Asset Liability Management modeling using multi-stage mixed-integer Stochastic Programming

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Abstract

A pension fund has to match the portfolio of long-term liabilities with the portfolio of assets. Key instruments in strategic Asset Liability Management (ALM) are the adjustments of the contribution rate of the sponsor and the reallocation of the investments in several asset classes at various points of time. We formulate a multistage mixed-integer stochastic program to model this ALM process. Special attention is paid to the use of binary variables.

1. Introduction

We present a multistage Asset Liability Management (ALM) model based on mixed-integer stochastic programming. Stochastic programming can be used for supporting decision making under uncertainty, while taking into account the probability distributions of uncertain parameters. Since it is a powerful tool, stochastic programming is used in many fields, for example in production planning, scheduling problems, location problems and electricity generation (see Birge and Louveaux [1], Kall and Wallace [8], Prékopa [13], Wets and Ziemba [17] and the bibliography Van der Vlerk [16]). It is also used in financial planning models, especially in Asset Liability Management, see for example Boender [2], Cariño et al. [3], Consigli and Dempster [4], Dert [5] and Mulvey and Ziemba [11]. In most of these models, only continuous variables appear. Only Dert uses binary variables in a model with chance constraints. In this paper we show that binary variables can be used to model a variety of realistic features. Next to the chance constraints, binary variables are used to model remedial contributions made by the sponsor of the fund after several periods of underfunding and to model conditional constraints. In addition, we present a flexible modeling of the contribution rate.

In this paper we focus on modeling issues. However, the inclusion of integer variables in stochastic programming models causes substantial computational difficulties, see the surveys Klein Haneveld and Van der Vlerk [10] and Stougie and Van der Vlerk [15]. How to deal with these difficulties, for example by using the special structure of the proposed model, will be the subject of our future research.

The contents of this paper can be summarized as follows. In Section 2, we explain what ALM for pension funds is and we also present basic (in)equalities of an ALM model. In Section 3, we propose extensions to these basic (in)equalities, leading to the introduction of binary variables. After modeling the feasible region, we define an objective function. Section 4 shows the dynamic character and the corresponding special structure of the model. In Section 5 we make some concluding remarks.

2. ALM for pension funds

A pension fund has the task of making benefit payments to participants who have ended their active income earning career. We assume that the pension fund has three sources of funding its liabilities: revenues from its asset portfolio, regular contributions made by the sponsor of the fund and remedial contributions made by the sponsor. The latter

payments may be called for if the value of the assets is too low compared to the value of the liabilities. The pension fund has to decide periodically how to distribute the investments over different asset classes and what the contribution rate should be in order to meet all its obligations. This decision process is called Asset Liability Management.

A pension fund has long term obligations, up to decades, and therefore its planning horizon is large, too. The main goal of ALM is to find acceptable investment and contribution policies that guarantee that the *solvency* of the fund is sufficient during the planning horizon. Usually, the solvency is characterized by the *funding ratio* F_t, defined by $F_t := A_t/L_t$, where A_t denotes the value of the assets and L_t is the value of the liabilities[1]. The subscript t denotes time t. *Underfunding* occurs when the funding ratio is less than one. Another way of characterizing underfunding is by saying that the *surplus* S_t at time t is negative, where $S_t := A_t - L_t$. The funding ratio changes over time, mainly because of uncertain developments in the liabilities and in the returns of the assets. Therefore, a pension fund rebalances its asset portfolio and adjusts its contribution policy regularly, in order to control the funding ratio as well as possible.

How much risk of underfunding is acceptable, depends on specific characteristics of the fund. For instance, if the number of active participants is large compared to the number of retired members, some temporary risk of underfunding might be acceptable since there is a possibility to recover later by adjusting the contribution rate if needed. On the other hand, if a limited degree of temporary underfunding is acceptable, this gives the opportunity to increase the investment in stocks and to decrease the investment in bonds. In the long run one may expect that this will increase the returns, since (historically) the mean return on a broadly diversified stock portfolio is higher than that on a broadly diversified bond portfolio; however, its variability is higher too. Of course, if the current funding ratio is much larger than one, there is less reason to worry about promising but risky investments.

In the remainder of this section we describe the basic elements of an ALM model. In order to be realistic we need a dynamic model, since when making decisions at one time, it is necessary to take into account possible adjustments of decisions later, based on observed realizations of uncertain parameters.

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1   It is not trivial to calculate L_t from the liability portfolio of the fund. We will not go into details on this aspect here.

2.1 ALM model for pension funds: scenarios and decisions

In this section we discuss basic ingredients of an ALM model. Extensions of this model, motivated by the need to incorporate realistic and flexible ways to deal with the risk of underfunding, follow in Section 3.

