Economics Bulletin

Volume 30, Issue 4

Two axioms for the majority rule

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Abstract

Two axioms are shown to characterize the relative majority rule when preferences are defined over two alternatives. According to one axiom, if all the individuals in a group are indifferent, then the associated group preference is indifference. The second axiom states that a group S prefers alternative a to alternative b if and only if there is a subgroup T whose members unanimously prefer a to b and such that, if S ≠ T, indifference represents the preference of the group S/T.

Financial support from the Secretaría de Estado de Investigación of the Spanish Ministerio de Ciencia e Innovación (research project SEJ2007-67580-C02-01) is gratefully acknowledged.
Citation: Antonio Quesada, (2010) "Two axioms for the majority rule", *Economics Bulletin*, Vol. 30 no.4 pp. 3033-3037.
Submitted: Nov 04 2010. **Published:** November 15, 2010.

1. Introduction

This note offers an axiomatization of the (relative) majority rule for the case in which preferences are defined over two alternatives. May (1952, p. 682), Fishburn (1973, p. 58; 1983, p. 33) and Llamazares (2006, p. 319) have suggested axiomatizations when the set of individuals is fixed and their preferences are variable. Aşan and Sanver (2002, p. 411), Woeginger (2003, p. 91; 2005, p. 9) and Miroiu (2004, p. 362) characterize the majority rule when both the set of individuals and their preferences are variable. Xu and Zhong's (2010, p. 120) axiomatization assumes the set of individuals to be variable but their preferences to be fixed.

The framework adopted in this note allows individuals and preferences to vary. The universal set of individuals may be finite, as in Miroiu (2004), or infinite, as in Aşan and Sanver (2002) and Woeginger (2003, 2005). The characterization just postulates two axioms. One, that groups constituted by indifferent individuals are indifferent. And two, that having a group with strict preference a is equivalent to having a decomposition of the group into two subgroups, one being indifferent or empty and the other consisting of individuals with preference a.

2. Definitions and axioms

Let I be a non-empty (finite or infinite) subset of the set of positive integers. The members of I designate individuals. A society (or group) is a finite non-empty subset of I. The set of alternatives is $\{\alpha, \beta\}$, with $\alpha \neq \beta$. A preference over $\{\alpha, \beta\}$ is represented by a number from the set $\{-1, 0, 1\}$. If the number is 1, α is preferred to β; if -1, β is preferred to α; if 0, α is indifferent to β. A preference profile for a society S is a function $p_S : S \to \{-1, 0, 1\}$ assigning a preference over $\{\alpha, \beta\}$ to each member of S. The set P is the set of all preference profiles for all societies in I. For positive integer r, P_r is the set of all preference profiles for societies with exactly r members.

For preference profile p_S and society $T \subset S$, p_T is the restriction of p_S to T, that is, the preference profile p_T for T such that, for all $i \in T$, $p_T(i) = p_S(i)$. For $p_S \in P$ and $i \in S$, p_i abbreviates $p_S(i)$. If $p_{S_1}, \dots, p_{S_r}$ are preference profiles for mutually disjoint societies $S_1, \dots, S_r$, then $(p_{S_1}, \dots, p_{S_r})$ is the preference profile for the aggregate society $S_1 \cup \dots \cup S_r$. Preference profile p_S is unanimous if there is $a \in \{-1, 0, 1\}$ such that, for all $i \in S$, $p_i = a$. The preference profile for S such that, for all $i \in S$, $p_i = a$, is denoted by (a^S). The preference profile $p_{\{i,j\}}$ such that $p_i = 1$ and $p_j = -1$ is also denoted by $(1^i, -1^j)$.

Definition 2.1. A social welfare function is a mapping $f : P \to \{-1, 0, 1\}$.

