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Stability of the adjustment process with the difference between the weighted average and the actual value

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Abstract

In this paper, we show a similar concept of stability of the adjustment process concerning the difference between the weighted average and the actual value. Because the adjustment process includes only one eigenvalue of zero, and thus one dimension of freedom, it is difficult to apply the Routh-Hurwitz theorem to the process, at least directly. To solve the one-dimensional freedom, we fully apply Goodwin's analysis of the weighted average. One-dimensional freedom is suitable for indeterminacy issues, or the characterization of price and the like, which has only relative value and no absolute value. From this analysis, we know that it is reasonable to be concerned about the difference between the average and the actual value.

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1. Introduction

This paper examines a similar concept to stability of the adjustment process with the difference between the weighted average and the actual value. To analyze the stability, it is usual to apply the Routh-Hurwitz theorem or the Hicksian method of stability with which we can derive the condition that all the roots of the nth degree equation with real coefficients have negative real parts. However, at least directly, we cannot use these methods for this process when the process includes only one eigenvalue of zero. To analyze the stability of the process, we must consider the eigenvalue of zero, which means one-dimensional freedom. The case is similar to the price adjustment process because price has essentially relative value and no absolute value. We analyze the stability of the process, modifying the concept of the Hicksian method of stability according to the process. We show that the process is linear approximation stable except choosing the absolute level, which we define for analyzing the stability of the price adjustment system.

In the stable case, Goodwin (1949) answered the similar case including the weighted average.[1] Essentially, we apply Goodwin (1949) fully to the adjustment system, which is a form of the linear approximation system.

Recently, more and more areas take in some affection from other economic agents. For example, the areas of social status and status preference take in the preference affecting the average wealth. The concept of strategic complement is widely applied in various areas. The findings that we present in this paper may thus have wide application to a variety of research endeavors like these areas. Since we analyze a case including an eigenvalue of zero, our findings may have implications for situations characterized by indeterminacy and have applications to the mathematical basification of endogenous growth models.

The paper is organized as follows: The following section explains and analyzes the model. The third section discusses several implications of our findings and the final section presents our conclusions.

2. Model and Analysis

One of the most important prices is the wage. Hereafter, we use the model with the form of the wage adjustment process. Suppose that the economy has n jobs, and the wage moves only the wage adjustment process. Let $w_k(t)$ be a wage of the kth job for any $k = 1, \cdots, n$. Let $\overline{w}_k$ be a positive initial wage for any $k = 1, \cdots, n$. Let b_{kj} be the weight parameter of

[1] Recently, Goodwin's findings have been applied to various fields of research; for example, they have been used in the studies of Punzo (1995), Brida *et al.* (2003), and Raghavendra (2006). In addition, his findings have been reconsidered in the studies of Velupillai (1998) and Punzo (2006).

the j th wage affecting the k th wage, satisfying $0 \le b_{kj} \le 1$ and $\sum_{j=1}^{n} b_{kj} = 1$. Specifically, in the arithmetic mean,

$$b_{kj} = \frac{1}{n} \text{ for any } k, j = 1, \cdots, n$$

holds. In the economy, the wage adjustment process with the difference between the weighted average and the actual value is as follows: For any $k = 1, \cdots, n$,

$$\dot{w}_k = \sum_{j=1}^{n} b_{kj} w_j - w_k, \quad w_k(0) = \overline{w}_k, \tag{1}$$

where $\dot{w}_k = \frac{dw_k}{dt}$. For simplicity, we use the following assumption:

Assumption 1 $b_{kk} < 1$ is satisfied for any $k = 1, \cdots, n$.

If $b_{kk} = 1$, then, w_k becomes constant, which we avoid.

(1) can be rewritten as follows:

$$\frac{d}{dt}\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = B \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad \begin{bmatrix} w_1(0) \\ w_2(0) \\ \vdots \\ w_n(0) \end{bmatrix} = \begin{bmatrix} \overline{w}_1 \\ \overline{w}_2 \\ \vdots \\ \overline{w}_n \end{bmatrix}, \quad B \overset{\text{dfn}}{=} \begin{bmatrix} b_{11}-1 & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22}-1 & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn}-1 \end{bmatrix}. \tag{2}$$

We consider the stability of (2). From the definition of b_{kj}, $B\mathbf{1} = \mathbf{0}$ holds, where $\mathbf{1}$ is the transpose of $[1, \cdots, 1]$ and $\mathbf{0}$ is the zero vector. Thus, $\det B = 0$, which implies that the coefficient matrix B is not linear approximation stable because B has at least an eigenvalue of zero. Therefore, we define a similar concept of stability. For this purpose, the fact, which we show without proof is important to define a concept similar to stability.

