On the Use of Density Forecasts to Identify Asymmetry in Forecasters' Loss Functions

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Abstract
We consider how to use information from reported density forecasts from surveys to identify asymmetry in forecasters' loss functions. We show that, for the three common loss functions - Lin-Lin, Linex, and Quad-Quad - we can infer the direction of loss asymmetry by just comparing point forecasts and the central tendency (mean or median) of the underlying density forecasts. If we know the entire distribution of the density forecast, we can calculate the loss function parameters based on the first order condition of forecast optimality. This method is applied to forecasts for annual real output growth and inflation obtained from the Survey of Professional Forecasters (SPF). We find that forecasters treat underprediction of real output growth more dearly than overprediction, reverse is true for inflation.

Key Words: Asymmetric loss function; Density forecasts; the Survey of Professional Forecasters;

1. Introduction

Although the importance of asymmetric loss function for model estimation, selection, prediction, forecast evaluation, and rationality test is widely recognized, few studies try to estimate the loss function from data directly. One exception is Elliott, Komunjer, and Timmermann (2005, 2008), hereafter referred to as EKT. They propose to estimate the parameters of loss function by GMM method based on moment conditions implied by forecast rationality. This method relies on two assumptions. First, the loss function parameters are constant over time; and second, forecasts are rational. As stated in EKT, they "back out the loss function parameters consistent with the forecast being rational".

In this paper, we consider how to use information from density forecasts to learn about the loss functions. Since forecasters form their point forecasts based on what they believe to be the data generating processes (density forecasts) and their loss functions, we can reverse this process and learn about forecasters' loss functions by comparing forecasters' point forecasts and density forecasts for the same target. The advantage of this method is that we can relax the two assumptions needed in EKT's GMM method: the point forecasts and density forecasts need not to be rational and the loss function parameters need not to be constant over time. Moreover, we do not need to know the actual values of the target variable. We just compare the two types of forecasts. We show how this method can be applied for the common loss functions in empirical work– Lin-Lin, Linex, and Quad-Quad loss functions.

The rest of the paper is organized as follows. In section 2, we discuss the data used in the paper - the Survey of Professional Forecasters (SPF) and the real time macro data. From these two data sets we construct point and density forecasts for the same target: annual inflation rate and real output growth rate. In section 3, we set up the general framework of optimal forecasts under asymmetric loss function. In section 4, we show that if we know only the central tendency (mean or median) of density forecasts, we can infer the direction of loss asymmetry by just comparing point forecasts and the central tendency of density forecasts for the three common loss functions mentioned above. In section 5, we show that if we know the entire density forecasts, we can learn not only the direction but also the degree of loss asymmetry. Section 6 concludes the paper.

2. Data

Two data sets are used in this paper. One is the Survey of Professional Forecasters (SPF), which provides density forecasts for annual inflation and real output growth, as well as point forecasts for quarterly output-price index and real output. The other is the real time macro data, which provides the actual values of quarterly price index and real output in real time. We combine these actual values with the SPF point forecasts for quarterly price index and real output to calculate the implied point forecasts for annual inflation and real output growth. Both the SPF data and the real time macro data are available from the Federal Reserve Bank of Philadelphia web site.

SPF was started in the fourth quarter of 1968 by the American Statistical Association and the National Bureau of Economic Research, and was taken over by the Federal Reserve Bank of Philadelphia in June 1990. The respondents are professional forecasters from the academia, government, and business. The survey is mailed four times a year, the day after the first release of the NIPA (National Income and Product Accounts) data for the preceding quarter. Most of the questions ask for point forecasts on a large number of variables for the current and the next four quarters, including the level of quarterly price index and real output. A unique feature of the SPF data set is that respondents are also asked to provide density forecasts for year-over-year growth rates in aggregate output and output price index in the current and following year. Or more specifically, they are asked to provide probabilities that the growth rates will fall in different intervals[1]. In this paper we use inflation forecasts during 1968Q4-2003Q4 and real output growth forecasts in 1981Q3-2003Q4. Forecasts on real output growth are not available before 1981Q3.

To calculate the implied point forecasts for annual inflation and real output growth, we need to know the actual values of price index and real output in quarters before the forecasting period. Theoretically, we should not use the most recent data because that was not available to forecasters when they made the forecasts. The real time data set provided by the Federal Reserve Bank of Philadelphia is a good choice[2]. This data set reports values of variables as they existed in the middle of each quarter from November 1965 to the present. Thus, for each vintage date, the observations are identical to those one would have observed at that time. Fortunately, this is also approximately the date when forecasters of SPF are asked to submit their forecasts. In addition, the definition of the price index and real output in this data set is consistent with that in the SPF data set.

[1] More descriptions of this data set can be found in Lahiri and Liu (2006).
[2] See Croushore and Stark (2001) for descriptions of this data set.

