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SECTORS MAY USE MULTIPLE TECHNOLOGIES SIMULTANEOUSLY: THE RECTANGULAR CHOICE-OF-TECHNOLOGY MODEL WITH BINDING FACTOR CONSTRAINTS

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Number 1006

October 2010

Sectors May Use Multiple Technologies Simultaneously: The Rectangular Choice-of-Technology Model with Binding Factor Constraints

Abstract. We develop the rectangular choice-of-technology model with factor constraints, or RCOT, for analysis of the economy of a single region, or of multiple regions in the context of a model of the world economy. RCOT allows for one or more sectors to operate more than one technology simultaneously, using the relatively lowest-cost one first and adding another if and when the preceding one encounters a binding factor constraint. The model is motivated by the evident fact that oil wells and mineral deposits of different qualities may be exploited simultaneously, as may the use of both primary and recycled sources for the same materials. RCOT generalizes Carter's choice-of-technology model, which allowed one of two choices to all sectors, for *up to q* choices and adds the factor constraints that allow several alternatives to operate simultaneously. The Appendix gives a numerical example.

Keywords: input-output model, choice of technology, binding factor constraints, nonrenewable resources

JEL codes: C67, O33, Q32

1. Introduction

Every good may be produced in different establishments using somewhat different technologies. However, for some goods the range of variation in simultaneously utilized input structures is especially wide, with significant economic and environmental implications. This is particularly true for resource extraction sectors exploiting deposits of heterogeneous qualities and limited supply. If a low-cost mine does not produce enough to satisfy demand, it may be necessary to supplement this output using a different technology at other mines with less accessible, lower grade deposits. An economy may rely on both low-input, rain-fed agriculture while also making use of higher-input, irrigated methods for producing the same crop on other land. Metal-processing industries may exploit recycled materials as available, supplemented by extraction of primary ores to fill their remaining needs. These choices among alternatives will reflect cost differences, and in the case of scarce factors several options may be simultaneously operational. It is especially important to capture wide variations in cost structures for producing a particular good, rather than relying on an average only, in the context of a model of the world economy based on comparative advantage, since in this framework it is the relatively highest-cost producer who sets the world price. Relying on an average technology in a producing country would set the world price artificially low.

One of the features of the basic input-output model, $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{y}$, is that each sector is assumed to produce a single characteristic output and to do so using a single average input structure, or technology. It is well understood that both of these are simplifying assumptions. Any model substantially simplifies the complexities of a real economy, and the widespread use of square input-output tables and matrices, not only by economists but increasingly also by industrial ecologists, suggests that the resulting representation is deemed adequate for most purposes. This is true because the variations among co-existing technologies within a sector are generally much smaller than those among sectors producing different outputs. But it can be important to represent the options among alternative input structures in selected cases, especially for the extractive sectors, which need to match technologies to the heterogeneous qualities of factor endowments (land, water, in-ground and offshore resources) and shift to different technologies, often with step increases in costs, as higher quality endowments are exhausted.

This paper develops what we call the rectangular choice-of-technology model with factor constraints, or RCOT model, and illustrates its use for analysis of a single region's economy and then for multiple regions in the context of a model of the world economy. The term rectangular refers to the additional columns, but not rows, that are inserted into the initially square **A** matrix to represent the additional technologies. The number of rows remains unchanged because the purchasers of the product of the sector are assumed to be indifferent to the technology used in its production. The model is compared to previous models, referred to in this paper as square models, that allow a choice of technology by means of multiple, square **A** matrices to account for a given number (the same number for all sectors in the case of square models) of additional technologies.

For the questions we set out to address, described below, we show that the rectangular model of a single region's options is superior to the square model because: (1) it obtains a well-defined, unique solution by contrast with the indeterminateness inherent in the square model, and (2) it eliminates the need to maintain and manipulate redundant information that characterizes the square model. Only in the special case when all sectors have the same number of distinct technological options are the two methods mathematically equivalent. In the general case when each sector may have any number of options, and many if not most have only one, a single rectangular **A** matrix with contiguous representation of all the options available to each sector has clear advantages: it is not only more compact but also conceptually and practically more compelling than the "trick" of using multiple square matrices with a lot of redundancy.

In Section 2 we review the small literature on square choice-of-technology models, where all sectors have exactly two options, and indicate the differences in both objectives and assumptions relative to the RCOT model. Section 3 presents the RCOT model, first without factor constraints as is the case in applications of the square model, and it then introduces factor constraints and describes their implications. Section 4 generalizes the square choice-of–technology model, first for the case where all sectors have an arbitrarily large but identical number of alternative technologies and then for the more general case where different sectors have different numbers of choices. The section concludes by comparing the rectangular model and square model. Section 5 extends the rectangular model from a single region to multiple regions, situating them in the context of an input-output model of the world economy, the World Trade Model, or WTM (Duchin 2005). Section 6 offers concluding comments and a discussion of data sources and next steps, and the Appendix contains a numerical example to illustrate the models.

