Convexities Related to Path Properties on Graphs; a Unified Approach

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18-09-2000

Abstract

Path properties, such as 'geodesic', 'induced', 'all paths' define a convexity on a connected graph. The general notion of path property, introduced in this paper, gives rise to a comprehensive survey of results obtained by different authors for a variety of path properties, together with a number of new results. We pay special attention to convexities defined by path properties on graph products and the classical convexity invariants, such as the Carathéodory, Helly and Radon numbers in relation with graph invariants, such as clique numbers and other graph properties.

*The research for this paper was mainly done while this author was visiting the Econometric Institute of Erasmus University, Rotterdam and the University of Groningen as a BOYSCAST fellow of the Department of Science and Technology (DST) of the Ministry of Science and Technology of India in 1998. The financial support of the DST, New Delhi, and the University of Groningen, and the hospitality of the Econometric Institute, Rotterdam, are greatly acknowledged.

1 INTRODUCTION

For any two vertices v_1 and v_2 in a graph, the vertices on a path between v_1 and v_2 can be considered as located between v_1 and v_2; vertices on other similar paths between v_1 and v_2 can be seen as being in the convex hull of v_1 and v_2. Convexities on graphs are studied by many authors (see [4], [5], [8], [9], [12], [13], [14], [17], and [21]) for a variety of specific path properties, such as 'geodesic'(see [5],[12],[15], and [21]), 'induced' (see [4], and [14]), and 'all paths' (see [1], and [17]). The main purpose of this paper is to offer a unifying approach based on the general notion of a 'path with property Φ'. This notion is used to introduce the so called Φ-convexity on a graph which satisfies the general notion of abstract convexity, introduced in among others [6] and [19]. This connection enables us to study the invariants of Carathéodory, Helly and Radon for connected graphs. We consider finite simple connected loopless graphs, denoted by $G = (V, E)$, with vertex set V and edge set E. By a $u - v$ path joining two vertices u and v in G, we mean a sequence of distinct vertices $u, u_1, u_2, ..., u_i, u_{i+1}, ..., u_n, v$ such that consecutive vertices in this sequence join an edge. A path property Φ is defined as a property which associates for each pair of vertices in G a path (called a Φ-path) with the property Φ, while a path in a graph isomorphic to G, corresponding to a Φ-path in G, is a Φ-path as well. A $u - v$ path with property Φ is called a $u - v$ Φ-path. For example, if Φ = 'geodesic', then a geodesic path between any two vertices u and v in G is a $u - v$ path of smallest length. If Φ = 'induced', then an induced path between u and v is a path without chords, where a chord of a path is an edge joining two nonconsecutive vertices in that path. The collection of all Φ-paths between two vertices u and v is denoted by $\mathcal{P}_{(u,v)}(\Phi)$.

Let $R : V \times V \to 2^V$ be a function satisfying the following two axioms:

$(t1)$ $u \in R(u, v)$ for all u and v in V,
$(t2)$ $R(u, v) = R(v, u)$ for all u and v in V.

R is said to satisfy the betweenness property if

$(b1)$ $x \in R(u, v)$, $x \neq v \Longrightarrow v \notin R(u, x)$,
$(b2)$ $x \in R(u, v) \Longrightarrow R(u, x) \subseteq R(u, v)$.

Because obvious properties of betweenness are not present in the case of all functions R in general, we follow the terminology of [13] and call these functions transit functions. Let R_1 and R_2 be transit functions. Then the relation '$\leq$' defined by: "$R_1 \leq R_2$ if and only if $R_1(u, v) \subseteq R_2(u, v)$ for all $u, v \in V$" is a partial ordering on the family of all transit functions on G. Moreover, $R_1 \wedge R_2$ defined as $(R_1 \wedge R_2)(u, v) = R_1(u, v) \cap R_2(u, v)$ and $R_1 \vee R_2$ defined as $(R_1 \vee R_2)(u, v) = R_1(u, v) \cup R_2(u, v)$ for all $u, v \in V$, are transit functions and hence the family of all transit functions is a lattice denoted by $L(R)$. For any path property Φ, define the function $R_\Phi : V \times V \to 2^V$ by

$$R_\Phi(u,v) \quad = \{z \in V | z \in P \text{ for some P} \in \mathcal{P}_{(u,v)}(\Phi)\}.$$

We can easily see that R_Φ is a transit function. Since R_Φ is defined by a Φ-path, R_Φ is called the Φ- path transit function associated with Φ. Also note that the subgraph induced by $R_\Phi(u,v)$ is a connected subgraph of G. If no confusion is likely, we call a Φ-path transit function a path transit function. If $R_{\Phi 1}$ and $R_{\Phi 2}$ are two path transit functions, then $R_{\Phi 1} \wedge R_{\Phi 2}$ need not be a path transit function. For example, if $\Phi_1 =$ 'geodesic' and $\Phi_2 =$ 'longest ', then $(R_{\Phi_1} \wedge R_{\Phi_2})(u,v)$ can be equal to $\{u,v\}$ which need not be a path. However, $R_{\Phi 1} \vee R_{\Phi 2}$ is always a Φ-path transit function, namely Φ is the path property that either Φ_1 or Φ_2 holds. Hence, the family of all path transit functions is a join semi-lattice of $L(R)$, denoted as $L(R_\Phi)$. Clearly, the "all paths" transit function defined by

$$A(u,v) \quad = \{z \in V(G) | z \text{ lies in some } u - v \text{ path in } G\},$$

is an universal upper bound of $L(R_\Phi)$.

