Nonrepresentative Representative Consumers*

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Abstract: Single consumer models are often used to focus attention on economic efficiency, leaving aside equity considerations. In general, these "representative consumer" models do not accurately portray the effects of changes in policies, endowments or technology in the multi-consumer economies they are meant to represent; but even when they do, they are not necessarily adequate for evaluating efficiency. The representative consumer can be Pareto inconsistent, preferring a situation B to A even though all the consumers in the represented economy prefer A to B. It is not clear from the literature how serious a defect this can be. The known examples of Pareto inconsistency are not robust. Small changes in the consumers' preferences remove the Pareto inconsistency. It has been an open question whether large robust Pareto inconsistencies are possible.

This paper shows that they are. In one example, the actual consumers require 56% more income than the representative consumer requires in order to be compensated for the doubling of a price. But such large Pareto inconsistencies require that there is a Giffen good for the representative consumer. We argue that the inconsistencies of representative consumers in most macroeconomic applications are likely to be small and we give conditions ruling out inconsistencies entirely.

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1. Introduction

A great deal of economic analysis treats aggregate community demand as if it were the demand of a single competitive "representative" consumer. Representative consumer models allow analysts to focus on economic efficiency, leaving equity considerations aside. However, the aggregation across consumers that is implicit in these models is often problematic. Aggregate community demand might violate revealed preference axioms that would be satisfied if there were just one consumer. As a result, representative consumer models could misrepresent the effects of changes in endowments, technology or policy on prices and aggregate consumption. But this is not the only problem. Even when aggregate community demand satisfies the strong axiom and therefore is indistinguishable from the demand of a single competitive consumer, the single-consumer model might not be adequate for evaluating efficiency. The representative consumer can be Pareto inconsistent, preferring an aggregate situation A to B even though all the actual consumers in the community prefer B to A (Jerison (1984) and Dow and Werlang (1988)).

Should representative consumers be banished from economic analysis because they might be Pareto inconsistent? That would be going too far. The inconsistencies might be very small or might arise only in unrealistic settings. In Jerison's (1984) example of inconsistency, a payment of less than 0.5% of aggregate income is enough bring the representative consumer into agreement with the actual consumers in the community. Dow and Werlang (1988) give an example of a larger Pareto inconsistency, but it is not robust. It requires a special discontinuity in the way the consumers' incomes vary depending on prices and aggregate income. Kirman (1992) gives another example of a large Pareto inconsistency. But the consumption vector chosen by his "representative individual" equals the aggregate consumption vector only in the two budget situations being compared, not in others. In the present paper, as in Samuelson (1956), Jerison (1984), and Dow and Werlang (1988), we restrict attention to the most favorable case for representative consumer analysis, the case in which there is a "positive" representative consumer whose demand is the same as the aggregate demand no matter what aggregate income and prices are. The positive representative consumer's preferences generate the *entire* aggregate demand *function*. Reasons for considering this case are given below.

This paper examines the conditions under which a positive representative consumer can be Pareto inconsistent, and the possible sizes of the inconsistencies. The adequacy of a representative consumer model depends on which communities and which policies or events are to be analyzed. If all the actual consumers are competitive and identical then there is a positive representative consumer model that makes accurate predictions and Pareto consistent welfare judgments for all possible policies and events. But this is certainly a limited case, considering the varied consumer behavior we observe. In order to apply the conclusions from representative consumer models to broader classes of communities, it is necessary to restrict the range of possible policies and events considered.

Since our goal is to study models that are used to analyze changes in endowments, technology and policy, we allow aggregate income to vary independently of prices. We also assume that the consumers' incomes are determined by aggregate income and prices. The function thus defined is called a "sharing rule" in the literature on collective choice (Bourguignon and Chiappori (1992)). We prefer to use the more neutral term "distribution rule" since "sharing" may evoke voluntary interaction among the community members. In this paper, the community could be all the consumers in a country rather than the members of a family, and the income distribution could be determined by taxes.

If no restriction is placed on the distribution of income, then a positive representative consumer exists only for very implausible communities. In that case, a positive representative consumer exists only if income redistribution has no effect on the aggregate demand vector. This implies that for each good, at all prices and income levels, all consumers' marginal propensities to consume are equal (Antonelli (1886), Gorman (1953) and Nataf (1953)).

This unrealistic restriction is avoided when we assume that incomes are determined according to a distribution rule. In that case, aggregate demand is a function of prices and aggregate income no matter what individual preferences are. A positive representative consumer is simply a utility function that generates the entire aggregate demand function. Pure income redistribution without any change in prices or aggregate income is ruled out a priori (or, if it occurs, the aggregate demand vector and the representative consumer's utility function are allowed to change).

In the language of the collective choice literature, existence of a positive representative consumer means that the community is "unitary," i.e., it behaves like a single competitive consumer. We will show that the preferences generating the aggregate demand function in a unitary model do not necessarily represent the preferences of the community members.

Lucas (1987) presents a striking example of a representative consumer that fits our framework. He considers a consumer with a discount factor that generates consumption choices matching certain features of aggregate U.S. demand. He shows that this consumer is willing to reduce its initial consumption by 42% in order to raise its consumption growth rate from 3% to 6%. Consumption in the 6% growth path remains less than in the 3% growth path for the first 16 years. Lucas uses the example to compare the welfare gains from faster growth to gains from steadier growth (removal of business cycles).

In our framework, the alternative consumption growth paths are viewed as optimal choices of a representative consumer in alternative intertemporal budget sets. We ask if Lucas' representative consumer's surprisingly strong preference for growth could be the result of Pareto inconsistency. Can the representative consumer's preferences differ substantially from those of the actual consumers in the represented community? We will see in section 6 that the answer is no. If the consumer in the growth example is a positive representative consumer then there must be consumers in the represented community willing to reduce their initial consumption by over 40% in order to raise the growth rate from 3% to 6%.

The main contributions of this paper are as follows. We first describe a simple way to construct robust examples of positive representative consumers with large Pareto inconsistencies. In one example, the actual consumers require 56% more income than the representative consumer requires in order to be compensated for the doubling of a price. However such large inconsistencies require that there is a Giffen good for some consumer in the community and for the representative consumer. We argue that Pareto inconsistencies must be small if the representative consumer has homothetic preferences, as in Lucas' (1987) example and in most representative consumer models in macroeconomics.

Pareto consistency is necessary in order for a representative consumer's preferences to have a social welfare interpretation. Dow and Werlang (1988) show that it is also sufficient. If the representative consumer is Pareto consistent, then its preferences coincide with preferences derived from a particular Bergson-Samuelson social welfare function. This raises the question which communities have Pareto consistent representative consumers.

A well-known sufficient condition for existence of a Pareto consistent representative consumer is for income to be distributed optimally according to some social welfare function no matter what prices prevail (Samuelson (1956), Chipman and Moore (1979)). We show that this condition is not necessary. We give an example of a Pareto consistent representative consumer in a community in which the distribution rule is not optimal with respect to any social welfare function. We also characterize the set of economies with two goods and a robustly Pareto consistent representative consumer (where the consistency does not disappear when the preferences are perturbed).

We use the terms "representative consumer" and "positive representative consumer" interchangeably, and we restrict attention to these representative consumers, whose demands equal the aggregate demand at all prices and aggregate income levels. One reason for doing so is that users of representative consumer models can always assume that they are working with a positive representative consumer. Each competitive consumer is a positive representative consumer for some class of communities, and it is interesting to know in which of these communities the representative consumer can be used to evaluate efficiency. Another reason for considering only positive representative consumers is that national aggregate consumption time series data rarely violate the strong axiom of revealed preference. Thus the limited aggregate data we have are often consistent with existence of positive representative consumers (Landsburg (1981), Varian (1982)).

The framework and notation are presented in the next section. Section 3 characterizes economies with representative consumers when the actual consumers' incomes are determined by prices and the aggregate income. Section 4 shows how to construct examples of large Pareto inconsistencies. Section 5 derives a bound on the size of the inconsistencies. Section 6 argues that there are tighter bounds in the most common macroeconomic representative consumer models. Section 7 examines the class of communities with Pareto consistent representative consumers and shows that it is larger than the class of communities with optimal income distribution. Section 8 discusses remaining open problems.

