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PRODUCTION ANALYSIS: ECONOMETRIC ISSUES

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This paper addresses some econometric issues that arise in production analysis. The first section presents the basic analytical framework in analyzing production decisions. The discussion serves as a background for the three subsequent sections which deal with the following issues: model selection, the use of panel data, and bias in nonrandom sampling.

Methodological Foundation

There are essentially two approaches in econometric productivity analysis, namely, the primal and the dual approach. The first approach is based on the idea that, given a transformation or production function and the assumption of profit maximization or cost minimization, factor demand and output supply may be derived based on the necessary conditions of optimization. While this approach may be easily worked out from simple production functions (e.g., factor demands are straightforwardly obtained from the Cobb-Douglas production function), it may not be workable when the production technology is complex.

The second approach provides an alternative in obtaining relationships describing producer behavior. By appealing to duality theory and the concept of cost and profit functions, it enables one to derive a system of input demand and output supply quite readily while preserving the complexity of the structure of the producer decisionmaking

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process. In the context of production studies, the duality theory provides the theoretical background that makes it possible to recover information about the production structure from a dual profit or cost structure. It theorizes that optimal input and output levels implied by the technical transformation function and the necessary conditions are equivalently obtained by the optimization of a dual profit or cost function. It ensures that when certain restrictions hold for the optimized dual function, they also hold for the transformation function. The general development of the application of duality theory and the concept of profit and cost functions is given in Section II of the overview paper. An application of these developments is illustrated as follows. A system model may be constructed for a farmer producing one output, say rice, by applying four variable factors of production, say labor, fertilizer, animal power and tractors. Consider that this farmer produces under perfect competition in both input and product markets. Assume that

$$g\,(Y, X_1, X_2, X_3, X_4\, S) = 0$$

is the technical transformation function that represents the farmer's production technology, where Y represents output, X_i represents the ith input, i = 1,2,3,4 and S is a vector of variables representing the structure of the farm as may be measured by size of farm, proportion of area planted with high-yielding varieties, irrigation investment, past investment in research and extension, and degree of rural electrification. Diewert (1974) proposed that, if the transformation function obeys certain regularity conditions, it is possible to obtain the maximum profits in production as

$$\pi = \pi\;(P,\; W_1,\, W_2,\; W_3,\, W_4,\;\, S)$$

where P is the price of the product and W_i, i = 1,2,3,4 are observed factor prices. This profit function can be viewed as a solution to the following constrained maximization problem

$$\pi\,(P, W, S) = \max\,(PY - W_1X_1 - W_2X_2 - W_3X_3 - W_4X_4 : \\ g(Y,\; X_1 \ldots X_4,\, S) = 0\,).$$

In this formulation it is assumed that, given an exogenously determined vector of input and output prices, farmers will choose the level of output and factor combination that maximizes profits subject to the constraint presented by the production technology. This is the basis of the concept of duality between maximized profits and the technical transformation function.

Given the maximized profits function, the application of the Shephard-Hotelling lemma (Varian 1978) produces a system of equations representing the output supply and input demand functions. According to the lemma, optimal input and output levels are obtained by differentiation of the profits function with respect to output and input prices, i.e.,

$$Y = \partial\pi / \partial P$$

$$X_i = \partial\pi / \partial W_i, \; i = 1,2,3,4.$$

Note that the maximized profit, π, are expressed as a function of prices and structure variables. The expression does not include endogenous or choice variables such as output quantities and variable factors of production. These are eliminated from the expression by substitution of "first-order conditions." Consequently, the derived input demand and output supply equations are expressed as a function of prices and structure variables.

The methodology described above has the following useful consequences. First, the procedure takes into account the nature of farmers' production decisions whereby choices of individual factor input and output levels are interdependent. The interrelationship among factor and product markets is captured in the derived system so that it provides a basis for a formal analysis of the simultaneous effects of policy changes on product supply and factor demand decisions. Second, the duality principle has allowed us to move from a transformation function which is a function of endogenous variables (quantities) to a profit function which is a function of exogenous variables (prices, fixed factors). As a consequence, simultaneity problems are avoided. Furthermore, multicollinearity problems are reduced since there is less covariance in prices than in quantities. Lastly, the functional forms for the systems can be linear and economical in parameters and can still be "flexible" in the sense that they do not impose some restrictions on the "curvature" of the primal functions.

Parameter estimates for the system of equations representing output supply and demand for various factors may be obtained by applying iterative versions of seemingly unrelated regressions (Zellner 1962;

Gallant 1975). Each equation in the system looks like an individua regression but the variances of parameters are reduced by simultaneously estimating the system of equations so that error interdependence is taken into account. The basic assumption in this procedure is related to serial independence and contemporaneous correlation of the residual terms. If e_{tj} is the residual term corresponding to the observation t in the regression j, and $e_{t'j'}$ is the same for observation t' in equation j', the following implication will be true with respect to the variance-covariance matrix

$$E(e_{tj}, e_{t'j'}) = 0 \text{ if } t = t'$$

$$\sigma_{jj}, \text{ if } t = t'$$

This method is an extension of the generalized least squares procedure. It is preferred because it permits convenient imposition of across equation restrictions.