For any transit function R, a subset A of V is said to be R-convex if $R(u,v) \subseteq A$ for all $u,v \in A$. The collection of all R-convex subsets of V is an abstract convexity, denoted by $\mathcal{C}_R$, in the sense that it is closed under both intersections and nested unions and also both $\emptyset$ and V are R-convex sets. Convexities defined by a transit function are called interval convexities, or interval spaces in e.g. [6] and [21]. For a detailed account on abstract convexities, see for example [21]. The smallest R-convex subset containing a subset A of V is denoted by $\langle A \rangle_{\mathcal{R}}$ and is called the R-convex hull of A. It is left to the reader to show that, in general, we do not have that $\langle A \rangle_{\mathcal{R}} = \bigcup_{u,v \in A} R(u,v)$. The family of all R-convexities on V is a lattice with meet and join, defined for any two R-convexities $\mathcal{C}_{R1}$ and $\mathcal{C}_{R2}$ by $\mathcal{C}_{R1} \wedge \mathcal{C}_{R2} = \mathcal{C}_{R1} \cap \mathcal{C}_{R2}$, and $\mathcal{C}_{R1} \vee \mathcal{C}_{R2} = \{B \cap C | B \in \mathcal{C}_{R1}, C \in \mathcal{C}_{R2}\}$. The lattice of all $R - \mathit{convexities}$ on V is denoted by L$(\mathcal{C}_R)$. We have the following theorem.

Theorem 1 $\mathcal{C}_{R1 \wedge R2} = \mathcal{C}_{R1} \vee \mathcal{C}_{R2}$ *and* $\mathcal{C}_{R1 \vee R2} = \mathcal{C}_{R1} \wedge \mathcal{C}_{R2}$

Proof. Since, $A \in \mathcal{C}_{R1 \vee R2} \iff [(R1 \vee R2)(u,v) \subseteq A \ \forall \ u,v \in A] \iff [R1(u,v) \cup R2(u,v) \subseteq A \ \forall \ u,v \in A] \iff [R1(u,v) \subseteq A$ and $R2(u,v) \subseteq A \forall u,v \in A] \iff [A \in \mathcal{C}_{R1}$ and $A \in \mathcal{C}_{R2}] \iff A \in \mathcal{C}_{R1} \cap \mathcal{C}_{R2}$,we have that $\mathcal{C}_{R1 \vee R2} = \mathcal{C}_{R1} \wedge \mathcal{C}_{R2}$. Moreover, since $A \in \mathcal{C}_{R1 \wedge R2} \iff [A = \cup\{(R1 \wedge R2)(u,v) | u,v \in A\}] \iff [A = \cup\{R1(u,v) \cap R2(u,v) | u,v \in A\}] \iff [A = B \cap C$ where $B = \cup\{R1(u,v) | u,v \in A\}$ and $C = \cup\{R2(u,v) | u,v \in A\}] \iff [A = B \cap C$ where $B \in \mathcal{C}_{R1}$ and $C \in \mathcal{C}_R] \iff A \in \mathcal{C}_{R1} \vee \mathcal{C}_{R2}$, we have that $\mathcal{C}_{R1 \wedge R2} = \mathcal{C}_{R1} \vee \mathcal{C}_{R2}$. ■

2 SPECIAL Φ-CONVEX SETS

In this section we analyze a number of specific Φ-path transit functions and the corresponding convexities. We introduce a number of new path transit functions. Throughout, we call R_Φ-convex sets shortly Φ-convex sets and write $\mathcal{C}_\Phi$ instead of $\mathcal{C}_{R_\Phi}$.

2.1 The geodesic transit function

The geodesic or shortest-path transit function I on a connected graph G is defined as follows. Let u,$v \in V$. Then,

$$I(u,v) \quad = \{w \in V | w \text{ lies on some shortest } u - v \text{ path in } G\}$$

In this definition 'shortest' is again in terms of the number of vertices on the path. The geodesic interval function I and the geodesic convexity of a connected graph G are important tools for the study of the metric properties of G, cf. e.g. [3, 12]. An example of a class of graphs where these tools are indispensable, is that of median graphs. Such graphs are defined by the property that, for any triple of vertices, the intervals between the pairs of the triple intersect in exactly one vertex. This class of graphs is well studied; see [10, 12]. This definition of I is in terms of the distance function of G. In Nebeský[15, 16] an axiomatic characterization of the geodesic interval function is given without any reference to metric notions. It may be noted that geodesic convex sets are very difficult to characterize.