2. Notation

We consider a group of $m \geq 2$ competitive consumers in an n-good economy. Each consumer i has a column-vector valued demand function $X^i(p, y_i)$ generated by a utility function u^i. The corresponding indirect utility function $v^i(p, y_i)$ is assumed to be twice continuously differentiable with strictly positive marginal utility of income, $\partial v^i(p, y_i)/\partial y_i > 0$. The expenditure function of consumer i is $e^i(p, u)$. At price vector p, with income y_i, consumer i has the *marginal propensity to consume* $M^i(p, y_i) \equiv \partial X^i(p, y_i)/\partial y_i$, the *average propensity to consume* $A^i(p, y_i) \equiv X^i(p, y_i)/y_i$ and the *Slutsky matrix*

$$S^i(p, y_i) \equiv \partial X^i(p, y_i)/\partial p + M^i(p, y_i)X^i(p, y_i)^T,$$

where the superscript T denotes the transpose.

An economic *situation* is represented by (p, y), where y is aggregate income (the sum of the consumers' incomes) and $p \gg 0$ is a vector of n prices.[1] A *mean situation* is a vector (p, z) where z represents the average of the consumers' incomes and p is a price vector. We allow aggregate income to vary independently of prices so that the model can apply to comparative static analyses of changes in endowments, technology or policy. The group members might be thought of as members of a family or as citizens of a small country so that their consumption has no effect on prices.

We will use the term *community* to mean the group of consumers along with a "distribution rule" specifying their incomes as functions of aggregate income and prices. A *distribution rule* is a continuously differentiable function $D = (D^1, \ldots, D^m) \geq 0$, homogeneous of degree 1, satisfying $\sum D^i(p, y) = y$ and $D^i_y(p, y) > 0$. A distribution rule is a smooth sharing rule in the terminology of Bourguignon and Chiappori (1992). Here and below, subscripts denote partial derivatives. $D^i(p,y)$ is the income of consumer i in situation (p,y). The *income share* of consumer i in situation $(p,y) \gg 0$ is $D^i(p,y)/y$. A distribution rule need not be determined by private ownership. It can incorporate the effects of redistributive policies. Since there are no consumption externalities in our model, allocation by means of distribution rules is essentially equivalent to allocation that is Pareto efficient in every situation. Homogeneity and smoothness of the distribution rule are not required for efficiency. We impose these weak restrictions in order to consider cases most favorable for the existence of a representative consumer.

A distribution rule D determines aggregate demand as a function of aggregate income and prices:

$$X^D(p,y) = \sum X^i(p, D^i(p,y)).$$

It also determines the vector of consumer utilities $V^D(p,y)$ with ith component $V^{Di}(p,y) \equiv v^i(p, D^i(p,y))$. We say that there is a (*positive*) *representative consumer* for D if the aggregate demand function X^D is generated by a utility function. When such a utility function exists, it determines preferences over situations and also over mean situations. We say that the representative consumer is *Pareto consistent* if it prefers one situation to another whenever all the consumers prefer the former to the latter. Formally, the representative consumer with indirect utility function $v^D(p,y)$ is *Pareto consistent* if $V^D(p,y) \gg V^D(q,z)$ implies $v^D(p,y) > v^D(q,z)$.

The distribution rule D is *optimal for* $w : \mathbb{R}^m \to \mathbb{R}$ if, for each $(p,y) \gg 0$ and each vector of incomes $(y_1, \ldots, y_m) \geq 0$ satisfying $\sum y_i \leq y$, we have

$$w(V^D(p,y)) \geq w(v^1(p,y_1), \ldots, v^m(p,y_m)).$$

The distribution rule is optimal for w if in every situation there is no alternative distribution of the aggregate income that yields a higher value of w. A *social welfare function* (in the given community) is a nondecreasing[2] real-valued function on $\mathbb{R}^m$ that is strictly increasing on the set of attainable utility vectors $\{V^D(p,y) | (p,y) \gg 0\}$. We call D *optimal* if it is optimal for *some* social welfare function. Note that a constant function cannot be a social welfare function, so distribution rules are not necessarily optimal. In fact, Jerison (1994) shows that a typical (i.e., generic) distribution rule is not optimal with respect to *any* social welfare function.

3. Existence of a Positive Representative Consumer

As noted in the introduction, existence of a positive representative consumer requires the Slutsky matrix of the aggregate demand function to be symmetric and negative semidefinite.[3] These conditions are not satisfied automatically because the Slutsky matrix of aggregate demand generally differs from the sum of the individual consumers' Slutsky matrices. The difference matrix can be interpreted as the covariance matrix of two vector valued random variables defined on the set of consumers. Under certain conditions the matrix can be estimated from cross section or time series data. This section defines the "covariance matrix" and shows that when it is symmetric and positive semidefinite a representative consumer exists. Symmetry of the covariance matrix is necessary for existence of a representative consumer whereas positive semidefiniteness is not. On the other hand, both conditions are necessary in order for the distribution rule to be optimal with respect to some social welfare function (Jerison (1994)). Section 7, below, shows that if the covariance matrix is everywhere nonzero and if there are only two goods and two consumers, then there is a Pareto consistent representative consumer.

It seems then that the covariance matrix should be of special interest to any user of representative consumer models for normative analysis. For this reason we offer a number of interpretations for the above restrictions on the covariance matrix along with examples of communities in which the properties are satisfied.

The jk component of the covariance matrix will be defined to be the covariance of the consumers' marginal propensities to consume good j and their "adjusted demands" for good k. The *adjusted demand* of consumer i at situation (p, y) is

$$X^{Di}(p, y) \equiv [1/D^i_y(p, y)][X^i(p, D^i(p, y)) - D^i_p(p, y)],$$

where the subscripts on D^i denote partial derivatives. When aggregate income is fixed, a consumer's adjusted demand vector is orthogonal to the consumer's indifference curve in price space (taking account of the way the consumer's income D^i is affected by price changes). Thus, the adjusted demand vector $X^{Di}(p, 1)$ is parallel to the vector of price derivatives $\partial V^{Di}(p, 1)/\partial p$. (This is easily verified using Roy's identity.) The homogeneity of D implies that $p \cdot X^{Di}(p, y) = y$, so the adjusted demand vector lies in the frontier of the aggregate budget set. The adjusted demand for good k is approximately equal to the change in *aggregate* income y required to compensate consumer i for a unit change in the price of good k taking account of the effect of the price change on the consumer's income.

The *covariance matrix* is defined to be

$$C^D(p, y) \equiv \sum_i D^i_y(p, y) M^i(p, D^i(p, y))[X^{Di}(p, y) - X^D(p, y)]^T.$$

It is the covariance matrix of the consumers' marginal propensities to consume and their adjusted demands, with consumers weighted by their marginal income shares $D^i_y(p, y)$. (This uses the fact that the mean of the adjusted demands is the aggregate demand: $\sum D^i_y(p, y) X^{Di}(p, y) \sum [X^i(p, D^i(p, y)) - D^i_p(p, y)] = X^D(p, y)$.)

Under the smoothness assumptions of our model, there is a representative consumer for the distribution rule D if and only if the Slutsky matrix of aggregate demand is symmetric and negative semidefinite (Richter (1979) Theorem 12). As a consequence we have

Proposition 3.1. *There is a positive representative consumer if the covariance matrix $C^D(p, y)$ is symmetric and positive semidefinite at each (p, y). Symmetry of this covariance matrix is necessary for existence of a positive representative consumer.*

Proof. Let $y_i \equiv D^i(p, y)$. The Slutsky matrix of aggregate demand is

$$\begin{aligned} S^D(p, y) &\equiv X^D_p(p, y) + X^D_y(p, y) X^D(p, y)^T \\ &= \sum_i [M^i(p, y_i) D^i_p(p, y) + X^i_p(p, y_i)] + \sum_i M^i(p, y_i) D^i_y(p, y) X^D(p, y)^T \\ &= \sum_i [X^i_p(p, y_i) + M^i(p, y_i) X^i(p, y_i)] \\ &\quad - \sum_i M^i(p, y_i)[X^i(p, y_i) - D^i_p(p, y)] + \sum_i D^i_y(p, y) M^i(p, y_i) X^D(p, y)^T \\ &= \sum_i S^i(p, y_i) - C^D(p, y). \end{aligned}$$

Since each consumer's Slutsky matrix $S^i(p, y_i)$ is symmetric and negative semidefinite, C^D must be symmetric for S^D to be symmetric. If in addition C^D is positive semidefinite, then S^D is negative semidefinite. □

Remark 3.2. *Requiring the covariance matrix $C^D(p, y)$ to be symmetric and positive semidefinite does not restrict the form of the demand function of any consumer.*

For example if the consumers have the same arbitrary demand function and equal income shares, then the covariance restrictions are satisfied with $C^D = 0$. On the other hand, *symmetry of the covariance matrix is not robust.* It is lost under perturbation of the consumers' preferences when there are at least three goods (Jerison (1994)). Thus, in a typical economy with more than two goods there is no representative consumer.

