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BARGAINING AND MARKETS: COMPLEXITY AND THE WALRASIAN OUTCOME

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Bargaining and Markets: Complexity and the Walrasian Outcome*

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Abstract

Rubinstein and Wolinsky (1990b) consider a simple decentralized market in which agents either meet randomly or choose their partners volunatarily and bargain over the terms on which they are willing to trade. Intuition suggests that if there are no transaction costs, the outcome of this matching and bargaining game should be the unique competitive equilibrium. This does not happen. In fact, Rubinstein and Wolinsky show that any price can be sustained as a sequential equilibrium of this game. In this paper, I consider Rubinstein and Wolinsky's model and show that if the complexity costs of implementing strategies enter players' preferences (lexicographically), together with the standard payoff in the game, then the only equilibrium that survives is the unique competitive outcome. This will be done both for the random matching and for the voluntary matching models. Thus the paper demonstrates that complexity costs might have a role in providing a justification for the competitive outcome.

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1. Introduction

In a competitive market agents take prices parametrically. This is usually justified by saying that agents are 'negligible'. In a dynamic game-theoretic context this can be sometimes be formalized by assuming that there is a continuum of anonymous agents. The equilibria of these models can be shown to coincide with the competitive equilibria of these models under some regularity conditions. On the other hand, the equilibria of dynamic games with a (large but) finite number of players can be shown to be radically different from those in a model with a continuum of players. While this paradox may seem narrow, the issue has broad economic significance. The rational for the continuum economy is that it is useful idealization for an economy with a large but finite number of agents. Clearly this idealization is of a limited value if the equilibria of finite economies are radically different from those of the continuum case. The reason for this paradox in dynamic games with a finite number of players is that players can choose history-dependent strategies. The possibility of conditioning behaviour on histories induces different expectations for future play depending on the history preceding the play. This allows one to construct a large number of (history-dependent) equilibria in which a single agent has a large effect. The best example of this is the Folk Theorem of the repeated game, which holds for an arbitrary number of players.[1] In these dynamic games a player has to consider the possible reaction of others. As a result, these equilibria will depart from the competitive outcome even in a frictionless market with a large but finite number of players. Thus even in environments in which competitive outcome might appear as the natural outcome (e.g. the case of one seller of an indivisible good who faces two buyers who bid for the unit), one can show that in general non-competitive outcomes might emerge as equilibria if the environment is modelled as a dynamic game.

One research strategy for dealing with the large number of equilibria in dynamic games is to consider explicitly bounds on human computational and storage abilities (bounded rationality). Such bounds impose restrictions on the way strategies can depend on history in what appears to be a natural way. For example, in recent years many have considered repeated games played by finite automata. (See Kalai (1990) for a survey).

In this paper, I investigate the effect of introducing complexity costs in the dynamic matching and bargaining games. In particular, I will show that complexity considerations (some elements of 'bounded rationality') can provide a game-theoretic foundation

[1]With a continuum of players, the Folk Theorem remains valid if the game is non-anonymous. With a finite number of players, Folk Theorem type results can survive even with anonymity. One needs noise (in strong form) and/or anonymity to eliminate history-dependent equilibria in repeated games with a large but finite number of players (see Green (1980), Sabourian (1989), Levine and Pesendorfer (1995), Gale (forthcoming) and Al-Najjar and Smordinsky (1999)).

for the competitive behaviour in decentralized markets with a finite number of agents. Also, I will show that in these models the introduction of complexity costs into players' preferences ensures that in equilibrium players choose history-independent (sometimes referred to as stationary or Markov) strategies.

There is already a large literature on dynamic matching models with explicit non-cooperative bargaining. (For example Rubinstein and Wolinsky 1985, Binmore and Herrero 1986, Gale 1986a and b, Mclennan and Sonnenschein 1989; also see the text on bargaining and markets by Rubinstein and Osborne (1990a) and a forthcoming book on the subject by Gale (2000)). By assuming a continuum of agents and/or by restricting the strategy sets to the stationary ones, such models have been used to provide a game-theoretic foundation to the competitive equilibrium . One of the few papers that deals with bargaining and matching with a finite number of players and with unrestricted set of strategies, is that of Rubinstein and Wolinsky (1990b) - henceforth referred to as RW. This paper considers a simple decentralized market in which agents either meet randomly or choose their partners voluntarily and bargain over the terms on which they are willing to trade. Intuition suggests that if there are no transaction costs, the outcome of bargaining should be the competitive equilibrium. This intuition turns out to valid if players are restricted to choosing history-independent strategies. However, RW demonstrate that if there are no restrictions on the set of strategies and thus players can condition their behaviour on past history of plays, then a continuum of non-competitive sequential equilibria emerges.

In sections 2 and 4 below, I shall summarise and discuss how the predictions of RW's model differ from that of the competitive behaviour. I shall then show that if player's attach some weight (lexicographically) to complexity of their strategies then the only equilibrium outcome that survives is the competitive one, and that the equilibrium strategies are stationary. This will be done both for the random matching (section 2) and for the voluntary matching (section 4) models of RW. The proofs of the results of this paper are in section 3 and Appendices A and B. Section 5 cntains some concluding remarks.

In the literature on dynamic games, the strategy of concentrating on the history-independent/stationary/Markov equilibria is very common. Very little justification is usually provided for this approach except for an occasional mention that without imposing such a restriction there is a large number of equilibria to deal with. Sometimes these Markov equilibria are proposed as "focal" points. Intuitively, one might argue that a player concerned with the cost of implementing complex strategies would choose a stationary strategy, where behaviour in each period is independent of payoff-irrelevant past history.[2,3]

[2] Stationarity is also often assumed in models of non-cooperative coalitional bargaining to get results - see, for example, Chatterjee et.al. (1993). In these models, simplicity is sometimes mentioned as a reason for selecting stationary equilibria in these models (Gul 1989).

[3] Osborne and Rubinstein (1994) have, however, provided arguments against such intuition. They argue that if equilibrium strategies are thought of as equilibrium in beliefs then it is not clear why

This paper, in addition to providing a justification for the competitive outcomes, attempts to formalise this intuition, in the context of dynamic matching and bargaining, by introducing complexity costs lexicographically with the standard payoff into the players' preference ordering as in Rubinstein (1986), Abreu and Rubinstein (1988), Piccione and Rubinstein (1993) and others. In these papers, players are modelled as finite-state automata involved in a two-player repeated game. Complexity is measured by the (arbitrarily small) cost of maintaining an additional "machine" state.

Here, I will also focus on the complexity of implementation rather than of computational complexity (see Papadimitriou, 1992) and model players as automata. But, because of the asymmetric nature of bargaining, my notion of complexity of strategies is somewhat different from that in the above literature. Informally, the measure of complexity adopted in the random matching model has the following property: if two strategies are otherwise identical except that in some instance the first strategy uses more information than that available in the current period of bargaining and the second uses only the information available in the current period then the first strategy is more complex than the second. This notion of complexity is a very weak measure of the complexity of response rules within a period. Thus, I shall refer to it as response-complexity. Chatterjee and Sabourian (1999, 2000) also use a similar complexity criterion to justify stationary equilibrium in alternating n-player bargaining games. This notion of complexity neither implies nor is implied by the 'counting states' notion of complexity. In the voluntary matching model of section 4, I shall use both the counting states measure together with response-complexity to select uniquely the competitive outcome.