So we can conveniently combine the SPF data and the real time macro data to calculate the implied point forecasts for annual inflation and real output growth.

To see how this is done, consider a density forecast $\pi_{i,th}$, which is made h quarters before the end of year t about the target variable (annual inflation or real output growth) in year t by forecaster i. A point forecast for the same target and with the same forecast horizon can be constructed as follows.

If the density forecast is made in the first quarter of year t, the corresponding point forecast for the value of the target variable in year t is

$$f_{i,th} = 100 \times \left[\frac{P_{i,t,1} + P_{i,t,2} + P_{i,t,3} + P_{i,t,4}}{A_{t-1,1} + A_{t-1,2} + A_{t-1,3} + A_{t-1,4}} - 1 \right] \quad (1)$$

where $P_{i,t,j}$ is respondent i's predicted value of the price index or real output in the j^{th} quarter of year t and $A_{t,j}$ is the real time "actual value" of the price index or real output in the j^{th} quarter of year t. Similarly if the density forecast is made in the second quarter of year t, the corresponding point forecast is

$$f_{i,th} = 100 \times \left[\frac{A_{t,1} + P_{i,t,2} + P_{i,t,3} + P_{i,t,4}}{A_{t-1,1} + A_{t-1,2} + A_{t-1,3} + A_{t-1,4}} - 1 \right] \quad (2)$$

Point forecasts in the third and fourth quarter can be constructed similarly.

3. Optimal Point Forecast under Asymmetric Loss Function

Suppose we want to forecast the value of y in year t, h quarters ahead. When viewed h quarters before the end of year t, y_t is a random variable and is best described by a density function conditional on the information available at time t-h, I_{th}. This density function is often referred to as forecast density and can be denoted as $f^{O}(y_t \mid I_{th})$, where the superscript "o" means that the forecast density describes the true data generating process. Forecasters may not know the true data generating process. What they believe to be the true data generating process, or their density forecasts for the same target, are denoted as $f^{S}(y_t \mid I_{th})$, which we call subjective density. Typically, people will report a point forecast to represent the future occurring random variable. If f_{th} is chosen, the resulting forecast error is $e_{th} = y_t - f_{th}$. The loss function associated with this error could be expressed as $L(e_{th};\eta_{th})$ [3], characterized by some shape parameter η_{th}. When a forecaster decides on his point forecast, he minimizes $E^{S}\left[L(e_{th};\eta_{th}) \mid I_{th}\right]$, i.e.

$$\min_{f_{th}} \int L(y_t - f_{th};\eta_{th}) f^{S}(y_t \mid I_{th}) dy_t \quad (3)$$

[3] We assume that the loss is a function of just the size and sign of the forecast error.

If everything is well behaved, we get the first order condition as $E^S(L'_e(e_{th};\eta_{th}) \mid I_{th}) = 0$, or

$$\int L'_e(y_t - f_{th};\eta_{th}) f^S(y_t \mid I_{th}) dy_t = 0$$

where L'_e denotes the derivative of L with respect to the error e. This first order condition is equivalent to

$$\int L'_e(e_{th};\eta_{th}) f^S(e_{th} \mid I_{th}) de_{th} = 0 \tag{4}$$

where $f^S(e_{th} \mid I_{th})$ is derived immediately from $f^S(y_t \mid I_{th})$ given that $e_{th} = y_t - f_{th}$ and f_{th} is just a constant given I_{th}. (4) is the moment condition implied by forecast optimality. For any variables in the information set I_{th}, say V_{th}, (4) implies

$$E^S(L'_e(e_{th};\eta_{th}) \cdot V_{th}) = 0\,. \tag{5}$$

Using constant as the instrument variable, (5) means that the optimal point forecast under the asymmetric loss function must satisfy

$$E^S(L'_e(e_{th};\eta_{th})) = 0\,. \tag{6}$$

Conversely, given the point forecast and density forecast for the same target we can use equation (6) to learn about the loss function parameters.

4. Comparison of Point Forecasts and the Central Tendency of Density Forecasts

4.1 Inference about the Direction of Loss Asymmetry

In most cases, we don't know the entire distribution of $f^S(y_t \mid I_{th})$. Therefore we cannot calculate the loss function parameters using equation (6). But we may have some information about the central tendency of $f^S(y_t \mid I_{th})$, such as its mean, or median, or mode. The following three theorems prove that, for the common loss functions (Lin-Lin, Quad-Quad and Linex) and any belief about the data generating process, a non-zero difference between the optimal point forecast and the mean (for Quad-Quad and Linex) or the median (for Lin-Lin) of the density forecast is both a sufficient and a necessary condition of loss function asymmetry. This means that we can determine if the loss function is asymmetric or not by just comparing the point forecast with the mean or the median of the underlying density forecast.