We split the planning horizon in T subperiods and denote the resulting time stages by an index t. Time $t = 0$ is the current time and $t = T$ is the length of the horizon. By period t ($t = 1, \ldots, T$), we mean the span of time $[t-1, t)$. At each time $t \in \mathcal{T}_0 := \{0, 1, \ldots, T-1\}$, the pension fund is allowed to make some decisions, for example it may change its asset mix.

One way of modeling uncertainty is through a large but finite number S of scenarios. Each scenario represents a possible realization of all uncertain parameters in the model. To be specific, let ω_t represent the vector of random parameters whose values are revealed in period t. Then the set of all scenarios is the set of all realizations $(\omega_1^s, \ldots, \omega_T^s)$, $s \in \mathcal{S} := \{1, 2, \ldots, S\}$, of $(\omega_1, \ldots, \omega_T)$. Each scenario s has a probability p^s, where $p^s > 0$ and $\sum_{s=1}^{S} p^s = 1$. Since in a dynamic model information on the actual value of the uncertain parameters is revealed in stages, a suitable representation of the set of scenarios is given by a scenario tree, such as in Figure 2.1. In this case, $T = 3$ and $S = 12$. Each path from $t = 0$ to $t = T$ represents one scenario. Any node of the tree, corresponding to time t, symbolizes a possible state of the world at time t, represented by the observed values of $\omega_1, \ldots, \omega_t$. The branches directly to the right of it symbolize the various values of ω_{t+1} (and their corresponding conditional probabilities) given the realization of $\omega_1, \ldots, \omega_t$. Obviously, all scenarios passing this node have the same history in periods $1, \ldots, t$. The status of the decision variables is related to the scenario tree, too. Basically, a decision at time t may depend on the observed part of the scenario at that time, but not on unknown values of future periods. That is, for each possible history (i.e. for each node at time t in the scenario tree) there is precisely one vector of decision variables representing the decisions at hand.

However, in our model formulation it is convenient to introduce a complete set of decision variables for each scenario separately. Therefore, so-called *nonanticipativity* or *information constraints* have to be added, in order to guarantee that decisions do not depend on values of random parameters that will be revealed in later periods.

Let us now introduce the random parameters and the decision variables of the ALM model. For $t \in \mathcal{T}_1 := \{1, 2, \ldots, T\}$, we define

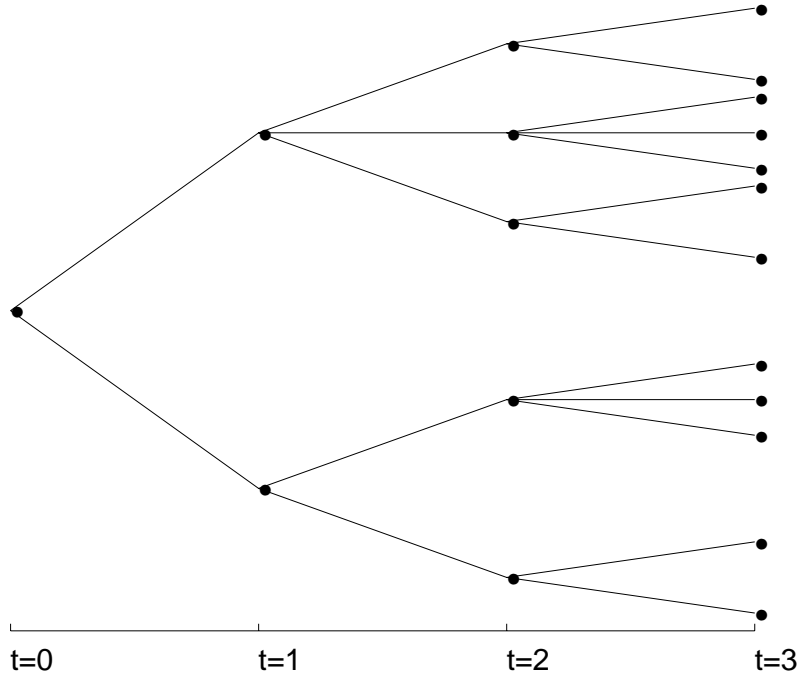$$\omega_t = (r_{1t}, r_{2t}, \ldots, r_{Nt}, W_t, P_t, L_t),$$

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Figure 2.1: Example of a scenario tree

where

$$
\begin{aligned}
r_{it} &= \text{return on asset class } i \text{ in period } t, i = 1, \ldots, N, \\
W_t &= \text{total wages of active participants in period } t, \\
P_t &= \text{total benefit payments in period } t, \\
L_t &= \text{total value of liabilities after period } t.
\end{aligned}
$$

Their realizations in scenarios $s \in \mathcal{S}$ are denoted by

$$
\omega_t^s = (r_{1t}^s, r_{2t}^s, \ldots, r_{Nt}^s, W_t^s, P_t^s, L_t^s).
$$

Obviously, $\omega_t^s = \omega_t^q$ if the scenarios s and q coincide up to and including period t.