Fact 1 Consider one variable x following the linear differential equation $\dot{x} = \gamma x$. Even if x starts at any initial value, the differential equation converges not only in the case that the real part of γ is negative but also in the case $\gamma = 0$. In the case $\gamma = 0$, the steady state $\lim_{t \to +\infty} x$ depends on the initial value. This is different from the case that the real part of γ is negative, where the steady state is 0 uniquely, which does not depend on the initial value.

With this fact, we define a similar concept to stability as follows.

Definition 1 A system of ordinal linear differential equations is *linear approximation stable* *except choosing the absolute value* if and only if the following conditions are satisfied:

1. Any eigenvalue of the coefficient matrix except one has a negative real part.
2. The coefficient matrix has an eigenvalue of zero.
3. The eigenvalue of zero has only a one-dimensional set of eigenvectors.
4. $\mathbf{1}$ is an eigenvector of the coefficient matrix about an eigenvalue of zero.

Therefore, if a system of ordinal linear differential equations is linear approximation stable except choosing the absolute level, the system converges except choosing the absolute level. This is a better characterization because this stability has a freedom in choosing the numeraire. We offer the following proposition:

Proposition 1 Under Assumption 1, the wage adjustment process system with the difference between weighted average and the actual value, (2), is linear approximation stable except choosing the absolute level.

Proof For the sake of simplicity, we only provide the proof of this proposition when $b_{kj} > 0$. $B\mathbf{1} = \mathbf{0}$ implies that the coefficient matrix has an eigenvalue of zero and $\mathbf{1}$ is an eigenvector of the coefficient matrix about the eigenvalue of zero. First, we show that the eigenvalue of zero has only a one-dimensional set of eigenvectors. The $n-1$ th successive principle minors of B have the same sign as $(-1)^{n-1}$, which is non-zero. Thus, $\mathrm{rank}B = n-1$ holds because $\det B = 0$. Therefore, the eigenvalue of zero has only a one-dimensional set of eigenvectors.

Second, we show that any eigenvalue of the coefficient matrix has a non-positive real part. For any $\varepsilon > 0$, define a new matrix $\tilde{B}(\varepsilon)$ as follows:

$$\tilde{B}(\varepsilon) \overset{\text{dfn}}{=} B - \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon \end{bmatrix}.$$

From the definition of b_{kj}, the l th successive principle minors of $\tilde{B}(\varepsilon)$ have the same sign as $(-1)^l$ for any $l = 1, \cdots, n$. Applying Theorem 4.D.3 of Takayama (1985, p.393) to this condition, the real parts of all eigenvalues of $\tilde{B}(\varepsilon)$ are negative. Let $\mathrm{Re}\,\lambda(\varepsilon)$ be the largest real part of eigenvalue of $\tilde{B}(\varepsilon)$. $\mathrm{Re}\,\lambda(\varepsilon) < 0$ for any $\varepsilon > 0$, which means that $\lim_{\varepsilon \downarrow +0} \mathrm{Re}\,\lambda(\varepsilon) \le 0$. Therefore, the real parts of all eigenvalues of B are non-positive.

Third, we show that any eigenvalue of the coefficient matrix except one has a negative real part. Suppose not. B has eigenvalues of $\pm ai$, where i satisfies $i^2 = -1$, and a is a non-zero real number. Then, $\det(B - aiI_n) = 0$ is satisfied, where I_n is the identity matrix. However, all elements of B are real numbers. Therefore, for any $k = 1, \cdots, n$,

$$|1 - b_{kk} - ai| = \sqrt{(1 - b_{kk})^2 + a^2} > |1 - b_{kk}| = \sum_{j \neq k} |b_{kj}|,$$

which implies that $B - aiI_n$ is a dominant diagonal matrix. Applying Theorem 4.C.1 of Takayama (1985, p.381) to this condition, $B - aiI_n$ is nonsingular, that is, $\det(B - aiI_n) \neq 0$ holds, which is in contradict to the equation that $\det(B - aiI_n) = 0$. Therefore, any eigenvalue of the coefficient matrix except one has a negative real part.