Examples of studies that used the above methodology are Lau and Yotopoulos (1972), Yotopoulos, Lau and Lin (1976), Sidhu and Baanante (1979, 1982), Quizon (1980), and Kalirajan and Flinn (1983).

The Problem of Model Selection

One issue that immediately comes up in the empirical application of duality theory is the choice of functional form for the statistical model of the unknown cost or profit functions. Statistically, we seek a form which gives close approximations to the parameters of interest. The problem is to determine which mathematical formulation can provide the flexibility required on the parameters of the function. To this end, the use of functional forms that are flexible, meaning that the parameters can take arbitrary values so that it does not necessarily impose restrictions on the curvature of the production technology, is widespread.

Flexibility is based on approximation theory. Two methods of approximation are usually used: Taylor series approximation and Fourier series approximation. In fact, the functional forms are referred to as either locally flexible (Taylor series) or globally flexible (Fourier series) according to the method of approximation used.

The notion of "local flexibility" in production analysis originated from Diewert's (1974) suggestion of the use of a second-order local approximation, say g, to the true function g^* at some price P^* as satisfaction of the following conditions:

$$g(P^*) = g^*(P^*)$$

$$\left.\frac{\partial g(P)}{\partial P_i}\right|_{P^*} = \left.\frac{\partial g^*(P)}{\partial P_i}\right|_{P^*} \qquad \forall_i$$

$$\left.\frac{\partial^2 g(P)}{\partial P_i\, \partial P_j}\right|_{P^*} = \left.\frac{\partial^2 g^*(P)}{\partial P_i\, \partial P_j}\right|_{P^*} \qquad \forall_{i,j}$$

These conditions can be understood in two ways. In the first place, it can be taken to mean that the first- and second-order derivatives of the actual function, $g^*(P^*)$, are equal to the derivatives of the approximating one, $g(P^*)$,, at the point P^*, for example, the point of profit maximization. The alternative criterion is that, in addition to the equality of derivatives at point P^*, the deviation between $g^*(P^*)$ in a defined neighborhood about the point P^* consists of only terms of the third or higher order. This implies that any second-order Taylor series expansion about point P^* is a second-order approximation, as the deviation is bounded by the remainder term. In general, functions that can be considered second-order Taylor series about the point of profit maximization are in fact second order approximations to the underlying flexible aggregator function at that point by either of the two meanings of the term given above (Lau 1974, pp. 183-184; Fuss et al. 1978, pp. 233-234).

It is important to note that Diewert's definition of flexibility does not require a quadratic form or a Taylor approximation — only a twice differentiable expression. The flexible forms commonly used for the dual relationships are themselves nonlinear but have linear derivatives. Forms with linear derivatives are not generally "globally convex," i.e., convex at all possible data points, but are convex over certain ranges. In practice, however, the most commonly used locally flexible forms are essentially quadratic forms and take a Taylor series interpretation. A description of selected functional forms is given in the appendix. For example, take the translog cost function i.e.

$$lnC = \beta_O + \Sigma_i \beta_i\, ln W_i + (1/2)\, \Sigma_i\, \Sigma_j\, \beta_{ij}\, ln\, W_i\, ln\, W_j.$$

This may be viewed as a second-order Taylor series approximation of some unknown cost function by comparing the above form with the following.

$$lnC = lnC\Big|_{lnWo} + \Sigma_i \frac{\partial lnC}{\partial lnW_i}\Big|_{lnWo} (lnW_i - lnW_o)$$

$$+ (1/2)\, \Sigma_i \Sigma_j \frac{\partial^2 lnc}{\partial lnWi\, \partial lnW_j}\Big|_{lnWo} (lnW_i - lnW_{io})\, (lnW_j - lnW_{jo}).$$

The standard practice is to regress *ln* of cost on *ln* of prices; regression coefficients are then interpreted as the coefficients in the Taylor series, i.e.,

$$lnC = \beta_o$$

$$\frac{\partial lnc}{\partial lnW_i} = \beta_i$$

$$\frac{\partial^2 lnc}{\partial lnW_i\, \partial lnW_j} = \beta_{ij}$$

A critique of the use of the Taylor series expansion as a basis for the construction of flexible functional forms is made by Gallant (1981). The argument is on two levels. In the first place, the Taylor expansion is only an approximation over a nonspecified (unknown and unknowable) region — its properties as one moves away from the point of approximation cannot be ascertained with any degree of confidence. In the second place, the econometric properties of the Taylor series are weak (Gallant 1981, p. 212).