A social welfare function transforms the preferences over $\{\alpha, \beta\}$ of all the members of any given society S into a collective preference over $\{\alpha, \beta\}$ (or, alternatively, a decision). Specifically, $f(p_S) = 1$ means that society S prefers α to β (or that α is the chosen alternative); $f(p_S) = -1$, that S prefers β to α (β is the chosen alternative); and $f(p_S) = 0$, that S is indifferent between α and β (a tie arises because no alternative is chosen).

Definition 2.2. The majority rule is the social welfare function $\mu : P \to \{-1, 0, 1\}$ such that, for all $p_S \in P$: (i) if $\sum_{i\in S} p_i > 0$, then $\mu(p_S) = 1$; (ii) if $\sum_{i\in S} p_i < 0$, then $\mu(p_S) = -1$; and (iii) if $\sum_{i\in S} p_i = 0$, then $\mu(p_S) = 0$.

A0. For every society $S \subseteq I$, $f(0^S) = 0$.

A1. For all $p_S \in P$ and $a \in \{-1, 1\}$, $f(p_S) = a$ if and only if there is $T \subseteq S$ such that $p_T = (a^T)$ and, if $S \neq T$, $f(p_{S\setminus T}) = 0$.

A0 is the unanimity principle for the particular case in which all the individuals are indifferent. A1 holds that $f(p_S) = a$ if and only if p_S can be partitioned into a unanimous profile (a^T) and the preference profile $p_{S\setminus T}$ of an indifferent society (with $S = T$ allowed). More specifically, A1 requires that if $f(p_S) = a \neq 0$, then the preference aggregation problem represented by p_S can be reduced to a unanimous preference aggregation problem by removing the preferences of some indifferent society. Conversely, A1 also requires that if the above reduction is possible, then the preference of society S is a.

3. Result

Proposition 3.1. A social welfare function f satisfies A0 and A1 if and only if f is the majority rule.

Proof. "$\Leftarrow$" The majority rule obviously satisfies A0. With respect A1, choose $p_S \in P$ and suppose that, for some $a \in \{-1, 1\}$, there is $T \subseteq S$ such that $p_T = (a^T)$ and, if $S \neq T$, $\mu(p_{S\setminus T}) = 0$. It must be shown that $\mu(p_S) = a$. If $S = T$, then $p_S = (a^S)$ and, evidently, $\mu(p_S) = \mu(a^S) = a$. If $S \neq T$, then $\mu(p_{S\setminus T}) = 0$ implies $\mu(p_S) = \mu(p_T)$. Since $p_T = (a^T)$, $\mu(p_T) = a$. Conversely, suppose that, for some $a \in \{-1, 1\}$, $\mu(p_S) = a$. It must be shown that there is $T \subseteq S$ such that $p_T = (a^T)$ and, if $S \neq T$, $\mu(p_{S\setminus T}) = 0$. If $p_S = (a^S)$, then the desired T is S itself. If $p_S \neq (a^S)$, then define $M = \{i \in S: p_i = -a\}$ and $A = \{i \in S: p_i = a\}$. Case 1: $M = \emptyset$. In this case, $p_S = (a^A, 0^{S\setminus A})$. As $\mu(0^{S\setminus A}) = 0$, the desired T is A. Case 2: $M \neq \emptyset$. Given that $\mu(p_S) = a$, there is $N \subset A$ having the same number of elements as M such that $p_S = (a^{A\setminus N}, a^N, -a^M, 0^{S\setminus(A\cup M)})$. Since $\mu(a^N, -a^M, 0^{S\setminus(A\cup M)}) = 0$, the desired T is $A\setminus N$.

"⇒" Part 1: for all $a \in \{-1, 0, 1\}$ and society $S \subseteq I$, $f(a^S) = a$. Let $a \in \{-1, 0, 1\}$. Consider any society $S \subseteq I$. If $a = 0$, then, by A0, $f(a^S) = a$. If $a \neq 0$, then, letting $p_S = a^S$, there is $T \subseteq S$ such that $p_T = (a^T)$: S itself. By A1, $f(a^S) = a$.