The decisions at time $t \in \mathcal{T}_0$ are denoted by the vector

$$
x_t = (V_t, Z_t, \Delta^+ X_{1t}, \ldots, \Delta^+ X_{Nt}, \Delta^- X_{1t}, \ldots, \Delta^- X_{Nt}, c_{t+1}),
$$

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where

$$V_t \quad = \quad \text{restitution to the sponsor at time } t,$$

$$Z_t \quad = \quad \text{remedial contribution by the sponsor at time } t,$$

$$\Delta^+ X_{it} \quad = \quad \text{value of assets in class } i \text{ bought at time } t, i = 1, \ldots, N,$$

$$\Delta^- X_{it} \quad = \quad \text{value of assets in class } i \text{ sold at time } t, i = 1, \ldots, N,$$

$$c_{t+1} \quad = \quad \text{contribution rate for period } t+1.$$

At the time horizon $t = T$, only the decisions V_T and Z_T occur. The precise meaning of the variables V_T and Z_T will be explained in the next section. Since we introduce decision variables for the same decision in different scenarios, we have the following set of decision variables: for $s \in \mathcal{S}$ and $t \in \mathcal{T}_0$ (or $\mathcal{T}_1$),

$$x_t^s = (V_t^s, Z_t^s, \Delta^+ X_{1t}^s, \ldots, \Delta^+ X_{Nt}^s, \Delta^- X_{1t}^s, \ldots, \Delta^- X_{Nt}^s, c_{t+1}^s).$$

The corresponding nonanticipativity constraints are, for $t \in \mathcal{T}_0$,

$$x_\tau^s = x_\tau^q \qquad \tau = 0, 1, \ldots, t,$$

for all scenarios s and q that coincide up to time t.

The following additional variables are important too. For each scenario s, and for $t \in \mathcal{T}_1$:

$$A_t^s \quad = \quad \text{total asset value at time } t$$

$$X_{it}^s \quad = \quad \text{value of investments in asset class } i, \text{ at the beginning of period } t.$$

These are state variables (together with L_t^s). They are determined by the parameters and the decision variables, but from an optimization point of view they are decision variables too, if one includes their definitions as constraints in the model, as we shall do. Next, we have to explain in more detail what we mean by 'time t' in the definition of A_t^s. We assume that at the end of period t, i.e., just before time t, the contribution of period t comes in and the benefit obligations of period t are paid. At the same time, the revenues of the assets of period t are revealed. After carrying over a possible restitution to the sponsor V_t, or receiving a remedial contribution from the sponsor Z_t, the total asset value A_t^s is calculated. In Table 1 we give an overview of incoming and outgoing cash flows at time t in scenario s.

Transaction costs, arising from the adjustment of the asset portfolio at time t, do not affect A_t^s (but they do influence $X_{i,t+1}^s$, as we will see). Therefore, the total value of the

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assets at time t in scenario s is given by

$$A_t^s = \sum_{i=1}^{N}(1 + r_{it}^s)X_{it}^s + c_t^s W_t^s - P_t^s + Z_t^s - V_t^s,$$

for $s \in \mathcal{S}$ and $t \in \mathcal{T}_1$. The value of the investments in asset class i, at the beginning of period $t + 1$ in scenario s, is recursively defined by

$$X_{i,t+1}^s = (1 + r_{it}^s)X_{it}^s - \Delta^- X_{it}^s + \Delta^+ X_{it}^s - k_i(\Delta^- X_{it}^s + \Delta^+ X_{it}^s), \quad t \in \mathcal{T}_0,$$

where k_i denotes the proportional transaction cost for asset class i. The initial position in each asset class i is given by the parameter X_{i1} (so that $X_{i1}^s = X_{i1}$ for each scenario $s \in \mathcal{S}$). Moreover, all assets should be allocated:

$$\sum_{i=1}^{N} X_{i,t+1}^s = A_t^s, \qquad s \in \mathcal{S}, t \in \mathcal{T}_0.$$

Realistic scenarios have to satisfy some characteristics. They should for example be consistent with macroeconomic and financial theory. Another aspect is that they should be arbitrage free. That is, it should not be possible to construct a portfolio with zero value such that the payoff in all possible future states is nonnegative and strictly positive in at least one future state.