It is easy to see that the covariance matrix is symmetric and positive semidefinite in the best-known cases where positive representative consumers exist. For example, if the consumers have identical homothetic preferences, or, more generally, have parallel Engel curves at each price vector p (the case considered by Antonelli (1886), Gorman (1953) and Nataf (1953)), then their marginal propensity to consume vectors $M^i(p, D^i(p, y))$ are equal and $C^D(p, y) = 0$. A positive representative consumer exists in much broader classes of consumption sectors if the income shares are fixed.

Remark 3.3. *If the consumers have fixed income shares θ_i so that $D^i(p, y) = \theta_i y$, then the covariance matrix is*

$$C^D(p, y) = \sum \theta_i M^i[(X^i/\theta_i) - X^D]^T = y \sum \theta_i M^i(A^i - A^D)^T, \tag{3.1}$$

aggregate income times the covariance matrix of the consumers' marginal and average propensities to consume, with consumers weighted by their income shares and with X^D and A^D evaluated at (p, y) and X^i, M^i and A^i evaluated at $(p, D^i(y, p))$.

When income shares are fixed, positive semidefiniteness of the covariance matrix means that consumers with larger than average budget shares for any good also tend to have higher marginal propensities to consume that good. If the consumers' preferences are homothetic, but not necessarily identical, then $C^D(p, y)$ equals y times the covariance matrix of the consumers' marginal propensity to consume vectors, hence is symmetric and positive semidefinite (Eisenberg (1961)). More generally, with fixed income shares and consumer demands of Muellbauer's (1976) PIGL form, $X^i(p, y_i) = y_i[a^i(p) + b^i(p) \ln y_i]$, where a^i and b^i are functions from $\mathbb{R}^n$ into $\mathbb{R}^n$, the covariance matrix becomes $C^D(p, 1) = \sum \theta_i(b^i + c^i)(c^i)^T - \sum \theta_i(b^i + c^i) \sum \theta_i(c^i)^T$, where b^i and c^i are evaluated at p with $c^i(p) \equiv a^i(p) + b^i(p) \ln \theta_i$. For symmetry and positive semidefiniteness of the covariance matrix it is sufficient, though not necessary, that the b^is or c^is are identical across consumers. PIGL demands include commonly used "flexible functional forms" such as the AID system of Deaton and Muellbauer (1980) and demands generated by translog indirect utility functions (Christensen, et. al. (1975)).

When the income shares are fixed, positive semidefiniteness of the covariance matrix means, roughly, that the consumers' demand vectors spread out when their incomes rise by equal amounts. To be more precise, consider a situation (p, y) and imagine a thought experiment in which all consumers are given an additional income transfer Δ. The demand of consumer i is then $X^i(y_i + \Delta, p)$, where $y_i = \theta_i y$. Letting x be a vector of length 1, the dispersion in the consumers' demand vectors in the direction x can be measured by the variance of the scalars $x \cdot X^i(p, y_i + \Delta)$, with consumers weighted equally:

$$\frac{1}{m} \sum x^T X^i(p, y_i + \Delta) X^i(p, y_i + \Delta)^T x - \frac{1}{m^2} \sum x^T X^i(p, y_i + \Delta) \sum X^i(p, y_i + \Delta)^T x.$$

To find the effect of a small income transfer, we differentiate this variance with respect Δ and evaluate at $\Delta = 0$. We obtain $x^T[\tilde{C}(p,y) + \tilde{C}(p,y)^T]x$, where

$$\tilde{C}(p,y) \equiv \frac{1}{m}\sum M^i(p,y_i)X^i(p,y_i)^T - \frac{1}{m^2}\sum M^i(p,y_i)X^D(p,y)^T.$$

We say that the community has *increasing dispersion* if for each (p,y), the matrix $\tilde{C}(p,y)$ is positive semidefinite on the hyperplane orthogonal to the aggregate demand vector $X^D(p,y)$. This means that the consumers' demand vectors become more dispersed (or at least not less dispersed) in all directions orthogonal to the aggregate demand when the consumers' incomes all rise by the same small amount. A community with fixed income shares has increasing dispersion if and only if its covariance matrix $C^D(p,y)$ is positive semidefinite for each (p,y) (Jerison (1994) Remark 3).

The hypothesis of increasing dispersion can be tested using the nonparametric statistical method of average derivatives. Härdle, et. al. (1991), Hildenbrand (1994) and Kneip (1993) show that French and U.K. consumer expenditure data are consistent with the hypothesis. In testing for increasing dispersion, the number of commodities matters. With very narrowly defined commodity categories there are likely to be inferior goods. For these goods, consumers may converge toward zero consumption as they become richer. The papers referred to above use data with goods grouped in nine to 14 commodity aggregates. They lend support for the hypothesis of increasing dispersion in analyses of policies or events for which this type of commodity grouping is appropriate (as it is for policies or events that do not affect relative prices of goods within any given commodity group).

Remark 3.4. *It is easy to verify that $C^D(p,y)p = C^D(p,y)^T p = 0$. In addition, the $n \times n$ symmetrized covariance matrix $C^D + (C^D)^T$ is the sum of $2m$ rank 1 matrices, so it cannot have rank greater than* $\max\{2m, n-1\}$. *When it has this maximal rank, positive semidefiniteness of the covariance matrix is a robust condition that is preserved under perturbation of the consumers' preferences.*

4. Nonrepresentative Representative Consumers

In this section we show how to construct examples of positive representative consumers with large Pareto inconsistencies. One reason why the examples are surprising is that positive representative consumers are Pareto consistent *locally* (at least to first order). To see why, consider a smooth curve $p(t)$ in price space, and let $\hat{y}_i(t)$ be the aggregate income needed so that the utility of consumer i, $V^{Di}(p(t), \hat{y}_i(t))$ remains constant, with $\hat{y}_i(0) = \bar{y}$. Differentiating with respect to t yields $v^i_y(p(t), D^i)[D^i_y \hat{y}'_i(t) + D^i_p p'(t)] + v^i_p(p(t), D^i)p'(t) = 0$ and, by Roy's identity, $D^i_y \hat{y}'_i(t) = X^i(p, D^i)p'(t) - D^i_p p'(t)$, where D^i and its derivatives are evaluated at $(p(t), \hat{y}_i(t))$. Summing over i, we obtain $\sum_i D^i_y \hat{y}'_i(t) = X^D(p(t), \hat{y}(t))p'(t) = \hat{y}'(t)$, where $\hat{y}(t)$ is the aggregate income needed to compensate the representative consumer so that $v^D(p(t), \hat{y}(t))$ stays constant, with $\hat{y}(0) = \bar{y}$. Thus, to first order, the compensation required by the representative consumer is a weighted average of the changes in aggregate income needed to compensate the actual members of the community. Consider a Pareto improving path $(p(t), y(t))$ $(0 \leq t \leq 1)$ with $y(0) = \bar{y}$. As aggregate income and prices move along the path, no consumer's utility falls, so $y(t) \geq \max_i \hat{y}_i(t)$ and $y'(0) \geq \max_i \hat{y}'_i(0) \geq \hat{y}'(0)$, which implies $dv(p(t), y(t))/dt|_{t=0} \geq 0$. To first order, the representative consumer's utility does not fall either.

In spite of this, representative consumers need not be Pareto consistent. The following result is useful for constructing examples of Pareto inconsistency.

Remark 4.1. *If all the consumers are indifferent between two situations but the representative consumer is not, then the representative consumer is Pareto inconsistent.*

To see this, we note first that if a representative consumer exists, it can be assumed to have a C^1 indirect utility function v. This follows from the fact that the aggregate demand function is C^1. Homogeneity of the consumers' indirect utility functions and of the distribution rule D implies that without loss of generality we can let aggregate income equal 1. Suppose that all members of the consumption sector are indifferent between the price vectors p and q, but the representative consumer prefers q, so that $V^D(p) = V^D(q)$ and $v(p, 1) < v(q, 1)$. This last inequality is preserved if all components of q are proportionally decreased by a small enough amount. But then every consumer prefers p to q even though the representative consumer prefers q.

