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Bagwell's paradox, forward induction and outside option games

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Abstract

In Stackelberg-like games there is an advantage of moving first. However, Bagwell (1995) shows that this result may not hold if the second player can make only imperfect observations. We explore whether this paradox also holds when the advantage comes from forward induction arguments in the class of outside option games. *JEL numbers*: C72.

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1. Introduction

Bagwell (1995) shows that the advantage of moving first may disappear when actions are not perfectly observed by the player moving second. When the advantage is due to forward induction reasoning, Ferreira (2010) finds that whether it is lost or not depends on the particular game. It is, therefore, of interest to know in which games the introduction of noise destroys the forward induction selection and in which games it does not. In this work we study with detail the class of outside option games.

The intuitive idea in forward induction says that, when a player moves first, she must be able to induce her preferred equilibrium when her action clearly indicates which one it is. However, it is hard to translate this idea into a formal, universal definition. Van Damme (1989) provides a definition for outside option games, but when noise is introduced in a game in this class, the new game does not belong to it and the definition no longer applies.

Thus, we use instead the concepts of iterative elimination of dominated strategies, *IEDS*, and of equilibrium evolutionarily stable set, *EESS*. We show that the selection made by *IEDS* in the outside option game is preserved in the noise game. Because whenever *IEDS* selects a unique equilibrium, it satisfies van Damme's definition we then have that the selection in the noise game also selects the “intuitive” forward induction compatible equilibrium. Next we show that the concept of *EESS* makes the same selection in the game without and with noise. This is interesting because *EESS* is always compatible with van Damme's definition in the game without noise.

These two results show that the advantage of forward induction is not lost with the addition of noise in outside option games.

2. Games with an outside option

Following van Damme, in an outside option game, Γ_o, Player 1 plays first by choosing between actions *I* and *O*. If she plays *O* the game ends. If she chooses *I* a simultaneous game between herself and Player 2 follows, in which A_i denotes Player *i*'s strategy set. Within this class of games, an equilibrium *s* in which Player 1 plays *O* is said to satisfy the forward induction criterion if the following does not occur: there is precisely one equilibrium in which Player 1 chooses *I* that gives this player a strictly better payoff than *s*.

We define the noise game associated with an outside option game, Γ_o^N, as follows. First Player 1 chooses between actions *I* and *O*, then nature produces a signal $\phi \in \{i, o\}$ with $\Pr(\phi = i | I) = \Pr(\phi = o | O) = 1 - \varepsilon$. If Player 1 has selected *I* she now has to choose a strategy for the simultaneous game, whereas if she selected *O* she does nothing else. Player 2 observes the signal ϕ but not the choice of Player 1, then he chooses a strategy within his strategy set in the simultaneous game after any observation. For instance, if Player 1 selected *O*, Player 2 still has to make his decision as he does not know for sure that the game ended. Notice that in the noise game the strategy set for Player 2 is $S_2 = A_2 \times A_2$, where $s_2 = a_2 a_2^{'} \in S_2$ indicates that Player 2 plays a_2 after observing

$\phi = i$, and $a_2^{'}$ after $\phi = o$. Finally, abusing notation we will denote by $u_i(.)$ the utility of player *i* in both games. The arguments in the utility function will clearly indicate the game of reference.

3. The iterative elimination of dominated strategies

The noise game defined after an outside option game does not have the structure of an outside option game and van Damme's definition of forward induction is not applicable. However we can still apply other definitions. First we explore *IEDS*, as it is well known that whenever *IEDS* selects one equilibrium in outside option games this equilibrium satisfies van Damme's criterion. Next we show that *IEDS* selects basically the same equilibria in the games with and without noise.

Definition 1. *A strategy s_i is dominated by $s_i^{'}$ if $u_i(s_i, s_{-i}) \le u_i(s_i^{'}, s_{-i})$ for all s_{-i}, and $u_i(s_i, s_{-i}^{'}) < u_i(s_i^{'}, s_{-i}^{'})$ for some $s_{-i}^{'}$.*

Denote by $\Gamma^{(i)}$ the game that results after eliminating all the dominated strategies in the game $\Gamma^{(i-1)}$, where $\Gamma^{(0)} = \Gamma$. We say that s_i survives *IEDS* in the game *Γ* if s_i is a strategy in the game $\Gamma^{(\infty)}$.