2. Applications of the Square Choice-of-Technology Model with Two Choices

In her important book-length study of technological change in the U.S. economy, Carter introduced into the basic, one-region input-output model the choice between two alternative technologies (Carter 1970). With the objective of explaining the technological changes that had taken place between 1947 and 1958, Carter provided to each sector in 1958 the choice between the input structure in the 1958 input-output table and that in the table for 1947. She formulated the problem as a linear program in terms of 2 coefficient matrices, $\mathbf{A_{1947}}$ and $\mathbf{A_{1958}}$, representing the old and new options, respectively. The program assured that 1958 final demand was satisfied at the lowest cost in terms of value-added, the latter being calculated as the sum of labor costs (labor inputs times a wage rate) and interest charges:

$$\text{Min } Z = \mathbf{v}'_{\mathbf{1947}}\mathbf{x}_{\mathbf{1947}} + \mathbf{v}'_{\mathbf{1958}}\mathbf{x}_{\mathbf{1958}}$$

$$\text{s.t. } (\mathbf{I}-\mathbf{A}_{\mathbf{1947}})\mathbf{x}_{\mathbf{1947}} + (\mathbf{I}-\mathbf{A}_{\mathbf{1958}})\mathbf{x}_{\mathbf{1958}} \geq \mathbf{y}. \qquad (1)$$

The optimal solution relied upon most sectors' selecting the 1958 technology. However, for several sectors including iron and petroleum mining, the 1947 technologies would have been more efficient choices in 1958 but were probably no longer plausible options. The superiority of the 1958 technologies turned out to be robust under a plausible range of wage rates and different assumptions about the interest rates in the two years.

Leontief (1986) used the same model with two alternatives for each sector to analyze the choice between an old technology, represented by a column in the U.S. input-output matrix for 1977, and a new technology, in a projected matrix for 2000. He examined how the choices would change under different assumptions about relative prices for capital and labor, employing an explicit matrix of capital coefficients and an index of real wages, and allowing for trade-offs between the two factor prices. He found that health-care and education sectors would choose the old options in all cases and explained that outcome by government subsidies to the newer option to enable quality improvements. At the other extreme, computers and semiconductors would adopt the new technologies under the full range of combinations of factor prices that were examined (ranging from zero to 40% as the rate of return on capital).

Duchin and Lange (1995) employed the same model with technological choices for 1963 and 1977, based on input-output tables for those two years, and then 1977 and 2000, using a projected matrix for 2000. They sought to distinguish whether opting for the newer technology in 1977 (or 2000) was advantageous for a sector if only that sector adopted the new technology, or if the benefit of that choice was dependent on other sectors' also adopting the newer technological options. They found that typically the newer technology was cost-saving in both cases, but that some sectors would have made clearly suboptimal choices in choosing the newer technology if other sectors – in particular the computer-related sectors -- did not also do so.

In each of these three studies, all sectors have exactly two distinct technologies to choose between, the options correspond to the average technologies available in two different years, and the alternatives are represented as the columns of two square matrices of dimension n x n. The solution selects one and only one of the two technologies to be employed in each sector. Since it is known in advance that the newer technology was in fact put in place in the case of an *ex post* analysis, the square model makes it possible to discover those instances where this decision does not appear to have been cost saving for individual sectors. Carter's linear programming formulation has the important property of allowing producers a choice of technologies, to which consumers of the homogeneous output are indifferent.

More recently, Julia and Duchin (2007) developed a square formulation that is more general in several ways. They allow for any number of alternative technologies for a given sector in a single region, where alternatives represent simultaneously available options and not all sectors are required to have the same number of options. In addition, individual regions make these choices *simultaneously* within the context of a model of the world economy. This framework for the first time offers both intra-regional and inter-regional choices among technological options. To study the effects of climate change on global agriculture, Julia and Duchin allow each region up to six options for crop production and six for livestock production, with the choices depending ultimately upon competition for different qualities of land. The representation for a single region takes the form of six region-specific **A** matrices of order n x n. (It also requires six region-specific matrices of factor requirements per unit of output. The representation of factor requirements and the imposition of factor constraints are discussed below.) For those sectors that have only a single technological option, the same column of coefficients is repeated in all matrices. Applying the World Trade Model (Duchin 2005) to twelve regions, the cost structures for all the alternative technologies in all regions are considered simultaneously to determine the lowest-cost international distribution of production and corresponding world prices. The WTM also imposes factor constraints, with implications described below.