We will consider consumption sectors with two goods and two consumers with equal income shares. If the consumers' preferences are homothetic, then there is a Pareto consistent representative consumer. We will start with a pair of homothetic consumers and modify their preferences in order to obtain a consumption sector with a Pareto inconsistent representative consumer. Let the price of good 2 be fixed at 1. We start with a homothetic consumer i (for $i = 1, 2$) with an indirect utility function v^i, expenditure function e^i and demand function $x^i(p, y_i)$ for good 1. The homothetic consumers 1 and 2 are indifferent between the mean situations $A = (q, 1)$ and $B = (\bar{p}, z)$, but their indifference curves through A and B do not intersect at any other points in the space of aggregate income and the price of good 1. Figures 1a and 1b show the consumers' indifference curves in that space. Consumer 1 has dotted (blue) indifference curves and consumer 2 has broken (red) curves. Letting u_i be the utility of consumer i in the mean situations A and B, we see that $e^1(q, u_1) = e^2(q, u_2) = 1$ and $e^1(p, u_1) = e^2(p, u_2) = z$. The corresponding representative consumer, with a solid indifference curve, is also indifferent between the mean situations A and B.

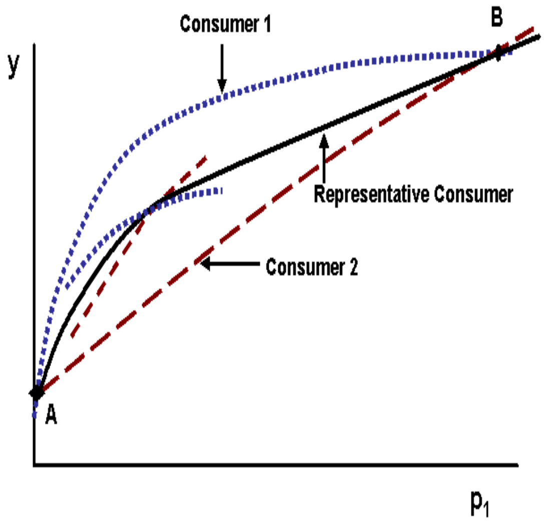

Figure 1a

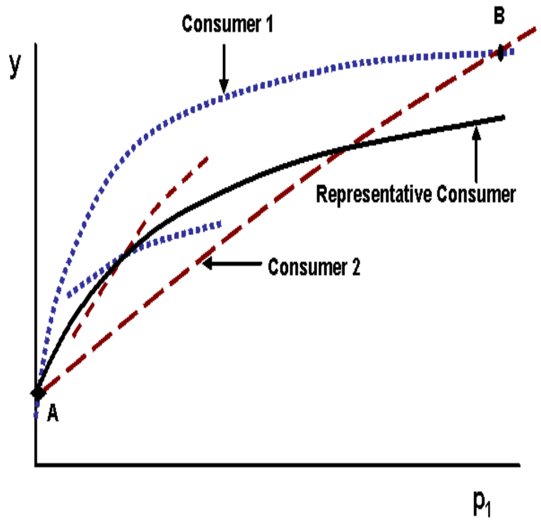

Figure 1b

In a two good economy, a smooth consumer demand function $x(p, y)$ for good 1 completely determines the consumer's indirect preferences as long as the demand function satisfies the Slutsky condition

$$\frac{\partial x(p, y)}{\partial p_1} + x(p, y)\frac{\partial x(p, y)}{\partial y} \leq 0.$$

Using this fact, we can specify new consumer preferences by changing consumer i's demand function for good 1 to $\hat{x}^i(p, y_i) = \phi_i(v^i(p, y_i) - u_i)x^i(p, y_i)$ for $i = 1, 2$, where $\phi_i : \mathbb{R} \to (0, 1)$ is a smooth function with $\phi_i'(t) > 0$ for $t < 0$ and $\phi_i(t) = 1$ for $t \geq 0$. The new demand for good 1 is the same as the old demand at points above the consumer's indifference curve through A and is strictly less than the old demand at points below that indifference curve. It is easy to verify that the new demand function for good 1 satisfies the Slutsky condition if the old one does. This implies that the new demand functions x^i determine a new consumption sector.

In the income and price space of Figures 1a and 1b, consumer i's indifference curve of utility level u_i^* is the graph of the expenditure function $e^i(\cdot, 1, u_i^*)$. Since consumer i's demand for good 1 at mean situation (p, y) equals $\partial e^i(p, u_i^*)/\partial p_i$ the modification of i's demand described above flattens i's indifference curves below the curve through A while leaving i indifferent between A and B. If a representative consumer exists in the new consumption sector then its indifference curve through A is flatter than that of the original representative consumer. Therefore it passes below B and so, by Remark 4.1, the representative consumer is Pareto inconsistent.

In order for the new consumption sector to have a representative consumer, the Slutsky matrix of aggregate demand must be symmetric and negative semidefinite. Symmetry holds automatically when there are only two goods. Slutsky negative semidefiniteness requires

$$\begin{aligned} 0 \;\geq\; & (1/2)(\phi_1' v_p^1 x^1 + \phi_2' v_p^2 x^2 + \phi_1 x_p^1 + \phi_2 x_p^2) \\ & +(1/4)(\phi_1 x^1 + \phi_2 x^2)(\phi_1' v_y^1 x^1 + \phi_2' v_y^2 x^2 + \phi_1 x_y^1 + \phi_1 x_y^1) \\ = \;& -(1/2)[\phi_1' v_y^1 (x^1)^2 + \phi_2' v_y^2 (x^2)^2 + \phi_1 x_p^1 + \phi_2 x_p^2] \\ & +(1/4)(\phi_1 x^1 + \phi_2 x^2)(\phi_1' v_y^1 x^1 + \phi_2' v_y^2 x^2 + \phi_1 x_y^1 + \phi_1 x_y^1). \end{aligned}$$

Here, v^i and x^i and their derivatives are evaluated at (p, y), and the subscripts y and p denote partial derivatives with respect to the first and second arguments, y and p_1. Both ϕ_i and ϕ_i' are evaluated at $v^i(p, y) - u_i$ for each i.

When each ϕ_i is sufficiently close to 1, the Slutsky condition above is satisfied. This follows from the fact that there is a representative consumer when the actual consumers have homothetic preferences. So the argument above shows that there can be a Pareto inconsistent representative consumer. The question is how inconsistent. If each ϕ_i is close to 1, then the modified consumer preferences are nearly homothetic and the representative consumer's indifference curve through A passes close to B. Thus the inconsistency is small. To obtain a large inconsistency we must make $\phi_i(t)$ approach 0 rapidly as t decreases starting from 0. But the Slutsky condition above is violated if ϕ_i' is too large for $i = 1$ or 2. For this reason, the representative consumer's indifference curve through A cannot be made arbitrarily flat. Still, it can be made to pass substantially below B, as shown by the following example.

Example 4.2. A Large Pareto Inconsistency

We start with two homothetic consumers, the first with Cobb-Douglas preferences and indirect utility function $v^1(p, y) = y p_1^{-a} p_2^{a-1}$ *and the second with Leontief preferences and indirect utility function* $v^2(p, y) = y/(p_1 + bp_2)$, *where* $a = [\ln(7/4)]/\ln 2$ *and* $b = 1/3$. *Let the consumers receive equal income shares. The consumers are indifferent between the mean situations* $(q, 1)$ *and* $(p, 1.75)$, *where* $q = (1, 1)$ *and* $p = (2, 1)$. *Let* x^i *be the demand function for good 1 corresponding to the indirect utility function* v^i. *As described above, we let consumer i have a demand for good 1 that is less than* x^i *at mean situations that yield* *utility less than* $v^i(q,1)$*. To be precise, let* $\phi_i(t) = \exp(-10^4t^2)$ *for* $t < 0$ *and* $\phi_i(t) = 1$ *for* $t \geq 0$*. Let consumer i's demand function for good 1 be* $\hat{x}^i \equiv \phi_i(v^i - v^i(q,1))x^i$*, for* $i = 1, 2$*.*

It can be verified that ϕ_i is C^∞ and satisfies $\phi_i'(t) > 0$ for $t < 0$. It can be checked numerically that aggregate demand satisfies Slutsky negative semidefiniteness, so the modified consumption sector has a representative consumer.