Theorem 1: *For Lin-Lin loss function, symmetry of loss function is a sufficient and necessary condition for the equality of optimal point forecast and the median of the underlying density forecast.*

Consider the following Lin-Lin loss function

$$L(y_t - f_{th}) = \begin{cases} \alpha_{th}|y_t - f_{th}|, & \text{if } y_t - f_{th} > 0 \\ (1-\alpha_{th})|y_t - f_{th}|, & \text{if } y_t - f_{th} \le 0 \end{cases}$$

$\alpha_{th} \in (0,1)$. This loss function is symmetric when $\alpha_{th} = 0.5$. The optimal point forecast f_{th} solves

$$\min_{f_{th}} E_{th}^S[L(y_t - f_{th})] =$$
$$\alpha_{th}\int_{f_{th}}^{\infty}(y_t - f_{th})f^S(y_t \mid I_{th})dy_t - (1-\alpha_{th})\int_{-\infty}^{f_{th}}(y_t - f_{th})f^S(y_t \mid I_{th})dy_t$$

The first-order condition is

$$-\alpha_{th}(1 - F^S(f_{th} \mid I_{th})) + (1-\alpha_{th})F^S(f_{th} \mid I_{th}) = 0$$

where $F^S(f_{th} \mid I_{th})$ is the cumulative distribution function. The above first-order condition is equivalent to

$$F^S(f_{th} \mid I_{th}) = \alpha_{th} \qquad (7)$$

If the Lin-Lin loss function is symmetric, i.e. if $\alpha_{th} = 0.5$, (7) implies that $F^S(f_{th} \mid I_{th}) = \alpha_{th} = 0.5$, or f_{th} is the median of the density forecast. This establishes the sufficient part of theorem 1. Conversely, if f_{th} is the median of the density forecast, $\alpha_{th} = F^S(f_{th} \mid I_{th}) = 0.5$. This establishes the necessary part of theorem 1.

We can also infer the direction of asymmetry of Lin-Lin loss function by comparing the optimal point forecast and the median of the underlying density forecast. For example, if f_{th} is less than the median, then $F^S(f_{th} \mid I_{th}) = \alpha_{th} < 0.5$. Similarly, if f_{th} is more than the median, then $F^S(f_{th} \mid I_{th}) = \alpha_{th} > 0.5$.

Theorem 2: *For the Quad-Quad loss function, symmetry of loss function is a sufficient and necessary condition for the equality of optimal point forecast and the mean of the underlying density forecast.*

Consider the following Quad-Quad loss function

$$L(y_t - f_{th}) = \begin{cases} \alpha_{th}(y_t - f_{th})^2, & \text{if } y_t - f_{th} > 0 \\ (1-\alpha_{th})(y_t - f_{th})^2, & \text{if } y_t - f_{th} \le 0 \end{cases}$$

$\alpha_{th} \in (0,1)$. This loss function is symmetric when $\alpha_{th} = 0.5$. The optimal point forecast f_{th} solves

$$\min_{f_{th}} E_{th}^S[L(y_t - f_{th})] =$$
$$\alpha_{th}\int_{f_{th}}^{\infty}(y_t - f_{th})^2 f^S(y_t \mid I_{th})dy_t + (1-\alpha_{th})\int_{-\infty}^{f_{th}}(y_t - f_{th})^2 f^S(y_t \mid I_{th})dy_t$$

The first-order condition is

$$\alpha_{th}\int_{f_{th}}^{\infty}(y_t - f_{th}) f^S(y_t \mid I_{th})dy_t + (1-\alpha_{th})\int_{-\infty}^{f_{th}}(y_t - f_{th}) f^S(y_t \mid I_{th})dy_t = 0 \qquad (8)$$

If the loss function is symmetric, i.e. if $\alpha_{th} = 0.5$, the first order condition (8) implies

$$\int_{f_{th}}^{\infty}(y_t - f_{th}) f^S(y_t \mid I_{th})dy_t + \int_{-\infty}^{f_{th}}(y_t - f_{th}) f^S(y_t \mid I_{th})dy_t$$
$$= \int_{-\infty}^{\infty}(y_t - f_{th}) f^S(y_t \mid I_{th})dy_t = 0 \Rightarrow f_{th} = E_{th}^S(y_t)$$

i.e., f_{th} is the mean of the density forecast. This establishes the sufficient part of theorem 2.

