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NEGISHI'S THEOREM AND METHOD*

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*This paper is written as a 'half-century' homage to two of the most important contributions to economic theory and mathematical economics: Negishi (1960) and Uzawa (1962). The former celebrated its Golden Jubilee last year; the latter will do so next year. Almost nothing in computable general equilibrium theory, and its many and varied applied variants, would be possible without the results in these two classics.

Abstract

Takashi Negishi's remarkable youthful contribution to welfare economics, general equilibrium theory and, with the benefit of hindsight, also to one strand of computable general equilibrium theory, all within the span of six pages in one article, has become one of the modern classics of general equilibrium theory and mathematical economics. Negishi's celebrated *theorem* and what has been called *Negishi's Method* have formed one foundation for *computable* general equilibrium theory. In this paper I investigate the computable and constructive aspects of the theorem and the method.

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1 Introduction and Motivation

David Luenberger's pithy characterisation of the defining achievements of post-war mathematical general equilbrium theory reflects, I believe uncontroversially, the general consensus among economic theorists of all shades of opinion:

> "The most remarkable achievements of modern microeconomic theory are the *proof of the existence* of an equilibrium and the First and Second Theorems of Welfare Economics which establish the relation between equilibria and Pareto efficiency. These results show that the structure of microeconomics has a desirable self-consistency, and they show that this structure can fruitfully address significant economic issues. Understandably, therefore, there has been much attention devoted to various alternative proofs, and extensions of these basic results."
> Luenberger (1994), p. 147; italics added.

I have, myself, tended to refer to these three celebrated achievements as *the crown jewels* of mathematical general equilibrium theory. The two kinds of mathematical theorems that underpin the method of proofs used in demonstrating these three results, to which I refer as the *pearls* of mathematical economics, are the fix point theorems of Brouwer (1910) and Kakutani (1941), on the one hand, and the separating hyperplane theorems – in particular *the Hahn-Banach Theorem* (cf. Debreu, 1984, in particular, p. 269) – on the other.

Negishi's theorem[1] and the *method of proof* used in deriving the theorem are two of the fundamental contributions to the further theoretical and applied development of the crown jewels of mathematical equilibrium theory (Negishi, 1960). Negishi himself had the following reflection, more than thirty years later, on the method of proof used in Negishi (1960):

> "The *method of proof* used in this essay [i.e., in Negishi (1960)] has been found useful also for such problems as equilibrium in infinite dimensional space and *computation of equilibria*."
>
> Negishi (1994), p. xiv; italics added."

My aim in this paper is the narrow one of studying the *computability* and *constructive* theoretic properties of Negishi's theorem and, hence, the methods of proof utilised in them. The reasons for this particular focus of investigation is easily stated. In an early and important contribution to a study of Negishi's Theorem, Diewert made the important observation:

> "If general equilibrium analysis is to become a useful tool in the construction of macroeconomic models, **effective algorithms** must

[1] There are actually five theorems in Negishi (1960). I shall concentrate on *Theorem 2* (ibid, p.5), which (I think) is the most important one and the one that came to play the important role justly attributed to it via the *Negishi Research Program* outlined by Young (2008).

> be developed in order to *compute* general equilibrium. ... Scarf ... has constructed an algorithm which will find a fixed point ... ."
>
> Diewert, 1973, p. 119; bold italics added.

This may be juxtaposed and read together with the claims of Shoven and Whalley on the achievements of Scarf's algorithm:

> "The major result of postwar mathematical general equilibrium theory has been to demonstrate the existence of such an equilibrium by showing the applicability of mathematical fixed point theorems to economic models. ... Since applying general equilibrium models to policy issues involves computing equilibria, these fixed point theorems are important: It is essential to know that an equilibrium exists for a given model before attempting to compute that equilibrium. .....
>
> ...
>
> The weakness of such applications is twofold. First, they provide *non-constructive rather than constructive proofs of the existence of equilibrium*; that is, they show that equilibria exist but do not provide techniques by which equilibria can actually be determined. Second, existence per se has no policy significance. .... Thus, fixed point theorems are only relevant in testing the logical consistency of models prior to the models' use in comparative static policy analysis; such theorems do not provide insights as to how economic behavior will actually change when policies change. *They can only be employed in this way if they can be made constructive* (i.e., be used to find actual equilibria). *The extension of the Brouwer and Kakutani fixed point theorems in this direction is what underlies the work of Scarf .... on fixed point algorithms ....*"
>
> Shoven & Whalley (1992), 12, 20-1; italics added

Diewert's desidaratum on 'effective algorithms' can only be given content, formally, in terms of *computability theory*; Shoven and Whalley, on the other hand, claim – explicitly – that Scarf's work on 'fixed point algorithms' are *constructive* (in the strict mathematical sense). In a recent series of papers on these issues I have demonstrated that the first desidaratum is impossible to achieve and the second claims are incorrect (cf., Velupillai, 2006, 2009).