All the models described in this section utilize multiple square **A** matrices in order to describe the alternative technologies. In the next section we replace these matrices by a single rectangular form of the **A** matrix. None of the previous studies considered the implications of constraints on factor availability, and in the absence of factor constraints each sector operates using a single, lowest-cost technology. Factor constraints, however, may require a sector to use more than one of its available technologies, starting with the cheapest technology until it runs into a binding constraint and then choosing the next best option. For each constraint that is actually binding, some sector will utilize one additional technology. All sectors compete for some factors, notably capital and labor. Other factors may be sector-specific: for example, the oil extraction sector will apply one technology (on average) to extract crude oil from land-based wells, limited by the amount of crude and the production capacity, and if more oil is needed, different technology for deep-sea offshore oil fields. No sector other than oil extraction will be directly constrained by these sector-specific capacity constraints. Other constraints may affect more than one but not all sectors. This may be the case if there is competition among all extraction sectors for, say, mining engineers who are in short supply. We introduce an explicit representation of factor requirements in the models that follow.

3. Rectangular Choice-of Technology Model in a Single Region

3.1. The Basic Input-Output Model with Given Technologies

An economy with n sectors, each using a single given technology, and k factors of production is described by

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{y} \text{ and}$$
$$\boldsymbol{\varphi} = \mathbf{F}\mathbf{x},$$

where the output vector **x** and final demand vector **y** are n x 1, the coefficient matrix **A** is n x n, the factor use vector **φ** is k x 1, and the factor requirement matrix **F** is k x n. With **y** given, **x** and **φ** can be uniquely determined as

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} \quad \text{and}$$
$$\boldsymbol{\varphi} = \mathbf{F}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}.$$

This model makes use of **F**, the matrix of factor requirements per unit of output, where the factors, including resources, will generally be measured in physical units. To maintain the representation of factors in physical units throughout the analysis, we proceed in the next section to introduce a vector of factor prices, $\boldsymbol{\pi}$; $\boldsymbol{\pi}'\mathbf{F}$ will replace the vector of value-added in equation (1) as the measure of factor costs.

3.2. Rectangular Model with Up to q Alternative Technologies for Each Sector

Say that sector i has the choice among t_i alternative technologies. Then

$$t = \sum_{i=1}^{n} t_i\ ,\ \ t_i \geq 1, \quad t \geq n$$

is the total number of technologies available to the economy as a whole. We can rewrite the basic model to accommodate these alternative technologies if we augment **x** by inserting additional components, and augment **I, A** and **F** by inserting additional columns, corresponding to the alternate technologies. The resulting model can be expressed formally as

$$(\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} = \mathbf{y}$$
$$\boldsymbol{\varphi} = \mathbf{F}^*\mathbf{x}^*$$

where $\mathbf{x^*}$ is t x 1, $\mathbf{I^*}$ and $\mathbf{A^*}$ are n x t, **y** remains n x 1, $\mathbf{F^*}$ is k x t, and $\boldsymbol{\varphi}$ remains k x 1. (In general, k can be expected to increase from the basic model if some technology-specific factors are included for the new options. For simplicity, we retain k instead of using a new notation.) Thus the i^{th} row of I* will contain as many 1's as there are alternative technologies for sector i. The rectangular structure of $\mathbf{A^*}$ means that in general there is no unique solution for $\mathbf{x^*}$ unless we specify a criterion for selecting among the infinite number of feasible solutions. We choose to minimize total factor use, where individual factors are weighted by factor prices: $\boldsymbol{\pi}'\,\boldsymbol{\varphi}$. The resulting linear program model is

$$\min Z = \boldsymbol{\pi}'\,\mathbf{F^*x^*}$$
$$\text{s.t. } (\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} \geq \mathbf{y}, \tag{2}$$

where the inequality imposes the production constraints.

Proposition

(a) Each of the n sectors will use at most one of its available technologies.
(b) The technology it uses will be its lowest-cost option.

Proof:

(a) Each element of **x*** corresponds to a unique technology. From linear programming theory, as there are *n* functional (final demand) constraints, at most *n* of the *t* elements of **x*** will be non-zero (Hillier and Liebermann, 2010, p. 96).
(i) Assuming non-zero demand (intermediate or final) for each of the *n* sectors, then each sector can contribute only one non-zero element to **x*** and thus can employ only one technology.
(ii) In the degenerate case, there is no demand for at least one sector's output, say sector m. Then the system may be reduced to $n - 1$ sectors with $n - 1$ functional constraints and $t - t_m$ technologies. This elimination can be repeated for each non-producing sector, until all remaining sectors are producing. The argument of part (i) then holds. In what follows, we do not consider the degenerate case.