The RW's dynamic matching and bargaining game is rather special. Other games might give different results. The point, here, is not that there is a right way of modelling competitive behaviour but to give example of what it takes, in terms of the primitives of the model, to obtain the competitive outcome. In particular, this paper demonstrates that complexity costs might have a role in providing a justification for a competitive equilibrium.

Gale (2000) also discusses how the introduction of 'bounded rationality' can provide a justification for the competitive equilibrium in RW's model. He obtains his results by either putting a bound on the complexity of the *strategy profiles* or by introducing noise in the implementation of the strategies. Our motivation is similar to that of Gale; however the approach taken in this paper is somewhat different from his.

2. Random matching model

RW has a model of a market with B identical buyers and S identical sellers. Let $\mathcal{B}$ denote the set of buyers and $\mathcal{S}$ denote the set of sellers. Each seller has one unit of an indivisible good. Each buyer wants to buy at most one unit of the good. The

players should believe that other players follow the same actions after histories which have involved highly non-stationary past plays.

valuations of the buyers and the sellers for one unit of the good are one and zero respectively. Throughout I assume that

$$B > S.$$

Time is discrete and each player has a discount factor $\delta \in [0,1]$. Thus if a seller sells one unit of the commodity to a buyer at a price p in any period $t = 0, 1, 2..$, the payoffs of the seller and that of the buyer are given by $\delta^t p$ and $\delta^t(1-p)$ respectively.

At each period t the agents remaining in the market are randomly matched in pairs of one seller and one buyer (all possible matches are equally likely). One member of each matched pair is then randomly chosen (with probability 1/2) to propose a price p between 0 and 1. Then the other agent accepts (A) or rejects (R) the offer. I shall denote such a match between a seller s and a buyer b with s as the proposer by the ordered pair (s, b) and a match between s and b with b as the proposer by the ordered pair (b, s). If a proposal is accepted by the responder, the parties implement it and leave the market. Rejection dissolves the match, in which case the agents proceed to the next matching stage. Any unmatched buyers are forced to remain inactive for the period.

Thus the game is such that Nature effectively chooses the random matching and the choice of proposer and responder. I shall denote a typical choice of Nature by η. Formally, I define the choice of Nature as follows. Let $q : \mathcal{S} \rightarrow \mathcal{B}$ stand for a one-to one function and let $\mathcal{Q} \equiv \{1, ..., Q\}$ be the set of all such one-to-one functions from $\mathcal{S}$ to $\mathcal{B}$. Thus at any period with S sellers and B buyers a match among the players corresponds to a choice of $q \in Q$ and $q(s)$ corresponds to s's partner in this match. Also for each match q, let $m_q(s) \in \{s, q(s)\}$ and $r_q(s) \in \{s, q(s)\}$ denote respectively the proposer and the responder when s is matched with $q(s)$. Then a choice η by Nature corresponds to $(q; \{m_q(s), r_q(s)\}_{s \in \mathcal{S}})$.

At each period t each agent has perfect information about all the past events of the game, including all the past play in matches in which he did not participate. However, when they choose their actions they do not know what actions are being simultaneously chosen by other agents.

2.1. RW's equilibrium characterisation with random matching for the case of $\delta = 1$.

The competitive model corresponds to the case in which the frictions and the transaction costs in the market are negligible. In RW's model this corresponds to the case in which the players do not discount the future. The main result of RW corresponds to this case.

Theorem 2.1. *(See RW) If $\delta = 1$ then for every price $\overline{p}$ between 0 and 1 and for every one to one function q from the set of sellers to the set of buyers there exists a sequential equilibrium in which each seller s sells one unit of the good to buyer $q(s)$ for a price $\overline{p}$.*

Thus, there is a continuum of prices that can be sustained as a sequential equilibria.[4] On the other hand, $p = 1$ is the unique competitive equilibrium because $B > S$. Therefore the competitive outcome is not the unique sequential equilibria of the matching and bargaining game with a finite number of agents (irrespective of the numbers B and S).

The intuition behind the proof for the case of $S = 1$ is the following. There is a distinguished buyer $\bar{b}$ who has the 'right' to buy the good of the single seller at $\bar{p}$ ($\bar{b}$ depends on the past history of play). The equilibrium strategies are such that whenever the seller meets the distinguished buyer, whichever is chosen as the proposer offers a price $\bar{p}$ and the responder accepts. Whenever the seller meets a buyer $b \neq \bar{b}$, the seller as a proposer offers the good at a price $p = 1$ and the buyer $b \neq \bar{b}$ offers to pay a price $p = 0$. In both cases the responders reject the offers. The outcome of these strategies is that the seller sells the good to buyer $\bar{b}$ at $\bar{p}$.

To show that it does not pay players to deviate from the above, the strategies further specify the following responses to any deviations. If the seller proposes a price different from the equilibrium price ($\bar{p}$ to $\bar{b}$ and 1 to $b \neq \bar{b}$) to any buyer $\widetilde{b}$ then this buyer rejects and he has the 'right' to buy the good at a price $\widetilde{p} = 0$. Thus, the continuation strategy is the same as above with the price $\widetilde{p}$ in place of $\bar{p}$ and the buyer $\widetilde{b}$ in place of $\bar{b}$.

If one of the buyers deviates from their equilibrium strategies then the seller rejects and another buyer $\widehat{b}$ gets the right to buy the good at a price $\widehat{p} = 1$. The continuation strategy is the same as before with the price $\widehat{p}$ in place of $\bar{p}$ and the buyer $\widehat{b}$ in place of $\bar{b}$.

Further deviations can be treated in exactly the same way.

It is easy to check that it does not pay any player to deviate from the above strategy after any history. Clearly, any initial deviator is no better off from deviating given the punishments. Also after any deviation any responder is at least as well off rejecting the proposed deviation and following the punishments than accepting the proposed deviation.

Notice that the strategies are quite complicated and the behaviour of each agent at any period depends on the history play up to that period - there are potentially an indefinite number of potential deviations and for each deviation the above strategy profile specifies a tailor-made response in order to deter the deviation. Thus the agents need a large amount of information to implement the above strategy profile. At the other extreme, one can assume that at any period the agents only have access to the history of play in that period and can not condition their behaviour on the previous history of plays. Thus for the purpose of comparison, one can consider history-independent (stationary or Markov) strategies. RW show that the only sta-

[4] When there is more than one seller, there is more than one match per period. As a result with more than one seller the game is one of imperfect information and the appropriate equilibrium concept is sequential equilibrium (or perfect Bayesian equilibrium). When there is only one seller, subgame perfect equilibrium will suffice.

tionary equilibrium outcome is the competitive one. In fact, their result is slightly stronger.