Scarf, however, was much more measured in his claims of what he had achieved:

> "In applying the algorithm it is, in general, *impossible* to select an ever finer sequence of grids and a convergent sequence of subsimplices. An algorithm for a digital computer must be basically finite and cannot involve an infinite sequence of successive refinements. .... *The passage to the limit is the nonconstructive aspect of Brouwer's theorem*, and we have no assurance that the subsimplex

determined by a fine grid of vectors on S contains or is even close to a true fixed point of the mapping."

Scarf (1973), p.52; italics added

To the best of my knowledge, no one has investigated the exact content of Negishi's Theorem or his Method(s) of Proof from either computable or constructive points of view. This is the main reason for the aims of this paper.

The paper is organised as follows. In the next section a brief background discussion of the economic theoretic basis of the theorem is outlined, with some minor clairifications and explanations.

2 Background Clarifications

What exactly was Negishi's *method of proof* and how did it contribute to the *computation of equilibria*?

A characterisation of the difference between the standard approach to *proving the existence* of an Arrow-Debreu *equilibrium*, and its *computatio*n by a tâtonnement procedure – i.e., algorithm – of a mapping from the price simplex to itself, and the alternative *Negishi method* of iterating the weights assigned to individual utility functions that go into the definition of a social welfare function which is maximised to determine – i.e., compute – the equilibrium, captures the key innovative aspect of the latter approach. Essentially, therefore, the difference between the standard approach to the proof of existence of equilibrium Arrow-Debreu prices, and their computation, and the *Negishi approach* boils down to the following:

- The standard approach proves the existence of Arrow-Debreu equilibrium prices by an appeal to a fixed point theorem and computes them – the equilibrium prices – by invoking the *Uzawa equivalence theorem* (Uzawa, 1962) and devising an algorithm for the excess demand functions that map a price simplex into itself to determine the fixed point (Scarf, 1973).

- The Negishi approach *proves*, given initial endowments, *the existence* of individual welfare weights defining a social welfare function, whose *maximization* (subject to the usual constraints) *determines* the identical Arrow-Debreu *equilibrium*. The standard mapping of excess demand functions, mapping a price simplex into itself to determine a fixed point, is replaced by a mapping from the space of utility weights into itself, appealing to the same kind of fixed point theorem (in this case, the Kakutani fixed point theorem) to prove the existence of equilibrium prices.

- In other words, the method of proof of existence of equilibrium prices in the one approach is replaced by the *proof of existence* of 'equilibrium utility weights', both appealing to traditional *fixed point theorems* (Brouwer, 1910, von Neumann, 1937, and Kakutani, 1941).

- In both cases, the computation of equilibrium prices on the one hand and, on the other, the computation of equilibrium weights, algorithms are devised that are claimed to determine (even if only approximately) the same fixed points.

Before proceeding any further, I should add that I am in the happy position of being able to refer the interested reader to a scholarly survey of Negishi's work. Takashi Negishi's outstanding 'contributions to economic analysis' are brilliantly and comprehensively surveyed by Warren Young in his recent paper (Young, 2008). Young's paper provides a particularly appropriate background to the issues I tackle here. It – Young's paper – is especially relevant also because his elegant summary of Negishi's 'contribution to economic analysis' identifies Negishi (1960) as one of the two crucial pillars[2] on which to tell a coherent and persuasive story of what he calls the Negishi 'research program' (ibid, p. 162; second set of italics, added):

> "To sum up, a number of major research programs can be identified, therefore, as emanating from Negishi's now *classic* papers, that of [Negishi, 1960] and [Negishi 1961], respectively. Negishi's 1960 paper forms the basis for both 'theoretical' and 'applied' research programs in general equilibrium analysis, and his 1961 paper ... has been *almost as influential* in demarcating ongoing research up to the present in the field of imperfect competition and non-tatonnement processes. These papers ... attest to Negishi's considerable influence on the development of modern economic theory and analysis."