(b) The cost of a sector's inputs are the costs of factors required directly or indirectly to produce its output. Since the objective function minimizes total factor use with individual factor inputs weighted by factor prices, each sector will use its lowest cost technology. End of proof.

A numerical example is provided in the Appendix.

3.3. Rectangular Model with Up to q Alternative Technologies for Each Sector and with Factor Constraints

If in addition to production constraints we impose constraints on factor availability, the model becomes

$$\begin{aligned} &\min Z = \boldsymbol{\pi}' \mathbf{F^*x^*} \\ &\text{s.t. } (\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} \geq \mathbf{y} \\ &\qquad \mathbf{F^*x^*} \leq \mathbf{f} \end{aligned} \tag{3}$$

where **x*** is *t* x 1, **I*** and **A*** are *n* x *t*, **y** remains *n* x 1, **F*** is *k* x *t*, and **f** is a *k* x 1 vector of factor endowments.

Theorem. The RCOT model has the following properties:

a) The total number of technologies used may be as many as n + k, depending on the number of factor constraints that are binding.
b) If there are any binding factor constraints, then one or more sectors may use multiple technologies.
c) If a sector uses one technology, it will use the lowest-cost option available; if it uses multiple technologies, it will do introduce them in the order that minimizes its production costs.

Proof:

a) Since there are $n + k$ functional constraints, as many as $n + k$ elements of **x*** can be non-zero. However, for each non-binding factor constraint, one of the slack variables is non-zero, reducing the possible number of non-zero elements of **x*** (the decision variables) by 1 (Hillier and Lieberman, 2010).

b) If none of the factor constraints is binding, then each producing sector uses one technology. Each binding constraint increases the possible number of non-zero elements in **x*** (part a), and thus technologies, by 1.

c) Both a) and b) follow from the Proposition. End of proof.

If a factor used by more than one sector becomes scarce (*i.e.*, the factor constraint is binding), and if there is still a feasible solution, any of the sectors using the scarce factor may need to operate two or more technologies simultaneously. If it is a sector-specific factor that is scarce, that sector may operate two technologies at once with both using the scarce factor. However, if a technology-specific factor is scarce, once it is fully utilized a second technology for that sector, not dependent on the scarce factor, will need to be put into operation.

4. Square Choice-of-Technology Model in a Single Region

The square model can be used to offer *exactly q* choices, or *up to q* choices, to each sector. Following Carter, these models are described in the absence of factor constraints. The Appendix provides a numerical example applying the square model using the same data as for the rectangular model of Section 3.

4.1. Square Model with Exactly q Distinct Technologies for all Sectors

First, consider the case for a single region, where every one of its *n* sectors has exactly *q* distinct production technologies, numbered from 1 to *q*. If the options correspond to different years, the order from 1 to *q* can be chronological; otherwise it will be arbitrary. Thus the total number of available technological alternatives is

$$d = n^*q, \quad d \geq n.$$

We form q n x n matrices $\mathbf{A^{(1)}}$ ...... $\mathbf{A^{(q)}}$, where the n columns of $\mathbf{A^{(j)}}$ consist of the j^{th} alternative for each of the n sectors. In similar manner we form the q k x n matrices $\mathbf{F^{(1)}}$ … $\mathbf{F^{(q)}}$, and the q n x 1 vectors $\mathbf{x^{(1)}}$ ... $\mathbf{x^{(q)}}$, where $x_i^{(j)}$, the i^{th} element of $\mathbf{x^{(j)}}$, is the output of sector i using its j^{th} technological alternative. The model is

$$\min \quad Z = \boldsymbol{\pi}'\tilde{\mathbf{F}}\tilde{\mathbf{x}} = \boldsymbol{\pi}'\left[\mathbf{F^{(1)}} \quad \mathbf{F^{(2)}} \quad .. \quad .. \quad \mathbf{F^{(q)}}\right]\begin{bmatrix}\mathbf{x^{(1)}} \\ \mathbf{x^{(2)}} \\ \vdots \\ \vdots \\ \mathbf{x^{(q)}}\end{bmatrix} \tag{4}$$

$$\text{s.t.} \qquad [\mathbf{I} - \mathbf{A}^{(1)} \quad \mathbf{I} - \mathbf{A}^{(2)} \ \ldots\ldots \ \mathbf{I} - \mathbf{A}^{(q)}] \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \\ \vdots \\ \vdots \\ \mathbf{x}^{(q)} \end{bmatrix} \geq \mathbf{y} .$$

