging-proofness* and *uniformity preserving multilateral merging-proofness*. By *merging-proofness* and Lemma 1, R satisfies *equal treatment of equals*. Then, by Young (1987, Theorem 1), R is a continuous parametric rule. Let $f:[a,b]\times\mathbb{R}_+\to\mathbb{R}$ be a representation of R. Then, we show that, for each $\lambda\in[a,b]$, $f(\lambda,\cdot)$ is superhomogeneous. By continuity, we only have to prove that for each $y_0\geq 0$ and each pair of integers l,m with $m\geq l>0$, $f(\lambda,my_0/l)\geq mf(\lambda,y_0)/l$. Suppose, by contradiction, that there exist $\lambda\in(a,b)$, $y_0>0$, and $m>l>0$ such that

$$f(\lambda,my_0/l)<mf(\lambda,y_0)/l. \tag{4}$$

Since f is nondecreasing in the first variable (and $f(a,y_0)=0$), there is $\lambda_*\in(a,\lambda]$ such that $f(\lambda_*,y_0)=f(\lambda,y_0)$ and for each $\lambda'\in(a,\lambda_*)$, $f(\lambda',y_0)<f(\lambda_*,y_0)$. Let $N\equiv\{1,\ldots,l,l+1,\ldots,m,m+1\}$, $S\equiv\{1,\ldots,l\}$, and $S'\equiv\{l+1,\ldots,m\}$. Let $y\in\mathbb{R}^N_+$ be such that $y_1=\cdots=y_m=y_{m+1}=y_0$. Let $y'\equiv(y'_1,\ldots,y'_l,y'_{m+1})$ be such that $y'_1=\cdots=y'_l=my_0/l$ and $y'_{m+1}=y_0$. Let $T\equiv lf(\lambda_*,my_0/l)+f(\lambda_*,y_0)$, $x\equiv R(N,y,T)$, and $x'\equiv R(N\backslash S',y',T)$. Clearly, $x'\equiv(f(\lambda_*,y'_i))_{i\in N\backslash S'}$. Since, by (4), $f(\lambda_*,my_0/l)\leq f(\lambda,my_0/l)<mf(\lambda,y_0)/l=mf(\lambda_*,y_0)/l$, then

$$T=lf(\lambda_*,my_0/l)+f(\lambda_*,y_0)<mf(\lambda_*,y_0)+f(\lambda_*,y_0). \tag{5}$$

Since $f(\cdot,y_0)$ is nondecreasing, there is $\lambda_0<\lambda_*$ such that $mf(\lambda_0,y_0)+f(\lambda_0,y_0)=T$. Then $x\equiv(f(\lambda_0,y_i))_{i\in N}$ and, therefore, $x_{m+1}=f(\lambda_0,y_{m+1})=f(\lambda_0,y_0)<f(\lambda_*,y_0)=x'_{m+1}$. Since $N\backslash\{m+1\}=S\cup S'$ and $\sum_{i\in N\backslash S'}x'_i=T=\sum_{i\in N}x_i$, then $\sum_{i\in S}x'_i<\sum_{i\in S\cup S'}x_i$, contradicting *uniformity preserving multilateral merging-proofness*.

Conversely, let R be a rule admitting a continuous parametric representation that is superhomogeneous in income. Then, by Corollary 1, R is *merging-proof*. Thus, it only remains to show that R satisfies *uniformity preserving multilateral merging-proofness*. Let $(N,y,T)\in\mathscr{D}$, $S\subseteq N$ and $S'\subseteq N\backslash S$ be such that for each $i\in S\cup S'$, $y_i=y_0$, for each $i\in S$, $y'_i=y_i+(|S'|/|S|)\,y_0(=(|S\cup S'|/|S|)\,y_0$, and $y'_{N\backslash S}=y_{N\backslash[S\cup S']}$. Let $\lambda\in[a,b]$. Define $T\equiv\sum_{i\in S}f(\lambda,y'_i)+\sum_{i\in N\backslash[S\cup S']}f(\lambda,y'_i)$. Then, by superhomogeneity,