> Taylor's theorem fails rather miserably as a means of understanding the statistical behavior of parameter estimates and test statistics. If one insists on using Taylor's theorem as a means of understanding statistical behavior one is led into an algebraic morass. The reason for this failure is that the statistical regression methods essentially expand the true function in a (general) Fourier series — not in a Taylor series. Due to this fact, Fourier series permit a natural transition from demand theory to statistical theory. The key fact which permits this transition is that the classical Fourier sine/cosine series expansion approximates not only the function to within arbitrary accuracy . . . but also its first derivatives.

Other authors criticize the practice (White 1980 and Barnett 1976, among others) by taking issue on the actual behavior of a Taylor series approximation in a regression setting. On the one hand, Taylor series approximants apply only locally about a point or about its neighborhood. The approximating error from Taylor series approaches zero as the point of approximation is reached but can become quite large away from the point of approximation. On the other hand, regression methods attempt to make errors in approximation small over the range of data, that is, the function is usually not fitted about a specific point, but over a domain or a constructed sample mean. This may entail that the function in fact is not a second-order approximation at any particular point. The relationship between the real and fitted functions, therefore, will contain an element of uncertainty as to the interpretation of some comparative static results. So, while the approximation method applies locally about a point, the estimation procedure applies globally over the range of data. The implication is that estimates obtained in this manner do not consistently approximate the parameters of the true function (unless the latter is of the approximating form). Elasticities obtained are as a consequence also inconsistent. If the true function is actually of the translog form, for example, then there is no problem; and the procedure is straightforward curve fitting of the cost function. If not, then one interprets the flexible functional form as an approximating function – the coefficients of which are interpreted as the derivatives in Taylor series expansion. As pointed out by Gallant, the empirical function is constructed to be an approximation about a certain point of the true aggregator function, but it may not maintain that desirable property as we move away from the original point of approximation. For other possible sets of positive finite price vectors outside a narrow range about the original, such crucial properties as convexity and monotonicity may fail. Since modelling implies extrapolating outside this narrow range, the possible loss of approximating characteristics in the empirical analyses may weaken the quantitative and perhaps even the qualitative results obtained.[1] Under certain conditions, the fitted function

1. For a discussion of these points, see Lau (1978), where he discusses the problems of convexity and monotonicity for the three functions forms used in this study and develops tests that can be used ex post. The article by Douglas Caves and Laurits Christensen (1980) cites the lack of theoretical discussion of global properties of the flexible functional forms, since the literature has so far only referred to

may in fact reject a test for the symmetry constraints, even though the constraint may be true for the actual function (Fuss, McFadden, and Mundlak 1978, pp. 233-234).[2]

These observations led to an alternative approach to the approximation problem: the Fourier form (Gallant 1981). It uses the Sobolev norm as the appropriate measure of approximating error. The Sobolev measure of the distance of an approximating function, $g(x)$, from the true $g^*(x)$ is

$$\|g^* - g\|_{m,P,w} = \Big(\sum_{|u|^* < m} \int |D^u g^* - D^u g|^P dw(x)\Big)^{(1/P)}, \quad 1 < P < \infty$$

where m denotes the largest order derivative of g which is of interest, D^u denotes partial differentiation, $w(x)$ is a distribution function, and u is the region of approximation. This means that, if g approximates g^* poorly or poorly approximates any of its derivatives to the order m, then the Sobolev norm will assign a large value to the approximation error (Gallant 1984).

The relevance of the use of the Fourier form in production analysis is the interest in approximating not only the unknown function but the first and second derivatives as well. For example, an estimate of the measure of the following cross elasticity of substitution

$$0 = 1 + \frac{\dfrac{\partial^2 C}{\partial W_i \, \partial W_j}}{\dfrac{\partial C}{\partial W_i} \quad \dfrac{\partial C}{\partial W_j}}$$

empirical tests of the various forms now currently being used (see, for example, Ernst Berndt and Mohammed Khaled 1979). Caves and Christensen carry out a discussion of these properties for the translog and linear functions as the regions over which the regularity conditions hold in the two cases are different. The discussion is carried out with respect to the variation in price elasticities and substitution, and give a graphical exposition of the comparison of these forms in the two and three commodity homethetic and nonbomothetic cases.

2. If the function is in fact fitted about a single on the actual function, and not at an estimated mean, then the approximating functions should confirm the symmetry condition and any other constraints which the actual function fulfills. See Lau (1978, pp. 409-11, 418-20).

(where C represents a cost function $C = C(W)$ and W is a vector of prices) is of economic interest. One can see from this expression that it is not enough that the cost function, C, approximate the true cost function closely; the first and second derivatives of the approximating function must also approximate the first and second derivatives of the true cost function closely. The Sobolev measure of approximation error takes into account also the need to approximate derivatives.

The Fourier series expansion has the global property of approximating a function throughout its domain. This lends itself naturally to the usual practice of running a regression to estimate parameters because now both the method of approximation and estimation procedure are global. Gallant (1981) shows that estimation based on a Fourier series produces consistent estimates of the parameters of interest.