However, as mentioned above, no one – to the best of my knowledge – has studied Negishi's *method of proof* from the point of view of *constructivity* and *computability*. Young's perceptive - and, in my opinion, entirely correct - identification of the crucial role played by Negishi (1960) in 'both "theoretical" and "applied" research program in general equilibrium analysis' is, in fact, about *methods of* ***existence*** *proofs* and *computable general equilibrium* (**CGE**), and its offshoots, in the form of *applied computable general equilibrium analysis* **ACGE**) – even leading up to current frontiers in computational issues in *Dynamic Stochastic General Equilibrium* models (cf., Judd (2005), pp. 52-57, for example).

Before I turn to these issues of the constructivity and computability of Negishi's method of existence proofs and the underpinning of some aspects computation in CGE and ACGE models in Negishi's approach (rather than, for example, in the standard approach pioneered by Scarf, 1973), there is one important economic theoretic confusion that needs to be sorted out. This is the question of the role played by the *fundamental theorems of welfare economics* in Negishi's method of the proof of the existence of a general (Walrasian) equilibrium.

[2] The other one being Negishi (1961). I am in full agreement with Young's important observation that it is Negishi (1960) that is more important, which is why I have added italics to the phrase 'almost as influential', in the above quote.

It is generally agreed that the *Negishi method of existence proof* is an applications of fixed point theorems on the *utility simplex*, in contrast to the 'standard' way of applying such theorems to the *price simplex* (cf., Cheng 1991, p. 138, and above). This fact has generated a remarkable confusion on the question of *which fundamental theorem of welfare economics underpins the Negishi method*! For a method that has been around for over half a century, it is somewhat disheartening to note that frontier research and researchers seem still to be confused on which of the two fundamental theorems of welfare economics is relevant in Negishi's method. Thus, we find Judd, as recently as only a few years ago (op.cit, pp. 52-3) claiming, unreservedly, that (italics added):

> "T*he Negishi method exploits the first theorem of welfare economics*, which states that any competitive equilibrium of an Arrow-Debreu model is Pareto efficient."

On the other hand, Warren Young (op.cit, p.152; italics added) equally confidentially stating that:

> In his pioneering 1960 paper, Negishi provided a completely new way of proving the existence of equilibrium, *via the Second Welfare Theorem*. He established equivalence between the equilibrium problem set out by Arrow-Debreu and what has been called 'mathematical programming', thereby developing a 'method' that has been used with much success by later economists working in both theoretical and applied general equilibrium modelling ... ."

Fortunately, Negishi himself returned to a discussion of the 'Negishi method, or Negishi approach' more recently (Negishi, 2008, p. 168) and may have helped sort out this conundrum (ibid, p. 167; italics added):

> "The so-called Negishi method, or Negishi approach, has often been used in studies of dynamic infinite-dimensional general equilibrium theory, and the numerical computation of such equilibria ... . This method is an application of the Negishi theorem (Negishi, 1960), which demonstrates the existence of a general equilibrium *using the first theorem of welfare economics*, which states that any competitive equilibrium of an Arrow-Debreu model is Pareto efficient. In other words, a general equilibrium of a competitive economy is considered as the maximization of a kind of social welfare function (i.e., the properly weighted sum of individual utilities), where the weights are inversely proportional to the marginal utility of income."

Negishi is one of those rare economists who is both a scholar of the history of economic theory and one of the most competent general equilibrium theorists and – *even if he had not been the originator of the Negishi method* – one may feel forced to reject Warren Young's claim[3]!

[3] The puzzle here is that the Young and Negishi articles appear 'back-to-back', in the same issue of the *International Journal of Economic Theory* and the two distinguished authors thank each other handsomely in their respective acknowledgements!

As a matter of fact, from a constructivist and recursion theoretic point of view, this conundrum is a non-problem for several reasons. First of all, both fundamental theorems of welfare economics are provably non-constructive and lead to uncomputable equilibria. Secondly, all – to the best of my knowledge – of the current algorithms utilised in CGE, ACGE and DSGE modelling appeal to undecidable disjunctions – essentially by an appeal to the Bolzano-Weierstrass theorem - and are effectively meaningless from computable and constructive points of view. Thirdly, and most importantly, *Negishi's theorem*(s) are, themselves, *proved* nonconstructively – the issue(s) to which I now turn.