2.2 The induced-path transit function

The induced-path transit function J on G is for each $u, v \in V$ defined as

$$J(u,v) \quad = \{z \in V(G) | z \text{ lies on some } u - v \text{ induced-path in } G\}$$

The induced-path transit function is also known in the literature as minimal path transit function; see for example [4]. The induced-path transit function J and the convexity generated by this transit function is an interesting concept and various authors have studied it, see e.g.[4, 8]. The analogue of median graphs in the case of the function J is studied in [14]. The characterization of this transit function in terms of transit axioms alone seems to be difficult, but its convex sets are nicely characterized. The following characterization of the induced-path convex hull is due to Duchet[4].

> In a connected graph G a vertex v belongs to the induced-path convex hull of a subset A of V if and only if no clique of $G \backslash v$ separates v and A.

2.3 The all-paths transit function

The all paths transit function for each $u, v \in V$ is defined as

$$A(u,v) \quad = \{z \in V(G) | z \text{ lies on some } u - v \text{ path in } G \ \}.$$

This transit function can be seen as the coarsest path transit function. From the convexity point of view, the convexity generated by the all-paths function A

has also been studied in [4, 17], where it is called the coarsest path interval. A characterization in terms of transit axioms is recently established in [1]. The all-paths function has a nice structure, reflecting the block cut-vertex structure of the graph.

2.4 The triangle-path transit function

Let R be a Φ-path transit function on G. Then $R^{\triangle}$ is the Φ-path transit function defined for each $u, v \in V$ by

$$R^{\triangle}(u,v) = \{z \in V(G) | z \text{ lies on some } \Phi\text{-path between } u \text{ and } v \text{ or } z \text{ is adjacent to two consecutive vertices on some } \Phi\text{-path between } u \text{ and } v \text{ in } G\}.$$

An $u - v$ path with property $\Phi^{\triangle}$ is either an $u - v$ path with property Φ, or an $u - v$ path of which some of its edges are replaced by triangles, meaning that edge ab on the Φ-path is replaced by the vertices az and bz with z some vertex not on the Φ-path. Hence for each $u, v \in V$, it holds that

$$R^{\triangle}(u,v) = \{z \in V(G) | z \text{ lies on some } \Phi^{\triangle}\text{-path between } u \text{ and } v \text{ in } G\}.$$

The transit function $R^{\triangle}$ is called the triangle Φ-path or simply triangle-path transit function. Recursively, define $R^{k\triangle}$ as $R^{0\triangle} = R$ and $R^{k\triangle} = \left(R^{(k-1)\triangle}\right)^{\triangle}$ for $k \geq 1$. Clearly, $R^{k\triangle}$ is a Φ-path transit function as well. We can easily see that the convexities defined by $R^{k\triangle}$ are the same for all $k \geq 1$. We can also see that $R^{(k-1)\triangle} \leq R^{k\triangle}$ for $k \geq 1$. Since the I-convex sets are difficult to characterize, we may expect that the $I^{\triangle}$-convex sets are also difficult to characterize. However, similar to the J-convex sets the $J^{\triangle}$-convex sets can be characterized as follows. The following statement can be found in [2].

> Let $G = (V, E)$ be a connected graph, and let $A \subseteq V$. Any vertex v does not belong to the $J^{\triangle}-$ convex hull of A if and only if there exists a clique M separating v and A in such a way that any two paths connecting v to two distinct vertices of M contain a chordless cycle of length at least 4.

2.5 The I_j-path transit function

For $j \geq 0$, the I_j-path interval is for any $u, v \in V$ defined by

$$I_j(u,v) = \{z \in V(G) | z \text{ lies on a path of length} \leq d(u,v) + j\}.$$

We can easily see that $I^{\triangle}(u,v) \leq I_1(u,v)$, since a vertex lying on a geodesic triangle-path between u and v should have length at most $d(u,v)+1$. Similarly, $I^{k\triangle}(u,v) \leq I_k(u,v)$ for every $k \geq 0$. Moreover, we have that $< I_k^{\triangle}(u,v) > \subseteq < I_k(u,v) >$.