$$\begin{aligned}
T &\equiv \sum_{i\in S}f(\lambda,y'_i)+\sum_{i\in N\backslash[S\cup S']}f(\lambda,y'_i)=\sum_{i\in S}f\left(\lambda,\frac{|S\cup S'|}{|S|}y_0\right)+\sum_{i\in N\backslash[S\cup S']}f(\lambda,y_i)\\
&\geq \sum_{i\in S}\frac{|S\cup S'|}{|S|}f(\lambda,y_0)+\sum_{i\in N\backslash[S\cup S']}f(\lambda,y_i)\\
&= |S\cup S'|f(\lambda,y_0)+\sum_{i\in N\backslash[S\cup S']}f(\lambda,y_i)=\sum_{i\in N}f(\lambda,y_i).
\end{aligned} \tag{6}$$

Therefore, there is $\lambda_0\in[a,b]$ such that $\lambda_0\geq\lambda$ and $\sum_{i\in N}f(\lambda_0,y_i)=T$. Then, $\sum_{i\in N\backslash[S\cup S']}f(\lambda_0,y_i)\geq\sum_{i\in N\backslash[S\cup S']}f(\lambda,y_i)=\sum_{i\in N\backslash[S\cup S']}f(\lambda,y'_i)$, which implies $\sum_{i\in S}f(\lambda_0,y_i)+\sum_{j\in S'}f(\lambda_0,y_j)\leq\sum_{i\in S}f(\lambda,y'_i)$. Since $R(N,y,T)\equiv(f(\lambda_0,y_i))_{i\in N}$ and $R(N\backslash\{S'\},y',T)\equiv(f(\lambda,y'_i))_{i\in N\backslash\{S'\}}$, then $\sum_{i\in S}R_i(N,y,T)+\sum_{j\in S'}R_j(N,y,T)\leq\sum_{i\in S}R_i(N\backslash S',y',T)$. □

5 Robustness of the axioms

We now study the robustness of our two main axioms by analyzing *operators* on the space of rules. An axiom is said to be *preserved* under an operator if any rule that satisfies the axiom is mapped by the operator into a rule that also satisfies the axiom. Considering the next operator introduced by Thomson and Yeh (2008), we will show that *progressivity* is slightly more robust than *merging-proofness*.

Given a problem (N, y, T) and an agent $i \in N$, suppose $T - \sum_{j \in N \setminus \{i\}} y_j > 0$. Then this part of the revenue cannot be covered even if everyone other than i pays her full income. Thus this part can be viewed as the minimal burden imposed on agent i. For each $i \in N$, let $m_i(N, y, T) \equiv \min\{0, T - \sum_{j \neq i} y_j\}$ be i's *minimal burden*. Let $m(N, y, T) \equiv (m_i(N, y, T))_{i \in N}$ and $M(N, y, T) \equiv \sum_N m_i(N, y, T)$. The *minimal-burden operator* associates with each rule R the rule R^m defined by the following two-step payment procedure. For each problem, first each agent pays her minimal burden; second, each agent pays his tax according to R applied to the revised problem obtained by reducing agents' incomes by their minimal burdens and the tax revenue by the sum of the minimal burdens. That is, for each $(N, y, T) \in \mathscr{D}$,

$$R^m(N, y, T) \equiv m(N, y, T) + R(N, y - m(N, y, T), T - M(N, y, T)).$$

The next proposition shows that progressivity is more robust to the minimal-burden operator than merging-proofenss.

Proposition 4. *The minimal-burden operator preserves progressivity. However, it does not preserve merging-proofness.*

The proof of the statement regarding *progressivity* is provided in the appendix. Example 1 below shows that the minimal-burden operator does not preserve *merging-proofness*.