Now, consider the necessary part. Use μ_{th} to denote the mean of the density forecast $E_{th}^S(y_t)$. If $f_{th} = E_{th}^S(y_t) = \mu_{th}$, equation (8) can be rewritten as

$$\alpha_{th}\int_{\mu_{th}}^{\infty}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t + (1-\alpha_{th})\int_{-\infty}^{\mu_{th}}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t = 0$$
$$\Rightarrow \alpha_{th}\int_{\mu_{th}}^{\infty}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t + \alpha_{th}\int_{-\infty}^{\mu_{th}}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t$$
$$-\alpha_{th}\int_{-\infty}^{\mu_{th}}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t + (1-\alpha_{th})\int_{-\infty}^{\mu_{th}}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t = 0$$
$$\Rightarrow \alpha_{th}\int_{-\infty}^{\infty}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t + (1-2\alpha_{th})\int_{-\infty}^{\mu_{th}}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t = 0$$

Note that the first term is equal to zero. So we have

$$(1-2\alpha_{th})\int_{-\infty}^{\mu_{th}}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t = 0 \qquad (9)$$

Since $\int_{-\infty}^{\mu_{th}}(y_t - \mu_{th}) f^S(y_t \mid I_{th})dy_t < 0$, (9) implies that $(1-2\alpha_{th}) = 0 \Rightarrow \alpha_{th} = 0.5$
This establishes the necessary part of Theorem 2.

As before, we can also infer the direction of asymmetry of Quad-Quad loss function by comparing the optimal point forecast and the mean of the density forecast.
Suppose the optimal point forecast $f_{th} = E_{th}^S(y_t) + m_{th} = \mu_{th} + m_{th}$. Let $x_{th} = y_t - f_{th} = y_t - \mu_{th} - m_{th}$, then $E_{th}^S(x_{th}) = E_{th}^S(y_t) - \mu_{th} - m_{th} = -m_{th}$.
The first-order condition (8) can be rewritten as

$$\alpha_{th}\int_{f_{th}}^{\infty}(y_t - f_{th})f^S(y_t \mid I_{th})dy_t + (1-\alpha_{th})\int_{-\infty}^{f_{th}}(y_t - f_{th})f^S(y_t \mid I_{th})dy_t = 0$$
$$\Rightarrow \alpha_{th}\int_{0}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} + (1-\alpha_{th})\int_{-\infty}^{0}x_{th}f_x(x_{th} \mid I_{th})dx_{th} = 0$$
$$\Rightarrow (2\alpha_{th} - 1)\int_{0}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} + (1-\alpha_{th})\int_{0}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th}$$
$$+ (1-\alpha_{th})\int_{-\infty}^{0}x_{th}f_x(x_{th} \mid I_{th})dx_{th} = 0$$
$$\Rightarrow (2\alpha_{th} - 1)\int_{0}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} + (1-\alpha_{th})\int_{-\infty}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} = 0$$
$$\Rightarrow (2\alpha_{th} - 1)\int_{0}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} - (1-\alpha_{th})m_{th} = 0$$

since $E_{th}^S(x_{th}) = \int_{-\infty}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} = -m_{th}$. So we have

$$(2\alpha_{th} - 1)\int_{0}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} - (1-\alpha_{th})m_{th} = 0 \tag{10}$$

If the optimal point forecast f_{th} is less than the mean of the density forecast μ_{th}, $m_{th} < 0$ by definition. (10) implies that

$$\begin{aligned}&(2\alpha_{th} - 1)\int_{0}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} - (1-\alpha_{th})m_{th} = 0\\ &\Rightarrow (2\alpha_{th} - 1)\int_{0}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} = (1-\alpha_{th})m_{th} < 0\end{aligned} \tag{11}$$

Since $\int_{0}^{\infty}x_{th}f_x(x_{th} \mid I_{th})dx_{th} > 0$, (11) implies that $2\alpha_{th} - 1 < 0 \Rightarrow \alpha_{th} < 0.5$.

Similarly, if the optimal point forecast f_{th} is more than the mean of the density forecast μ_{th}, we can show that $\alpha_{th} > 0.5$.

Theorem 3: *For the Linex loss function, symmetry of loss function is a sufficient and necessary condition for the equality of optimal point forecast and the mean of the underlying density forecast.*

Consider the following Linex loss function

$$L(y_t - f_{th}) = \exp[\alpha_{th}(y_t - f_{th})] - \alpha_{th}(y_t - f_{th}) - 1, \quad \alpha_{th} \neq 0$$

Assuming that we may interchange the expectation and differentiation operators, the first-order condition for the optimal point forecast, f_{th}, under the Linex loss function takes the form

$$E_{th}^{S}\left[\frac{\partial L(y_t - f_{th})}{\partial f_{th}}\right] = \alpha_{th} - \alpha_{th} E_{th}^{S}\left[\exp\{\alpha_{th}(y_t - f_{th})\}\right] = 0$$
$$\Rightarrow E_{th}^{S}\left[\exp\{\alpha_{th}(y_t - f_{th})\}\right] = 1 \qquad (12)$$

Since exp(.) is a convex function, by Jensen's inequality and assume that the density forecast of the target variable, y_t, is not degenerate, we have

$$1 = E_{th}^{S}\left[\exp\{\alpha_{th}(y_t - f_{th})\}\right] > \exp\left[E_{th}^{S}\{\alpha_{th}(y_t - f_{th})\}\right] \qquad (13)$$