Theorem 2.2. *(See RW) If at any time each player's information consists only of the set of players that are present in the market at time t and the time itself then the unique sequential equilibrium price is the competitive price of 1.*

The above informational restriction prevents agents from punishing a deviator since the deviator is not remembered. For example, in the proof of Theorem 2.1, any deviation by the seller was rejected by the responder because the rejection led to a reward for the buyer. In Theorem 2.2 with stationary strategies the buyer could not be rewarded because the deviation of the seller could not be observed.

2.2. Complexity, equilibrium selection and the competitive outcome

Before introducing the notion of complexity used in this paper I need some further notation.

An outcome of a match at any period is described by an ordered four-tuple (i, j, p, l) where $i \in \mathcal{B} \cup \mathcal{S}$ is the proposer in this match, $j \in \mathcal{B} \cup \mathcal{S}$ is the responder, $p \in [0, 1]$ is the proposal by i and $l \in \{A, R\}$ is the response by j. I also denote a history of outcomes in a period of the game by e. Thus e consists of outcomes of S different matches, one for each seller; it describes everything that happens at every period of the game. For example e could be

$$\{q; (m_q(s),\ r_q(s),\ p_s, l_s)_{s \in S}\} \qquad \text{for some } p_s \in [0,1] \text{ and for some } l_s \in \{A, R\};$$

namely that each seller s was matched with buyer $q(s)$, the proposer and the responder in this match were $m_q(s)$ and $r_q(s)$, the proposal in the match involving seller s was p_s and the response was l_s. Let E be the set of such outcomes. The history of outcome at any time t is denoted by e^t and the history of outcomes of the game up to the beginning of each period t consists of a sequence of outcomes $h^t = (e^0, ..., e^{t-1})$. I shall denote the set of such t-period history of outcomes that do not result in an agreement by H^t.

At each date t, in addition to history of the outcomes h^t of the preceding periods, players also receive information about the preceding moves by Nature and/or other players during the current period. I also need notation to describe these partial descriptions of outcomes (partial history) a player receives within a bargaining period. I shall denote such a partial history by d and the set of such partial histories by D. Thus $d \in D$ is either the ordered pair (i, j) describing the match between player i and j with i as the proposer, or the ordered triplet (i, j, p) describing the match between players i and j followed by a price offer p by i. If $d = (i, j)$ the bargaining is just beginning and an offer has yet to be made by i to j, and if $d = (i, j, p)$ it is player j's turn to respond to an offer price of p by player i. Also, I shall denote the information sets (the sets of partial histories) for player i in any stage by D_i. Thus

$$D_i \equiv \{d \in D \mid \text{it is } i\text{'s turn to play after } d\}$$

Let

$$C = [0,1] \cup A \cup R$$

and denote the set of choices available to a player i, given a partial description $d \epsilon D_i$, by $C_i(d)$. Thus

$$C_i(d) = \begin{cases} [0,1] & \text{if } d \text{ is such that } i \text{ is the proposer} \\ \{A,\ R\} & \text{if } d \text{ is such that } i \text{ is the responder to some offer } x. \end{cases}$$

Let $H^\infty = \cup_{t=0}^{\infty} H^t$ be the set of all possible finite histories of periods. (H^0 is assumed to be empty). Then a strategy for player i is a function $f_i : H^\infty \times D_i \longrightarrow C$, where $f_i(e^0, .., e^{t-1}, d) \in C_i(d)$ for any $(e^0, .., e^{t-1}) \in H(t)$ and for any partial history $d \in D_i$. I shall denote the set of strategies for player i by F_i. Also for any strategy f_i and for any history $h \in H^\infty$, I shall define the strategy induced by f_i after h by $\langle f_i \mid h \rangle$.

Given any strategy profile $f = \{f_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$, the equilibrium path is a stochastic process because of Nature's moves (the random matching and random choice of proposers). I shall denote the expected payoff to each player i if strategy profile $f = \{f_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$ is chosen by $\pi_i(f)$. Since I only allow for pure strategies the expectation is with respect to the moves of Nature.[5]

2.2.1. Automata and Complexity

Any strategy in the game can be implemented by an automaton (machine) consisting of a set of states (not necessarily finite), an initial state, a terminal state, an output function describing the output of the machine as a function of its current state (and its current input) and a transition function determining the next state of the machine as a function of its current state and current input (the outcome in the current period).

In the literature on automata in repeated one-shot games, there is a natural specification of a machine. Here, we are dealing with a repeated extensive form game. Moreover, since each player has to play a different role (of a proposer and a responder) the extensive form bargaining game in each period has a certain degree of asymmetry built in. As a result, I can choose to specify a machine to implement a particular strategy in several different ways. In this paper, I shall assume

(i) the states of the machine do not change during each period of the game and transitions from a state to another state in the same player's machine take place at the end of a period.

[5]Formally, one needs to define an underlying probability space and expectation is taken with respect to the appropriate probability measure.

(ii) each state of the machine would specify an action for every role of the player concerned, with the action chosen depending on d - the partial history of the period.

A referee (called "Master of the Game" by Piccione and Rubinstein, (1993)) would activate each player's machine when needed.

I now set down the formal definition.

Definition 1. *A machine M_i is a five-tuple $(Q_i, q_i^1, T, \lambda_i, \mu_i)$, where*

Q_i is a set of states;

q_i^1 is a distinguished initial state belonging to Q_i;

T is a distinguished terminal state (T for "Termination");

$\lambda_i : Q_i \times D_i \to C$, describes the output function of the machine given the state of the machine and given the partial history that has occurred during the current period of the game before i's move, such that $\lambda_i(q_i, d) \in C_i(d)$, $\forall q_i \in Q_i$ and $\forall d \in D_i$;

$\mu_i : Q_i \times E \to Q_i \cup T$ is the transition function, specifying the state of the machine in the next period of the game as a function of the current state and the realised history of the current period.[6]

The fact that the game is identical at the beginning of each period (though behaviour could be different depending on past histories as encapsulated in the state) provides the basic rationale for using this specification of a machine. Thus, with this specification, the nature of the output and transition maps remain the same in each period. Other definitions are possible: for example the state of the machine changes before a player has to move or player has different sub-automaton to play different roles. But these definitions do not have the "game-stationarity" features that the current one does.

Remark 1. *If we denote the set of strategies for a player i in any period of the bargaining game by*

$$\mathcal{G}_i \equiv \{g : D_i \to C \,|g(d) \in C_i(d) \; \forall d \in D_i\}$$

then the output function λ_i in Definition 1 can be thought of as a mapping $\widetilde{\lambda}_i : Q_i \to \mathcal{G}_i$ where $\widetilde{\lambda}_i(q_i)(d) = \lambda_i(q_i, d)$. Thus each q_i specifies a mapping $\widetilde{\lambda}_i(q_i) \in \mathcal{G}_i$ from the information set within a period to the set of choices.