Example 1. For each $(N, y, T) \in \mathscr{D}$, let $R(N, y, T)$ coincide with the leveling tax R^L when $T \geq 10$ and with the flat tax R^F when $T < 10$. Since both the leveling tax and the flat tax are *merging-proof*, R is also *merging-proof*. However, R^m is not *merging-proof*. To show this, consider the problem $(N, y, T) = (\{1, 2, 3\}, (5, 55, 100), 70)$. Then,

$$R^m(N, y, T) = (0, 0, 10) + R^L(\{1, 2, 3\}, (5, 55, 90), 60) = \left(0, \frac{25}{2}, \frac{115}{2}\right).$$

Consider now the problem that results when agents 2 and 3 merge their incomes and are represented by agent 3, i.e., $(N \setminus \{2\}, y', T) = (\{1, 3\}, (5, 155), 70)$. Then,

$$R^m\left(N \setminus \{2\}, y', T\right) = (0, 65) + R^F(\{1, 3\}, (5, 90), 5) = \left(\frac{5}{19}, \frac{1325}{19}\right).$$

Consequently,

$$R^m_3(N \setminus \{2\}, y', T) < R^m_2(N, y, T) + R^m_3(N, y, T),$$

which shows that R^m is not *merging-proof*.

Note that R is *progressive*. By Proposition 4, so is R^m. Therefore, R^m is an example showing that *progressivity* does not imply *merging-proofness*, as claimed in Remark 1.

For rules satisfying the following mild axiom, however, we show that the minimal-burden operator preserves *merging-proofness*. Suppose that an agent donates part of her income and that the donation is used to finance tax revenue. Then both the donor's income and the tax revenue go down by the amount of the donation. The next axiom says that the donor's total payment (tax plus donation) should not be lower than her total payment without donation.

No Donation Paradox.[12] For each $(N,y,T) \in \mathscr{D}$, each $i \in N$ and each $t \in [0, \min\{T, y_i\}]$,

$$R_i(N,y,T) \leq t + R_i(N,(y_i - t, y_{-i}), T - t).$$

Note that the rule in Example 1 violates *no donation paradox*. To show this, consider the problem $(N,y,T) = (\{1,2\},(3,15),11)$. Then, $R(N,y,T) = (0,11)$ and $R(N,(3,13),9) = (27/16, 117/16)$. Thus, $R_2(N,y,T) = 11 > 2 + 117/16 = 2 + R_2(N,(3,13),9)$.

Most of the well-known rules satisfy *no donation paradox*, which shows that it is indeed a mild condition. Nevertheless, it is enough to guarantee that the minimal-burden operator preserves *merging-proofness*, as shown in the next result.

Proposition 5. *Within the family of rules satisfying no donation paradox, the minimal-burden operator preserves merging-proofness.*

The proof is provided in the appendix.

6 Concluding remarks

We have shown an intimate connection between the ethical principle of *progressive* taxation and the strategic principle of *merging-proofness*. Our results give rise to dual byproducts, which also reveal an intimate connection between the dual notions of these two principles; namely, *regressivity* (for any pair of agents, the one with lower income should face a tax rate at least as high as the rate the other faces) and *splitting-proofness*.[13]

Formally, for any given rule R, the *dual rule of* R, denoted as R^*, associates with each $(N,y,T) \in \mathscr{D}$, $R^*(N,y,T) \equiv y - R(N,y,Y-T)$. A rule is *self-dual* if it coincides with its dual. For any given property α, α^* is the *dual property of* α if for each rule R, R satisfies α if and only if its dual rule R^* satisfies α^*. *Consistency*, *continuity* and *anonymity* are self-dual properties (see Table 3.2 in p.229 of Thomson, 2006). *Progressivity* and *regressivity* are dual to each other; so are *splitting-proofness* and *merging-proofness* (e.g., de Frutos, 1999). It is not difficult to show that changing the direction of the inequalities in the definitions of *multilateral merging-proofness* and *uniformity perserving multilateral merging-proofness*, we can define their dual properties: *multilateral splitting-proofness* and *uniformity preserving multilateral splitting-proofness*. It is known that a rule is characterized by a set of properties if and only if its dual rule is characterized by the corresponding set of dual properties (e.g., Herrero and Villar, 2001). Similarly, if α and β are two properties and α^* and β^* are their corresponding dual properties, α implies β if and only if α^* implies β^* (e.g., Moreno-Ternero and Villar, 2006). Combining all these facts with our main results, we obtain the