Taking natural logarithm of both sides of (13), we obtain

$$E_{th}^{S}\left[\alpha_{th}(y_t - f_{th})\right] < 0 \Rightarrow \alpha_{th}\left[E_{th}^{S}(y_t) - f_{th}\right] < 0 \qquad (14)$$

From (14), we can easily prove that loss symmetry is both a sufficient and a necessary condition for the equality of point forecast and the mean of the density forecast. By (14)

$$\alpha_{th} > 0 \Leftrightarrow E_{th}^{S}(y_t) < f_{th} \quad \text{and} \quad \alpha_{th} < 0 \Leftrightarrow E_{th}^{S}(y_t) > f_{th}$$

Theorem 1 shows that, for Lin-Lin loss function, we can infer the direction of loss asymmetry by comparing the optimal point forecast and the median of the density forecast; Theorem 2 and 3 show that, for Quad-Quad and Linex loss function, we can infer the direction of asymmetry of the loss function by comparing the optimal point forecast and the mean of the density forecast.

4.2 Empirical Results

To compare the point forecasts and the mean or median of the underlying density forecasts for annual inflation and real output growth, we follow the method proposed by Engelberg, Manski and Williams (2009). They compare the point forecasts with the central tendency (mean, median and mode) of the underlying density forecasts in the SPF data. Their sample period is 1992Q1-2004Q4, excluding 1996Q1. They calculate the point forecasts for annual inflation rate and real output growth by using the annual level of output-price index and real output in the previous year (provided by the SPF surveys), and the point forecasts for the same two variables in the current year and the next year. These data are available from the SPF only since 1992Q1. As discussed in section 2, we calculate the point forecasts for annual inflation and real output growth by combining the SPF data and the real time macro data. This allows us to calculate the point forecasts for a longer sample period. To find the relationship between forecasters' point forecasts and the central tendency of their density forecasts, Engelberg, Manski and Williams (2009) employ both nonparametric analysis and parametric analysis. The nonparametric analysis does not assume density forecasts to have any specific shape but the parametric analysis assume that each density forecast has a Beta or isosceles-triangle shape. We will focus on the nonparametric analysis here. Engelberg, Manski and Williams (2009) notice that the SPF density forecasts report the subjective probabilities that real output growth or inflation will lie in given intervals. Thus, these forecasts do not fully reveal the subjective distributions that respondents hold and, hence, the central tendency cannot be calculated precisely. However, they do imply bounds on the subjective means and medians. By assuming that the mode is contained in the interval with the greatest probability mass, they also suggest a way to find the bounds on the mode.

Having computed the bounds on the central tendency of density forecast, Engelberg, Manski and Williams (2009) check if the point forecast lies within the bounds on the central tendency. If not, they reject the hypothesis that the point forecast is the central tendency examined. They also counted how many times the point forecasts are below the lower bounds and how many times the point forecasts are above the upper bounds. We apply their method to our longer sample period and the results are presented in Table 1. Our finding is similar to theirs. First, most point forecasts are consistent with the central tendency of density forecasts (falling within the bounds on the central tendency). But still for a significant fraction of observations, they are not. This fraction is usually between 5% and 25% and varies over forecast horizons and across different measures of the central tendency. Second, forecasters who skew their point forecasts tend to present rosy scenarios. For real output growth, forecasters are more likely to report a point forecast that is above the upper bound on the central tendency; for inflation, however, forecasters are more likely to report a point forecast that is below the lower bound on the central tendency. Engelberg, Manski and Williams (2009) do not provide an explanation for this phenomenon. A possible one may be that the associated loss functions are asymmetric. Based on section 4.1 and findings in table 1, we may infer that, for real output growth, the cost of underprediction may be higher than overprediction. As a result, forecasters tend to report an optimistic forecast. The opposite may be true for inflation. Third, Table 1 also shows that as the forecast horizon shortens, the point forecasts are more consistent

Table 1: Percentage of Point Forecasts Falling below Lower Bounds, inside Bounds and above Upper Bounds of Various Moments of Density Forecasts

Real Output Growth (1981Q3-2003Q4)			
	Mean	Median	Mode
4Q Ahead Forecast	421/0.04/0.82/0.14	557/0.10/0.73/0.17	555/0.06/0.85/0.09
3Q Ahead Forecast	495/0.04/0.84/0.13	660/0.07/0.78/0.15	656/0.04/0.87/0.08
2Q Ahead Forecast	518/0.04/0.87/0.08	616/0.09/0.82/0.09	614/0.05/0.89/0.06
1Q Ahead Forecast	570/0.02/0.94/0.03	646/0.08/0.87/0.05	645/0.04/0.92/0.04
Inflation (1968Q4-2003Q4)			
	Mean	Median	Mode
4Q Ahead Forecast	930/0.22/0.71/0.07	1110/0.25/0.66/0.09	1108/0.16/0.77/0.07
3Q Ahead Forecast	992/0.20/0.76/0.05	1200/0.20/0.73/0.07	1195/0.13/0.82/0.05
2Q Ahead Forecast	861/0.14/0.77/0.10	1008/0.16/0.70/0.13	1005/0.12/0.79/0.10
1Q Ahead Forecast	627/0.10/0.88/0.03	717/0.11/0.83/0.07	714/0.08/0.88/0.04