There have been questions about how good are the approximations that these flexible forms provide. Guilkey and Lovell (1980) and Guilkey, Lovell and Sickles (1982) have shown that, if a translog approximation has elasticities of substitution that depart from unity, the quality of the form deteriorates markedly. Guilkey and associates explain that there is a tradeoff between flexibility and global behavior:

> When selecting a functional form for use in empirical work, one is confronted by a choice between forms that exhibit global behavior and those that possess flexibility.

The use of flexible functional forms based on Taylor's expansions implies the possibility of not satisfying the regularity conditions globally unless the true function is identical to the chosen representation. If one wants models that are flexible, there is the risk of having forms that do not obey those conditions over all observations. One good example of this tradeoff can be demonstrated by the translog specification. A sufficient condition for global convexity is to have all second-order coefficients equal zero; however, this leads to an inflexible Cobb-Douglas functional form.

With respect to the use of Fourier forms, substantial tradeoffs between **bias** and **instability** have been indicated. Chalfant (1984) shows that the Fourier form, with desirable properties concerning bias, features much greater oscillation in estimated elasticities than does the generalized Box-Cox (which nests the generalized Leontief and generalized square-root quadratic forms as special cases and the translog as a limiting case). Further research is called for to adjust estimation procedures and model specification to improve the attainable levels of unbiasedness and stability.

On the Use of Panel Data

A data base that uses time-series and cross-section information is oftentimes useful in production analysis. They are important to analysts because they contain information that is necessary to deal with both intertemporal dynamics and individual effects of entities being investigated. In addition, time-series information alone or cross-section information alone may not provide enough variation or is insufficient to provide enough degrees of freedom for fitting especially in cases where more complicated flexible functional forms are involved. Pooled time-series and cross-section information (panel data) may thus make econometric estimation feasible.

The most common approach in the analysis of panel data is the use of the covariance or dummy variable model. The basic approach is to identify cross-sections by dummy variables and then to apply the generalized least squares estimation method (Zellner 1962) to obtain the parameter estimates. This method is computationally appealing and yields asymptotically unbiased and consistent estimators (Wallace and Hussain 1969). In principle, the weakness of covariance approach lies in the fact that inter-cross-section variation is ignored, leading to inefficient estimation when this variation is known (Fuss, 1977). For a complete review of the current statistical methodology on dealing with pooled cross-section and time-series data, see Dielman (1983).

One item of serious concern in empirical work is the problem of errors in variables. Consider the equation,

$$Y_{it} = \mu_i + (\beta X_{it} - \beta \nu_{it}) + n_{it} = \mu_i + \beta X_{it} + e_{it} \quad (1)$$

where μ_i's are unobserved individual effects;

ν_{it} is an i.i.d. measurement error;

and n_{it} are the standard best case disturbance term $\sim (0, \sigma^2\, I)$.

Parameter estimates obtained from using OLS estimation of the above equation which involves erroneously measured right-hand variables will be biased for two reasons: (1) because of the correlation of the X_{it} with the left-out variable effects; and (2) because of the negative correlation between the observed X_{it} and the composite disturbance term $(n_{it} - \beta \nu_{it})$.

Griliches and Hausman (1984) proposed a clever way of identifying the magnitude of the "true" parameter in a panel data context. The key idea is that there are different ways of transforming the data to eliminate the first source of bias; and that different transformations imply different and deducible changes for the magnitudes for the second type of bias. Such changes can be used in identifying the magnitude of the true parameters. They propose the use of difference models of the form.

$$dy = d X \beta - d v \beta + d n = d X \beta + de \quad (2)$$
$$= d^2 y = d^2 X \beta - d^2 v\beta + d^2 n = d^2 X\beta + d^2 e$$
$$d^m y = d^m X\beta - d^m v\beta + d^m n = d^m X\beta + d^m e$$

where

$$d^n yt = Y_{it} - Y_{it-n}$$

to obtain implicit estimates of the bias. For panel data with a time series of length T, one can construct $T/2$ difference models from which there are $T/2$ independent consistent estimates. The strategy is to take advantage of these alternative consistent estimates, they are compared to obtain estimates of bias.

The general procedure suggested is given in three steps. First, estimate equation (1) by generalized least squares (variance components) and by within estimation. Do a test for equality of the estimates using a Hausman (1978) or Hausman-Taylor (1981) type test. Second, if you reject the hypothesis regarding the equality of estimates, then calculate some differenced estimates by OLS. If they differ significantly, errors in measurement may well be present. A joint test of all the differenced estimates can be made by using GLS on the system of equations in (2). Lastly, estimate the equations in (2) using the instrumental variable technique. In this respect, the X's provide the instrumental variables for each equation where all X's not involved in the difference are used as instruments. Then do a specification test of the no correlation assumption in the errors in measurement. If the different instruments of β do not differ significantly, then the magnitude of the true parameter estimate is obtained. If they do differ significantly, one of the following is called for: the specification of specific correlated errors in measurement process, the use of outside instruments, or the respecification of the original model.