In the modified consumption sector the representative consumer's indifference curve through $(q,1)$ is the graph of the function $c(t)$ satisfying $c(1) = 1$ and the differential equation $c'(t) = \hat{x}(t,1,c(t))$, where $\hat{x} \equiv (1/2)(\hat{x}^1 + \hat{x}^2)$ is the mean demand function. Numerical solution of this differential equation shows that $c(2) < 1.1203$. Thus, consumers 1 and 2 require 75% more income in order to be compensated for a doubling of the price of good 1. But the representative consumer requires only 12% more income. The inconsistency ratio is more than 1.56. In order to be compensated for the price rise, the actual consumers require over 56% more than the representative consumer requires.

In Example 4.2, the consumer's preferences are transformed without changing their indifference curves through the points A and B. Such a transformation is possible because in the initial consumption sector the set of mean situations that yield the consumer utility vector (u_1, u_2) (with the price of good 2 fixed) is disconnected. Jerison (1984) shows that as long as the mean situations that yield a given vector of consumer utilities form a connected set, the representative consumer must be indifferent among them. The disconnectedness in Figure 1 comes from the fact that for consumer 2 the goods are perfect complements (a large change in relative prices has little effect on the ratio of demands), whereas for consumer 1 the goods are substitutes. There is no obvious reason for ruling out such preference profiles.

The same perturbation argument used to construct example 4.2 shows that every utility function can be viewed as the utility of a Pareto inconsistent representative consumer for some consumption sector. To be more precise, for every smooth indirect utility function v there exists a consumption sector with a Pareto inconsistent representative consumer whose indirect utility function is v. This can be seen by starting with a consumption sector in which all the consumers have the indirect utility function v and equal incomes. Then it is possible to perturb the consumers' preferences so that there are two consumer types, with indifference curves that differ from those of the representative consumer and that cross twice like those of consumers 1 and 2 in Figure 1a. Then a slight flattening of the indifference curves of consumer 1 as in Figure 1b makes the representative consumer Pareto inconsistent, with all goods normal for all consumers.

The large Pareto inconsistency in example 4.2 may be considered pathological since it contains a Giffen good. The reason for the Giffen effect can be seen in Figure 1b. By comparing the slopes of the indifference curves of consumer 1 at a low price p_1, we see that a rise in income raises the consumer's demand for good 1 rapidly. The budget constraint implies that the demand for good 2 falls, so good 2 is inferior in that region. The indifference curves are also nearly linear, so the substitution effect is small and good 2 is a Giffen good. When Giffen goods are ruled out, the lower indifference curve of consumer 1 cannot be as flat as it is in Figure 1b, and this restricts the size of the Pareto inconsistency. In the next two sections we will consider other sources of bounds on Pareto inconsistencies.

5. A Bound on the Pareto Inconsistency of a Representative Consumer

Representative consumers' Pareto inconsistencies cannot be arbitrarily large. This is implied by the following fact, which is of independent interest. It is impossible for all the consumers to prefer situation A to B if B is revealed preferred to A for the representative consumer.

Lemma 5.1. *If (p, y) is strictly revealed preferred to (q, z) for the representative consumer then at least one consumer prefers (p, y) to (q, z).*

Proof. Suppose that $pX^i(q, D^i(q, z)) \geq D^i(p, y)$ for every i. Summing over i yields $pX(q, z) \geq y$. Thus if (p, y) is strictly revealed preferred to (q, z) by the representative consumer ($pX(q, z) < y$) then $pX^i(q, D^i(q, z)) < D^i(p, y)$ and $V^i(q, z) < V^i(p, y)$ for some i. □

If all the consumers are at least as well off at (q, z) as at (y, p) then aggregate income y cannot be too high. Lemma 1 implies that $y \leq pX(q, z)$. Similarly, if all the consumers are at least as well off at (p, y) as at (q, z) then there is a lower bound on y determined by $qX(p, y) \geq z$. Let $y^*(q, p)$ be the minimum y satisfying $qX(p, y) \geq 1$. It follows that the inconsistency ratio for a move from $(q, 1)$ to the price vector p lies in the interval $[y^*(q, p)/e^*(q, p), pX(q, 1)/e^*(q, p)]$, where $e^*(q, p) \equiv e(p, v(q, 1))$.

The interval determines bounds that are not tight. Better bounds can probably be found. But for commonly used models and normal price variation, the interval above is rather small. Consider, for example, a Cobb-Douglas representative consumer in a two-good consumption sector. Let the representative consumer's utility function be $u(x_1, x_2) = x_1^\alpha x_2^\beta$, with $\alpha + \beta = 1$, and let $q = (1, 1)$. Then $pX(q, 1) = \alpha p_1 + \beta p_2$, $e^*(q, p) = p_1^\alpha p_2^\beta$ and $y^*(q, p) = p_1 p_2/(\alpha p_2 + \beta p_1)$. The weakest bound on the inconsistency ratio occurs when $\alpha = \beta = 1/2$. (This is the case in which $pX(q, 1)/y^*(q, p)$ is maximized.) In this case, the inconsistency ratio lies in the interval $[2\sqrt{p_1 p_2}/(p_1 + p_2), (p_1 + p_2)/(2\sqrt{p_1 p_2}]$. If, starting at the price vector $q = (1, 1)$, the price of good 1 doubles and the price of good 2 does not change, then the inconsistency ratio lies in the interval $[.9428, 1.0607]$. So the compensation required by the Cobb-Douglas representative consumer will not differ by more than about 6% from that required by the actual consumers when the price of good 1 doubles.

6. Nonrepresentative Representative Consumers in Macroeconomics

The preferences of the nonrepresentative representative consumer in Example 4.2 are far from homothetic. This is not an accident. It comes from the twisting of the indifference curves in Figure 1b. In this section we show that a homothetic representative consumer can be Pareto inconsistent, but the inconsistency bounds obtained above are especially confining in that case. We will argue that the homothetic representative consumers most often used in macroeconomics are unlikely to have large Pareto inconsistencies, given the range of price variation in typical applications.

We focus on communities with two consumers and fixed income shares. In that case, the consumers' demands and indirect utilities have the special functional forms in Lemma 6.1, which make it possible to compute inconsistency ratios. We conjecture that the inconsistencies cannot be much larger in communities with more consumers and price-dependent income shares.

Lemma 6.1. *In a two consumer community with income shares θ_i and a homothetic representative consumer, the consumer demands have the form*

$$X^1(p, y_1) = y_1 K(p) + \mu(y_1/\theta_1, p) B(p) \quad \text{and} \quad X^2(p, y_2) = y_2 K(p) - \mu(y_2/\theta_2, p) B(p),$$

where τ is scalar valued. The indirect utilities have the form $v^i(p, y_i) = \nu^i(\alpha(p), \beta(p), y_i)$, where α and β are scalar valued and homogeneous of degree 1.

Proof. Define $F(p,y) \equiv (1/\theta_1)X^1(p,\theta_1 y) - (1/\theta_2)X^2(p,\theta_2 y)$ and note that $F_y(p,y) = M^1(p,\theta_1 y) - M^2(p,\theta_2 y)$. In the given community, $C^D(p,y) = y\theta_1\theta_2 F_y(p,y)F(p,y)^T$. Existence of a representative consumer implies that the matrix $F_y(p,y)F(p,y)^T = F(p,y)F_y(p,y)^T$, so there is a scalar valued function μ such that $F_y(p,y) = \mu(p,y)F(p,y)$ wherever $F(p,y) \neq 0$. Then $F(p,y) = \tau(p,y)B(p)$ for some scalar valued τ and vector valued B. So $(1/\theta_2)X^2(p,\theta_2 y) = (1/\theta_1)X^1(p,\theta_1 y) - \tau(p,y)B(p)$. Since the representative consumer is homothetic, aggregate demand is linear in income: $X^1(p,\theta_1 y) + X^2(p,\theta_2 y) = yK(p)$, and therefore $X^1(p,\theta_1 y) = yK(p) - (\theta_2/\theta_1)X^1(p,\theta_1 y) + \theta_2\tau(p,y)B(p)$ and $X^1(p,\theta_1 y) = \theta_1 yK(p) + \theta_1\theta_2\tau(p,y)B(p)$ and $X^2(p,\theta_2 y) = \theta_2 yK(p) - \theta_1\theta_2\tau(p,y)B(p)$. These are the forms above with $\mu = \theta_1\theta_2\tau$. The forms of the indirect utilities of these "rank two" demands were derived by Gorman (1981). (See also Lewbel (1991).) □