3 PATH TRANSIT FUNCTIONS ON GRAPH PRODUCTS

In this section we discuss path transit functions on graph products. Given connected graphs G_1 and G_2, the Cartesian product of G_1 and G_2, denoted by $G_1 \times G_2$ is defined as the graph with vertex set $V(G_1) \times V(G_2)$ and edge set as follows; two vertices $(u_1, u_2), (v_1, v_2)$ in $V(G_1) \times V(G_2)$ form an edge if either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$. Obviously, if Φ_1 and Φ_2 are two path properties on G_1 and G_2, respectively, then a $\Phi_1\Phi_2$-path is a path in $G_1 \times G_2$ with the property that its projections on G_1 and G_2 are the Φ_1- and Φ_2-paths, respectively. For any two vertices $(u_1, v_1), (u_n, v_m) \in V(G_1 \times G_2)$, there exist a Φ_1-path between u_1 and u_n, say $u_1, u_2, ..., u_n$ in G_1, and a Φ_2-path between v_1 and v_m, say $v_1, v_2, ..., v_m$ in G_2. Now the path defined as $(u_1, v_1), (u_2, v_1), ..., (u_n, v_1)(u_n, v_2), ..., (u_n, v_m)$ is a $(u_1, v_1), (u_n, v_m)$-path in $G_1 \times G_2$ of which the projection on G_1 is the Φ_1-path $u_1, u_2, ..., u_n$ and the projection on G_2 is the Φ_2-path $v_1, v_2, ..., v_m$. Hence there is a natural way of defining a path property in the Cartesian product of graphs, given two path properties in the component graphs. We have,

Theorem 2 *Let Φ_1 and Φ_2 be path properties on the connected graphs G_1 and G_2, respectively. Then $\Phi_1\Phi_2$ is a path property on $G_1 \times G_2$.*

Clearly, since $R_{\Phi_1\Phi_2} = R_{\Phi_1} \times R_{\Phi_2}$, $R_{\Phi_1\Phi_2}$ is a path transit function. In general, the product convexity, denoted by $\mathcal{C}_1 \times \mathcal{C}_2$, on the convexity spaces $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined by $\mathcal{C}_1 \times \mathcal{C}_2 = \{A \times B | A \in \mathcal{C}_1, B \in \mathcal{C}_2\}$; see, e.g. [18, 19]. Instead of $\mathcal{C}_{R\Phi}$, we will write $\mathcal{C}_\Phi$.

Theorem 3 *For any two path properties Φ_1 and Φ_2, it holds that $\mathcal{C}_{\Phi_1\Phi_2} = \mathcal{C}_{\Phi_1} \times \mathcal{C}_{\Phi_2}$.*

Proof. Take any $C \in \mathcal{C}_{\Phi_1\Phi_2}$. We need to show that $C = \Pi_1 C \times \Pi_2 C$, with $\Pi_1 C \in \mathcal{C}_{\Phi_1}$ and $\Pi_2 C \in \mathcal{C}_{\Phi_2}$. To that end, take any u_1,$u_2 \in \Pi_1 C$. Then there exist v_1,$v_2 \in \Pi_2 C$ such that (u_1, v_1),$(u_2, v_2) \in \mathcal{C}$. Since $R_{\Phi_1\Phi_2}((u_1, v_1), (u_2, v_2)) = (R_{\Phi_1} \times R_{\Phi_2})((u_1, v_1), (u_2, v_2)) = R_{\Phi_1}(u_1, u_2) \times R_{\Phi_2}(v_1, v_2) \subseteq C$, it follows that $R_{\Phi_1}(u_1, u_2) \subseteq \Pi_1 C$ and $R_{\Phi_2}(v_1, v_2) \subseteq \Pi_2 C$. Hence $\Pi_1 C \in \mathcal{C}_{\Phi_1}$. Similarly $\Pi_2 C \in \mathcal{C}_{\Phi_2}$. Since $C \subseteq \Pi_1 C \times \Pi_2 C$, we only need to show that $\Pi_1 C \times \Pi_2 C \subseteq C$. Take any $u_1 \in \Pi_1 C$ and $v_1 \in \Pi_2 C$. Then there exist $u_2 \in \Pi_1 C$, $v_2 \in \Pi_2 C$ such that $(u_2, v_1) \in C$ and $(u_1, v_2) \in C$. Hence, $(u_1, v_1) \in R_{\Phi_1}(u_1, u_2) \times R_{\Phi_2}(v_1, v_2) = R_{\Phi_1\Phi_2}((u_1, v_1), (u_2, v_2)) \subseteq C$. ■

As a specialization of the above discussion, one may ask the questions: under what conditions do we have the following situations?