[12] In the problem of adjudicating conflicting claims, this axiom is introduced by Thomson and Yeh (2008). It is the dual of "claims monotonicity" (see p.100 and p.161 in Thomson, 2006). See also Moreno-Ternero (2006) and Moreno-Ternero and Villar (2006)

[13] *Regressivity* may not be an interesting property from a fairness point of view. Nevertheless, it is an almost ubiquitous feature in the optimal taxation literature, initiated by Mirrlees (1971), whose classical model, which assumes agents with identical preferences (on consumption and working time) but with different earning abilities, typically prioritizes the labour-discouraging effects of the income tax over its hypothetical redistributive benefit.

next corollary:

Corollary 2. *The following statements hold*:
1. *Splitting-proofness and consistency together imply anonymity.*
2. *A consistent and continuous rule satisfies regressivity* (*splitting-proofness*) *if and only if it has a continuous parametric representation that is subhomogeneous* (*subadditive*) *in income.*[14]
3. *Within the domain of rules satisfying consistency and continuity, regressivity implies splitting-proofness.*
4. *Multilateral splitting-proofness and consistency together imply merging-proofness.*
5. *A rule satisfies multilateral splitting-proofness and consistency if and only if it is the flat tax.*
6. *A rule satisfies regressivity if and only if it satisfies splitting-proofness and uniformity preserving multilateral splitting-proofness.*

Besides obtaining logical relations between these principles, and their implications, we have also studied their robustness to the so-called minimal-burden operator. We have shown that *progressivity* is slightly more robust than *merging-proofness*, as we find an additional but mild axiom that helps the latter to be preserved by the operator. Three other operators have been considered in the literature (e.g., Thomson and Yeh, 2008) but none of them allows us to distinguish between the two principles. More precisely, both *progressivity* and *merging-proofness* are preserved under the so-called *convexity operator* (which maps a list of rules into the convex combination of these rules) whereas none of them is preserved under the so-called *duality operator* (which maps each rule into its dual rule), or under the so-called *truncation operator* (which maps each rule R into the rule defined, for each problem, by applying R after each income has been truncated at the tax revenue). Regarding the dual notions, it is a straightforward consequence of the previous results that both *regressivity* and *splitting-proofness* are preserved under the *convexity operator* whereas none of them are preserved under the *duality operator*, or under the *minimal-burden operator*. In this case, it is the *truncation operator* that allows us to distinguish between both axioms. The truncation operator preserves *regressivity* but not *splitting-proofness*. *Splitting-proofness* can be preserved after the addition of *income monotonicity* (if an agent increases her income, ceteris paribus, she cannot pay less taxes), the dual property of *no donation paradox*.[15]

Finally, we acknowledge that our analysis has focused on the simplest model of taxation that exists in the literature. This model, introduced by O'Neill (1982), provides an extremely useful framework to discuss ethical and strategic principles although it has some shortcomings as a model of taxation. It is left for further research to extend our results to a more general model of taxation in which incomes would result from economic choices and negative taxation (i.e., subsidies) would be allowed. An instance of such a model has been recently studied by Fleurbaey and Maniquet (2006), who look for the optimal tax on the basis of efficiency and fairness principles (and under incentive-compatibility constraints). In their model, agents have unequal skills (and, therefore, unequal earning abilities) and heterogeneous preferences over consumption and leisure (and, therefore, unequal labour time