Note that when all n sectors have exactly q technologies then the rectangular model **I* - A*** has the exact same dimensions, $n \times nq$, as $[\mathbf{I} - \mathbf{A}^{(1)} \ \mathbf{I} - \mathbf{A}^{(2)} \ \ldots.. \ \mathbf{I} - \mathbf{A}^{(q)}]$. Similarly, F***** has the same dimensions as $\tilde{\mathbf{F}}$ and **x*** has the same dimensions as $\tilde{\mathbf{x}}$. Reordering the columns in **I* - A*** and **F***, and the corresponding rows in **x***, leads to optimization problems 2 and 4 being identical. Thus in this case there is no advantage for the RCOT model.

4.2. Square Model with Up to q Alternative Technologies for Each Sector

If sectors are offered the technological choices that were available at two or more specific times, and assuming that each sector has only one technology available to it at each time, then each industry would have the same number of technological alternatives. If, by contrast, we seek to consider the desirability of actual contemporaneous alternatives, different sectors will typically consider different numbers of alternatives, and for some a single average technology will be adequate. For an economy with n sectors, where the i^{th} sector has t_i technological alternatives, let

$$q = \max_i \{t_i\} \text{ and, as before, } t = \sum_{i=1}^{n} t_i .$$

Then, since the model is formulated in terms of square coefficient matrices, $d = n*q \geq t$ remains the number of output variables.

Say that sector i has fewer than q technological options. In order that all the $\mathbf{A}^{(j)}$, $j = 1 \ldots q$, remain square (n x n) matrices, any one of its t_i distinct technology options may be *replicated* $q - t_i$ times. The model will remain mathematically in the form of equation (4). However, the n x d matrix

$$\left[\mathbf{I} - \mathbf{A}^{(1)} \quad \mathbf{I} - \mathbf{A}^{(2)} \quad \ldots \quad \ldots \quad \mathbf{I} - \mathbf{A}^{(q)}\right]$$

will have the column representing technology option i *repeated* $q - t_i + 1$ times. $\mathbf{A}^{(q)}$ contains new column options for only those sectors with q distinct options; for a sector with fewer than q distinct options it contains the same column as appeared in $\mathbf{A}^{(q-1)}$, which may in turn be the same as in one or more preceding matrices.

As in the rectangular case, in the absence of factor constraints, there can at most be n non-zero sector outputs. This means that again each of the n sectors may produce output using one technology only. The objective function assures that it will be the one with the lowest factor costs. However, if this low-cost technology is one that is replicated, the system described by equation (4) will be indeterminate: the solution is not unique, and the output for a sector with a technological option that is replicated may be attributed to the output variable for any one of these repetitions. Operationally this indeterminacy does not pose a major problem, requiring us only to be aware of which output variables refer to repetitions of the same technology. However, this requirement complicates interpretation of the results and reflects an inefficient representation. The ratio *t/d* is a rough measure of the relative efficiency of the square model relative to the rectangular one, equaling 1.0 when all sectors have the same number of distinct options. However, if an economy has 100 sectors, five of which are mining sectors with four technological options each while each remaining sector has one average technology, *t/d* = (95*1 + 5*4)/(100*5) = 115/500 = 0.23. In this case, which is the general case for analysis of scenarios about the future, the rectangular model is substantially more efficient.

5. Multiple Regions: The World Trade Model (WTM)

We return to the earlier discussion of Julia and Duchin (2007), who introduced into the World Trade Model the choice of up to q technologies for each sector in each of m individual regions in square-matrix format, for the simultaneous choice among intra-regional as well as inter-regional options. The WTM imposes factor constraints, and the model that they solved is as follows:

$$
\begin{aligned}
&Min\ Z = \sum_{i=1}^{m}\sum_{j=1}^{q} \boldsymbol{\pi}_{\mathbf{i}}^{(\mathbf{j})\prime}\mathbf{F}_{\mathbf{i}}^{(\mathbf{j})}\mathbf{x}_{\mathbf{i}}^{(\mathbf{j})} \\
&s.t. \sum_{i=1}^{m}\sum_{j=1}^{q}(\mathbf{I}-\mathbf{A}_{\mathbf{i}}^{(\mathbf{j})})\mathbf{x}_{\mathbf{i}}^{(\mathbf{j})} \geq \sum_{i=1}^{m}\mathbf{y}_{\mathbf{i}} \qquad (5)\\
&\quad \mathbf{F}_{\mathbf{i}}^{(\mathbf{j})}\mathbf{x}_{\mathbf{i}}^{(\mathbf{j})} \leq \mathbf{f}_{\mathbf{i}}^{(\mathbf{j})} \quad j=1,.....,q,\ i=1,.....,m
\end{aligned}
$$

Because technological options were introduced only for agricultural sectors, and these options have sector-specific factor constraints, each agricultural sector in a region may utilize several technologies simultaneously. That is, some crops can be produced using each land quality, starting with the relatively lowest-cost choice and proceeding to the next lowest-cost as the more productive land is fully utilized.