3 Nonconstructive and Uncomputable aspect of Negishi's Theorem and the Negishi Method

A brief preamble on why the Brouwer fix point theorem, which lies at the basis of the proofs of both Kakutani's theorem and Slater's results (see below) may make the content of this section reasonably self-contained. Before I do provide the 'brief preamble' it may also be useful to begin it with the following important observation by Brouwer himself:

> "This is a specimen of intuitionist reasoning in topology, and in particular an illustration of the consequences of the invalidity of the Bolzano-Weierstrass theorem in intuitionism, for *the validity of the Bolzano-Weierstrass theorem would make the classical and intuitionist forms of the fixed-point theorms equivalent.*"
> Brouwer (1952), p. 1; italics added.

Brouwer, in the above quote, is – of course – referring to his celebrated fixed-point theorem, widely used in mathematical economics in its original form, or in one or another of its 'generalizations', by Kakutani, Knaster-Kuratwoski-Mazurkiewicz (KKM), etc. On the other hand, just because a fixed-point theorem is invalid from an intuitionistic point of view[4] does not necessarily mean that it is non-constructive or uncomputable from mathematical points of view claiming allegiance to other forms of constructivism and varieties of computability theories. The point here, however, is the role of the Bolzano-Weierstrass theorem and its intrinsic undecidable disjunctions, which make any theorem invoking it in its proof fundamentally non-constructive and uncomputable from *any* (known) mathematical point of view.

An algorithm, by definition, is a finite object, consisting of a finite sequence of instructions. Just being a finite object does not automatically make it *effective*. Any one of the finite sequence of instructions has the potential to invoke undecidable disjunctions, as in Scarf's algorithm. However, such a finite object is perfectly compatible with 'an infinite sequence of successive refinements'

[4]We are 'advised', in a recent advanced textbook in **Real Analysis with Economic Applications** (Ok, 2007, p. 279, footnote 47), 'If [we] want to learn about intuitionism in mathematics', to do so 'in [our] spare time, please'! The footnote in which this 'advice' appears contains elementary mathematical and biographical errors (on Brouwer).

(Scarf, 1973, p. 52), provided a stopping rule associated with a clearly specified and verifiable approximation value is part of the sequence of instructions that characterize the algorithm. Moreover, it is *not* 'the passage to the limit [that] is the nonconstructive aspect of Brouwer's [fix point] theorem' (ibid, p.52)[5]. Instead, the sources of non-constructivity are the undecidable disjunctions - i.e., appeal to the *law of the excluded middle* in *infinitary* instances - intrinsic to the choice of a convergent subsequence in the use of the Bolzano-Weierstrass theorem[6] and an appeal to the *law of double negation* in an infinitary instance during a *retraction.* The latter reliance invalidates the proof in the eyes of the Brouwerian constructivists; the former makes it constructively invalid from the point of view of every school of constructivism, whether they accept or deny intuitionistic logic.

Brouwer's proof of his celebrated fix point theorem was indirect in two ways: he proved, first, the following:

Theorem 1 *Given a continuous map of the disk onto itself with no fixed points,* $\exists$ *a continuous retraction of the disk to its boundary.*

Having proved this, he then took its *contrapositive:*

Theorem 2 *If there is no continuous retraction of the disk to its boundary then there is no continuous map of the disk to itself without a fixed point.*

Using the logical principle of equivalence between a proposition and its contrapositive (i.e., logical equivalence between theorems 1 & 2) and the law of double negation ($\nexists$ a continuous map with **no** fixed point = $\exists$ a continuous map with a fixed point) Brouwer demonstrated the existence of a fixed point for a continuous map of the disk to itself. This latter principle is what makes the proof of the Brouwer fix point theorem via retractions (or the non-retraction

[5] In Scarf (1982), p. 1024, Scarf is more precise about the reasons for the failure of constructivity in the proof of Brouwer's fix point theorem:

> "In order to demonstrate Brouwer's theorem completely we must consider a sequence of subdivisions whose mesh tends to zero. Each such subdivision will yield a completely labeled simplex and, *as a consequence of the compactness* of the unit simplex, there is a convergent subsequence of completely labeled simplices all of whose vertices tend to a single point x^*. (This is, of course, the non-constructive step in demonstrating Brouwer's theorem, rather than providing an approximate fixed point)."

There are two points to be noted: first of all, even here Scarf does not pinpoint quite precisely to the main culprit for the cause of the non-constructivity in the proof of Brouwer's theorem; secondly, nothing in the construction of the algorithm provides a justification to call the value generated by it to be an approximation to x^*. In fact the value determined by Scarf's algorithm has no theoretically meaningful connection with x^* (i.e., to p^*) for it to be referred to as an approximate equilibrium.