[14] A parametric representation $f\colon [a,b]\times\mathbb{R}_+ \to \mathbb{R}$ is *subhomogeneous in income* if for each $\lambda \in [a,b]$, each $y_0 \in \mathbb{R}_+$, and each $\alpha \geq 1$, $f(\lambda, \alpha y_0) \leq \alpha f(\lambda, y_0)$. A parametric representation $f\colon [a,b]\times\mathbb{R}_+ \to \mathbb{R}$ is *subadditive in income* if for each $\lambda \in [a,b]$ and each pair $y_0, y_0' \in \mathbb{R}_+$, $f(\lambda, y_0 + y_0') \leq f(\lambda, y_0) + f(\lambda, y_0')$. It is worth noting that both properties are also invariant with respect to the choice of the representation.

[15] This follows from our Propositions 4 and 5, and Theorem 6 in Thomson and Yeh (2008).

1	2	3
a	a	
a	a	0
	a	a
0	a	a
a	a	

(a) Income profiles

1	2	3
x_1	x_2	
x_1	x_2	0
	x'_2	x'_3
0	x'_2	x'_3
x_1	x_2	

(b) Tax profiles

Table 1: Proof of Lemma 1.

choices).

A Proofs

Proof. [**Proof of Lemma 1**] Chambers and Thomson (2002, Lemma 3) show that *consistency* and *equal treatment of equals* together imply *anonymity*. Thus, we only have to show that *merging-proofness* and *consistency* imply *equal treatment of equals*. Now, since *equal treatment of equals* is lifted by *consistency* from the two-agent case (e.g., Hokari and Thomson (2008, Theorem 1)), we can restrict ourselves to that case. More precisely, let $(N,y,T) \in \mathscr{D}$ where $N = \{i,j\}$ and $y_i = y_j = a$. We prove in two steps that the tax amounts of i and j in this problem are the same.

Step 1. For each $k \in \mathbb{N}\backslash\{i,j\}$, $R_j(\{i,j\},(a,a),T) = R_j(\{j,k\},(a,a),T)$ *and* $R_i(\{i,j\},(a,a),T) = R_k(\{j,k\},(a,a),T)$.

Without loss of generality, suppose $i = 1$, $j = 2$, and $k = 3$. Let $x \equiv R(N,y,T)$ (the problem (N,y,T) is illustrated in the first row of Table 1-a, whereas x is illustrated in the first row of Table 1-b) and $x' \equiv R(\{2,3\},(a,a),T)$. Consider now the problem in which agent 3, with zero income, joins the original problem (N,y,T). Then, we get the three-agent problem $(N',(a,a,0),T)$, where $N' \equiv \{1,2,3\}$ (second row of Table 1-a). By *boundedness*, $R_3(N',(a,a,0),T) = 0$. By *balance* and *consistency*, $R_N(N',(a,a,0),T) = R(N,y,T)$ (second row of Table 1-b). When agents 1 and 3 in this three-agent problem merge their incomes into the income of agent 3, we obtain a new problem $(\{2,3\},(a,a),T)$ (third row of Table 1-a) and, by *merging-proofness*, $x'_3 \geq x_1$. The reverse inequality $x'_3 \leq x_1$ is obtained after using the symmetric argument as above, switching the roles of agents 1 and 3 (start from $(\{2,3\},(a,a),T)$, the third row of Table 1-a, and work through the problems in the fourth and the last rows applying the same reasoning as above).

Step 2. $R_i(\{i,j\},(a,a),T) = R_j(\{i,j\},(a,a),T)$.