3. ALM model for pension funds: constraints and objective

In the previous section we defined the scenarios and the decision variables of the model. Now we will introduce a variety of possible constraints and the objective function. Some constraints are already mentioned before: the nonanticipativity constraints and the definitions of the state variables. In addition, nonnegativity is required for the value

incoming cash flows	outgoing cash flows
$c_t^s W_t^s$	P_t^s
$\sum_{i=1}^{N} r_{it}^s X_{it}^s$	V_t^s
Z_t^s	

Table 2.1: Incoming and outgoing cash flows at time t in scenario s

of each asset class i and for restitutions and remedial contributions

$$\Delta^+ X_{it}^s \geq 0, \; \Delta^- X_{it}^s \geq 0, \; V_t^s \geq 0, \; Z_t^s \geq 0, \quad s \in \mathcal{S}, t \in \mathcal{T}_0.$$

There are also fund-dependent lower and upper bounds for the asset mix:

$$w_i^l \sum_{j=1}^N X_{jt}^s \leq X_{it}^s \leq w_i^u \sum_{j=1}^N X_{jt}^s \quad s \in \mathcal{S}, t \in \mathcal{T}_1,$$

where w_i^l and w_i^u are parameters that specify upper and lower bounds on the value of asset class i as a fraction of the total asset portfolio.

Moreover, usually policy constraints are included. For instance, upper and lower bounds on the contribution rate are given by

$$c^l \leq c_t^s \leq c^u, \quad s \in \mathcal{S}, t \in \mathcal{T}_1,$$

where the numbers c^l and c^u are fund dependent.

The most important constraints, of course, deal with the goal of the pension fund: in all circumstances the funding ratio must be sufficiently high. An obvious way to incorporate this goal in the model, is to add the constraints

$$A_t^s \geq \alpha L_t^s, \quad s \in \mathcal{S}, t \in \mathcal{T}_1,$$

where α is a policy parameter, usually greater than or equal to one. For example, if $\alpha = 1$, the condition asks for nonnegative surpluses $S_t^s = A_t^s - L_t^s$ for all scenarios $s \in \mathcal{S}$ and times $t \in \mathcal{T}_1$. However, such an implementation might be too strict: it is quite possible that, due to unfavorable developments in the stock market, underfunding is unavoidable, or that underfunding can only be avoided at unrealistic funding costs. For this reason, we relax these conditions in the next subsections. In the first one, chance constraints are explained. They model so-called Surplus at Risk and some variants of it. Next, we model the possibility of remedial cash flows from or to the sponsor to react on a situation of repeated underfunding or overfunding. This is a novelty in ALM modeling; as for chance constraints, binary variables are needed to model it. Similarly, binary variables are used in the third subsection to model the possibility, that certain actions (e.g. taking a position in derivative securities) are allowed, but only in precisely described circumstances. In Section 3.4 flexible goal constraints are formulated with respect to the contribution policy of the pension fund. In the last subsection the objective function is defined: given the constraints, the expected discounted contributions are minimized, together with artificial penalty costs for undesirable decisions such as remedial contributions or rapidly changing contribution rates. We note that the most

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important goals of the pension fund are represented in the constraints, as is the case in many other models. The objective function acts as a criterion, which enables the use of optimization algorithms to find a good strategy. Penalty parameters in the objective function do not necessarily have an exact financial interpretation. By varying them, one may try to find feasible solutions that are appreciated by the management of the pension fund.

3.1 Chance constraints

The management of a pension fund wishes to satisfy the goal constraints $A_t^s \geq \alpha L_t^s$ at all times t and for all scenarios s, for some $\alpha \geq 1$. As explained above, this should not be modeled as a hard constraint. A more realistic model states that the funding ratio in the next period should be sufficiently large with a certain prescribed probability ϕ_t:

$$P(A_t^s \geq \alpha L_t^s) \geq \phi_t, \quad t \in \mathcal{T}_1. \tag{1}$$

In these chance constraints, the probability distribution used is the conditional distribution of ω_t given the observed values of $\omega_1, \ldots, \omega_{t-1}$. The value of the parameter ϕ_t, the minimum required reliability at time t, is set by the decision makers. It should not be set too low, because then it will lose its meaning of modeling a goal. On the other hand, solving models with $\phi_t = 1$ leads to expensive solutions or to infeasibilities. Also note that ϕ_t may be time dependent. In earlier periods, it may be even less desirable to have a low funding ratio.