Bias from Non-Random Sampling

Randomization in the selection of sample units ensures two things. It eliminates subjective selection bias and provides a basis for statistical inference. This process, however, is oftentimes not realized in the implementation of practical survey work. In some cases, sampling units are simply selected for convenience without randomization. In other cases, as in the case of complex surveys, it is economical to select existing natural groupings of observations. These are characterized by relative homogeneities within the groups that negate the assumption of independence of sample elements.

In a regression setting, the situation is equivalent to the failure of satisfying the assumption of independence betweeen observations and the bias in parameter estimates that follows. Consider a view of the problem taken by Kish and Frankel (1974):

> (i) Consider a finite population of size N. Associated with each of these elements is a vector of $k + 1$ values Y_i , X_{li} , S_{2i} , ... , X_{ki}; $P((Y_i, X_{li}, X_{2i}, \ldots, X_{ki}))$. (ii) Our parameters are numbers B_j such that $\Sigma_i^N (Y_i - \Sigma_j^k \beta_j X_{ji})^2$ is minimum subject to $\Sigma_i^N (Y_i - \Sigma_j^k \beta_j X_{ji}) = 0$. (iii) Given a sample of N vectors from the population of N vectors, our desire is to estimate the parameter β_j.
>
> The regression model stated in (ii) does not in practice correspond exactly (or even closely) to the complex relationship among the actual population of vectors. The error term measures (usually in a least-squares sense) the extent to which the model departs from the actual complex relations among the population of vectors.
>
> Statistical theory of regression assumes a basic structure of relationships. Letting $X_i = (X_{li}, \ldots, X_{ki})$ and $\beta = (\beta_l, \ldots, \beta_k)$, it uses the model $Y_i = X\beta + e_i$ and then makes several strong assumptions:
>
> A. linearity
>
> B. homoscedasticity
>
> C. independence between observations
>
> D. normality for the e_i.
>
> Assumptions (A), (B), and (D) concern the basic structure of the universe of the model, whereas (C) involves independent selections from it. This (or a similar) well-specified model yields desirable results; the standard least squares b are minimum variance, linear, unbiased, normal, etc. Literature and textbooks are written about this pretty model; this is what researchers find but they find very little to reconcile this model with the real population they are investigating.

Specifically, assumption (C) fails to hold when correlation between observations is induced by the sampling scheme or sampling implementation.

Recognizing the problem as such, it looks like the correction for bias may be implemented by standard econometric methodology.

Directions in tackling the problem are indicated by the studies of Holt, Smith and Winter (1980) and Kish and Frankel (1974). In Holt, Smith and Winter, they assume $Y = X\beta + e$; $E(e/X) = 0$; $B = (X'X)^{-1} (X'Y) = \beta + 0 (1/\sqrt{N})$ in a finite population of size N, where β is the corresponding parameter on the infinite population from which N came. If (π_i, $i = 1, \ldots, N$) denote the inclusion probabilities, they find that an element of $\Sigma_i^N X_{ij} X_{ik}$ of $X'X$ and $\Sigma_i^N X_{ij} Y_i$ of $X'Y$ have probability weighted estimates, $\Sigma X_{ij} X_{ik} / \pi_i$ and $\Sigma X_{ij} Y_i / \pi_i$. Hence a probability weighted estimator of β is

$$b^* = (X'D'X')^{-1} \quad (X'D^{-1} y)$$

where
$$D = \begin{bmatrix} \pi_1 & & \\ & \pi_2 & 0 \\ & 0 & \\ & & \pi_n \end{bmatrix}$$

In the case of Kish and Frankel, they ...nd

$$b = \frac{\Sigma_{js} X_j Y_j / \pi_j}{\Sigma_{js} X_j^2 / \pi_j}$$

where π_j is defined as selection probabilities instead of inclusion probabilities.

The direction that is suggested by the above studies is to correct the bias analytically by incorporating known or assumed probabilities of selection of sampling units. Econometrically, this may be implemented in two ways: (1) reparameterize the model to integrate the available information; and (2) incorporate the information into the error structure of the model.

For example, for the model $Y = X\beta + e$, use the fact that, for every positive semidefinite matrix D, there exists a matrix P such that $(P'P)^{-1} = D$. Thus, one can construct a reparameterization matrix P

which integrates information about the sampling scheme, e.g., known or assumed selection probabilities. In the problem considered by Holt et al. (1980) and Kish and Frankel (1974), the matrix P may be constructed as

$$P = \begin{bmatrix} \frac{1}{\sqrt{\pi_1}} & & 0 \\ & \frac{1}{\sqrt{\pi_2}} & \\ 0 & & \frac{1}{\sqrt{\pi_n}} \end{bmatrix}$$

so that the model may be reparameterized as

$$Y^* = X^*\beta + e^*$$

where $Y^* = PY$
$X^* = PX$

and $$(P'P)^{-1} = D = \begin{bmatrix} \pi_1 & & \\ & \pi_2 & 0 \\ 0 & & \\ & & \pi_n \end{bmatrix}$$

In effect, the observation matrix is weighted by the inclusion probabilities. With the reparameterized model, GLS estimation would give

$$\beta = (X' D X)^{-1} \quad (X'D Y).$$

The author notes that, in the studies cited, no explicit derivations or explanations were given. The results, however, are consistent with such reparameterization.