Note: For each entry, the first number is the total number of observations. The second number is the percentage of point forecasts falling below the lower bounds of mean/median/mode. The third number is the percentage of point forecasts falling between the lower bounds and upper bounds of mean/median/mode. The fourth number is the percentage of point forecasts falling above the upper bounds of mean/median/mode.

with the central tendency of the density forecasts. For example, for the real output growth forecasts, when the forecast horizon is 4 quarters, 82% of the point forecasts are consistent with the means of the density forecasts. As the forecast horizon falls to 1 quarter, this faction increases to 94%, implying that at longer forecast horizon, loss function is more likely to be asymmetric. Above preliminary analysis provides some information about asymmetry in loss functions (when the loss function is Lin-Lin, Linex or Quad-Quad) associated with the forecasts for real output growth and inflation, and can be used to check the validity of various methods of estimating loss function parameters.

5. Combining the Point Forecasts and the Density Forecasts to Calculate the Loss Function Parameters

5.1 Calculation of Common Loss Function Parameters

If we know the entire distribution of $f^S(y_t \mid I_{th})$, we can derive the entire distribution of $f^S(e_{th} \mid I_{th})$. We can then calculate the loss function parameters using equation (6). After some manipulation, we can show that the loss function parameter can be calculated as $\hat{\alpha}_{th} = F^S(f_{th} \mid I_{th}) = \frac{1}{2}\left[1 - E_{th}^S\left(\frac{e_{th}}{|e_{th}|}\right)\right]$ for Lin-Lin loss function, $\hat{\alpha}_{th} = \frac{1}{2}\left[1 - \frac{E_{th}^S(e_{th})}{E_{th}^S|e_{th}|}\right]$ for Quad-Quad loss function, and solution of $E_{th}^S\left[\exp(\alpha_{th}e_{th})\right] = 1$ for Linex loss function.

Since the SPF density forecasts are not continuous and forecasters report just the probabilities with which the target variables fall in different intervals, the calculated distributions of forecast errors are also histograms telling us the probabilities with which the forecast errors fall in different intervals. To calculate the loss function parameters, we could follow two methods: nonparametric analysis and parametric analysis as in Engelberg, Manski and Williams (2009). The nonparametric analysis does not assume the probability distributions of the target variables and forecast errors to have any specific shape. As a result, we cannot calculate the loss function parameters exactly. However, for Lin-Lin loss function we can calculate bounds on the loss function parameter[4]. The calculation of bounds on the Quad-Quad and Linex loss function parameter is more complicated and not considered in this paper.

We also do parametric analysis by making some assumptions about the distributions underlying the histograms of the target variables, or equivalently the forecast errors. We considered two types of distributions: (1) Assume within each interval, the probability falls on the midpoint of that interval; (2) The underlying distributions of the target variables, or equivalently the forecast errors are normal. For the first type of distributions, the calculation of the moments of the forecast errors (and exponential of forecast errors) is straightforward. For the normal distribution, it can be shown that the estimated $\hat{\alpha}_{th}$ depends only on the mean and variance of the subjective density of the forecast error for the Lin-Lin, Quad-Quad and Linex loss functions[5]. For example, suppose that $f^S(y_t \mid I_{th}) = N(\mu_{th}, \sigma_{th}^2)$ where μ_{th} and σ_{th}^2 are the mean and variance of the density forecast of y_t when forecast horizon is *h*. Then the subjective density of the forecast error is $f^S(e_{th} \mid I_{th}) = N(b_{th}, \sigma_{th}^2)$, where $b_{th} = \mu_{th} - f_{th}$ is the expected forecast bias. Then the estimated $\hat{\alpha}_{th}$ for Lin-Lin loss function is

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[4] Since $\hat{\alpha}_{th} = F^S(f_{th} \mid I_{th})$, if the point forecast f_{th} falls within the interval [a, b], the lower bound of $\hat{\alpha}_{th}$ would then be $F^S(a \mid I_{th})$ and the upper bound of $\hat{\alpha}_{th}$ would be $F^S(b \mid I_{th})$.