Proposition 1. *An outside option game has only one equilibrium (s_1, a_2) that survives IEDS if and only if all the equilibria than survive IEDS in the noise game when ε is small enough are of the form $(s_1, a_2 a_2^{'})$.*

Proof:

We proceed in four steps.

(i) *The strategy profile (s_1, a_2) is an equilibrium in Γ_o if and only if $(s_1, a_2 a_2^{'})$ is an equilibrium in Γ_o^N for ε small enough.*

Consider $u_1(s_1, a_2 a_2^{'}) \ge u_1(s_1^{'}, a_2 a_2^{'})$ for all $s_1^{'} \in S_1$. By definition of Γ_o^N it is equivalent to $(1-\varepsilon)u_1(s_1, a_2) + \varepsilon u_1(s_1, a_2^{'}) \ge (1-\varepsilon)u_1(s_1^{'}, a_2) + \varepsilon u_1(s_1^{'}, a_2^{'})$ for all $s_1^{'} \in S_1$, which holds if and only if $u_1(s_1, a_2) \ge u_1(s_1^{'}, a_2)$ for all $s_1^{'} \in S_1$ if ε is small enough.

The proof for Player 2 is entirely similar.

(ii) *The strategy profile (s_1, a_2) contains no dominated strategies in Γ_o if and only if $(s_1, a_2 a_2)$, $(s_1, a_2 a_2^{'})$ and $(s_1, a_2^{'} a_2)$ contain no dominated strategies in Γ_o^N for all $a_2^{'} \in A_2$ if ε is small enough.*

Assume $s_1 \in S_1$ is dominated by $s_1^{'} \in S_1$ in Γ_o, then $u_1(s_1, a_2) \le u_1(s_1^{'}, a_2)$ for all $a_2 \in A_2$ and $u_1(s_1, a_2^{'}) < u_1(s_1^{'}, a_2^{'})$ for some $a_2^{'} \in A_2$. Then, $(1-\varepsilon)u_1(s_1, a_2) + \varepsilon u_1(s_1, a_2^{''}) \le (1-\varepsilon)u_1(s_1^{'}, a_2) + \varepsilon u_1(s_1^{'}, a_2^{''})$ for all $a_2, a_2^{''} \in A_2$ and

$(1-\varepsilon)u_1(s_1,a_2^{'})+\varepsilon u_1(s_1,a_2^{'})<(1-\varepsilon)u_1(s_1^{'},a_2^{'})+\varepsilon u_1(s_1^{'},a_2^{'})$ for $a_2^{'}\in A_2$. The last inequalities are equivalent to saying that $s_1 \in S_1$ is dominated in Γ_o^N.

To proceed with Player 2's strategy, consider that $a_2 \in A_2$ is dominated by $a_2^{'} \in A_2$ in Γ_o, then $u_2(s_1,a_2)\le u_2(s_1,a_2^{'})$ for all s_1, and $u_2(s_1^{'},a_2)<u_2(s_1^{'},a_2^{'})$ for some $s_1^{'}$. Clearly $s_1^{'}\in I\times A_1$ as $u_2(Oa_1,s_2)$ does not depend on s_2. Since $u_2(s_1,a_2)=u_2(s_1,a_2a_2)$ for all a_2 it is immediate that a_2a_2 is dominated by $a_2^{'}a_2^{'}$. To see that $a_2a_2^{'}$ is dominated assume that $s_1 \in I\times A_1$ (if not, the weak inequality follows immediately) and write $u_2(s_1,a_2a_2^{'})=(1-\varepsilon)u_2(s_1,a_2)+\varepsilon u_2(s_1,a_2^{'})\le u_2(s_1,a_2^{'}),$ for all s_1 if and only if $u_2(s_1,a_2)\le u_2(s_1,a_2^{'}),$ for all s_1. The inequality is strict for the same $s_1^{'}$ that satisfies $u_2(s_1^{'},a_2)<u_2(s_1^{'},a_2^{'})$ as long as $\varepsilon>0$. To show that $a_2^{'}a_2$ is dominated by $a_2^{'}a_2^{'}$ proceed similarly and notice that in this case it must be that $\varepsilon<1$.