Next I need to define the strategy that is implemented by a given machine. Before addressing this, with some abuse of notation, denote the state of machine $M_i = (Q_i, q_i^1, T, \lambda_i, \mu_i)$ after any history $h = (e^1, ..., e^t)$ by $\mu_i(q_i^1, h)$. Thus $\mu_i(q_i^1, h)$ can be defined inductively by

$$\mu_i(q_i^1, e^1, ..., e^\tau) = \mu_i(\mu_i(q_i^1, e^1, ..., e^{\tau-1}), e^\tau)) \text{ for any } 1 < \tau \leq t \qquad (2.1)$$

[6] Henceforth, I shall not always explicitly refer to the terminal state T. I am assuming that if an offer is accepted then the machine of each participant to this agreement enters state T and shuts off. Thus $\mu_l\,(q_l, i, j, p, A) = T$ for any state q_l, any player $l = i, j$ and any price p. Also, I shall simply refer to the members of the set Q_i as the states of the machine.

Definition 2. *For any machine* $M_i = (Q_i, q_i^1, T, \lambda_i, \mu_i)$ *for player* i*, the strategy* $f_i \in F_i$ *implemented by* M_i *is defined by*

$$f_i(h, d) = \lambda_i(\mu_i(q_i^1, h), d) \text{ for all } h \in H^\infty \text{ and for all } d \in D_i$$

The complexity of a machine (or of a strategy) can be measured in many different ways. In the literature on repeated games played by automata the number of states of the machine is often used as a measure of complexity. Henceforth I shall refer to this measure of complexity by state-complexity (or simply by s-complexity). This is because the set of states of the machine can be regarded as a partition of possible histories. (See footnote 7 below and Kalai and Stanford 1989)

Definition 3. *(State-complexity) A machine* $M_i' = (Q_i', q_i^{1'}, T, \lambda_i', \mu_i')$ *is more s-complex than another machine* $M_i = (Q_i, q_i^1, T, \lambda_i, \mu_i)$*, denoted by* $M_i^{'} \succ^s M_i$*, if*
$|Q_i'| > |Q_i|$
where, for any set W*,* $|W|$ *refers to the cardinality of the set* W*.*
I shall also use $M_i^{'} \succeq^s M_i$ *to denote "*$M_i^{'}$ *is at least as s-complex as* M_i*".*

In terms of the underlying strategy, the number of (non-redundant) states of a machine M_i measures the number of induced of strategies $|\{f_i \mid \langle h \rangle \, | h \in H^\infty\}|$ after different histories. Thus, I could also define s-complexity in terms of the underlying strategies in the game as in Kalai and Stanford (1989).

Definition 4. *A strategy* f_i' *is more s-complex than* f_i *(denoted by* $f_i^{'} \succ^s f_i$*) if* [7]

$$|\{f_i' \mid \langle h \rangle \, | h \in H^\infty\}| > |\{f_i \mid \langle h \rangle \, | h \in H^\infty\}|$$

Since, the states of a machine do not change during a period of the game, counting the number of states does not fully measure the complexity of the machine during a period, specifically the complexity of different choices following the same partial history. More formally, the above definition of complexity measures for each d the cardinality of the domain of $\lambda_i(., d)$ (or that of $\widetilde{\lambda}_i(.)$) but it does not capture the complexity of the range of the mapping $\lambda_i(., d)$. To illustrate the point further consider the following examples.

Example 1. *There are two machines* M_i *and* M_i'*. Both machines have two states* q_i^1 *and* q_i^2*. Both are in state* q_i^1 *in the odd periods and in state* q_i^2 *in the even periods (thus they have the same transition functions). Also as a proposer, in state* q_i^l $(l = 1, 2)$*, both machines offer price* p_i^l *to any seller. As a responder,* M_i *always rejects all offers. Machine* M_i'*, on the other hand, responds differently to the same proposal by any player* j *(by conditioning on the two states* q_i^1 *and* q_i^2*). In particular, for any offer* p *by* j*,* M_i' *rejects* p *in the odd periods and accept* p *in the even periods.*

[7] Any f_i defines a partition (call it s-partition) on H^∞ given by an equivalence relation

$$h \sim^s h' \text{ if and only if } f_i \mid \langle h \rangle = f_i \mid \langle h' \rangle$$

S-complexity simply reflects the size of these partitions.

Example 2. *There are two machines M_i and M_i'. Both machines have two states q_i^A and q_i^R. Both are in state q_i^A in the odd periods and in state q_i^R in the even periods (thus they have the same transition functions). Also as a responder, in state q_i^A both machines accept any price offer and in state q_i^R both machines reject any price offer. As a proposer, M_i always offers a price p. Machine M_i', on the other hand, makes different proposal to any player j (by conditioning on the two states q_i^A and q_i^R). In particular, M_i' proposes p in the odd periods and p' in the even periods.*

According to s-complexity M_i and M_i' are of equal complexity in both examples, despite the fact that in the first example the strategy that machine M_i' implements has the additional complexity of different responses to the same offer and in the second example the strategy that machine M_i' implements has the additional complexity of making different proposals in different periods. This is not a desirable property.

A plausible (and minimal) way of capturing the complexity of strategy *during a period* - complexity of different behaviour after the same partial history - is to assume that the complexity criterion satisfies the following two conditions.

(i) If two machines (and therefore two strategies) M_i and $M_i^{'}$ are otherwise identical machines for player i except that as a responder to some price offer p by some player j, M_i always responds the same way (always accepts or always rejects) whereas $M_i^{'}$ some times accepts and sometimes rejects the offer p by j, then M_i' should be considered as being more complex than M_i.

(ii) If M_i' makes at least two different proposals p and p' to some player j depending on the history of actions before the current period and if M_i is otherwise identical to M_i' except that as a proposer to player j it drops the offer p' in favour of p (after all histories at which M_i' proposes p' to j M_i proposes p) , then M_i' should be considered as being more complex than M_i.

I call such notion of complexity response-complexity (r-complexity). Similar definition can be found in Chatterjee and Sabourian (1999, 2000). The formal definition of r-complexity consists of a partial order (the weakest) on the set of machines that captures (i) and (ii) above.

Definition 5. *(Response complexity) A machine $M_i' = \left\{Q_i', q_i^{1\prime}, T, \lambda_i', \mu_i^{'}\right\}$ is more r-complex than another machine $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$, denoted by $M_i^{'} \succ^r M_i$, if the machines M_i and $M_i^{'}$ are otherwise identical except that given some (non-empty) partial history $d' \in D_i$, the response of M_i to d' is simpler than that of $M_i^{'}$. Formally, $M_i^{'} \succ^r M_i$ if $Q_i = Q_i^{'}, q_i^1 = q_i^{1\prime}$, $\mu_i = \mu_i^{'}$ and there exists a (non-empty) partial history $d' \in D_i$ and a set of states $\overline{Q}_i \subset Q_i (= Q_i')$ such that*

$$\left.\begin{array}{ll} \lambda_i^{'}(q_i, d) = \lambda_i(q_i, d) & \text{if } d \neq d' \text{ or if } q_i \notin \overline{Q_i} \\ \lambda_i(q_i, d') = \lambda_i(q_i', d') & \forall q_i, q_i^{'} \in \overline{Q}_i, \\ \lambda_i^{'}(q_i, d') \neq \lambda_i^{'}(q_i', d') & \text{for some } q_i, q_i^{'} \in \overline{Q}_i \\ \lambda_i^{'}(q_i, d') \neq \lambda_i^{'}(q_i', d') & \forall q_i \in Q_i/\overline{Q}_i \text{ and } \forall q_i^{'} \in \overline{Q}_i \end{array}\right\} \quad (2.2)$$

I shall use $M_i^{'} \succeq^r M_i$ to refer to M_i is not more r-complex than M_i'.