Let $k,l \in \mathbb{N}\backslash\{i,j\}$ be such that $k \neq l$. Applying Step 1 twice, first to the two problems $(\{k,l\},(a,a),T)$ and $(\{i,l\},(a,a),T)$ and then to the two problems, $(\{i,l\},(a,a),T)$ and $(\{i,j\},(a,a),T)$, we obtain

$$R_k(\{k,l\},(a,a),T) = R_i(\{i,l\},(a,a),T) = R_i(\{i,j\},(a,a),T).$$

Now switching the order of i and j above and applying Step 1 twice more, we obtain

$$R_k(\{k,l\},(a,a),T) = R_j(\{j,l\},(a,a),T) = R_j(\{i,j\},(a,a),T).$$

Therefore, $R_i(\{i,j\},(a,a),T) = R_j(\{i,j\},(a,a),T)$.[16] □

Proof. [**Proof of Proposition 4**] Let R be a rule satisfying *progressivity*. Let $(N,y,T) \in \mathscr{D}$ and $x^m \equiv R^m(N,y,T)$. Assume, without loss of generality, that $N = \{1,2,...,n\}$ and $y_1 \le y_2 \le \cdots \le y_n$. Let $k \in N$ be the first agent whose minimal burden is strictly positive, i.e., $y_{k-1} \le Y - T < y_k$. Then, $m_1(N,y,T) = \cdots = m_{k-1}(N,y,T) = 0 < m_k(N,y,T) \le m_{k+1}(N,y,T) \le \cdots \le m_n(N,y,T)$. For each $i \ge k$, $m_i(N,y,T) = y_i - Y + T$. Let $y' \equiv y - m(N,y,T) = (y_1,\dots,y_{k-1},Y-T,\dots,Y-T)$ and $T' \equiv T - \sum_{i=1}^n m_i(N,y,T) = T - \sum_{i=k}^n (y_i - Y + T)$. Let $x' \equiv R(N,y',T')$. Then

$$x_i^m = \begin{cases} x_i' \text{ if } i \le k-1; \\ y_i - Y + T + x_i' \text{ if } i \ge k. \end{cases} \tag{7}$$

Let $i,j \in N$ be such that $y_i \le y_j$. There are three cases.

Case 1: $y_i \le y_j < y_k$. By *progressivity* of R at (N,y',T'), $x_i^m/y_i = x_i'/y_i' \le x_j'/y_j' = x_j^m/y_j$.
Case 2: $y_k \le y_i \le y_j$. By *equal treatment of equals* of R at (N,y',T') (implied by the *progressivity* of R), $x_i' = x_j' = a$. By *boundedness,* $x_i' = x_j' = a \le Y - T$ and so $Y - T - x_i' = Y - T - x_j' = Y - T - a \ge 0$. Therefore, since $y_i \le y_j$,

$$\frac{x_i^m}{y_i} = \frac{y_i - Y + T + x_i'}{y_i} = 1 - \frac{Y-T-a}{y_i} \le 1 - \frac{Y-T-a}{y_j} = \frac{y_j - Y + T + x_j'}{y_j} = \frac{x_j^m}{y_j}.$$

Case 3: $y_i < y_k \le y_j$. By *progressivity* of R at (N,y',T'),

$$\frac{x_i'}{y_i} \le \frac{x_j'}{Y-T}. \tag{8}$$

Now, since $Y - T < y_j$ and, by *boundedness*, $x_i' \le y_i$, then $x_i'(y_j - Y + T) \le y_i(y_j - Y + T)$. Rearranging, we get

$$x_i' \le \frac{y_i(y_j - Y + T) + x_i'(Y-T)}{y_j}. \tag{9}$$

Therefore, combining (8) and (9),

$$\frac{x_i^m}{y_i} = \frac{x_i'}{y_i} \le \frac{y_i(y_j - Y + T) + x_i'(Y-T)}{y_i y_j} \le \frac{y_j - Y + T + x_j'}{y_j} = \frac{x_j^m}{y_j}.$$

□

To prove Proposition 5, we need the following additional axiom and lemma.

No donation paradox and *merging-proofness* together imply the following useful property, as shown in the next lemma. The property says that after an agent i donates her income to the taxation authority, the total payment faced by any pair of agents involving i should not be lowered.