APPENDIX

The most commonly used flexible functional forms used in studies to date are the transcendental logarithmic function (translog), the normalized homogenous quadratic function, the generalized Leontief or linear function, the Generalized Box-Cox, and the Fourier flexible form.

A. The Translog Function

This function was first introduced by Christensen. Jorgenson and Lau (1973) and developed further by Lau (1978)[1]. The unrestricted profit function form is written as

$$ln \pi = \Sigma_i b_i \, ln P_i + \Sigma_i \Sigma_j b_{ij} \, ln P_i \, ln P_j$$

Since we are concerned with the short-run profit maximizing behavior of farmers, we want to use the restricted form. In addition, certain restrictions have to be imposed to attain the property of linear homogeneity in prices. It has been shown that the translog function is linearly homogeneous if and only if (Diewert 1974, p. 139):

$$\Sigma_i b_i = 1, \text{ and}$$
$$\Sigma_i b_{ij} = 0 \text{ (and by extension } \Sigma_j b_{ij} = 0)$$

The restricted translog function is then given by

$$ln \, \pi = \Sigma_i b_i \, ln P_i + \Sigma_i \Sigma_j b_{ij} \, ln P_i \, ln P_j + \Sigma_i \Sigma_k b_{ik} \, ln P_i Z_k + \Sigma_i b_{it} \, t \, ln P_i$$

It turns out that this function is globally convex only if all $b_{ij} = 0$, that is, the function degenerates to a Cobb-Douglas form with unitary elasticity of substitution and constant returns to scale (Lau 1974, pp. 182-83; Lau 1978 , p. 240; Diewert 1974, p. 115). In most cases, however, it is possible to obtain a set of constraints on the b_{ij} coefficients which leaves the function locally convex.

When deriving the input demand and output supply functions in the translog case, the functions are expressed in value-share form, since

$$\partial ln \, \pi / \partial ln P_i = X_i (P_i / \pi_i) = S_i$$

The sum of the value shares comes to zero since both inputs and outputs are included so only $(n\text{-}1)$ equations are linearly independent. Using the homogeneity constraint we therefore have:

$$\partial \, ln \, \pi / \partial \, ln P_i = S_i = b_i + \Sigma_j^{n-1} b_{ij} (ln P_i - ln P_j) + \Sigma_k b_{ik} Z_k + b_{it} t$$

1. In some instances, all the three functional forms given here are written with a constant term. This is, strictly, not correct, since the functions then are no longer linearly homogeneous with respect to prices — some exogenous income is assumed.

The coefficients b_{ij} do not have any explicit economic interpretation, but allow for easy computation of the price elasticites we want. In general we have[2]:

$$n_{ij} = \pi_{ij} (P_j / X_i)$$

In the translog case, the cross price elasticities (CPE) and own price elasticities (OPE) therefore become:

$$n_{ij} = b_{ij} / S_i + S_j \qquad \text{(CPE)}$$
$$n_{ij} = b_{ii} / S_i + S_i - 1 \qquad \text{(OPE)}$$

Because of the fact that value shares must sum to zero, S must be estimated residually.

B. The Normalized Homogenous Quadratic Function

The normalized homogeneous quadratic function has the homogeneity constraint imposed by normalizing profits on the nth commodity. This commodity can be either an input or output – the normalization does not affect the optimum levels of inputs and outputs derived via Shephard's lemma. The function in its unrestricted form reads:

$$\pi / P_n = \Sigma_i^{n-1} b_i (P_i / P_n) + (1/2) \Sigma_i^{n-1} \Sigma_j^{n-1} b_{ij} (P_i / P_n) (P_j / P_n)$$

If we use the convention that the tilde "~" above a symbol indicates normalization on commodity n, the normalized homogenous quadratic function in its restricted form is

$$\tilde{\pi} = \Sigma^{n-1} b_i \tilde{P}_i + (1/2) \Sigma^{n-1} b_{ij} \tilde{P}_i \tilde{P}_j + \Sigma_i^{n-1} \Sigma_k b_{ik} \tilde{P}_i Z_k +$$
$$\Sigma^{n-1} b_{it} tp \tilde{P}_i$$

The necessary and sufficient condition for global convexity is that the symmetric matrix is positive semidefinite (Lau 1974, p. 182). This property can be tested for using Lau's approach (Lau 1978).