The r-complexity definition is a very weak *partial (local)* concept -partial in the sense that M_i' and M_i are everywhere identical except in response to some d', M_i always takes the same action in all states $q_i^{'} \in \overline{Q}_i$ whereas M_i' does not.[8] Since states of a machine encapsulate past history this is equivalent to saying that the behaviour of M_i' and M_i are everywhere identical except that in response to some d' machine M_i is conditioning less on history than M_i' does.

Remark 2. *Clearly, condition (2.2) above implies that*

$$\left.\begin{array}{ll} \lambda_i(q_i, d) = \lambda_i'(q_i, d) & \forall q_i \text{ and } \forall d \neq d' \\ \lambda_i(Q_i', d') \subset \lambda_i'(Q_i, d') & \end{array}\right\} \quad (2.3)$$

The results of this paper on equilibrium selection remain valid if we use the stronger condition (2.3) instead of (2.2) in the definition of r-complexity.

I could also define r-complexity in terms of the underlying strategies in the game.

Definition 6. *A strategy f_i' is more r-complex than f_i, denoted by $f_i^{'} \succ^r f_i$, if there exists a (non-empty) partial history $d' \in D_i$ and a set of histories $\overline{H} \subset H^\infty$ such that*

$$\left.\begin{array}{ll} f_i^{'}(h, d) = f_i(h, d) & \text{if } d \neq d' \text{ or if } h \notin \overline{H} \\ f_i(h, d') = f_i(h, d') & \forall h, h^{'} \in \overline{H}, \\ f_i^{'}(h, d') \neq f_i^{'}(h', d') & \text{for some } h, h' \in \overline{H} \\ f_i^{'}(h, d') \neq f_i^{'}(h', d') & \forall h \in H^\infty / \overline{H} \text{ and } \forall h' \in \overline{H} \end{array}\right\} \quad (2.4)$$

I shall also use $f_i^{'} \succeq^r f_i$ to refer to f_i is not more r-complex than f_i'.[9]

Notice that, given the specification of automata adopted here and given that we are dealing with a repeated *extensive form* game, s-complexity and r-complexity measure the complexity of different aspects of behaviour - the number of induced strategies at the beginning of each period versus the complexity of behaviour within a period.

For the results in this section with random matching, I only need to introduce this minimal notion of response-complexity into the standard game-theoretic set-up. In section 4, I use both complexity concepts (called response-state complexity).

[8]The first three conditions in (2.2) capture precisely the idea that M_i' and M_i are everywhere identical except in response to some d', M_i always takes the same action in all states $q_i^{'} \in \overline{Q}_i$ whereas M_i' does not. The fourth condition is imposed so that the partial order $\succ^s$ is not reflexive. If the fourth condition in (2.2) is not assumed, it is possible that $M_i \succ^s M_i'$ and $M_i' \succ^s M_i$.

[9]Clearly, condition (2.4) above implies that

$$\left.\begin{array}{ll} f_i(h, d) = f_i'(h, d) & \forall h \text{ and } \forall d \neq d' \\ f_i(H^\infty, d) \subset f_i'(H^\infty, d) & \end{array}\right\} \quad (2.5)$$

Definition 7. *(Response-state complexity)A machine M_i' is more rs-complex than M_i (denoted by $M_i' \succ^{rs} M_i$) if either $M_i' \succ^{s} M_i$ or $M_i' \succ^{r} M_i$. Also, I shall use $M_i' \succeq^{rs} M_i$ to refer to M_i is not more rs-complex than M_i'.*

Similarly, I could define rs-complexity in terms of the underlying strategies in the game.

Definition 8. *A machine f_i' is more rs-complex than f_i (denoted by $f_i' \succ^{rs} f_i$) if either $f_i' \succ^{s} f_i$ or $f_i' \succ^{r} f_i$. I shall use $f_i' \succeq^{rs} f_i$ to refer to f_i is not more r-complex than f_i'.*

Before, I define *Nash equilibrium* of the game with complexity cost denote, with some abuse of notation, the expected payoff to player i if machine profile M is chosen by $\pi_i(M)$.

Definition 9. *A profile $M = \{M_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$ constitutes a Nash equilibrium* machine *profile with complexity cost l (denoted by NECl machine profile) if for each player i the following two conditions hold*

$$\begin{array}{lll} \pi_i(M_i,\, M_{\sim i}) \geq \pi_i(M_i', M_{\sim i}) & \forall M_i' & \\ \text{if } \pi_i(M_i,\, M_{\sim i}) = \pi_i(M_i',\, M_{\sim i}) & \text{then} & M_i' \succeq^{l} M_i. \end{array}$$

Also, a profile f constitutes a Nash equilibrium strategy *profile with complexity cost l (denoted by NECl strategy profile) if it can be implemented by a NECl machine profile M.*

Remark 3. *Clearly, one could define NECl strategy profile independently of the machine specification by appealing directly to complexity criterion defined over the strategy set (Definitions 4, 6 and 8). Thus a profile $f = (f_i, f_{-i})$ is a NECl strategy profile if*

$$\begin{array}{l} \pi_i(f_i,\, f_{\sim i}) \geq \pi_i(f_i', f_{\sim i}) \qquad \forall f_i' \in F_i \\ \text{if } \pi_i(f_i,\, f_{\sim i}) = \pi_i(f_i',\, f_{\sim i}) \text{ for some } f_i' \in F_i \text{ then } f_i' \succeq^{l} f_i. \end{array}$$

Complexity costs are treated lexicographically in the definition of NECl above. I could also have introduced complexity directly into the payoff function as a small fixed costs of choosing a more complex strategy and defined a Nash Equilibrium with a fixed complexity cost as follows.

Definition 10. *A machine profile $M = \{M_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$ constitutes a Nash equilibrium with a (small) fixed $l-$complexity cost $c > 0$ if for each player i*

$$\pi_i(M_i,\, M_{\sim i}) \geq \pi_i(M_i', M_{\sim i}) + c\gamma(M_i, M_i') \qquad \forall M_i' \tag{2.6}$$

where

$$\gamma(M_i, M_i') = \begin{cases} 1 & \text{if } M_i \succ^{l} M_i' \\ -1 & \text{if } M_i' \succ^{l} M_i \\ 0 & \text{otherwise} \end{cases}$$

Such a payoff function would induce at least as much economy as the lexicographic criterion. For any fixed complexity costs $c > 0$, the results of this paper are also valid if complexity costs enter the payoff functions as in (2.6) .