[16] It is clear from the proof that a much weaker version of consistency, known as null (incomes) consistency (after agents with zero income disappear, the tax amounts of others should not be affected; also called limited consistency by Thomson 2006; see also Ju et al. 2007) would suffice to prove equal treatment of equals *for two-agent problems*. However, to obtain equal treatment of equals *for a general population* and also anonymity, we appeal to the results by Chambers and Thomson (2002) and Hokari and Thomson (2008), where full consistency plays a critical role. Hence, (full) consistency in our result cannot be replaced with the weaker axiom of null consistency.

Donation-Proofness. For each $(N,y,T) \in \mathscr{D}$ and each pair $i,j \in N$ with $T \geq y_i$,

$$R_i(N,y,T) + R_j(N,y,T) \leq y_i + R_j\left(N\backslash\{i\}, y_{N\backslash\{i\}}, T - y_i\right).$$

Lemma 3. *Merging-proofness and no donation paradox together imply donation-proofness.*

Proof. Let R be a rule satisfying *merging-proofness* and *no donation paradox*. Let $(N,y,T) \in \mathscr{D}$ and $i,j \in N$ such that $T \geq y_i$. By *merging-proofness*,

$$R_i(N,y,T) + R_j(N,y,T) \leq R_j(N\backslash\{i\}, \left(y_i + y_j, y_{N\backslash\{i,j\}}\right), T).$$

By *no donation paradox*, applied to agent j with donation y_i at $\left(N\backslash\{i\}, \left(y_i + y_j, y_{N\backslash\{i,j\}}\right), T\right)$,

$$R_j(N\backslash\{i\}, \left(y_i + y_j, y_{N\backslash\{i,j\}}\right), T) \leq y_i + R_j(N\backslash\{i\}, \left(y_j, y_{N\backslash\{i,j\}}\right), T - y_i).$$

Combining the two inequalities, we obtain

$$R_i(N,y,T) + R_j(N,y,T) \leq y_i + R_j\left(N\backslash\{i\}, y_{N\backslash\{i\}}, T - y_i\right),$$

which shows *donation-proofness*. □

Now we are ready to prove Proposition 5.

Proof. [**Proof of Proposition 5**] Let R be a rule satisfying *no donation paradox* and *merging-proofness*. By Lemma 3, R satisfies *donation-proofness.* Let $(N,y,T) \in \mathscr{D}$. Assume, without loss of generality, that $N = \{1,2,...,n\}$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. Let $k \in N$ be the first agent whose minimal burden is strictly positive, i.e., k is such that $y_{k-1} \leq Y - T < y_k$. Let $i,j \in N$ and $\hat{y} \in \mathbb{R}_+^{N\backslash\{i\}}$ be such that $\hat{y}_j = y_i + y_j$ and $\hat{y}_{N\backslash\{i,j\}} = y_{N\backslash\{i,j\}}$. Let $x \equiv R(N,y,T)$ and $\hat{x} \equiv R(N\backslash\{i\},\hat{y},T)$. Let $x^m \equiv R^m(N,y,T)$ and $\hat{x}^m \equiv R^m(N\backslash\{i\},\hat{y},T)$. We show next that $x_i^m + x_j^m \leq \hat{x}_j^m$.

Let $M \equiv M(N,y,T)$ and $\hat{M} \equiv M(N\backslash\{i\},\hat{y},T)$. Let $y' \equiv (y_1,\ldots,y_{k-1},Y-T,\ldots,Y-T)$ and $x' \equiv R(N,y',T-M)$. We distinguish five cases.