[5] For more general distribution and loss function, higher order moments are also needed.

$$\hat{\alpha}_{th} = F^S(f_{th} \mid I_{th}) = \Phi\left(\frac{f_{th} - \mu_{th}}{\sigma_{th}}\right) = 1 - \Phi(b_{th}/\sigma_{th}) \tag{15}$$

where, $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. For Quad-Quad loss function, the estimated $\hat{\alpha}_{th}$ is

$$\begin{aligned}\hat{\alpha}_{th} &= \frac{1}{2}\left[1 - \frac{E^S_{th}(e_{th})}{E^S_{th}\left|e_{th}\right|}\right] = \frac{1}{2}\left[1 - \frac{b_{th}}{2\sigma_{th}\phi(b_{th}/\sigma_{th}) + b_{th}\left[2\Phi(b_{th}/\sigma_{th}) - 1\right]}\right] \\ &= \frac{\sigma_{th}\phi(b_{th}/\sigma_{th}) + b_{th}\Phi(b_{th}/\sigma_{th}) - b_{th}}{2\sigma_{th}\phi(b_{th}/\sigma_{th}) + b_{th}\left[2\Phi(b_{th}/\sigma_{th}) - 1\right]} = \frac{D_{th} - b_{th}}{2D_{th} - b_{th}}\end{aligned} \tag{16}$$

where $D_{th} = \sigma_{th}\phi(b_{th}/\sigma_{th}) + b_{th}\Phi(b_{th}/\sigma_{th})$.

For Linex loss function, since e_{th} is normally distributed, $\alpha_{th}e_{th}$ is also normally distributed as $f^S(\alpha_{th}e_{th} \mid I_{th}) = N(\alpha_{th}b_{th}, \alpha^2_{th}\sigma^2_{th})$. $\exp(\alpha_{th}e_{th})$ then follows a lognormal distribution with mean $\exp\left(\alpha_{th}b_{th} + \frac{1}{2}\alpha^2_{th}\sigma^2_{th}\right)$. Based on the moment condition (12) $E^S_{th}\left[\exp(\alpha_{th}e_{th})\right] = 1$, we have

$$\exp\left(\alpha_{th}b_{th} + \frac{1}{2}\alpha^2_{th}\sigma^2_{th}\right) = 1 \Rightarrow \alpha_{th}b_{th} + \frac{1}{2}\alpha^2_{th}\sigma^2_{th} = 0 \Rightarrow \alpha_{th} = -\frac{2b_{th}}{\sigma^2_{th}} \tag{17}$$

5.2 Empirical Results

Figure 1 and Figure 2 show the distribution of calculated Lin-Lin loss function based on (15) across forecasters by forecast horizon[6]. Figure 1 is for real output growth forecasts. Figure 2 is for inflation forecasts. As shown in the figures, for inflation, there are more forecasters with the loss function parameter to be less than 0.5, which means that overprediction is more costly than underprediction. For real output growth, there are more forecasters with the loss function parameter to be more than 0.5, which means that overprediction is less costly than underprediction. Figure 1 and Figure 2 do not show clearly that the loss functions are more likely to be symmetric as the forecast horizon shortens. But note that the estimation is based on the assumption that the density forecasts are normal. This assumption may be not valid, see Lahiri and Teigland (1987).

[6] The results for Quad-Quad and Linex loss function, and the assumption that probability mass falls on the midpoint of each interval are similar and not reported here to save space.

Figure 1. Distribution of Estimated Lin-Lin Loss Function Parameter across Forecasters (Real Output Growth)