Example 6.2 Consider a two consumer community with fixed income shares and a representative consumer with homothetic, stationary, completely separable preferences for flows of consumption expenditures over an infinite time horizon, as is commonly assumed in macroeconomic applications. The utility function of such a representative consumer has the form

$$u(x) = (\sum_{t=0}^{\infty} \delta^t x_t^{1-\sigma})^{1/(1-\sigma)},$$

where $x = \{x_t\}_{t=0}^{\infty}$ and where x_t is consumption expenditure in period t. The corresponding indirect utility and expenditure functions are

$$v(p,y) = y(\sum_{t=0}^{\infty} \delta^{t/\sigma} p_t^{\epsilon})^{-1/\epsilon} \quad \text{and} \quad e(p,\bar{u}) = \bar{u} \cdot (\sum_{t=0}^{\infty} \delta^{t/\sigma} p_t^{\epsilon})^{1/\epsilon},$$

where $\epsilon \equiv (\sigma - 1)/\sigma$, and y is interpreted as lifetime wealth. If the consumer can borrow or save at a constant interest rate r then the price of consumption expenditure in period t can be taken to be $p_t = (1+r)^{-t}$, and optimal consumption (when it exists) grows at the rate g, where

$$\delta(1+r) = (1+g)^{\sigma}.$$

By Roy's identity, $X^i(p,y) = (1/v_y^i(p,y))v_p^i(p,y)$, at each (p,y), the two consumers' demand vectors and the aggregate demand are contained in the span of the two gradient vectors $\partial\alpha(p)$ and $\partial\beta(p)$, where α and β are the functions in the indirect utility functions in Lemma 6.1. It follows that the representative consumer's indirect utility function can be written as $v(p,y) = \nu(\alpha(p),\beta(p),y)$, and hence that $(\sum_{t=0}^{\infty} \delta^{t/\sigma} p_t^{\epsilon})^{-1/\epsilon}$ is a function of $\alpha(p)$ and $\beta(p)$. This implies that α and β are separable in each price, and have the CES form. Numerical computations suggest that the largest Pareto inconsistencies arise when α is a function of prices $p_1,\ldots,p_T$ for some T and β is a function of the remaining prices. In that case, α and β can be chosen to be

$$\alpha(p) = (\sum_{t=0}^{T} \delta^{t/\sigma} p_t^{\epsilon})^{1/\epsilon} \quad \text{and} \quad \beta(p) = (\sum_{t=T+1}^{\infty} \delta^{t/\sigma} p_t^{\epsilon})^{1/\epsilon},$$

and the representative consumer's expenditure function is $e(p,\bar{u}) = \tilde{e}(\alpha(p),\beta(p))\bar{u}$, where $\tilde{e}(a,b) \equiv (a^{\epsilon} + b^{\epsilon})^{1/\epsilon}$.

The functions α and β can be interpreted as price indices for commodity aggregates representing early and late consumption. The preferences of the representative and the two actual consumers are determined by their preferences for these commodity aggregates. The analysis is thus reduced to the two good case in Example 4.2. The worst possible

Pareto inconsistencies arise in cases when the actual consumers' indifference curves have very different curvature, as they have in Figure 1b. We obtain extreme examples by letting the utility function of consumer 2 (for the two aggregate commodities) be $u^2(x_1, x_2) \equiv (x_1 + x_2)/(s-1)$ if $sx_1 + x_2 < s$ and $u^2(x_1, x_2) \equiv [1/(s-1)] + (x_2/s)$ otherwise, where $s > 1$ is a fixed scalar. It is easy to verify that u^2 is continuous, quasiconcave and nondecreasing. Consumer 2 demands strictly positive amounts of both commodity aggregates whenever $sb > y > a > b$, where y is the consumer's income and a and b are the price indices of early and late consumption. On this region consumer 2 has the indirect utility function $v^2(a, b, y) = [(y-a)/(sb-a)] + s$ and the demand function $x_2 = (sb-y)/(sb-a)$ for good 1. The demands of consumer 2 and the representative consumer determine those of consumer 1. At prices a and b for the aggregate commodities the demand for good 1 by consumer 1 is $x_1(a, b, y) = 2y[a^{\epsilon-1}/(a^\epsilon + b^\epsilon)] - [(sb-y)/(sb-a)]$. Suppose that we fix $\bar{u}$ and a_L and let $y_L \equiv e^1(a_L, 1, \bar{u})$ and $y(a) \equiv e^1(a, 1, \bar{u})$, where e^1 is the expenditure function for consumer 1 in terms of the aggregated commodities. Then $y(\cdot)$ is the income compensation function that solves the differential equation $y'(a) = P(a)y(a) - Q(a)$, with the initial condition $y(a_L) = y_L$, where $P(a) = 2a^{\epsilon-1}/(a^\epsilon + 1)$ and $Q(a) = s/(s-a)$. The graph of $y(\cdot)$ is the indifference curve for consumer 1 of utility level $\bar{u}$ in the space of income and the price of "early consumption" when the price of late consumption is fixed at 1. The solution to this differential equation is

$$y(a; y_L) = \frac{(a^\epsilon + 1)^{2/\epsilon}}{(s-a)}[y_L \frac{s - a_L}{(a_L^\epsilon + 1)^{2/\epsilon}} - s \int_{a_L}^{a} (\alpha^\epsilon + 1)^{-2/\epsilon} d\alpha]. \tag{6.1}$$

In the case of Cobb-Douglas preferences, $\sigma = 1$ and $\epsilon = 0$. Then equation (6.1) is replaced by

$$y(a; y_L) = \frac{1}{s-a}\Big(\frac{a}{a_L}\Big)^{2m}\big[(s-a_L)y_L + \frac{sa_L}{1-2m}\big] - \frac{sa}{(1-2m)(s-a)}.$$

To construct examples of Pareto inconsistency, we fix T, the date separating early and late consumption, and specify alternative values of the interest rate. These interest rates determine alternative prices of early and late consumption (a_j, b_j), for $j = L, H$. We then find y_L and y_H such that both consumers 1 and 2 are indifferent between the budget situations $(a_L/b_L, 1, y_L)$ and $(a_H/b_H, 1, y_H)$. The y_L and y_H are the unique solutions to the equations $y_H = y(a_H/b_H; y_L)$ and $v^2(a_L/b_L, 1, y_L) = v^2(a_H/b_H, 1, y_H)$. The solution y_H is the wealth required by both consumers to compensate for a rise in the relative price of early consumption from a_L/b_L to a_H/b_H. The wealth required by the representative consumer is $y_R = y_L[((a_H/b_H)^\epsilon + 1)/((a_L/b_L)^\epsilon + 1)]^{1/\epsilon}$. If $y_R \neq y_H$ then the representative consumer is Pareto inconsistent, and the inconsistency ration is y_H/y_R.

We consider Pareto inconsistencies that can arise in the example of Lucas (1987) who compared a budget situation in which the representative consumer's consumption expenditure grows at 3% to a situation in which the consumption growth rate is 6%.

Case A. Let $\delta = .97$ and $\sigma = 2$, so that $\epsilon = 1/2$.

These are parameters taken as the base case by Stokey and Rebelo (1995) in their analysis of growth under alternative capital and labor tax rates. The optimal (constant) rate of consumption growth for the representative consumer changes from 3% to 6% if the interest rate changes from approximately 9.37% to 15.8%. The interest rate determines relative prices of early and late consumption for each value of T. It turns out that the inconsistency ratio is maximized when $T = 59$. For this T, a rise in the interest rate from 9.37% to

15.8% raises the relative price of early consumption, a/b, from 5.43 to 14.67. Taking $s = 1.001(a_H/b_H) \approx 14.68$, the inconsistency ratio is approximately 1.0217.

Case B. Let $\delta = .95$ and $\sigma = 1$.

This is the case of Cobb-Douglas utility used by Lucas (1987). The rate of consumption growth for the representative consumer rises from 3% to 6% when the interest rate changes from 8.4% to 11.58%. Again, taking $s = 1.001(a_H/b_H)$, the largest inconsistency ratio is slightly below 1.034, and it occurs with $T = 30$.