Case 1: $y_i + y_j \leq Y - T$. Then $y_i, y_j \leq Y - T$ and so $x_i^m = x_i'$ and $x_j^m = x_j'$. Note that $M = \hat{M}$. Then, $R_j^m(N\backslash\{i\},\hat{y},T)$ equals j's award under R at the problem obtained from y' after merging i and j's incomes. Therefore, by *merging-proofness* of R, $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 2: $y_i, y_j > Y - T$. Without loss of generality, suppose $y_i \leq y_j$. In this case,

$$x_i^m + x_j^m = \left(\begin{array}{c} y_i - (Y-T) + R_i(N, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ +y_j - (Y-T) + R_j(N, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \end{array}\right),$$

$$\hat{x}_j^m = y_i + y_j - (Y-T) + R_j(N\backslash\{i\}, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}).$$

Since $\hat{M} = M + Y - T$, then, by *donation-proofness*,

$$R_i(N, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) + R_j(N, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M)$$
$$\leq (Y-T) + R_j(N\backslash\{i\}, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}).$$

Therefore, $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 3: $y_i \leq Y-T < y_j$. Note that $\hat{M} = M + y_i$ and

$$x_i^m + x_j^m = \begin{pmatrix} R_i(N, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) + y_j - (Y-T) \\ +R_j(N, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \end{pmatrix},$$
$$\hat{x}_j^m = y_i + y_j - (Y-T) + R_j(N\backslash\{i\}, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-\hat{M}).$$

Then, by *donation-proofness* applied to agents i and j,

$$\begin{pmatrix} R_i(N, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ +R_j(N, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \end{pmatrix}$$
$$\leq y_i + R_j(N\backslash\{i\}, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-\hat{M}).$$

Therefore, $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 4: $y_j \leq Y-T < y_i$. Note that $\hat{M} = M + y_j$ and

$$x_i^m + x_j^m = \begin{pmatrix} y_i - (Y-T) + R_i(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ +R_j(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \end{pmatrix},$$
$$\hat{x}_j^m = y_i + y_j - (Y-T) + R_j(N\backslash\{i\}, y_1, \ldots, \underset{j^{\text{th}}\text{ income}}{\underset{\uparrow}{Y-T}}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}).$$

By *merging-proofness* applied to agents i and j,

$$\begin{pmatrix} R_i(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ +R_j(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \end{pmatrix}$$
$$\leq R_j(N\backslash\{i\}, y_1, \ldots, \underset{j^{\text{th}}\text{ income}}{\underset{\uparrow}{y_j + Y - T}}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-M).$$

By *no donation paradox* applied to agent j with donation y_j,

$$\begin{aligned}
&R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}}\text{ income}}}{y_j + Y - T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-M) \\
&\leq y_j + R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}}\text{ income}}}{Y-T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}).
\end{aligned}$$

Combining the two inequalities, we obtain

$$\begin{aligned}
&\left(\begin{array}{c} R_i(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ +R_j(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \end{array}\right) \\
&\leq y_j + R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}}\text{ income}}}{Y-T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}),
\end{aligned}$$

which implies $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 5: $y_i, y_j \leq Y - T$ and $y_i + y_j > Y - T$. Then $\hat{M} = M + T - (Y - (y_i + y_j))$ and

$$\begin{aligned}
x_i^m + x_j^m &= \left(\begin{array}{c} R_i(N, y_1, \ldots, y_i, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ +R_j(N, y_1, \ldots, y_i, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M), \end{array}\right) \\
\hat{x}_j^m &= T - (Y - (y_i + y_j)) + R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}}\text{ income}}}{Y-T}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-\hat{M}).
\end{aligned}$$

By *merging-proofness* applied to agents i and j,

$$x_i^m + x_j^m \leq R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}}\text{ income}}}{y_i + y_j}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M).$$

Since $T - \hat{M} = T - M - (T - (Y - (y_i + y_j)))$, then applying *no donation paradox* to j with donation $T - (Y - (y_i + y_j))$,

$$\begin{aligned}
&R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}}\text{ income}}}{y_i + y_j}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\
&\leq T - (Y - (y_i + y_j)) + R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}}\text{ income}}}{Y-T}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-\hat{M}).
\end{aligned}$$

Therefore, $x_i^m + x_j^m \leq \hat{x}_j^m$. □

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