The converse, if a_2a_2, $a_2a_2^{'}$ and $a_2^{'}a_2$ are dominated by $a_2^{'}a_2^{'}$ then a_2 is dominated by $a_2^{'}$, is trivial.

(iii) *If $a_2a_2^{'}$ is not dominated in Γ_o^N then a_2a_2 and $a_2^{'}a_2^{'}$ are not dominated either.*

This is straightforward as the conditions for domination of $a_2a_2^{'}$ are a convex combination of the conditions for a_2a_2 and $a_2^{'}a_2^{'}$.

(iv) Steps (ii) and (iii) before guarantee that, once a strategy $s_i \in S_i$ is eliminated in Γ_o it is also eliminated in Γ_o^N, and vice versa. This way, by induction, the *IEDS* eliminates the same strategies in $(\Gamma_o)^{(t)}$ and in $(\Gamma_o^N)^{(t)}$ (the induction argument is just the repetition of (ii) and (iii) for $t>1$). To complete the proof note now that, if there exists an equilibrium that survives *IEDS* in Γ_o^N that is of the form $(s_1,a_2^{''}a_2^{'})$ with $a_2^{''}\neq a_2$, then, by (i), $(s_1,a_2^{''})$ is an equilibrium in Γ_0 and, also by (i) $(s_1,a_2^{''}a_2^{''})$ is an equilibrium in Γ_o^N. By (ii) it follows that $(s_1,a_2^{''}a_2^{''})$ survives *IEDS* in Γ_o^N and, again by (ii) so does $(s_1,a_2^{''})$ in Γ_0, in contradiction with the fact that (s_1,a_2) is the only equilibrium in Γ_0. Q.E.D.

4. The equilibrium evolutionarily stable set

Next we consider the equilibrium evolutionarily stable set, *EESS*. In the language of evolutionary games, an *EESS* has to be immune to entry by a small proportion of mutants whose strategy is a best reply to the strategy of the post-entry population. More precisely, in a two-player game, let $\Sigma=\Sigma_1\times\Sigma_2$ denote the set of mixed strategy profiles and let $\mathrm{B}(s)=\mathrm{B}_1(s)\times\mathrm{B}_2(s)$ denote the set of mixed best replies against *s*, then we can write the following definition.

Definition 2 (Swinkels, 92). *The set $\Theta\subset\Sigma$ is an EESS if it is a minimal closed and non-empty set of Nash equilibria that satisfies*

(S) there exists $\delta'>0$ such that for all $\delta \in (0,\delta')$, for all $s \in \Theta$, and for all $s' \in \Sigma$

$$s' \in \mathrm{B}((1-\delta)s+\delta s') \Rightarrow (1-\delta)s+\delta s' \in \Theta .$$

In the class of generic outside option games where the only equilibrium that satisfies van-Damme's definition of forward induction requires Player 1 not to choose "out", Hauk and Hurkens (2002) show that *EESS* also selects this equilibrium (this is not true for other refinements). Let us denote this class of games by Γ_o. The next proposition shows that the same outcome is obtained by *EESS* if noise is added.