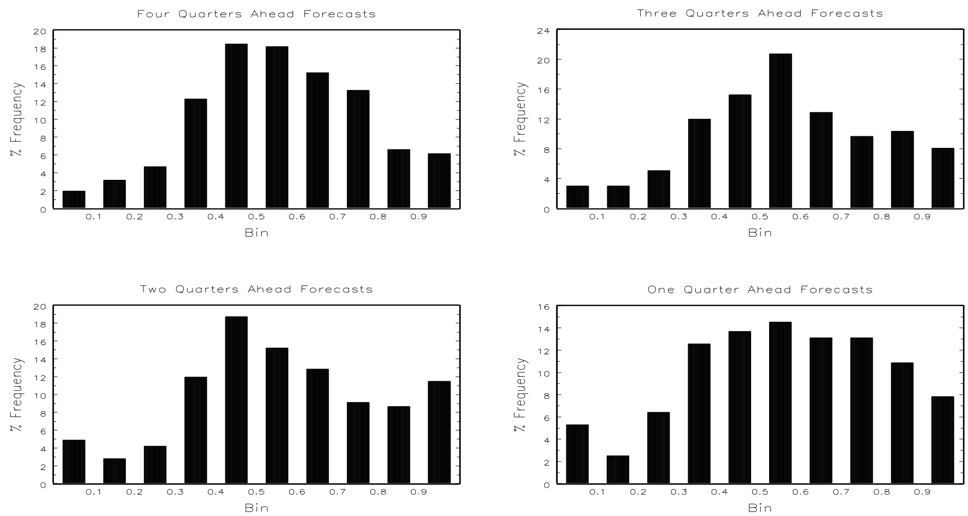
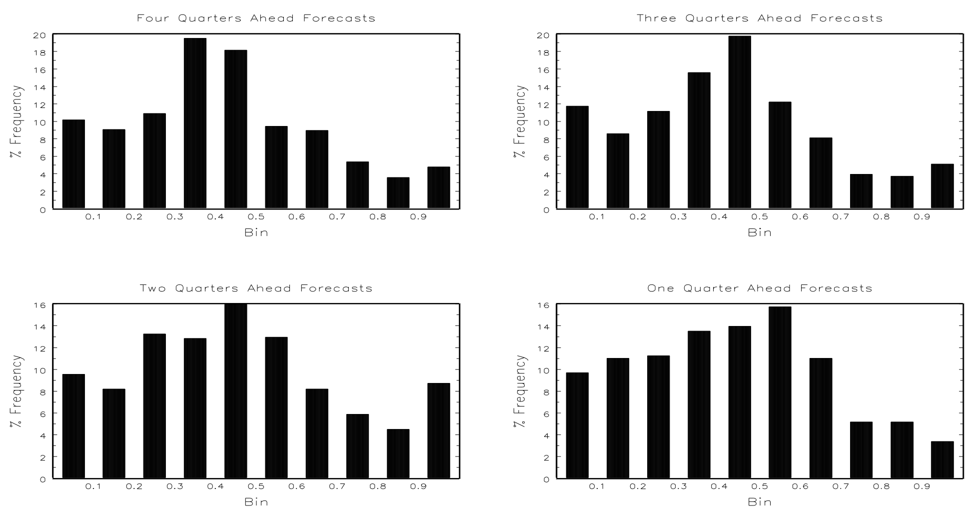


Figure 2. Distribution of Estimated Lin-Lin Loss Function Parameter across Forecasters (Inflation)



Alternatively we conduct a nonparametric analysis for Lin-Lin loss function as discussed in section 5.1. The result of this analysis is reported in table 2. We counted how many times the calculated bounds on loss function parameters cover 0.5 -- the value for symmetric loss function, and how many times 0.5 is less than the lower bounds and how many times 0.5 is larger than the upper bounds. Our findings are summarized as follows. First, in most cases, the bounds cover 0.5. But still for a significant fraction of observations, they do not, implying asymmetric loss functions. This fraction is usually between 10% and 25% and varies over forecast horizons. Second, for real output growth, there are more cases that 0.5 is less than the lower bounds than the cases that 0.5 is larger than the upper bounds. This means that forecasters are more likely to have a loss function parameter larger than 0.5. For inflation, the opposite is true. This finding is consistent with what we found in section 4. Third, table 2 also shows that as forecast horizon shortens, bounds are more likely to cover 0.5, or in other words, loss functions are more likely to be symmetric. For example, for real output growth forecasts, when the forecast horizon is 4 quarters, 82% of bounds cover 0.5. As the forecast horizon falls to 1 quarter, this fraction increases to 90%. This is consistent with our finding in section 4.2.

Table 2: Percentage of 0.5 Falling below Lower Bounds, inside Bounds and above Upper Bounds of Lin-Lin Loss Function Parameter

Real Output Growth (1981Q3-2003Q4)			
4Q Ahead Forecast	3Q Ahead Forecast	2Q Ahead Forecast	1Q Ahead Forecast
407/0.12/0.82/0.06	434/0.12/0.84/0.05	426/0.07/0.89/0.04	358/0.04/0.90/0.06
Inflation (1968Q4-2003Q4)			
4Q Ahead Forecast	3Q Ahead Forecast	2Q Ahead Forecast	1Q Ahead Forecast
834/0.06/0.77/0.17	859/0.05/0.80/0.15	733/0.09/0.79/0.12	444/0.05/0.87/0.08

Note: For each entry, the first number is the total number of observations. The second number is the percentage of cases that 0.5 falling below the lower bounds of loss function parameters. The third number is the percentage of cases that 0.5 falling between the lower bounds and upper bounds of the loss function parameters. The fourth number is the percentage of cases that 0.5 falling above the upper bounds of the loss function parameters.

6. Conclusion

In this paper, we consider how to use information from density forecasts to recover the loss function parameters. We prove that for Lin-Lin, Linex and Quad-Quad loss functions we could infer about the existence and direction of asymmetry by comparing the point forecasts with different measures of central tendency of the underlying density forecasts. When we know the entire distribution of the density forecast, we can calculate the loss function parameters based on the first order condition of forecast optimality. This method is applied to forecasts for annual real output growth and inflation obtained from the Survey of Professional Forecasters (SPF). We find that forecasters treat underprediction of real output growth to be more costly than overprediction; the reverse is true for inflation. Thus, for both variables, forecasts tend to be optimistic. In addition, as forecast horizon shortens, loss functions are more likely to be symmetric.

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