$(a)\quad R_{\Phi\Phi}(G_1 \times G_2) = R_\Phi(G_1 \times G_2),$
$(b)\quad R_{\Phi\Phi}(G_1 \times G_2) \leq R_\Phi(G_1 \times G_2),$
$(c)\quad R_{\Phi\Phi}(G_1 \times G_2) \geq R_\Phi(G_1 \times G_2),$

with G_1 and G_2 connected graphs. (We have included the product $G_1 \times G_2$ in the above formula for clearness sake.) So actually, the question is: when do we obtain the path property Φ on the product where on the two components of the product Φ holds. If $R_\Phi = A$, the all-paths transit function, then for connected graphs G_1 and G_2 with at least two vertices, the $A(G_1 \times G_2)$-convexity is the trivial convexity consisting of the empty set $\emptyset$, all single vertices and the whole set $V(G_1 \times G_2)$. Thus evidently $A(G_1) \times A(G_2) \leq A(G_1 \times G_2)$. Similarly for $R = J$, we have the same inequality, namely $J(G_1) \times J(G_2) \leq J(G_1 \times G_2)$, since for a vertex $z = (z_i, z_j) \in J(u_1, u_n) \times J(v_1, v_m)$ we have that z_i lies on an induced-path, say $u_1, u_2, \ldots, u_n$ and z_j lies on an induced-path $v_1, v_2, \ldots, v_m$. Hence, the path defined by $(u_1, v_1), (u_2, v_1), \ldots, (z_i, v_1), (z_i, v_2), \ldots, (z_i, z_j), \ldots, (z_i, v_m), (z_{i+1}, v_m), \ldots, (u_n, v_m)$ is an induced-path in $(G_1 \times G_2)$, all of whose vertices belong to $J(u_1, u_n) \times J(v_1, v_m)$. Since the distance function d of $(G_1 \times G_2)$ is the sum of the distance functions d_1 and d_2 of the components G_1 and G_2, respectively, we have that the geodesic transit function I satisfies $I(G_1) \times I(G_2) = I(G_1 \times G_2)$. The following theorem provides an example related to case(c).

Theorem 4 *Let $j, k \geq 1$ and let G_1 and G_2 be two connected graphs with I_j and I_k path transit functions on G_1 and G_2 respectively. Then,*

$$I_j(G_1) \times I_k(G_2) = I_{j+k}(G_1 \times G_2).$$

Proof. Take any $u_1, u_n \in V(G_1)$ and $v_1, v_m \in V(G_2)$. Then $z = (z_i, z_j) \in I_j(u_1, u_n) \times I_k(v_1, v_m) \iff z_i \in I_j(u_1, u_n)$ and $z_j \in I_k(v_1, v_m \iff z_i$ lies on a path of length $\leq d_1(u_1, u_n) + j$ and z_j lies on a path of length $\leq d_2(v_1, v_m) + k \iff z$ lies on a path of length $\leq d(u, z) + j + k$ in $G_1 \times G_2 \iff z \in I_{j+k}(u, v)$, where $u = (u_1, v_1)$ and $v = (u_n, v_m)$. ■

It follows directly from this theorem that $I_1(G_1 \times G_2) \subseteq I_1(G_1) \times I_1(G_2) = I_2(G_1 \times G_2) \subseteq I_3(G_1 \times G_2) \subseteq I_n(G_1 \times G_2) \ldots$. Moreover, the product convexity of two I_j-convexities is precisely the convexity generated by the path transit function I_{2j} on $G_1 \times G_2$, i.e, $I_j(G_1) \times I_j(G_2) = I_{2j}(G_1 \times G_2)$. Also note that $I(G_1) \times I(G_2) = I(G_1 \times G_2)$, but that $I^\triangle(G_1 \times G_2) \neq I^\triangle(G_1) \times I^\triangle(G_2)$. However, in the case of the $\mathcal{C}_{I^{k\triangle}}$-convexity we have the following theorem.

Theorem 5 *If G_1 and G_2 are two connected graphs, and $k \geq 0$, then $\mathcal{C}_{I^{k\triangle}}(G_1) \times \mathcal{C}_{I^{k\triangle}}(G_2) = \mathcal{C}_{I^{k\triangle}}(G_1 \times G_2)$.*

Proof. To show that $\mathcal{C}_{I^{k\triangle}}(G_1) \times \mathcal{C}_{I^{k\triangle}}(G_2) \subseteq \mathcal{C}_{I^{k\triangle}}(G_1 \times G_2)$ for each $k \geq 0$, take any $k \geq 0$, and $A \times B \in \mathcal{C}_{I^{k\triangle}}(G_1) \times \mathcal{C}_{I^{k\triangle}}(G_2)$. Moreover, take any $u, v \in A \times B$, and

$z \in R_{I^{k\triangle}}(u,v)$. Then there is a $I^{k\triangle}$-path in $G_1 \times G_2$ between u and v, say $P_{u,a,z,b,v}$ which includes $z = (z_1, z_2)$. Then either $\triangle^k_{a,b,z} \subseteq G_1 \times \{t\}$ or $\triangle^k_{a,b,z} \subseteq \{s\} \times G_2$ for some $(s,t) \in A \times B$, say $\triangle^k_{a,b,z} \subseteq G_1 \times \{t\}$. Since I is the geodesic transit function, we have that $\Pi_1 P_{u,a,z,b,v}$ (Π_1 is projection on G_1) is a $I^{k\triangle}$-path between $\Pi_1(u) = u - 1$, $\Pi_1(v) = v_1$, $\Pi_1(a) = a_1$, $\Pi_1(b) = b_1$, and $\Pi_1(z) = z_1$. Hence $z_1 \in P_{u_1,a_1,z_1,b_1,v_1} \subseteq R_{I^{k\triangle}}(u_1, v_1) \subseteq A$. Similarly it follows that $z_2 \in B$. Hence $z \in A \times B$, so that $A \times B$ is in fact $I^{k\triangle}$-convex in $G_1 \times G_2$.