In formulation (1), $P(A_t^s \geq \alpha L_t^s)$ is called the reliability and $1 - P(A_t^s \geq \alpha L_t^s)$ is called the risk of infeasibility. Decisions that are insufficiently reliable (with respect to the next decision moment) are not accepted. This restricts the feasible region.

The above formulation is closely related to Surplus at Risk (SaR). This can be seen as follows. SaR is defined as the estimate of the maximum amount that could be lost in one time period with a prespecified probability due to unfavorable market circumstances. The amount is calculated on the basis of the surplus of the pension fund. Therefore, for any time t the chance constraints in (1) with $\alpha = 1$, describe the condition that the probability that at that time the SaR is negative is at most $1 - \phi_t$.

In the optimization model, condition (1) acts as a constraint on the decisions at time t, in terms of consequences at time $t + 1$. As a matter of fact, although the representation in (1) does not show this explicitly, there are many chance constraints of this type at time t. Basically, there is a chance constraint for every node in the scenario tree corresponding to time $t \in \mathcal{T}_0$. Since inequalities (1) cannot be handled directly in a

linear programming framework, they have to be rewritten. This can be done as follows. At time t, we observe the realization of ω_t, and therefore know the actual state of affairs $s_t^k = (\omega_1, \ldots, \omega_t)$. The set of all posible states of affairs at time t will be denoted by K_t. Defining $p_{t+1}(s|s_t^k)$ as the conditional probability that scenario s occurs at time $t + 1$ given the state of affairs s_t^k, we can write the chance constraints as

$$\sum_{s=1}^{S} p_{t+1}(s|s_t^k) I_{\{A_{t+1}^s < \alpha L_{t+1}^s\}}(s) \leq 1 - \phi_t, \quad k \in K_t, t \in \mathcal{T}_0$$

where $I_{\{A_t^s < \alpha L_t^s\}}(s) = 1$ if $A_t^s < \alpha L_t^s$ and 0 otherwise. Now, we are able to write inequalities (1) in a mixed-integer linear programming formulation. We introduce binary variables $\delta_{1t}^s, s \in \mathcal{S}, t \in \mathcal{T}_1$. They play the role of shortfall indicators: δ_{1t}^s must get the value 1 at time t in scenario s if it holds that $A_t^s < \alpha L_t^s$, and it must be 0 otherwise. From an optimization point of view, these binary variables are decisions at time t, so they must satisfy the corresponding nonanticipativity constraints. In terms of these additional decision variables, the chance constraints can be written as linear inequalities:

$$M_1 \delta_{1,t+1}^s \geq \alpha L_{t+1}^s - A_{t+1}^s, \quad s \in \mathcal{S}, t \in \mathcal{T}_0 \tag{2}$$

$$\sum_{s=1}^{S} p_{t+1}(s|s_t^k) \delta_{1,t+1}^s \leq 1 - \phi_t, \quad k \in K_t, t \in \mathcal{T}_0.$$

Here, M_1 is a sufficiently large number.

This formulation was introduced by Dert [5]. He developed a heuristic which first obtains a feasible solution of an ALM model satisfying such chance constraints. Next, this solution is improved, while staying feasible. It is not guaranteed, however, that the heuristic finds an optimal solution.

3.2   Remedial contribution after several periods of underfunding

Pension funds want to avoid situations of underfunding, since these are a highly undesirable: not all future benefit payments can be guaranteed. One possibility to deal with underfunding is that the sponsor immediately has to pay a remedial contribution, such that the funding ratio is at least α again. However, it is quite possible, that such a radical measure is not really necessary. For instance, if there is a quick recovery in the stock market after a correction, it may not be necessary to have a remedial contribution from the sponsor to the fund. If no remedial contribution needs to be paid, the total cost of funding will be reduced.

Instead of requiring a remedial contribution as soon as $A_t^s < \alpha L_t^s$, we propose to base this decision on the funding ratios of the last b periods, where b is a natural number chosen by the management of the pension fund or stipulated by the supervisor of the fund. If in at least a of these periods the funding ratio is less than α, the situation of the pension fund is weak and the sponsor has to pay a remedial contribution to restore the funding ratio. Obviously, a direct payment in case of a funding ratio less than α is a special case of our more general formulation. Requiring a remedial contribution as soon as $A_t^s < \alpha L_t^s$ is achieved by setting $a = b = 1$. This new modeling is important for at least two reasons. The supervisor of pension funds in the Netherlands seems to judge the solvency position of funds partly on the basis of the chance of underfunding in three successive years ([9]). A second advantage of the above modeling is that it may lead to a lower total cost of funding, which is positive for all active participants, since it may lead to a lower contribution rate.