Because of the normalization, this form also has only (n-1) independent supply and demand equations of the form

$$X_i = \partial \tilde{\pi} / \partial \tilde{P}_i = b_i + \Sigma^{n-1} b_{ij} \tilde{P}_j + \Sigma_k b_{ik} Z_k + b_{it} t$$

2. Note that since we in fact have random (estimated) variables, both numerator and denominator, we can strictly speaking opt not to use methods based on linear estimation techniques for testing the significance of the elasticities (t-statistics, F-statistics).

The nth equation must be computed residually, setting $X_n = \tilde{\pi} - \Sigma_i^{n-1} X_i \tilde{P}_i$, so that it becomes

$$X_n = -\Sigma_i^{n-1} \Sigma_j^{n-1} b_{ij} \tilde{P}_i \tilde{P}_j$$

The elasticities for the normalized homogenous quadratic function are thus

$$n_{ij} = b_{ij} (\tilde{P}_j / X_i) \quad i \neq j, \; i \neq n \qquad \text{(CPE)}$$
$$n_{ij} = b_{ii} (\tilde{P}_i / X_i) \quad i \neq n \qquad \text{(OPE)}$$

The elasticities for the nth equation are estimated residually, using the homogeneity constraint $\Sigma_j n_{ij} = \Sigma_i n_{ij} = 0$:

$$n_{in} = -\Sigma_j^{n-1} b_{ij} (\tilde{P}_j / X_i) \qquad \text{(CPE)}$$

$$n_{nn} = -\Sigma_j^{n-1} n_{nj} \qquad \text{(OPE)}$$

C. *The (Truncated) Linear/Leontief Function*

The General Linear/Leontief Function is written in the following form:[3]

$$\pi = \Sigma_i b p_i (1/2) = +\Sigma_i \Sigma_j b_{ij} P_i (1/2) \, P_j (1/2)$$

The General Linear/Leontief function poses a problem, as it is a Taylor series expansion in the square root of prices, and hence does not exhibit the linear homogeneity done with the homogeneous quadratic. Here we will use the truncated version instead, by setting all b=0, as this function still exhibits the second order approximation conditions (Diewert 1971, 1973). The derived demand and supply equations are then:

$$X_i = \partial\pi / \partial P_i \neq bij + \Sigma_{j \neq i} b_{ij} (P_j / P_i)^{(1/2)} + \Sigma_k b_{ik} Z_k + b_{it} t$$

by imposing the side conditions for the restricted profit function as for the other two functions. The elasticities in this case become:

3. See Lau (1974, p. 184). Diewert (1973, pp. 295-300) between the generalized linear, which he writes in a form similar to the one used here, and the generalized Leontief, which he writes as $\pi = \Sigma_i \Sigma_j b_{ij} P_i^{1/2} P_j^{1/2}$. In this 1971 article, Diewert represents both of them in the truncated version (Diewert 1971, pp. 497-505). Lau gives the more general version, as do Fuss, McFadden, and Mundlak (1978, p. 238) and it is only this generalized version which is a proper Taylor Series form.

$$n_{ij} = (b_{ij}/2X_i)\ (P_j/P_i)^{1/2} \qquad j \neq i \qquad \text{(CPE)}$$

$$n_{ij} = -\Sigma_j\ (b_{ij}/2X_i)\ (P_j/P_i)^{1/2} \qquad j \neq i \qquad \text{(OPE)}$$

because of the homogeneity constraint, N is estimated residually, as no independent means of calculating it are available.

D. *The Generalized Box–Cox*

Of the class of second-order polynomial flexible functional forms, the generalized Box–Cox is the most general to date. It includes the generalized Leontief and generalized square root quadratic forms as special cases, and the translog appears as a limiting case. In various forms it has been applied by Denny (1974), Kiefer (1976), and Berndt and Khaled (1979). The presentation here is taken from Berndt and Khaled.

The expression for total cost is

$$C = [1 + \lambda\ G(P)]^{1/\lambda} yB(Y,P)$$

with

$$G(P) = \alpha_0 + \Sigma_{i=1}^{n}\ \alpha_i P_i\ (\ \ ) + (1/2)\ \Sigma_{i=1}^{n}\ \Sigma_{j=1}^{n}\ \gamma_{ij}\ P_i\ (\lambda)\ P_j\ (\lambda),$$

P = vector of input prices,

and $B(Y,P) = \beta + (\Theta/2)\ ln\ Y + \Sigma_{i=1}^{n}\ \delta_i\ ln P_i,$

$$P_i\ (\lambda) = \frac{P_i^{\lambda/2} - 1}{\lambda/2}$$

with the restriction that the cost function is linearly homogenous in input prices, the following restrictions are introduced:

$$\Sigma_{i=1}^{n}\ \alpha_i = 1 + \lambda\alpha_0\ ,$$

$$\Sigma_{j=1}^{n}\gamma_{ij} = (\lambda/2)\ \alpha_i$$

and

$$\Sigma_{j=1}^{n}\ \delta_i = 0.$$

The cost function then reduces to

$$C = [\ (2/\lambda)\ \Sigma_{i=1}^{n}\ \Sigma_{j=1}^{n}\ \gamma_{ij}\ \ P_i^{\ \lambda/2}\ P_j^{\ \lambda/2}\ ]^{1/\lambda}\ \gamma B\ (Y, P).$$