I shall now define formally stationary behaviour and a minimal complex machine in the context of this model.

Definition 11. *A strategy f_i is stationary if*

$$f_i(h, d) = f_i(h', d) \qquad \forall h, h' \in H^\infty \text{ and } \forall d \in D_i$$

Thus the behaviour of such strategies at any time may depend on the past outcomes in the current period but not on the previous history of the game before the current period.

Definition 12. *A machine M_i is minimal l-complex (for $l = r$ or s) if $M_i' \succeq^l M_i$ for any machine M_i'.*

Clearly, a minimal l-complex machine always implements a stationary strategy. Also a stationary strategy can be implemented by a minimal machine. Thus, a stationary Nash equilibrium strategy profile of the bargaining game is a NECl of the game with complexity cost.

Definition 13. *An automaton M_i is finite if it has a finite number of states. A strategy is finite if it can be implemented by a finite machine. A profile of machines (strategies) is finite if each of its components is finite.*

NECl strategy (machine) profiles are not necessarily 'credible' for the usual reasons. To ensure credibility, I could, as in Chatterjee and Sabourian (1999,2000), introduce noise into the system and consider extensive form trembling hand equilibrium with complexity costs. This will ensure that strategies are optimal after all histories that occur with a positive probability. (See also the section 5.) A more direct, and simpler, approach of introducing credibility would be to consider NEC-l strategy profiles that are perfect Bayesian equilibria (subgame perfect equilibrium for the case of the one seller) of the game without complexity costs.[10]

Definition 14. *A profile f constitutes a perfect Bayesian equilibrium strategy profile with complexity cost l (denoted by PBECl strategy profile) if f is both a NECl strategy profile and a perfect Bayesian equilibrium of the underlying game.* [11]

[10] Here, with more than one seller, a perfect Bayesian equilibrium refers to a profile of strategies for each player such that, at every information set (h, d), each player's strategy maximizes the player's expected continuation payoff given the strategies of the others, where expectation is with respect the choice of nature. Of course, in this set-up, with uncertainty over the choice of nature η, perfect Bayesian equilibrium is equivalent to sequential equilibrium; however, I shall use the former concept because it is easier to define.

[11] A perfect Bayesian equilibrium with a fixed complexity cost $c > 0$ can be defined in a similar fashion.

Similarly, a profile M constitutes a perfect Bayesian equilibrium machine profile with complexity costs l if it is both a NECl machine profile and the strategy implemented by M is a perfect Bayesian equilibrium of the underlying game.

Clearly, a PBECl strategy (machine) profile exists. Consider the following stationary profile of strategies (machines): all players always offer 1, seller accepts an offer if and only if the offer is 1, buyers accept all offers. This profile induces the competitive outcome. Trivially, it also constitutes a perfect Bayesian equilibrium and is stationary (has minimal l-complexity). Therefore this profile is a PBECl.

The next result demonstrates that the credible equilibria of the game with r-complexity costs induce the unique competitive outcome and are stationary.

Theorem 2.3. *Consider any PBECr strategy profile $f = \{f_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$. If each strategy f_i is finite then $\pi_s(f) = 1$ for all s, $\pi_b(f) = 0$ for all b, the unique induced price is the competitive price of 1 and each f_i is stationary.*[12]

The proof of the above Theorem for the case of a market with one seller can be found in the next section. The proof for markets with arbitrary number of sellers is by induction on the number of sellers. The proof for this more general case can be found in Appendix A.

Here, I shall first provide some intuition for the role of complexity by explaining why the strategies used by RW in the proof of Theorem 2.1 to support non-competitive outcomes cannot constitute a PBECr. First, notice that these strategies (machines) are non-stationary. In particular, all those buyers who do not have any 'rights' to any good (buyers who do not end up buying the goods on the equilibrium path constructed in the proof of Theorem 2.1) also follow non-stationary (complex) strategies. But such buyers receive zero payoff on the equilibrium path. But then these strategies could not be an PBECr because such buyers could always obtain at least a zero payoff by following a simpler strategy that always makes the same offer and accepts all offers.

The actual proof of Theorem 2.3 for the case of a single seller s uses a similar reasoning, but applied to the continuation payoffs of the buyers. The proof in this case basically consists of establishing the following three steps.

Step 1 (see Lemma 1, 2, 3): If any PBECr strategy profile results in a payoff for the seller that is less than 1 (the outcome is non-competitive), then there does not exist a history after which the seller reaches an agreement at a price of 1 with some buyer (Lemma 2 and 3).

This step is demonstrated by showing that if there is an agreement at a price of 1 then there is the possibility of economizing on r-complexity.

[12]The finiteness of f_i assumption - namely that f_i can implemented by a *finite* automaton - in the above Theorem is only needed because complexity costs enters the players' preferences lexicographically in the definition of PBEC-r. If postive fixed complexity cost is assumed, as in Definition 10, it can be shown that Theorem 2.3 holds without such an assumption (finiteness) on the set of PBECr strategies.

Step 2 (see Lemma 5): If any PBECr strategy profile results in a payoff for the seller that is less than 1 (the outcome is non-competitive), then for any buyer b and for any history h the continuation payoff to b after the ordered triple (h, b, s) is positive.

The intuition for this step is as follows. Since there is no agreement at a price of 1 (step 1), it follows that the continuation payoff to the seller is always less than 1 (Lemma 4). This, together with the finiteness of the PBECr strategy profile, imply that, after any history, if a buyer has the opportunity to make an offer to the seller he can obtain a positive payoff by offering a price that is both less than 1 and more than the continuation payoff of the seller.

Step 3 (see Lemma 7): This involves showing that for any finite subgame perfect equilibrium, there exists a buyer b and a history h such that the buyer's continuation payoff after (h, b, s) is zero.

This step follows from considering histories at which the continuation payoff of the seller is minimized. At such histories, competition between buyers ensures that the continuation payoff of at most one buyer is positive.

Steps 2 and 3 contradict each other unless the PBECr strategy profile results in a payoff of 1 for the seller. This establishes the result.

3. Proof of Theorem 2.3 for the case of one seller

The proof is in several Lemmas. The first Lemma, in fact, holds for an arbitrary number of sellers.