Note that, since only agricultural sectors have choices among alternative technologies, this is an inefficient and indeterminate representation. We now rewrite this model in rectangular form as equation (6), which is an input-output model of the world economy with choice of technology in individual regions using the RCOT formulation. The factors of production may be sector-specific or even technology-specific in some cases, while in other cases many or all sectors, and technologies, will compete for limited endowments of labor, capital, fresh water, land, and other resources. The model is completely general in that the numbers of choice may differ both by region and by sector.

$$Min\ Z = \sum_{i=1}^{m} \boldsymbol{\pi}_{\mathbf{i}}^{*\prime} \mathbf{F}_{\mathbf{i}}^{*} \mathbf{x}_{\mathbf{i}}^{*}$$

$$s.t.\ \ \sum_{i=1}^{m} (\mathbf{I}^{*} - \mathbf{A}_{\mathbf{i}}^{*}) \mathbf{x}_{\mathbf{i}}^{*} \geq \sum_{i=1}^{m} \mathbf{y}_{\mathbf{i}} \qquad (6)$$

$$\mathbf{F}_{\mathbf{i}}^{*} \mathbf{x}_{\mathbf{i}}^{*} \leq \mathbf{f}_{\mathbf{i}} \quad i = 1, \ldots\ldots, m.$$

6. Conclusions and Next Steps

The key feature of the rectangular choice-of-technology model for a single region, whether for use in a one-region model or for one region within a model of the world economy, is the replacement of the familiar square matrices **I** and **A**, once for each of *q* options, by a single instance of the rectangular, non-square matrices **I*** and **A***. In the limit, where only one sector has *q* alternatives and the other *n*-1 sectors have only one each, still *q n* x *n* **A** matrices (and *q k* x *n* **F** matrices) would be required, with *n*-1 of their *n* columns identical. The use of **I*** and **A*** as defined in this paper removes all redundancies and provides a determinate solution while also rendering the logic of the solution algorithm transparent.

The choice-of-technology model's reliance on rectangular, non-square input-output matrices offers another, less evident advantage. In the past most extensions of the basic input-output model – from the so-called dynamic inverse that makes investment endogenous, to the first input-output model of the world economy with endogenous imports and exports, to the Social Accounting Matrix that provides closure for households and government -- have relied exclusively on square, invertible coefficient matrices. The desirable features of the rectangular choice-of-technology model may demonstrate the potential fruitfulness of other departures from the legacy of square, invertible matrices only. There is no reason to restrict input-output economics to the analysis of an inverse matrix and its properties.

While all sectors have to take account of the qualitative attributes of their inputs, manufacturing and service sectors rely on relatively standardized commodities. By contrast, technologies suitable for extractive sectors depend heavily upon the heterogeneous quality of resource endowments. The RCOT model described in this paper was developed to facilitate scenario analysis applying the World Trade Model to a new, environmentally-extended, input-output database for the world economy (Tukker et al. 2009) to analyze scenarios about future resource supplies and prices subject to constraints on different qualities of resource endowments. We employ the RCOT model for individual economies to consider the prospect of mining lower and higher cost mineral deposits – or using both primary and recycled metals -- simultaneously.

The RCOT model is intended for analyzing alternative scenarios about the future. For this purpose, one typically starts from a database that was compiled for a baseline year and makes projections of the changes in question, in particular of the input structures for given sectors. The RCOT model offers the possibility to add columns specifying technological alternatives for sectors that were represented in the baseline database by a single average technology. For scenarios about entirely new technologies not already in use in the base year, one would derive the additional columns of coefficients from technological information.