To show that $\mathcal{C}_{I^{k\triangle}}(G_1 \times G_2) \subseteq \mathcal{C}_{I^{k\triangle}(G_1)} \times \mathcal{C}_{I^{k\triangle}(G_2)}$, let $C \in \mathcal{C}_{I^{k\triangle}}(G_1 \times G_2)$. Then $C = \langle A \rangle_{I^{k\triangle}}$ for some $A \subseteq V(G_1 \times V(G_2)$. Let $z \in C$. If $z \in A$, then $z \in \Pi_1(A) \times \Pi_2(A) \subseteq \langle \Pi_1(A) \rangle_{I^{k\triangle}} \times \langle \Pi_2(A) \rangle_{I^{k\triangle}} \in \mathcal{C}_{I^{k\triangle}}(G_1) \times \mathcal{C}_{I^{k\triangle}}(G_2)$, where Π_1 and Π_2 are the projections on $V(G_1)$ and $V(G_2)$ respectively, and we are done. If $z \notin A$, then $z \in I^{k\triangle}(a,b)$ for two vertices $a = (a_1, a_2)$, $b = (b_1, b_2) \in A$ and $k \geq 1$ implies that $z \in \left\langle I^{k\triangle}(a_1, b_1) \right\rangle \times \left\langle I^{k\triangle}(a_2, b_2) \right\rangle \in \mathcal{C}_{I^{\triangle}(G_1)} \times \mathcal{C}_{I^{\triangle}(G_2)}$. ■.

4 CONVEXITY INVARIANTS

In this section we discuss the classical convexity invariants such as Carathéodory, Helly, Radon and Exchange numbers in combination with the Rank and the Hull number. We will discuss these parameters for the well known convexities defined by path transit functions, and give improvements on the bounds of these parameters. We start with shortly recalling the various definitions. Let Φ be a path property. A Φ-copoint of a point p of V is a maximal Φ-convex subset of V not containing p. The *Carathéodory number* c of the convexity space $\mathcal{C}$ is the smallest integer (if exists) such that for any finite subset S of V, $\langle S \rangle_{\mathcal{C}} = \bigcup\{\langle F \rangle_{\mathcal{C}} \,|F \subseteq S, |F| \leq c\}$. The *Exchange number* e of $\mathcal{C}$ is the smallest integer (if exists) such that for any subset F of V with $|F| \geq e$ and any point p in F, $\langle F \backslash \{p\} \rangle_{\mathcal{C}} \subseteq \bigcup\{\langle F \backslash \{a\} \rangle_{\mathcal{C}} \,|a \in F \backslash \{p\}\}$. The *Helly number* h of $\mathcal{C}$ is the smallest integer (if exists) such that every family of convex sets with an empty intersection contains a subfamily of at most h members with an empty intersection. Equivalently, h is the smallest natural number such that $\bigcap_{s \,\in\, S} \langle S \backslash \{s\} \rangle_{\mathcal{C}} \neq \emptyset$ for every $(h+1)$-element subset S of V. The *Radon number* r of $\mathcal{C}$ is the smallest integer (if exists) such that every r-element set $A \subseteq V$ admits a Radon partition, that is, a partition $A = A_1 \cup A_2$, $(A_1 \cap A_2 = \emptyset)$ with $\langle A_1 \rangle_{\mathcal{C}} \cap \langle A_2 \rangle_{\mathcal{C}} \neq \emptyset$. The m^{th} *Radon number*, denoted by r_m, is the smallest number (if exists) such that every r_m-element set $A \subseteq V$ admits a Radon m-partition, that is a partition of A into m pair wise disjoint subsets $A_1, A_2, ..., A_m$ such that $\langle A_1 \rangle_{\mathcal{C}} \cap \langle A_2 \rangle_{\mathcal{C}} \cap .... \cap \langle A_m \rangle_{\mathcal{C}} \neq \emptyset$. A subset $I \subseteq V$ is called an *independent set* if $x \notin \langle I \backslash \{x\} \rangle_{\mathcal{C}}$ for every $x \in I$. The *Rank* of $\mathcal{C}$ is the supremum of the cardinalities of the independent subsets of V. The *Hull number* u of $\mathcal{C}$ is the infimum of the cardinalities of subsets S of V such that $\langle S \rangle_{\mathcal{C}} = V$.