The inconsistencies in these examples (less than 3.4%) are well below the inconsistency bounds implied by Lemma 5.1 (9.6% for Case A and 11.8% for Case B). Those bounds are not tight, whereas the examples themselves appear to be close to the worst possible given the range of relative price variation when the representative consumer has the utility u. The specification of the preferences of consumer 2 (through the choice of s) depends on the range of price variation. Over this range, good 1 is inferior for consumer 2. Still, the degree of Pareto inconsistency is small. If both goods were normal for both consumers, the inconsistency would be even smaller. We conclude that if the representative consumer in Lucas' example accurately represents aggregate demand behavior in the positive sense, there must be consumers willing to pay nearly as much as the representative consumer pays for higher growth. If many consumers are not willing to pay that much, then there must be many consumers willing to pay more. The derivation above points to the possibility of testing macroeconomic representative consumer models by comparing their welfare judgments to the preferences of real consumers.

7. Normative Representative Consumers

The Pareto inconsistencies illustrated above show that representative consumer models might not be adequate for normative analysis even if the representative consumer perfectly represents aggregate demand behavior. On the other hand, Dow and Werlang (1988) show that if a representative consumer *is* Pareto consistent, its preferences have a welfare interpretation: they are the same as preferences generated by a particular Bergson-Samuelson social welfare function. To state this result precisely, we say that a representative consumer (for D) *has a welfare interpretation* if its preferences over situations are represented by $w[V^D(p, y)]$ for some nondecreasing function w that is strictly increasing on the set of attainable consumer utility vectors $\{V^D(p, y)|y \geq 0, p \gg 0\}$.

Proposition 7.1. *A representative consumer has a welfare interpretation if and only if it is Pareto consistent.*

For a proof, see Dow and Werlang (1988) or Jerison (1994). Note that a social planner might have a social welfare function that is different from the w in the previous paragraph. Then the planners' preferences would differ from those of the representative consumer even if the latter had a welfare interpretation.

It is an open question under what conditions a consumption sector has a Pareto consistent representative consumer. One well known sufficient condition is that the income distribution rule is optimal with respect to a social welfare function.

Proposition 7.2. *If the income distribution rule is optimal then the consumption sector has a Pareto consistent representative consumer.*

Chipman and Moore (1979) and Dow and Sonnenschein (1986) show that when D is optimal, there is a utility function that generates a correspondence that contains the aggregate demand function as a selection. Jerison (1994) shows that since the aggregate demand function is smooth, the utility function generates the aggregate demand *function* itself. The distribution rule need not be uniquely optimal, but every other optimal rule determines the same aggregate demand.

The converse of Proposition 7.2 is false.

Proposition 7.3. *The income distribution rule need not be optimal for a consumption sector to have a Pareto consistent representative consumer.*

To prove Proposition 7.3 we use the following property of consumption sectors with optimal income distribution, proved by Jerison (1994).

Proposition 7.4. *If the income distribution rule D is optimal then for each situation (p, y) the covariance matrix $C^D(p, y)$ is symmetric and positive semidefinite.*

It follows that if the covariance matrix fails to be symmetric or positive semidefinite, then there is *no* social welfare function for which the distribution rule is optimal. We prove Proposition 7.3 by exhibiting a consumption sector with a Pareto consistent representative consumer and a covariance matrix that is not positive semidefinite.

Example 7.5. Pareto Consistency Without Optimal Income Distribution

Consider a consumption sector with two goods and two consumers with equal income shares. Consumer 1 has Cobb-Douglas utility and demand function

$$X_1^1(p, y_1) = y_1/(2p_1), \quad X_2^1(p, y_1) = y_1/(2p_2).$$

Consumer 2 has the demand function

$$X_1^2(p, y_2) = (y_2/2p_1) + (p_1/2y_2), \quad X_2^2(p, y_2) = (y_2/2p_2) - [p_1^2/(2p_2y_2)].$$

The demand vector $X^2(p, y_2)$ of consumer 2 is nonnegative whenever $y_2 \geq p_1$. We restrict attention to mean incomes and prices in this region.

The consumers' indirect utility functions are

$$v^1(p, y_1) \equiv y_1^2/(p_1p_2) \text{ and } v^2(p, y_2) \equiv (y_2^2/p_1p_2) - (p_1/p_2).$$

The two consumers receive equal shares of aggregate income, so when they have income y at prices p, aggregate demand is

$$X(p, 2y) = X^1(p, y) + X^2(p, y) = (\frac{y}{p_1} + \frac{p_1}{2y}, \frac{y}{p_2} - \frac{p_1^2}{2p_2y}).$$

An indirect utility function for this aggregate demand function is $v(p, y) = [y^2/(p_1p_2)] - (p_1/p_2)$. It is easy to show that v^1, v^2 and v are quasiconvex in p over the region where $X^2 \gg 0$.

To show that the distribution rule is not optimal with respect to any social welfare function it suffices to show that the covariance matrix of average and marginal properties to consume is not positive semidefinite. By (3.1), the upper left component of this covariance matrix is a positive multiple of $(M_1^2 - M_1^1)(A_1^2 - A_1^1)$ with each function evaluated at (p, y). But we have

$$M_1^2 - M_1^1 = \frac{1}{2p_1} - \frac{p_1}{2y^2} - \frac{1}{2p_1} = -\frac{p_1}{2y^2} < 0$$

$$A_1^2 - A_1^1 = \frac{1}{2p_1} + \frac{p_1}{2y^2} - \frac{1}{2p_1} = \frac{p_1}{2y^2} > 0.$$

Thus the covariance matrix is not positive semidefinite, and the income distribution cannot be optimal.

To show that the representative consumer is Pareto consistent, let each consumer i have income y, and suppose that the price vector p is Pareto superior to q. Then

$$\frac{y^2}{p_1p_2} \geq \frac{y^2}{q_1q_2} \qquad \text{and} \qquad \frac{y^2}{p_1p_2} - \frac{p_1}{p_2} \geq \frac{y^2}{q_1q_2} - \frac{q_1}{q_2}$$

with at least one strict inequality. Adding the second inequality to three times the first, we see that

$$v(p, 2y) = \frac{4y^2}{p_1p_2} - \frac{p_1}{p_2} > \frac{4y^2}{q_1q_2} - \frac{q_1}{q_2} = v(q, 2y),$$

so the representative consumer prefers the price vector p to q at aggregate income $2y$. This shows that the representative consumer is Pareto consistent.

Mas-Colell et. al. (1995) call a representative consumer "normative" if the distribution rule is optimal for some social welfare function. Proposition 7.1 and Example 7.6 suggest this terminology is unduly restrictive. A representative consumer's preferences can coincide with social welfare in all aggregate situations even though the distribution rule is not optimal. It seems more appropriate to say that a representative consumer is *normative* if it has a welfare interpretation. The term then applies in a broader class of communities.

It remains an open question how to characterize consumption sectors with Pareto consistent representative consumers. The next result provides a sufficient condition in a special case.

Proposition 7.6. *In a community with two goods, two consumers and a representative consumer, the representative consumer is Pareto consistent if for each* $p \gg 0$ *the matrix of covariances* $C^D(p, 1)$ *is nonzero.*

In a two-good economy, the matrix of covariances is symmetric. If it is positive semidefinite, then a representative consumer exists. If in addition the matrix is nonzero at every p, then, by Proposition 7.6, the representative consumer is Pareto consistent.

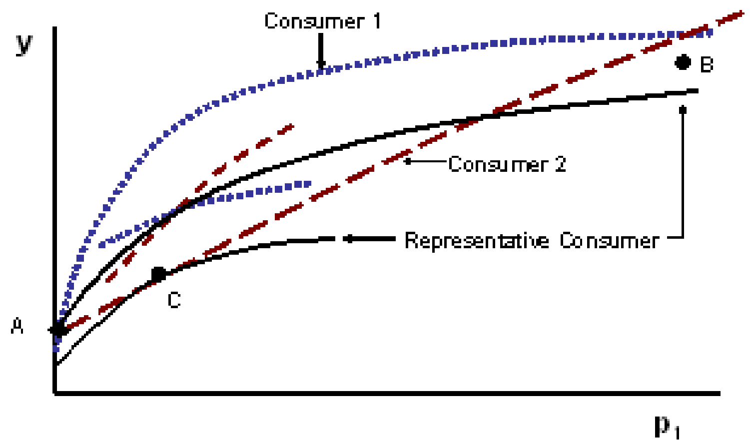

Figure 2

Figure 2 shows the main idea in the proof. In the figure, the actual consumers, 1 and 2, prefer A to B, but the representative consumer prefers B to A. By minimizing the utility of the representative consumer along the indifference curve of consumer 2 through A, we obtain a point C at which the consumers' indifference curves are tangent to each other. At such a point, the matrix of covariances is 0. Therefore, as long as the covariance matrix is nonzero everywhere, the representative consumer is Pareto consistent.