[6] Just for ease of reading the discussion in this section I state, here, the simplest possible statement of this theorem:

> **Bolzano-Weierstrass Theorem:** Every bounded sequence contains a convergent subsequence

theorem) essentially unconstructifiable. Scarf's attempt to discuss the relationship between these two theorems [i.e., between the non-retraction and Brouwer fix point theorems] and to interpret [his] combinatorial lemma [on effectively labelling a restricted simplex] as an example of the non-retraction theorem is incongruous. This is because Scarf, too, like the Brouwer at the time of the original proof of his fix-point theorem, uses the full paraphernalia of non-constructive logical principles to link the Brouwer and non-retraction theorems and his combinatorial lemma[7].

The two relevant theorems, in the context of the background provided in this paper, in Negishi (1960), are theorems 1 & 2. I shall concentrate on *Theorem 2* (ibid, p.5), which (I think) is the more important one and the one that came to play the important role justly attributed to it via the *Negishi Research Program* outlined by Young (op.cit)[8].

Proposition 3 *The Proof of the Existence of Maximising Welfare Weights in the Negishi Theorem is Nonconstructive*

Proof. (Sketch) Negishi's proof relies on satisfying the Slater (Complementary) Slackness Conditions (Slater, 1950[9]). Slater's proof[10] of these conditions invoke the Kakutani fixed point theorem (Theorem 1 in Kakutani, 1941), and Kakutani's Min-Max Theorem (Theorem 3, ibid). These two theorems, in turn, invoke Theorem 2 and the Corollary (ibid, p.458), which are based on Theorem 1 (ibid, p. 457). This latter theorem is itself based on the validity of the Brouwer fixed point theorem, which is not just nonconstructive, but also non-constructifiable (cf., Brouwer, 1952). ■

Proposition 4 *The vector of maximising welfare weights, derived in the Negishi Theorem, is uncomputable*

Proof. A straightforward implication of Proposition 1 ■

Discovering the exact nature and source of appeals to nonconstructive modes of reasoning, appeals to undecidable disjunctions and reliance on nonconstructive mathematical entities in the formulation of a theorem is a tortuous exercise. The nature of the pervasive presence of these three elements – i.e., nonconstructive modes of reasoning, primarily the reliance on *tertium non datur*, undecidable disjunctions and nonconstructive mathematical entities – in any standard theorem and its proof, and the difficulties of discovering them, is elegantly outlined by Fred Richman (1990, p. 125; italics added):

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[7] Scarf uses, in addition, proof by contradiction where, implicitly, LEM (*tertium non datur*) is also invoked in the context of an infinitary instance (cf. Scarf (1982), pp. 1026-7).

[8] To demonstrate the *nonconstructive* elements of Theorem 1 (ibid, p.5), I would need to include almost a whole tutorial on constructive mathematics to make clear the notion of *compactness* that is *legitimate in constructive analysis*.

[9] Slater (1950) must easily qualify for inclusion in the class of pioneering articles that remained forever in the '*samizdat*' status of a *Discussion Paper*!

[10] I should add that the applied general equilibrium theorists who use Negishi's method to 'compute' (uncomputable) equilibria do not seem to be fully aware of the implications of some of the key assumptions in Slater's complementary slackness conditions. That Negishi (1960) is aware of them is clear from his *Assumption 2* and *Lemma 1*.

> "Even those who like algorithms have remarkably little appreciation of the thoroughgoing algorithmic thinking that is required for a constructive proof. This is illustrated by the nonconstructive nature of many proofs in books on numerical analysis, the theoretical study of practical numerical algorithms. I would guess that most realist mathematicians are unable even to recognize when a proof is constructive in the intuitionist's sense.
>
> *It is a lot harder than one might think to recognize when a theorem depends on a nonconstructive argument. One reason is that proofs are rarely self-contained, but depend on other theorems whose proofs depend on still other theorems. These other theorems have often been internalized to such an extent that we are not aware whether or not nonconstructive arguments have been used, or must be used, in their proofs*. Another reason is that the law of excluded middle [LEM] is so ingrained in our thinking that we do not distinguish between different formulations of a theorem that are trivially equivalent given LEM, although one formulation may have a constructive proof and the other not."

These are further reasons to pay close attention to Richman's carefully spelled out constructive thoughts. For, a supreme mathematical economic theorist like Takashi Negishi, who also happens to be mathematically very able, could use words like 'computation', even if referring to application by others of his theorem(s), in an otherwise wholly nonconstructive setting and not suspect that 'proofs are rarely self-contained, but depend on other theorems whose proofs depend on still other theorems. These other theorems have often been internalized to such an extent that we are not aware whether or not constructive arguments have been used, or must be used, in their proofs.'