A *clique* of G is a subset of pair wise adjacent vertices. The *Clique number* ω is the cardinality of the largest clique in G. By a *clique separator* of G, we mean a clique whose removal disconnects G; an *atom* of G is a maximal connected subgraph of G containing no clique separator. The *atom-clique separator tree*

$T(G)$ of G is the intersection graph with the vertex set consisting of atoms and clique separators of G, and two vertices adjacent if one of them is an atom and the other is a clique separator intersecting with the atom. It can be easily verified that $T(G)$ is a tree.

In the literature we can find a variety of relationships between the mentioned numbers; see e.g.[19]. One of the open problems is showing that the so called Eckhoff-Jamison inequality $r \leq c(h-1)+2$ is sharp, meaning that there is a convexity space with (arbitrary) Carathéodory, Helly and Radon numbers c, h, and r, respectively, satisfying $r = c(h-1)+1$. For the Φ-convexities of this paper this inequality holds. Another famous open problem is the so-called Eckhoff conjecture (see [7] and [11]. This conjecture reads that for any convexity space the m^{th} Radon number satisfies the inequality $r_m \leq (r_2-1)(m-1)+1$ for $m \geq 2$. For the specific Φ-convexities in this paper the Eckhoff conjecture holds, but remains a challenging open problem.

4.1 Geodesic convexity

The geodesic convexity is in some sense "universal" with respect to the above mentioned invariants, namely in Duchet [5] it is observed that for every convexity on a finite set V, with Helly, Radon and m^{th} Radon numbers h,r and r_m, respectively, there is a finite connected graph G whose geodesic convexity has Helly number h, Radon number r and m^{th} Radon number at least r_m. So far no relationships between the invariants Carathéodory, Helly and Radon numbers and any known graph parameter are known. Observe that the n-cube Q_n has $h = 2$, $c = n$ and $r = \lceil \log_2(n+1) \rceil + 2$.

4.2 Induced-path convexity

For the induced-path convexity, Duchet has determined in [4] the relationships between the Helly and Radon numbers and the Clique number. It is also shown there that the Carathéodory number $c \leq 2$. Using the inequality $e \leq c+1$ [18], it follows that the Exchange number $e \leq 3$. Duchet's result is as follows.

> The J-convexity has Carathéodory number $c \leq 2$, Helly number $h = \omega$ and Radon number $r = \omega + 1$ if $\omega \geq 3$ and $r \leq 4$ if $\omega \leq 2$.

In the following theorem we characterize the cases $r = 3$ and $r = 4$ for triangle free graphs, i.e. graphs with $\omega \leq 2$.

Theorem 6 *The Radon number r of the J-convexity in a triangle free connected graph G is 3 if and only if either G is a simple path or G is 2-connected and the atom-clique separator tree of G is a simple path; $r = 4$ for all other triangle free graphs.*

Proof. If G is an induced simple path then clearly $r = 3$. Suppose G is two connected and the atom-clique separator tree T of G be a simple path. Then G consists of a chain of atoms connected by clique separators. For any three vertices, u, v, w in G, each of u, v and w will lie on atoms. If at least two of them lie on the same atom, then one of them will lie on the J-convex hull of the other two. If they lie on pair wise different atoms, then there is an atom containing one of u, v, w, say w, which lies in between the atoms containing u and v and hence the J-convex hull $\langle J(u, v)\rangle$ will contain the atom containing w, and hence u, v, w has a Radon partition. Suppose G be any other triangle free graph. If G is not 2-connected, then G has a cut vertex v having at least three neighbours. Any three vertex subsets of the set of neighbours with two vertices belonging to distinct components of $G\backslash v$ has no Radon partition. If G is two connected, then every clique-separator of G is an edge and since the atom-clique separator tree T of G is not a simple path, there exist at least three end vertices for T. The three vertices lying on three distinct atoms corresponding to the three end vertices of T has no Radon Partition. ■

From the definition of the J-convex hull, we have

Theorem 7 *For any connected graph G and any vertex p, it holds that any two distinct copoints of p are non-intersecting..*

Proof. Consider two distinct copoints K_p and L_p of vertex p of G. Since K_p and L_p are distinct J-convex sets, they are separated by a clique separator and hence have no vertex in common. Therefore K_p and L_p are non-intersecting. ■

Let $m, k \geq 1$. A convexity $\mathcal{C}$ on V has the $\mathcal{C}$-copoint intersection property $CIP(m, k)$ iff for each p in V, it holds that any set of m distinct $\mathcal{C}$-copoints at p contains a k-subset with an empty intersection. In Jamison-Waldner [9] the following result is shown.

> Let the convexity $\mathcal{C}$ on V satisfy $CIP(3, 2)$ and has finite Helly number h. Then for each $m \geq 1$, $r_m \leq 2m$ if $h = 2$, and $r_m = (m-1)h + 1$ if $h \geq 3$.