The term $B\ (Y, P)$ involves interactions between prices and output, thereby producing a nonhomothetic technology. Homotheticity requires that each δ_i be zero as in the translog case, and homogeneity of degree $(1/\beta)$ again follows when O equals zero.

To introduce technical change into the generalized Box-Cox cost function, Berndt and Khaled multiply by the term

$$e^{T(\tau,\ P)}$$

where

$$T(t,P) = \tau + \Sigma_{i=1}^{n} \quad \tau_i \ln P_i\, t.$$

The analogy to the translog specification is evident; t_i represents technical change bias and so on.

Special cases of the generalized Box-Cox are obtained by fixing the Box-Cox parameter λ. When λ equals one, the Berndt and Khaled specification reduces to the generalized Leontief. The generalized square root quadratic is obtained by setting λ equal to two. Finally, the translog is a limiting case as λ approaches zero.

Expressions for factor shares in the generalized Box–Cox cost function are produced by differentiation according to Shephard's lemma. The share of factor i is given by

$$S_i = \frac{P_i^{\lambda/2}\,[\Sigma_{j=1}^{n}\gamma_{ij}\, P_j^{\lambda/2}] + \delta_i \quad \ln Y + \tau_i\, t.}{[\Sigma_{i=1}^{n}\,\Sigma_{j=1}^{n}\eta_{ij}\, P_i^{\lambda/2}\, P_j^{\lambda/2}]}$$

Substitution elasticities given by Berndt and Khaled (1979) can be obtained by differentiating once more and rescaling:

$$\delta_{ij} = 1 - \lambda + \frac{\gamma_{ij}\,(P_iP_j)^{\lambda/2}}{S_i\, S_j}\, C^{-\lambda} + \lambda \frac{F_j(\gamma,\ t)}{S_j}$$

$$+ \lambda\left[1 - \frac{F_j(\gamma,\ t)}{S_j}\right] \frac{F_i(Y,\ t)}{S_i} \quad ,\ i \neq j$$

Setting λ equal to zero, this reduces to the translog expressions for elasticities of substitution.

E. *Fourier Flexible Form*

The logarithmic version of the Fourier flexible form introduced in Gallant (1982) takes a cost function of the form.

$$g_k(X/\theta) = \mu_o + b'X + (1/2)\, X'CX + \Sigma_{\alpha=1}\{\mu_{o\alpha} + \Sigma_{j=1}$$

$$[\mu_{j\alpha}\cos\ (\, j\lambda_s k'_\alpha X) - V_{j\alpha}\sin(j\lambda_s k'_\alpha\ X)]\}$$

where $X = \begin{bmatrix} P_1 \\ P_2 \\ Y \end{bmatrix}$

$$C = -\lambda_S^2 \, \Sigma_{\alpha=1} \, \mu_{o\alpha} k_\alpha k'_\alpha$$

K^t	=	number of parameters, possibly a function of the sample size determined by choice A and J
X	=	vector of exogenous variable
θ	=	vector of parameters
b	=	vector of own elasticities
C	=	matrix of cross price elasticities
A	=	number of multi-indices

Prior to estimation, it is necessary to rescale [4] the data because the Fourier function is periodic, so data must fall within $(0, 2\pi)$.

Differentiation of the logarithmic Fourier flexible form produces share equations of the form

$$\nabla g_{k_t}(X/\theta) = b - \lambda_S \Sigma^A_{\alpha=1} \{ \mu_{o\alpha} \lambda_S k'_\alpha X + 2\Sigma^J_{j=1} \, j [\mu_{j\alpha} \sin (j \lambda_S k'_\alpha X) + v_{j\alpha} \cos (j \lambda_S k'_\alpha X)]\} k_\alpha.$$

where ∇ denotes the gradient vector representing factor shares formed by differentiating with respect to the logged input prices. With the cost function and $n-1$ share equations formed, parameters may be estimated with the seemingly unrelated regressions technique.

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4. Recalling is to be accomplished by the following procedure: First, from each member of the logged series of exogenous variables, subtract the minimum of that series and then add some $\in$ say 10^{-5}. Next rescale any covariate such as output by a scalar

$$\lambda_c = \frac{\text{max. rescaled price} + \in}{\ln Y_{max} - \ln Y_{min} + \in}$$

Finally, all data are rescaled by

$$\lambda_s = \frac{6}{\text{maximum rescaled price} = + \in}$$

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