4 Concluding Notes

It may be appropriate to conclude this brief exercise with some comments on the uncomputable and non-constructive underpinnings of the two fundamental theorems of welfare economics.

The *First Fundamental Theorem of Welfare Economics* asserts that *a competitive equilibrium is Pareto optimal*. The theorem is proved non-constructively, using an uncomputable equilibrium price vector to compute an equilibrium allocation. Therefore, the contradiction step in the proof requires a comparison between an uncomputable allocation and an arbitrary allocation, for which no computable allocation can be devised. Moreover, the theorem assumes the intermediate value theorem in its non-constructive form. Finally, even if the equilibrium price vector is computable, the contradiction step in the proof invokes the law of the excluded middle and is, therefore, unacceptable constructively (because it requires algorithmically undecidable disjunctions to be employed in the decision procedure).

The *Second Fundamental Welfare Theorem* establishes the proposition that *any Pareto optimum can, for suitably chosen prices, be supported as a competitive equilibrium.* The role of the Hahn-Banach theorem in this proposition is in establishing *the suitable price system.*

The Hahn-Banach theorem does have a constructive version, but only on subspaces of *separable* normed spaces. The standard, 'classical' version, valid on nonseparable normed spaces depends on *Zorn's Lemma* which is, of course, equivalent to the axiom of choice, and is therefore, non-constructive[11].

Schechter's perceptive comment on the constructive Hahn-Banach theorem is the precept I wish economists with a numerical, computational or experimental bent should keep in mind (ibid, p. 135).:

> "[O]ne of the fundamental theorems of classical functional analysis is the *Hahn-Banach Theorem; ...* some versions assert the existence of a certain type of linear functional on a normed space X. The theorem is inherently nonconstructive, but a constructive proof can be given for a variant involving normed spaces X that are *separable* – i.e., normed spaces that have a countable dense subset. *Little is lost in restricting one's attention to separable spaces*[12]*, for in applied math most or all normed spaces of interest are separable.* The constructive version of the Hahn-Banach Theorem is more complicated, but it has the advantage that it actually *finds* the linear functional in question."

So, one may be excused for wondering, why economists rely on the 'classical' versions of these theorems? *They are devoid of numerical meaning and computational content. Why go through the rigmarole of first formalizing in terms of numerically meaningless and computationally invalid concepts to then seek impossible and intractable approximations to determine uncomputable equilibria, undecidably efficient allocations, and so on?*

Thus my question is: why should an economist *force* the economic domain to be a normed vector space? Why not a *separable normed vector space*? Isn't this because of unfamiliarity with constructive mathematics and a carelessness about the nature and scope of fundamental economic entities and the domain over which they should be defined?

On the other hand, the first fundamental theorem of welfare economics fails constructively and computably on three grounds: the dependence on the intermediate value theorem (non-constructive), the use of an uncomputable equilibrium price vector in the proof by contradiction (uncomputability) and the

[11] This is not a strictly accurate statement, although this is the way many advanced books on functional analysis tend to present the Hahn-Banach theorem. For a reasonably accessible discussion of the precise dependency of the Hahn-Banach theorem on the kind of axiom of choice (i.e., whether countable axiom of choice or the axiom of dependent choice), see Narici & Beckenstein (1997). For an even better and fuller discussion of the Hahn-Banach theorem, both from 'classical' and a constructive points of view, Schechter's encyclopedic treatise is unbeatable (Schechter, 1997)).

[12] However, it must be remembered that Ishihara, Ishihara (1989), has shown the constructive validity of the Hahn-Banach theorem also for uniformly convex spaces.

use of the law of the excluded middle in the proof by contradiction (non-constructivity).

Thus, although I still subscribe to Luenberger's – and the general mathematical economics and economic theoretic profession's – view that the proof of existence of equilibrium and the two fundamental theorems of welfare economics are the 'crown jewels' of the subject, and the two most important mathematical tools – fixed point theorems and the Hahn-Banach theorem – are the pearls of the subject, I now feel they lose some of their lustre, when viewed computable and constructively. To that extent the fundamental theorems and methods of proof of Negishi and Scarf, when invoked in computable contexts or endowed with constructivity properties, must be used with the care their originators always bestowed upon them. This care is less rigorously observed by the many applied economists who, understandably, want to make the theory useful in an empirical, policy oriented sense, for which the links that these theorems and methods of proof with the equilibrium existence theorem and the fundamental theorems of welfare economics are irresistible magnets.

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