Next we model the payment of a remedial contribution after several periods of underfunding as mixed-integer restrictions. We introduce binary variables δ_{1t}^s and δ_{2t}^s, which are used (and needed) to count events. The binary variables in the first inequalities, which are the same as in inequalities (2), indicate whether $F_t^s < \alpha$, in which case $\delta_{1t}^s = 1$. The latter inequalities below force δ_{2t}^s to equal 1 if in the last b periods the funding ratio is at least a times less than α.

The modeling of remedial contributions as proposed above can be modeled by mixed-integer restrictions as follows:

$$M_1 \delta_{1t}^s \geq \alpha L_t^s - A_t^s$$

$$b \delta_{2t}^s \geq \sum_{u=(t-b)^+ + 1}^{t} \delta_{1u}^s - a + 1.$$

As before, M_1 is a sufficiently large number, and $(x)^+ := \max\{x, 0\}$ for every $x \in R$.

The sponsor has to pay a remedial contribution if $\delta_{2t}^s = 1$:

$$Z_t^s \geq \alpha L_t^s - A_t^s - M_2(1 - \delta_{2t}^s),$$

where M_2 is a sufficiently large number.

If the sponsor has to pay a remedial contribution at time t, we still count period t as a period of underfunding ($\delta_{1t}^s = 1$). Therefore, it is possible that the sponsor has to pay remedial contributions in successive periods. For the time being, we have chosen this formulation, since it is easier to model and – from a practical point of view – it is more

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cautious.

If the funding ratio F_t^s is greater than or equal to β, the pension fund may make a restitution V_t^s to the sponsor. In this situation, we can use a reasoning similar to the one given above. Instead of a direct restitution as soon as $A_t^s > \beta L_t^s$, the pension fund may make a restitution only after several periods in which $A_t^s > \beta L_t^s$. If in these periods the funding ratio falls below β again, no restitution will be made. Otherwise it will be made immediately after these periods.

3.3 Conditional constraints

Conditional constraints are of the form: only if a certain event happens, it is allowed to undertake certain actions. To model conditional constraints, binary variables are needed: they take the value 1 if the condition is satisfied and 0 otherwise. The following example illustrates how such conditional constraints can be useful in ALM modeling.

From research on returns on stocks and stock indices, Drijver and Otter [7] found that if there is a large negative return in one period, the probability that in the next period there is also a large negative return is statistically greater than would be expected under the assumption that returns are independent and identically distributed.

In an ALM model, this information can be exploited as follows. Since the planning horizon of a pension fund is large, there is usually no reason to consider short-term instruments, although according to Dert [6], pension funds should invest more in derivative securities. However, if a large drop in returns is observed, it appears to be desirable to allow the fund to take a position in derivative securities, in order to protect the asset portfolio from a further decline.

This extension can be modeled using binary variables in linear constraints. In our example, the binary variable δ_{3it} equals 1 if and only if a drop of at least $\xi_{it} \times 100\%$ occurs in asset class i in period t.

$$M_3 \delta_{3it}^s \geq (1 - \xi_{it}) X_{i,t-1}^s - X_{it}^s$$

$$M_4 (1 - \delta_{3it}^s) \geq X_{it}^s - (1 - \xi_{it}) X_{i,t-1}^s$$

$$\delta_{3it}^s \in \{0, 1\}$$

with M_3 and M_4 sufficiently large numbers. In that case, a position in derivative securities D_{it}^s is allowed (of at most $D_i^{\max}$), which is modeled by the constraints $D_{it}^s \leq D_i^{\max} \delta_{3it}^s$.

3.4   Flexible modeling of the contribution rate

In the introduction of this section, hard upper and lower bounds c^l and c^u on the contribution rate c_t^s were given. Here, we consider a refinement. Not only the level of the contribution rate is important, pension funds (and the sponsor of the fund) also take into account the stability of the contribution rate, since too much variability is undesirable. We can model this as

$$-\rho \leq c_t^s - c_{t-1}^s \leq \rho \qquad\qquad (3)$$

for a sufficiently small value of ρ, a parameter to be specified by the decision makers. When the value of ρ is too large, it loses its meaning in modeling a contribution rate which does not change too fast.

On the other hand, if the funding ratio is relatively low for a longer period, it may be better to increase the contribution rate than to ask for remedial contributions for a number of successive periods. This may lead to an increase which is greater than ρ. Hence, it would be better to specify (3) as a goal constraint: changes greater than ρ are allowed, but they are penalized in the objective function.