Proof of Proposition 7.6: Suppose that all the consumers in the community prefer (q, z) to $(\bar{y}, \bar{p})$, but the representative consumer prefers $(\bar{p}, \bar{y})$ to (q, z). We will show that this leads to a contradiction. By the homogeneity of D and of each consumer's indirect utility function, there is no loss of generality in assuming that $q_2 = \bar{p}_2$. If $q_1 = \bar{p}_1$ then there cannot be a Pareto inconsistency, since the utility of each consumer (including the representative) is strictly increasing in aggregate income. Assume that $\bar{p}_1 > q_1$.

We begin by showing that $X_1^{D1}(q, z) \neq X_1^{D2}(q, z)$. Recall that $\sum_i D_y^i = 1$ and $\sum_i D_y^i X^{Di} = X^D$, with $D_y^i > 0$ for each i. Also, for each i, $X^{Di}(p, y)$ and $X^D(p, y)$ satisfy the same budget identity, $px = y$. Since there are only two goods, if $X_1^{D1}(q, z) = X_1^{D2}(q, z)$ then $X^{D1}(q, z) = X^{D2}(q, z) = X^D(q, z)$. But then $C^D(q, z) = 0$, contradicting the hypothesis. This proves that $X_1^{D1}(q, z) \neq X_1^{D2}(q, z)$, and without loss of generality we can assume that $X_1^{D1}(q, z) > X_1^{D2}(q, z)$. It follows that $X_1^D(q, z) > X_1^{D2}(q, z)$, since $\sum D_y^i X^{Di} = X^D$.

Consider the minimization problem

$$\min_{p,y} v(p, y) \qquad \text{s.t.} \qquad y \leq \bar{y} + 1, \quad \bar{p}_1 \geq p_1 \geq q_1, \quad p_2 = \bar{p}_2,$$
$$\text{and} \qquad V^{D2}(p, y) = V^{D2}(q, z).$$

The constraint set is compact and contains (q, z), so the problem has a solution denoted (p^*, y^*). If $p_1^* = \bar{p}_1$ then $p^* = \bar{p}$ and $y^* > \bar{y}$, since $V^{D2}(\bar{p}, \bar{y}) < V^{D2}(q, z) = V^{D2}(p^*, y^*)$. But then $v(p^*, y^*) > v(\bar{p}, \bar{y}) > v(q, z)$, which contradicts the assumption that (p^*, y^*) solves the minimization problem. Thus $p_1^* < \bar{p}_1$. Since v is nonincreasing in prices and nondecreasing in aggregate income, the first two constraints in the minimization problem hold with strict inequality. With these constraints not binding, the necessary first order conditions are $v_y - \lambda V_y^{D2} = 0$ and $v_{p_1} - \lambda V_{p_1}^{D2} - \mu = 0$, for nonnegative scalars λ and μ, with $\mu(p_1^* - q_1) = 0$, where the subscripts denote partial derivatives and all functions are evaluated at (p^*, y^*).

Note that $-v_p/v_y = X^D$ and $-V_p^{D2}/V_y^{D2} = X^{D2}$. So the first order conditions above imply that $X_1^D(p^*, y^*) \leq X_1^{D2}(p^*, y^*)$, with equality if $p_1^* > q_1$ (since $p_1^* > q_1$ implies $\mu = 0$). It follows that $(p^*, y^*) \neq (q, z)$ since $X_1^D(q, z) > X_1^{D2}(q, z)$. Therefore $p_1^* > q_1$ and $X^D(p^*, y^*) = X^{D2}(p^*, y^*)$. This implies that $X^{D1} = X^{D2} = X^D$ at (p^*, y^*), since $\sum D_y^i X^{Di} = X^D$. But then $C^D(p^*, y^*) = 0$, which contradicts the hypothesis. This proves that there cannot be a Pareto inconsistency of the form described above with $\bar{p}_1 > q_1$.

If $\bar{p}_1 < q_1$ we arrive at a similar contradiction by showing that $X_1^D(\bar{p}, \bar{y})$ is strictly between $X_1^{D1}(\bar{p}, \bar{y})$ and $X_1^{D2}(\bar{p}, \bar{y})$, and then by minimizing $V^{Di}(p, y)$ subject to $v(p, y) = v(\bar{p}, \bar{y})$, $\bar{p}_1 \leq p_1 \leq q_1$ and $p_2 = \bar{p}_2$, where consumer i has the larger $X_1^{Di}(\bar{p}, \bar{y})$. □

8. Conclusion

Positive representative consumers are Pareto consistent locally, but not necessarily globally. The problem of characterizing communities with globally Pareto consistent representative consumers remains open. In the examples above, the Pareto inconsistencies arise because the degree of substitutability among commodities is significantly different for different consumers. Figures 1a and 1b suggest that a "single crossing" property for the consumers' indifference curves ensures Pareto consistency. This is the type of condition implied by the matrix of covariances being nonzero in Proposition 7.6; however, that proposition does not generalize beyond the case of two goods and two consumers.

The matrix of covariances must be positive semidefinite in order for the distribution rule to optimal with respect to some social welfare function (Jerison (1994)). If the consumers receive fixed shares of aggregate income, then positive semidefiniteness of the matrix of covariances is essentially equivalent to the requirement that the consumers' demand vectors become more dispersed when their incomes all rise by the same amount (Jerison (1994)). This "increasing dispersion" is plausible and has empirical support in some contexts (Härdle, et. al. (1991), Hildenbrand (1994)). But it follows from an example of Schlee (2005) that positive semidefiniteness of the covariance matrix is not sufficient for the distribution rule to be optimal with respect to a social welfare function. Schlee's results may also help to determine whether increasing dispersion, along with symmetry of the covariance matrix C^D, is sufficient for existence of a Pareto consistent representative consumer when income shares are fixed.

We have argued that a positive representative consumer of the CES form common in macroeconomics is unlikely to be very Pareto inconsistent. In Lucas' (1987) growth example, if the actual consumers prefer one growth path A to B and the representative consumer prefers B to A, it does not take much compensation to make the representative accept A. For the cases considered in section 6, if the representative consumer is willing to give up 42% of its initial consumption in order to raise its consumption growth rate by three percentage points, then some consumer in the represented community must be willing to give up more than 40% of its initial consumption for the same rise in consumption growth. The argument in section 6 is just illustrative, but it suggests that one might test commonly used macroeconomic models by asking whether the preferences of their representative consumers differ greatly from those of real consumers.

The Pareto inconsistencies of positive representative consumers in macroeconomics may be small, but this does not extend the applicability of such models very far. A typical community does not have a positive representative consumer at all. Furthermore, even if there is a Pareto consistent representative consumer, its preferences need not be useful for policy evaluation. The representative consumer's preferences are a special form of compensation criterion (Jerison (1990)). They attach greater weight to richer consumers who consume more. It is possible that the representative consumer agrees with only a tiny minority of the community in its evaluation of the relevant policy alternatives. For these reasons, it may not be appropriate to identify a representative consumer's preferences with social welfare even if doing so entails no logical inconsistency.

FOOTNOTES

1. For vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ we write $x \gg y$ [resp. $x \geq y$] if $x_j > y_j$ [resp. $x_j \geq y_j$] for $j = 1, \ldots, n$.
2. A function w is *nondecreasing* if $u \geq r$ implies $w(u) \geq w(r)$ for every u and r in the domain of w. The function w is *strictly increasing on a set* if $w(u) \gg w(r)$ whenever $u \gg r$ for u and r in the set.
3. An $n \times n$ matrix M (not necessarily symmetric) is *positive* [respectively, *negative*] *definite on a set* X if $x^T M x >$ [<]0 for every $x \neq 0$ in X. The matrix M is *positive* [respectively, *negative*] *semidefinite on* X if $x^T M x \geq$ [≤]0 for every $x \in X$. We omit is reference to X if it is $\mathbb{R}^n$.

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