Part 2: for all $a \in \{-1, 1\}$, $i \in I$ and $j \in I\setminus\{i\}$, $f(a^i, -a^j) = 0$. Suppose not: $f(a^i, -a^j) = b \neq 0$. Letting $p_{\{i,j\}} = (a^i, -a^j)$, by A1, there is a non-empty $T \subseteq \{i, j\}$ such that $p_T = (b^T)$ and, if $\{i, j\} \neq T$, $f(p_{\{i,j\}\setminus T}) = 0$. Clearly, T cannot be $\{i, j\}$. If $T = \{i\}$, then $\{i, j\}\setminus T = \{j\}$ and $0 = f(p_{\{i,j\}\setminus T}) = f(p_j) = f(-a^j)$, which contradicts part 1. If $T = \{j\}$, then $\{i, j\}\setminus T = \{i\}$ and $0 = f(p_{\{i,j\}\setminus T}) = f(p_i) = f(a^i)$, which contradicts part 1.

Part 3: $f = \mu$ on $P_1 \cup P_2$. By parts 1 and 2, it suffices to show that, for all $a \in \{-1, 1\}$, $f(a^i, 0^j) = a$, where $i \in I$ and $j \in I\setminus\{i\}$. By A0, $f(0^j) = 0$. Hence, by A1, $f(a^i, 0^j) = a$.

Part 4: $f = \mu$. Taking part 3 as the base case of an induction argument, choose $r \geq 3$ and assume that $f = \mu$ on $P_1 \cup \ldots \cup P_{r-1}$. To show that $f = \mu$ on P_r, choose $p_S \in P_r$. Case 1: $\mu(p_S) = 0$. Case 1a: for all $i \in S$, $p_i = 0$. By A0, $f(p_S) = 0 = \mu(p_S)$. Case 1b: for some $i \in S$, $p_i \neq 0$. As a result, there must be $i \in S$ and $j \in S$ such that $p_i = 1$ and $p_j = -1$. To prove that $f(p_S) = 0$, suppose otherwise: $f(p_S) = a \neq 0$. By A1, there is $T \subseteq S$ such that $p_T = (a^T)$ and $f(p_{S\setminus T}) = 0$. That $S \neq T$ follows from the fact that, for some $i \in S$ and $j \in S$, $p_i = 1$ and $p_j = -1$. Since $\mu(p_S) = 0$ and $p_T = (a^T)$, $\mu(p_{S\setminus T}) = -a$. By the induction hypothesis, $f(p_{S\setminus T}) = \mu(p_{S\setminus T}) = -a$: contradiction.

Case 2: $\mu(p_S) = a \neq 0$. Define $M = \{i \in S: p_i = -a\}$ and $Z = \{i \in S: p_i = 0\}$, so $p_S = (a^{S\setminus(M\cup Z)}, -a^M, 0^Z)$. Case 2a: $M = \emptyset$. If $Z = \emptyset$, by part 1, $f(p_S) = f(a^S) = a = \mu(p_S)$. If $Z \neq \emptyset$, then $p_S = (a^{S\setminus Z}, 0^Z)$. By A0, $f(0^Z) = 0$. Given this, by A1, $f(p_S) = a = \mu(p_S)$. Case 2b: $M \neq \emptyset$. Since $\mu(p_S) = a$, there is a partition $\{T, R\}$ of $\{i \in S: p_i = a\}$ such that R has the same number of members as M and $T \neq \emptyset$. By the induction hypothesis, $f(p_{S\setminus T}) = \mu(p_{S\setminus T}) = \mu(a^R, -a^M, 0^Z) = 0$. By A1, $p_S = (a^T, a^R, -a^M, 0^Z)$ and $\mu(a^R, -a^M, 0^Z) = 0$ imply $f(p_S) = a = \mu(p_S)$.■

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