Lemma 1. *For any NECr profile of strategies $f = (\{f_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$, any buyer $b \in \mathcal{B}$ and any seller $s \in \mathcal{S}$ the following holds:*

$$f_b(h, s, b, 1) = f_b(h', s, b, 1) \quad \text{for all } h \text{ and } h' \in H^\infty \tag{3.1}$$

$$f_s(h, b, s, 1) = f_s(h', b, s, 1) \quad \text{for all } h \text{ and } h' \in H^\infty \tag{3.2}$$

Proof. To show that condition (3.1) holds, suppose otherwise. Then for some (h, s, b) and for some (h', s, b) the following holds

$$f_b(h, s, b, 1) = A \text{ and } f_b(h', s, b, 1) = R$$

Now consider another strategy f'_b for player b such that for all (h'', d)

$$\begin{array}{ll} f'_b(h'', d) = R & \text{if } d = (s, b, 1) \\ f'_b(h'', d) = f_b(h'', d) & \text{if } d \neq (s, b, 1) \end{array}$$

Clearly, the only difference between f_b' and f_b is that in some instance f_b accepts an offer of 1 and f_b' does not; thus f_b' induces at least the same payoff as f_b and moreover it is less r-complex than f_b according to Definition 6. But, by Remark 3, this is a contradiction.

Using a similar reasoning as above, I now show that condition (3.2) holds. Suppose not; then for some (h, b, s) and for some (h', b, s) the following holds

$$f_s(h, b, s, 1) = A \text{ and } f_s(h', b, s, 1) = R$$

Now consider another strategy f_s' for player s such that for all (h'', d)

$$\begin{array}{ll} f_s'(h'', d) = A & \text{if } d = (b, s, 1) \\ f_s'(h'', d) = f_s(h'', d) & \text{if } d \neq (b, s, 1) \end{array}$$

Clearly, the only difference between f_s and f_s' is that in some instance f_s' accepts an offer of 1 and f_s does not; thus f_s' induces at least the same payoff as f_s and it is less r-complex than f_s according to Definition 6. But, by Remark 3, this is a contradiction. ■

The next set of results will be demonstrated for the case of a market with one single seller s.

Lemma 2. *Suppose $S = 1$. Then for any NECr profile f that does not result in a payoff of 1 for the seller s we have $f_b(h, s, b, 1) = R$ for all b and for all h.*

Proof. Suppose not; then $f_b(h, s, b, 1) = A$ for some b and for some h. By Lemma 1 this implies that

$$f_b(h, s, b, 1) = A \text{ for all } h \tag{3.3}$$

Now consider any strategy f_s' for s that always proposes 1 and rejects all offers. Since the ordered pair (s, b) occurs with probability 1 in a finite time, it follows from (3.3) that f_s' can guarantee s a payoff of 1; but this is a contradiction. ■

Lemma 3. *Suppose $S = 1$. Consider any NECr f; if f does not result in a payoff of 1 for the seller s then there does not exist a buyer b and a history h such that the ordered pair (b, s) reaches an agreement at a price of 1 after h. Formally, for any NECr f, if $\pi_s(f) < 1$ then for all b and for all h*

$$\textit{either} \qquad f_b(h, b, s) \neq 1 \qquad \textit{or} \qquad f_s(h, b, s, 1) = R.$$

Proof. Suppose not; then there exists b and h such that $f_b(h, b, s) = 1$ and $f_s(h, b, s, 1) = A$. By Lemma 1 this implies that

$$f_s(h, b, s, 1) = A \text{ for all } h \tag{3.4}$$

Also since f results in a payoff less than 1 there exists h' such that $f_b(h', b, s) = p' \neq 1$ (otherwise s could always obtain a payoff of 1. This can be achieved by following a strategy that always proposes 1 and accepts an offer if and only if b offers a price of 1; then either some buyer accepts the offer of 1 by s or by the law of large numbers s will eventually be matched with b and will receive an offer of 1 from b). Now consider a strategy f_b' such that

$$\begin{array}{ll} f_b'(h', d) = p' & \text{for all } (h, d) \text{ such that } f_b(h, d) = 1 \\ f_b'(h, d) = f_b(h, d) & \text{otherwise} \end{array}$$

Clearly, the only difference between f_b and f_b' is that in some instance f_b proposes an offer of 1 (and this is accepted) and f_b' does not; thus $f_b^{'}$ induces as much payoff as f_b and is less r-complex than f_b according to Definition 6. But, by Remark 3, this is a contradiction. ■

Lemma 4. *Suppose $S = 1$. Then for any NECr profile f such that $\pi_s(f) < 1$, we have*

$$\pi_s(\langle f \mid h \rangle) < 1 \text{ for all } h \in H^\infty$$

Proof. This follows from b never accepting an offer of 1 (Lemma 2) and from the ordered pair (b, s) never reaching an agreement at a price of 1 after any history (Lemma 3). ■

Lemma 5. *Suppose $S = 1$. Then for any PBECr-r f such that $\pi_s(f) < 1$, we have*

$$\pi_b(\langle f \mid h, b, s \rangle) > 0 \text{ for all } h \text{ and for all } b$$

Proof. Suppose not; then for some h and for some b we have $\pi_b(\langle f \mid h, b, s \rangle) = 0$. Now, since $\pi_s(f) < 1$, then for all h we have, by Lemma 4, $\pi_s(\langle f \mid h \rangle) < 1$. Thus K, defined by

$$K \equiv \max_h \pi_s(\langle f \mid h \rangle),$$

is less than 1 (K is well-defined because f is finite - can be implemented by a finite automaton). Since f constitutes a subgame perfect equilibrium, if after the match (h, b, s) the buyer b offers a price $K + \epsilon$ for some ϵ such that $K + \epsilon < 1$, it will be accepted by s (otherwise s obtains at most K). Thus b can always obtain at least $1 - K - \epsilon > 0$; but this contradicts the supposition. ■

Now for any strategy profile f let

$$m_s^b(f) = \min_{h \in H^\infty} \pi_s(\langle f \mid h, b, s \rangle)$$

$$m_b^b(f) = \min_{h \in H^\infty} \pi_s(\langle f \mid h, b, s \rangle)$$

$$z(b, f) = \max_{h \in H^\infty} \pi_b(\langle f \mid h \rangle) \tag{3.5}$$

$$\overline{b}(f) \equiv \text{ arg min}_b \ (m_b^b(f) + m_s^b(f)) \tag{3.6}$$

Note that if f is finite then $z(b, f)$ and $m_i^b(f)$ are well defined for $i = b, s$.

For the rest of this section, I fix a profile f and refer to $m_s^b(f)$, $m_b^b(f)$, $z(b; f)$ and $\overline{b}(f)$ by m_s^b, m_b^b, $z(b)$ and $\overline{b}$ respectively.

Lemma 6. *Suppose* $S = 1$. *Then for any finite subgame perfect equilibrium strategy profile* f *we have* $m_{\bar{b}}^{\bar{b}} \geq m_s^{\bar{b}}$.