We can conclude that the wage adjustment process system with the difference between the weighted average and the actual value, (2), is linear approximation stable except choosing the absolute level. (Q.E.D.)

Proposition 1 shows that the adjustment process is essentially stable except the absolute level. The adjustment process satisfies the following: If the actual value is higher/lower than the weighted average, the value decreases/increases. Therefore, seeing the long term, the actual value converges to the weighted average. Moreover, if the actual value starts from the proposal of higher values, the steady state to which the system converges is also higher than the original convergent point.

At least in the standard stability theory, the focus point is such that any real part of the eigenvalue is negative, that is, the convergent point is unique, which prohibits freedom in the choice of the absolute level. However, the convergent cases are such that not only is the real part of the eigenvalue negative but also the eigenvalue is zero, without any imaginary part. The case that the eigenvalue is zero is such that the differential equation stops, at least in some sense. This quality brings to the adjustment process of the freedom of one dimension.

The freedom is particularly important for the price adjustment process when the price shows only relative value, not absolute value. Therefore, the adjustment process with the difference between the weighted average and the actual value is a suitable process because it satisfies stability essentially and includes the freedom to choose the numeraire.

The most important case is that where the weight is equivalent to all variables, that is, the case of the orthodox arithmetic mean. In this case, Assumption 1 is satisfied. We can state the following corollary from Proposition 1 straightforwardly.

Corollary 1 The wage adjustment process system with the difference between arithmetic mean and the actual value is linear approximation stable except choosing the absolute level.

Therefore, even if we use the adjustment process to minimize the difference between the arithmetic mean and the actual value, the system becomes linear approximation stable except choosing the absolute level, that is, essentially stable except choosing the absolute level.

3. Discussion

In this section, we discuss some potential for applications and some backgrounds.

Firstly, we mention the sketch of the adjustment system in the case of wage or price. In the case of wage, if a company shows a lower wage than an average one, then many workers will avoid to choose the job. Maybe, the company will raise the wage. Conversely, consider the case that a company shows a higher wage than an average one in the similar (but a little different) qualities of workers. Then, another company can collect the similar qualities of workers with lower wage than that of the company. The company has a probability to think as follows: Even if we show a little lower wage than the original one, we can collect workers. Thus, the company may decrease the wage. The similar mechanism can be shown in the case of price adjustment system.

Second, as previously discussed, Goodwin's findings (1949) have been widely applied to various areas. However, many studies have solely focused on Goodwin's analysis of discrete time lags and have neglected considering his analysis of the weighted average. In the case of simple discrete time, if all variables choose the arithmetic mean, the system will stop immediately, which is non-sense. However, in the continuous time case, the adjustment system is affected by an indeterminacy issue essentially, which implies that we must argue the convergence without point of closures. Indeterminacy issues are now widely spread to various dynamic areas. Especially, endogenous growth models have essentially the indeterminacy issues to grow their own engine. This study focuses on the potential of the overlooked point of Goodwin (1949) to apply to these areas.

Thirdly, we should stress that the system can be applied not only price or wage adjustment system but also various systems considering an average. Recently, the area of social status has developed.[2] In this area, people care about some kind of status like average wealth. Some models are directly introduced the difference between actual wealth.[3] This paper has a potential to the analysis of people who worrying the around. Moreover, the system is suitable for strategic complementarities, which are well-known as the cases if all players' variables mutually reinforce one another in game theory. Therefore, the finding of this paper has the potential of applying to cases satisfying strategic complementarities.

4. Conclusion

In this paper, we examined the stability of the adjustment process with the difference between the weighted average and the actual value. The orthodox stability is not satisfied because the

[2] See for example, Easterlin (1995) and Clark and Osward (1996).
[3] See for example, Corneo and Jeanne (2001) and Fisher and Hof (2005).

adjustment process matrix includes the eigenvalue of zero. However, to consider the concept of linear approximation stability except choosing the absolute level, we showed that the adjustment process is essentially stable without choosing the absolute level. The difference between this stability and orthodox stability is useful for a price adjustment system because price shows only relative value, not absolute value. The freedom of one dimension enables us to choose the numeraire.

Goodwin's analysis includes the one-dimensional freedom of the weighted average, at least in the static case. Therefore, our research essentially adds a new economic application of Goodwin's analysis to the various adjustment processes with the difference between the weighted average and the actual value.

Although we considered the weighted average value to be constant, our model remains applicable even when the weighted average value adjusts over time.

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