We can model this in a linear programming formulation by the introduction of additional decision variables c_{t+}^s, representing the amount by which the increase in the contribution rate exceeds ρ at time t. The second inequality in (3) is replaced by

$$c_{t+}^s \geq c_t^s - c_{t-1}^s - \rho$$

$$c_{t+}^s \geq 0.$$

In the objective function, we penalize $c_{t+}^s = (c_t^s - c_{t-1}^s - \rho)^+$, which is positive if $c_t^s > c_{t-1}^s + \rho$.

We can use an analogous reasoning in the situation if the funding ratio is relatively high for a number of successive periods. In this situation it may be desirable to lower the contribution rate in two successive periods by more than ρ instead of making restitutions to the sponsor in several consecutive periods.

We can model this by the introduction of additional decision variables c_{t-}^s, representing the amount by which the decrease of the contribution rate in time period t exceeds ρ:

$$c_{t-}^s \geq c_{t-1}^s - c_t^s - \rho$$
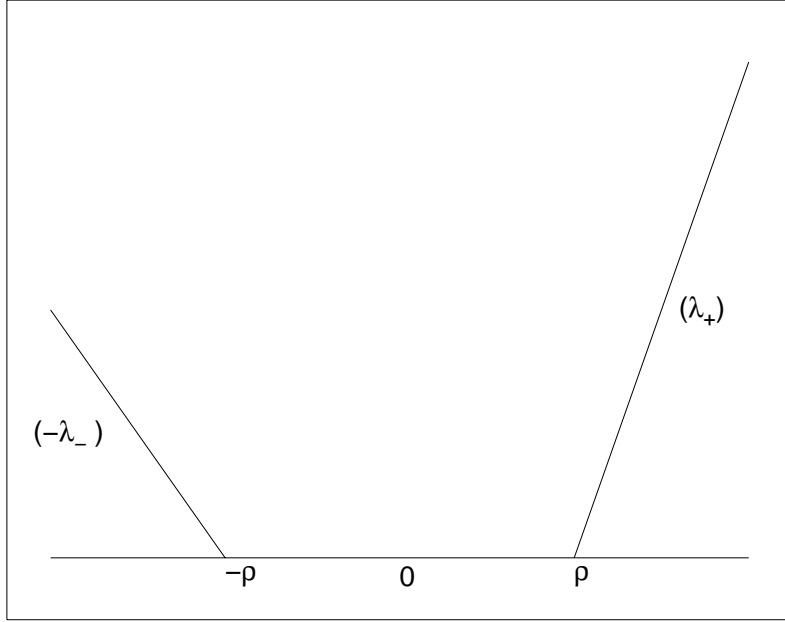
$$c_{t-}^s \geq 0.$$

Figure 3.1: Penalization for the change in the contribution rate in two successive periods.

In the objective function, we penalize $c_{t-}^s = (c_{t-1}^s - c_t^s - \rho)^+$, which is positive if $c_t^s < c_{t-1}^s - \rho$.

We penalize c_{t+}^s and c_{t-}^s by positive parameters λ_+ and λ_- (usually $\lambda_- \leq \lambda_+$), whereas no penalty is imposed if $|c_t^s - c_{t-1}^s| \leq \rho$. Figure 3.1 shows an example of such a penalty function.

In stochastic programming, this structure, that models piecewise linear increasing costs for shortages and surpluses, is known as multiple simple recourse.

3.5 Objective function

A pension fund wants to minimize the total cost of funding, i.e., the contribution rate for the active participants. Moreover, penalty costs are assigned to the undesirable events discussed above: remedial contributions, large increases and decreases in the contribution rate, and restitutions to the sponsor. The latter penalty can also be interpreted as a cost: a lower contribution rate would have been sufficient for a healthy position (in

terms of the funding ratio) of the pension fund.

All these components together constitute the objective function

$$\sum_{s=1}^{S}\Big(\sum_{t=0}^{T} p^s \gamma_t^s (c_t^s W_t^s + \lambda_Z Z_t^s + \lambda_V V_t^s + \lambda_+ c_{t+}^s W_t^s + \lambda_- c_{t-}^s W_t^s)\Big), \qquad (4)$$

where p^s is the probability of scenario s, γ_t^s is the discount factor for a cash flow in period t in scenario s, and λ_Z, λ_V, λ_+ and λ_- are penalty parameters. The definition of the objective function reflects that the pension fund seeks to minimize the total discounted funding costs.

4.   Solving the ALM model

We will use stochastic programming to find a good feasible solution of the ALM model described in Sections 2 and 3. The reason is the following. The ALM model is described with linear (in)equalities and a linear objective function. Since stochastic parameters, like returns on the different asset classes, inflation rates and the total wages of the participants, play a crucial role in the ALM model, we need a solution technique that explicitly takes into account these uncertainties. Stochastic programming models are well-suited for our problem, since they allow for progressive revelation of information through time matched by multiple decision stages, where each decision is adapted to the information available at that time. The pension fund will update the portfolio as information about the random parameters becomes available.