By Theorem 7 the J-convexity satisfies $CIP(3, 2)$. Therefore we have the following theorem.

Theorem 8 *The J-convexity on a connected graph satisfies $r_m \leq 2m$ if $\omega = 2$ and $r_m = (m-1)\omega + 1$ if $\omega \geq 3$.*

Consider the atom-clique separator tree $T(G)$ of G by taking all the minimal clique separators (A minimal clique separator is one such that no proper subset of it is not a clique separator). An end atom of G is an atom corresponding to an end vertex or external vertex of $T(G)$. If an end atom is itself a clique of size $m \geq 2$, then join $m-1$ vertices by an edge in $T(G)$ to the vertex corresponding to the minimal clique separator separating the end atom with the rest of G and the tree thus obtained is denoted by $T'(G)$. The set of end vertices of $T(G)$, $T'(G)$ is called the periphery of $T(G)$, respectively $T'(G)$ denoted as $P(T(G))$, respectively $P(T'(G))$. For each $x \in P(T'(G))$ take a corresponding vertex in G and the set of vertices thus obtained forms an independent set of $V(G)$. We have a straight forward theorem.

Theorem 9 *For the J-convexity of a connected graph G, it holds that the Hull number u is equal to the cardinality of $P(T(G))$ if the end atoms of G are not cliques and is equal to the cardinality of $P(T'(G))$ otherwise. That is, $u = |P(T(G))|$ or $u = |P(T'(G))|$.*

We have that every clique of G forms an independent set. The cardinality of $P(T'(G))$ or $P(T(G))$ is dependent on the number of external atoms and not dependent on ω; there can be graphs with $\omega \geq u$ and $u \geq \omega$. Therefore we have

Theorem 10 *The rank of J-convexity is given by $max(h, u)$.*

4.3 Triangle-path convexity

As in the the case of the geodesic convexity, no bound or relationship between the invariants of the $I^{\triangle}$-convexity and any other known graph parameter is known. But, for the $J^{\triangle}$-convexity, the bounds of the invariants are known. The following result can be found in [2]. The $J^{\triangle}$-convexity has the Carathéodory number $c = 2$, the Exchange number $e = 3$, the Helly number $h = 2$ and the Radon number r satisfying $3 \leq r \leq 4$.

Clearly, Theorem 7 formulated for J-convexities, also holds for $J^{\triangle}$- convexities for the triangle free connected graphs, because $J = J^{\triangle}$. Since any two $J^{\triangle}$-copoints are separated by clique separators, as in the case of the J-convexity, we have

Theorem 11 *For the $J^{\triangle}$-convexity, given any vertex p of G, any two distinct copoints of p are non-intersecting*

The $J^{\triangle}$-convexity has helly number $h = 2$ and by the previous theorem, it satisfies $CIP(3, 2)$. Therefore as a corollary of the theorem of Jamison-Waldner [9] we have the following theorem

Theorem 12 *Let $m \geq 1$. The m^{th} Radon number for the $J^{\triangle}$-convexity satisfies: $r_m \leq 2m$*

In order to discuss the Hull number and Rank, we need a small modification for the atom-clique separator $T(G)$ of G in case of the $J^{\triangle} - convexity$; denoted as $T^{\triangle}(G)$. The vertex set of $T^{\triangle}(G)$ is the family of all atoms and cliques not separating a $J^{\triangle}$-convex subset and an atom A is adjacent to a clique separator M in $T^{\triangle}(G)$ if M separates A such that any two paths connecting a vertex in A

to two distinct vertices of M contain a chordless cycle of length at least equal to 4. Here also $T^{\Delta}(G)$ is a tree and as in the case of the $P(T(G))$, the subset of V corresponding to the set of elements of $P(T^{\Delta}(G))$, forms an independent set of $V(G)$. We have a similar straightforward theorem to that of the J-convexity.

Theorem 13 *The Hull number of J^{Δ}-convexity is equal to the cardinality of $P(T^{\Delta}(G))$.*

Unlike the J-convexity, the clique vertices (vertices which forms a clique) of G are J^{Δ}-convexly dependent. Hence we have

Theorem 14 *The rank of the J^{Δ}-convexity is the Hull number u.*

The Carathéodory, Helly and Radon numbers for the all-paths convexity (A-convexity) are investigated in [17]. By considering the block-cut vertex tree $B(G)$ instead of the $T(G)$, and following similar arguments to that of the J^{Δ}-convexity, we can obtain the Hull number u and the Rank of A-convexity. We summerize these results in the following theorem.

Theorem 15 *The following statements hold for the all-paths convexity. The Carathéodory number $c = 2$, the Exchange number $e = 3$, the Helly number $h = 2$ and the Radon number r satisfies $3 \leq r \leq 4$. The m^{th} Radon number $r_m \leq 2m$. The Hull number and the Rank are the same and equal to the cardinality of the set $P(B(G))$.*

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