It may happen, however, that the scenarios are concerned with the changing relative importance of technologies that already are in use in a sector in the base year. These technologies have been aggregated into the average technology of the square matrix by weighting each alternative by its relative importance in the base year. As ten Raa has pointed out for the case of multi-product industries (ten Raa 1994), the input-output modeler can obtain valuable information about substitution possibilities by examining the rectangular supply and use matrices that are compiled by statistical offices as the data underlying the square (or "symmetric") input-output matrix. Our present concern is not with an industry's by-products, those that it may produce in addition to its characteristic product, but rather with the prospect that different industries may produce the same characteristic product but by different means. Industries are defined as the collection of establishments that "do similar things in similar ways" (BLS, North American Industry Classification System, http://www.bls.gov/bls/naics.htm), and we are interested precisely in distinguishing industries that produce the same product but in different ways.

The product-by-industry use matrix is the right conceptual place to accommodate this information, provided that the industrial classification system is sufficiently detailed to make the relevant distinctions among industries. For example, the North American Industrial Classification System distinguishes fossil fuel electric power generation from nuclear electric power generation but does not distinguish conventional from offshore extraction of crude petroleum. In the former case, there is a single sector in the square input-output matrix, corresponding to a weighted average (with the weights derived from the industry-by-product supply matrix) of the input structures of the relevant industries. For analysis with the RCOT model one could maintain the distinct columns from the use matrix, and add columns representing other options in addition, such as extraction of crude oil from tar sands. Here is the key new capability offered by the rectangular formulation: instead of assuming the base year's weights from the supply matrix for the technological options, the RCOT model can select among the options using its cost-based optimization criterion.

We plan to make use of this capability to internalize the choice between using primary materials or secondary materials that are recovered from wastes, first in a one-region framework and then in the context of the global economy. The latter research involves integrating the basic features of Waste Input-Output analysis (Nakamura and Kondo 2002) with the World Trade Model.

Acknowledgments

The authors wish to thank an anonymous reviewer for two very helpful insights: first that electric power is regularly generated using a mix of rather different technologies, and second that the use matrix is a conceptually appropriate source of this kind of information.

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Appendix: Numerical Example

We consider an example with n = 3 and k = 2, where

$$\mathbf{A} = \begin{bmatrix} 0.35 & 0.15 & 0.26 \\ 0.25 & 0.22 & 0.22 \\ 0.20 & 0.26 & 0.31 \end{bmatrix},\ \mathbf{F} = \begin{bmatrix} 2.1 & 3.2 & 1.2 \\ 1.2 & 2.2 & 1.3 \end{bmatrix},\ \boldsymbol{\pi} = \begin{bmatrix} 1.0 \\ 0.9 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 20 \\ 25 \\ 22 \end{bmatrix}.$$

A. No alternative technologies

The basic model has the following solution:

$$\mathbf{x} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} = \begin{bmatrix} 85.693 \\ 84.496 \\ 88.562 \end{bmatrix},$$

$$\boldsymbol{\varphi} = \mathbf{Fx} = \begin{bmatrix} 556.62 \\ 403.85 \end{bmatrix},$$

and $Z = \boldsymbol{\pi}'\mathbf{Fx} = 920.083$.

B. Rectangular model for alternative technologies with no factor constraints

We introduce one alternative technology for sector 2 and two alternative technologies for sector 3. Therefore, $t_1 = 1$. $t_2 = 2$, and $t_3 = 3$, so $q = 3$ and $t = 1+2+3 = 6$. Applying equation (2):

$$\mathbf{I^*} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A^*} = \begin{bmatrix} 0.35 & 0.15 & 0.23 & 0.26 & 0.28 & 0.24 \\ 0.25 & 0.22 & 0.16 & 0.22 & 0.21 & 0.25 \\ 0.20 & 0.26 & 0.30 & 0.31 & 0.33 & 0.30 \end{bmatrix}$$

$$\mathbf{F^*} = \begin{bmatrix} 2.1 & 3.2 & 1.9 & 1.2 & 0.8 & 1.4 \\ 1.2 & 2.2 & 1.3 & 1.3 & 1.1 & 1.1 \end{bmatrix}$$

Note that the pattern of 1's in the i^{th} row of **I*** indicates which columns in **A*** and **F*** are associated with the i^{th} sector.

Therefore,

$$\min Z = \boldsymbol{\pi}' \mathbf{F^*x^*}$$
$$\text{s.t. } (\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} \geq \mathbf{y}$$

leads to

$$\mathbf{x}^* = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_2^{(2)} \\ x_3^{(1)} \\ x_3^{(2)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 106.331 \\ 0 \\ 87.328 \\ 0 \\ 103.679 \\ 0 \end{bmatrix},$$

$$\boldsymbol{\varphi}^* = \mathbf{F^*x^*} = \begin{bmatrix} 472.16 \\ 355.17 \end{bmatrix},$$

and Z = 791.814.