Note that, although we are modeling a multistage decision problem, only the first-stage decisions will be implemented. The outcomes of the model for later stages are scenario dependent, and therefore not implementable. Moreover, new information becomes available before decisions on second-stage variables need to be taken. At that time an adapted version of the model is solved; this process is repeated, resulting in a rolling horizon approach.

As a first step towards solving our multistage mixed-integer stochastic program, it can

be written as a (very) large-scale mixed-integer linear program:

$$
\begin{aligned}
\min \quad & \sum_{s=1}^{S} p^s (d_0^s x_0^s + \ldots + d_T^s x_T^s) \\
\text{s.t.} \quad & G_0 x_0^s && = b_0 \\
& H_t^s x_{t-1}^s + G_t^s x_t^s && = b_t^s, && s \in \mathcal{S}, \ t \in \mathcal{T}_1 \\
& C x_t && = 0, && t \in \mathcal{T}_0 \\
& x_t^s \in \mathcal{X}_t, && && s \in \mathcal{S}, \ t \in \mathcal{T}_0.
\end{aligned}
\tag{5}
$$

The objective function in (5) is a reformulation of (4). The matrix G_0 and vector b_0 define deterministic constraints on the first-stage decision x_0. For $t \in T_1$, G_t, H_t and b_t define feasible regions for the recourse decisions $x_1^s, \ldots, x_T^s$. Of course, here the vector x_t^s is larger than has been defined in Section 2: also decision variables that are introduced later, such as the binary decision variables, are supposed to be included. The matrices G_t and H_t and vector b_t may depend on $(\omega_1, \ldots, \omega_t)$ but not on $(\omega_{t+1}, \ldots \omega_T)$. The equalities $C x_t = 0, t \in \mathcal{T}_0$, where $x_t = (x_t^1, \ldots, x_t^S)$, define the nonanticipativity constraints. Upper and lower bounds on the decision variables x_t, and also integrality restrictions for the binary decision variables, are modeled by $\mathcal{X}_t = \mathcal{X}_t(\omega_1, \ldots, \omega_t)$, $t \in \mathcal{T}_0$.

This model embodies the main features of a decision problem under uncertainty. At time $t = 0$, the decision makers have to select a decision, taking into account future realizations of the underlying multidimensional stochastic process. After $t = 0$, decisions are allowed to be functions of the observed realization of the stochastic parameters and the past decisions. At each time stage, previous decisions affect current problems through the stochastic matrices H_t.

Problem (5) is a large-scale structured mixed-integer linear program. Since the decision makers are faced with uncertainties at each time stage, the number of scenarios (and the number of variables) grows exponentially with the time horizon if in each node the number of successors is at least two. This means that ALM models of realistic size are too large to solve by means of standard solution techniques. It is well known that mixed-integer programming problems are NP-hard (see e.g. [14]); therefore we need to exploit the special structure in (5) even to solve the model approximately.

Because of the size of the problem and the integrality restrictions, the ALM model cannot be solved in reasonable time by standard solution techniques. Therefore, research is needed to get a good feasible solution for the ALM model. We will work on this problem and see how for example valid inequalities, Benders decomposition

and (stochastic) branch and bound (see [12]) can be used (in a heuristic) to get a good feasible solution for the ALM model with binary variables.

5.   Conclusions

We developed an Asset Liability Management (ALM) model which contains some new and important aspects which we did not encounter in other ALM models, thus allowing more realistic models of the ALM process for pension funds. The first new aspect is the flexible modeling remedial contributions after several periods of underfunding. This new modeling feature is important, since according to Klein Haneveld [9] the superivisor of pension funds in The Netherlands uses this criterion to judge the solvency position of a pension fund.

Also new in ALM modeling is the introduction of conditional constraints. For example, only if a certain drop in the asset portfolio is observed, a position in derivative securities may be taken.

Both modeling issues require the introduction of binary variables. These variables are also needed in modeling chance constraints in a linear programming formulation. We used chance constraints, which are not new in ALM, to impose the condition that the probability of underfunding in the next time period is less than a prespecified parameter.

ALM is concerned with making all promised payments at minimum costs. In the objective function these costs appear, together with penalties of undesirable events.

Since binary variables appear in the model, it is very difficult to obtain even a feasible solution for ALM models of realistic size. In future research we will try to obtain a good feasible solution for the multistage mixed-integer stochastic program.

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