Lemma. *Let $s'=(s_1^{'},s_2^{'})$ be a strategy profile in game Γ_o, and let $(s_1^{'},s_2^{'}s_2^{''})$ be a strategy profile in game Γ_o^N. Then*

$$s'=(s_1^{'},s_2^{'}) \in \mathrm{B}((1-\delta)(s_1,s_2)+\delta(s_1^{'},s_2^{'}))$$

if and only if

$$(s_1^{'},s_2^{'}s_2^{'}) \in \mathrm{B}((1-\delta)(s_1,s_2s_2^{'})+\delta(s_1^{'},s_2^{'}s_2^{'})) .$$

Proof. It is straightforward to check that the implications hold for the best response of Player 1. For Player 2, the first relation means

$$u_2((1-\delta)s_1+\delta s_1^{'},s_2^{'}) \geq u_2((1-\delta)s_1+\delta s_1^{'},s_2^{''}) \tag{1}$$

for all $s_2^{''} \in \Sigma_2$. The second relation means

$$u_2((1-\delta)s_1+\delta s_1^{'},s_2^{'}s_2^{'}) \geq u_2((1-\delta)s_1+\delta s_1^{'},s_2^{''}s_2^{'''}) \tag{2}$$

for all $s_2^{''}s_2^{'''} \in \Sigma_2 \times \Sigma_2$. The right hand sides of inequalities (1) and (2) have the same maximum. The left hand side of (2) is calculated as

$$(1-\varepsilon)u_2((1-\delta)s_1+\delta s_1^{'},s_2^{''})+\varepsilon u_2((1-\delta)s_1+\delta s_1^{'},s_2^{'''}) .$$

It is now trivial to check that both inequalities hold at the same time. Q.E.D.

Proposition 2. *Let $\Theta=\{(s_1,s_2)\}$ constitute the only EESS of Γ_o, then the set $\{(s_1,s_2s_2)\}$ is the only EESS of Γ_o^N.*

Proof. Consider a set Θ^N that is an *EESS* of Γ_o^N and suppose that $(s_1,s_2s_2) \in \Theta^N$. If, in addition, we have that

$$(s_1^{'},s_2^{'}s_2^{''}) \in \mathrm{B}((1-\delta)(s_1,s_2s_2)+\delta(s_1^{'},s_2^{'}s_2^{''})) , \tag{3}$$

then, by (S) it must also be that

$$(1-\delta)(s_1, s_2 s_2) + \delta(s_1^{'}, s_2^{'} s_2^{''}) \in \Theta^N .$$

Notice that $s_2^{''}$ is played with probability ε . Then, by the closedness of Θ^N this implies

$$(1-\delta)(s_1, s_2 s_2) + \delta(s_1^{'}, s_2^{'} s_2^{'}) \in \Theta^N .$$

This in turn implies $(1-\delta)(s_1, s_2) + \delta(s_1^{'}, s_2^{'}) \in \Theta$ by Lemma 1. But this is a contradiction if $(s_1, s_2) \neq (s_1^{'}, s_2^{'})$ as $\Theta = \{(s_1, s_2)\}$. Thus, the only possibility for strategy $s^{'}$ to satisfy (3) is to be of the form $s^{'} = (s_1, s_2 s_2^{'''})$. But now, if

$$(s_1, s_2 s_2^{'''}) \in \mathrm{B}((1-\delta)(s_1, s_2 s_2) + \delta(s_1, s_2 s_2^{'''})) ,$$

it is also true that

$$(s_1, s_2 s_2) \in \mathrm{B}((1-\delta)(s_1, s_2 s_2) + \delta(s_1, s_2 s_2^{'''})) ,$$

in contradiction with the genericity of the game Γ_o (that is translated into the game Γ_o^N) if $s_2^{'''} \neq s_2$. Hence we have shown that if $(s_1, s_2 s_2) \in \Theta^N$ then $\{(s_1, s_2 s_2)\} \in \Theta^N$, and condition (S) is trivially satisfied, and so are closedness and minimality. Finally, no other *EESS* can exist in Θ^N as, repeating the arguments above, this would contradict the uniqueness of Θ . Q.E.D.

5. Discussion

The addition of noise may destroy the appeal of forward induction arguments, as shown in Ferreira (2010). However, for outside option games we have shown that this is not the case. The formal definition of forward induction for this class of games cannot be applied to the noise game. To make sense of the statement that “forward induction is not lost in the outside option games” we work with definitions that are compatible with forward induction in this class of games. We show that these definitions, namely *IEDS* and *EESS*, when applied in the noise game choose the counterpart of the equilibria that are compatible with forward induction in the original game.

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