In the absence of factor constraints each sector uses only its least cost technology. The reduced value of Z compared to the case with n alternatives is the result of sector 2 switching to the second of its two technologies and sector 3 switching to the second of its three technologies. As a result the requirements for both factors are substantially reduced.

C. Rectangular model for alternative technologies with factor constraints

With the addition of factor constraints, we have equation (3):

$$\min Z = \boldsymbol{\pi}' \mathbf{F^*x^*}$$

$$\text{s.t. } (\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} \geq \mathbf{y}$$
$$\mathbf{F^*x^*} \leq \mathbf{f}.$$

Assuming for the two factor endowments

$\mathbf{f} = \begin{bmatrix} 540 \\ 342 \end{bmatrix}$ leads to

$$\mathbf{x}^* = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_2^{(2)} \\ x_3^{(1)} \\ x_3^{(2)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 99.788 \\ 0 \\ 87.536 \\ 0 \\ 26.644 \\ 71.953 \end{bmatrix},$$

$$\boldsymbol{\varphi}^* = \mathbf{F}^*\mathbf{x}^* = \begin{bmatrix} 498.92 \\ 342.00 \end{bmatrix},$$

and $Z = 805.724$.

Note that while there are now five constraints only four of the constraints are binding. The first of the two factor constraints retains a surplus of 540 – 498.92 = 42.08. As a result, there are only four non-zero outputs. Due to the binding constraint on factor 2, sector 3 is simultaneously using its third as well as its second technology, and the higher cost of this technology is reflected in the increased value of Z as compared to the model with no factor constraints.

D. Square model for alternative technologies with no factor constraints

This example illustrates model discussed in Section 4.2. Again, $t_1 = 1$. $t_2 = 2$, and $t_3 = 3$, $q = 3$ and $t = 6$. We have

$$\mathbf{A}^{(1)} = \begin{bmatrix} 0.35 & 0.15 & 0.26 \\ 0.25 & 0.22 & 0.22 \\ 0.20 & 0.26 & 0.31 \end{bmatrix},\ \mathbf{A}^{(2)} = \begin{bmatrix} 0.35 & 0.23 & 0.28 \\ 0.25 & 0.16 & 0.21 \\ 0.20 & 0.30 & 0.33 \end{bmatrix}, \text{ and } \mathbf{A}^{(3)} = \begin{bmatrix} 0.35 & 0.23 & 0.24 \\ 0.25 & 0.16 & 0.25 \\ 0.20 & 0.30 & 0.30 \end{bmatrix},$$

as well as

$$\mathbf{F}^{(1)} = \begin{bmatrix} 2.1 & 3.2 & 1.2 \\ 1.2 & 2.2 & 1.3 \end{bmatrix},\ \mathbf{F}^{(2)} = \begin{bmatrix} 2.1 & 1.9 & 0.8 \\ 1.2 & 1.3 & 1.1 \end{bmatrix}, \text{ and } \mathbf{F}^{(3)} = \begin{bmatrix} 2.1 & 1.9 & 1.4 \\ 1.2 & 1.3 & 1.1 \end{bmatrix}.$$

Then

$$\min Z = \boldsymbol{\pi}'\tilde{\mathbf{F}}\tilde{\mathbf{x}} = \boldsymbol{\pi}'\,[\,\mathbf{F}^{(1)}\ \ \mathbf{F}^{(2)}\ \ \mathbf{F}^{(3)}]\,\tilde{\mathbf{x}}$$

$$\text{s.t. } [\mathbf{I} - \mathbf{A}^{(1)}\ \ \ \mathbf{I} - \mathbf{A}^{(2)}\ \ \ \mathbf{I} - \mathbf{A}^{(3)}]\,\tilde{\mathbf{x}} \geq \mathbf{y}$$

leads to

$$\tilde{\mathbf{x}} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \\ x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 103.679 \\ 106.331 \\ 87.328 \\ 0 \end{bmatrix},$$

$$\tilde{\boldsymbol{\varphi}} = \tilde{\mathbf{F}}\tilde{\mathbf{x}} = \begin{bmatrix} 472.16 \\ 355.17 \end{bmatrix},$$

and $Z = 791.814$.

Note that, since sector 1 has only a single technology, the assignment of sector 1's output to $x_1^{(3)}$ rather than to $x_1^{(1)}$ or $x_1^{(2)}$ is arbitrary, reflecting the indeterminate nature of the formulation. Similarly, the output of $x_2^{(3)}$ could have equally well been assigned to $x_2^{(2)}$. Finally, note that the results of the rectangular (equation 2) and square models (equation 4) are equivalent in the information contained in the results, but the rectangular representation is more compact and offers a determinate solution.