Proof. Suppose not; then $m_{\bar{b}}^{\bar{b}} < 1/2(m_s^{\bar{b}} + m_{\bar{b}}^{\bar{b}})$. Now it follows from the definition of $m_{\bar{b}}^{\bar{b}}$ that there exists a history h such that $m_{\bar{b}}^{\bar{b}} = \pi_s(\langle f \mid h, \bar{b}, s\rangle)$. Now suppose p is the offer of $\bar{b}$ after $(h, \bar{b}, s)$. Now if s rejects p after $(h, \bar{b}, s)$ he can get at least $1/2(m_s^{\bar{b}} + m_{\bar{b}}^{\bar{b}})$ next period. But this exceeds $m_{\bar{b}}^{\bar{b}} = \pi_s(\langle f \mid h, \bar{b}, s\rangle)$. This contradicts the supposition that f is a subgame perfect equilibrium. ■

Lemma 7. *Suppose* $S = 1$. *Then for any finite subgame perfect equilibrium strategy profile* f *there exists a buyer* b *and a history* h *such that* $\pi_b(\langle f \mid h, b, s\rangle) = 0$.

Proof. Consider any subgame perfect equilibrium f and let $\bar{b}$ be defined as in (3.6). First, I establish that

$$z(\bar{b}) \geq 1 - m_s^{\bar{b}} \tag{3.7}$$

To show this, suppose otherwise; then

$$m_s^{\bar{b}} < 1 - z(\bar{b}) - \epsilon \text{ for some } \epsilon > 0. \tag{3.8}$$

Now consider any history h and suppose that players s and $\bar{b}$ are matched and s makes a price offer of $(1 - z(\bar{b}) - \epsilon)$ to $\bar{b}$ after $(h, s, \bar{b})$. By the definition of $z(\bar{b})$, given in condition (3.5), this offer will be accepted by $\bar{b}$. Thus $m_s^{\bar{b}} \geq 1 - z(\bar{b}) - \epsilon$. But this contradicts condition (3.8). Therefore, condition (3.7) holds.

Now it follows from the definition of $z(\bar{b})$ that there exists a h such that $\pi_{\bar{b}}(f \mid h) = z(\bar{b})$. Therefore,

$$z(\bar{b}) \leq \frac{1}{B}\left\{1/2(1 - m_s^{\bar{b}}) + 1/2(1 - m_{\bar{b}}^{\bar{b}})\right\} +$$
$$\textstyle\sum_{b \neq \bar{b}} \frac{1}{B}\left\{1/2(1 - \pi_s(\langle f \mid h, s, b\rangle) - \pi_b(\langle f \mid h, s, b\rangle)) + 1/2(1 - \pi_s(\langle f \mid h, b, s\rangle) - \pi_b(\langle f \mid h, b, s\rangle))\right\}$$

(The expression in the RHS of the above inequality gives an upper bound on $z(\bar{b})$. The term $1/2(1 - m_s^{\bar{b}}) + 1/2(1 - m_{\bar{b}}^{\bar{b}})$ bounds $\bar{b}$'s expected payoff in the event that he meets the seller in the next period and it is weighted by the probability, $1/B$, of that event. The second term on the RHS of the last inequality is the weighted sum of the payoff of $\bar{b}$ in the event that in the next period the seller meets one of the other buyers, weighted by the probability of each such event.) Therefore, from the definitions of m_s^b and m_b^b we have

$$z(\bar{b}) \leq \frac{1}{B}\left\{1/2(1 - m_s^{\bar{b}}) + 1/2(1 - m_{\bar{b}}^{\bar{b}})\right\} +$$
$$\textstyle\sum_{b \neq \bar{b}} \frac{1}{B}\left\{1/2(1 - m_s^b - \pi_b(\langle f \mid h, s, b\rangle)) + 1/2(1 - m_b^b - \pi_b(\langle f \mid h, b, s\rangle))\right\}$$

The last condition together with condition (3.7) imply that

$$m_s^{\bar{b}} \geq \frac{1}{2B}\left\{(m_s^{\bar{b}} + m_{\bar{b}}^{\bar{b}}) + \sum_{b\neq\bar{b}}\left[(m_s^b + m_b^b) + (\pi_b(\langle f \mid h, s, b\rangle) + \pi_b(\langle f \mid h, b, s\rangle))\right]\right\}$$

But this together with the definition of $\bar{b}$ imply that

$$m_s^{\bar{b}} \geq \frac{1}{2B}\left\{B(m_s^{\bar{b}} + m_{\bar{b}}^{\bar{b}}) + \sum_{b\neq\bar{b}}(\pi_b(\langle f \mid h, s, b\rangle) + \pi_b(\langle f \mid h, b, s\rangle))\right\}$$

Therefore, it follows from Lemma 6 that

$$m_s^{\bar{b}} \geq \frac{1}{2B}\left\{2Bm_s^{\bar{b}} + \sum_{b\neq\bar{b}}(\pi_b(\langle f \mid h, s, b\rangle) + \pi_b(\langle f \mid h, b, s\rangle))\right\}$$

Hence,

$$0 \geq \frac{1}{2B}\left\{\sum_{b\neq\bar{b}}(\pi_b(\langle f \mid h, s, b\rangle) + \pi_b(\langle f \mid h, b, s\rangle)\right\}$$

Since the continuation payoffs are always non-negative, it follows from the previous inequality that

$$\pi_b(\langle f \mid h, b, s\rangle) \leq 0 \text{ for all } b \neq \bar{b} \tag{3.9}$$

This completes the proof of this Lemma. ■

Now Lemmas 5 and 7 imply that $\pi_s(f) = 1$ for any PBECr f. This implies that for any PBECr profile f with probability 1 the seller s reaches an agreement at $p = 1$ with some buyer and $\pi_b(f) = 0$ for all b. Therefore, f_b is stationary for all b (otherwise, b could economize on complexity and obtain at least a zero payoff). Since some buyer b accepts an offer of 1 after some history, it follows from stationarity of f_b that b accepts offer of 1 after all histories. This implies that f_s is stationary (otherwise, s could economize on r-complexity and obtain a payoff of 1 by always proposing 1 and rejecting anything less than 1).

4. Voluntary matching, discounting and complexity

The no discounting assumption is important in establishing the existence a of continuum of sequential equilibrium prices in Theorem 2.1. Theorem 2.1 works because after any history there are special 'relationships' between buyers and sellers - after every history for each unit of the good of a seller a buyer has the right to buy it at a particular price. Each deviation from the equilibrium strategies is deterred by the creation of a new relationship. With random matching, with probability one, the two sides of the new relationship will meet in a finite time. With no discounting, the length of the period it will take for the two sides to meet is unimportant. However, with discounting the cost of maintaining these relationships may be high. In particular, if there is a large number of players, it can take a long time for the designated buyers and sellers to meet each other. Thus, with discounting it may not be optimal for players to play the appropriate punishments needed to support the equilibria in Theorem 2.1. Therefore, discounting eliminates a large number of equilibria. For the one seller model RW have the following result.

Theorem 4.1. *(See RW) Suppose that $S = 1$ and $\delta \in (0, 1)$. Then there exists a unique subgame perfect equilibrium in which trade takes place at $t = 1$. Moreover, as $B \to \infty$ or as $\delta \to 1$ the unique equilibrium converges to the competitive price of 1.*[13]

The above result (in particular the one on convergence of the equilibrium prices to the competitive one as $\delta \to 1$) seems to throw some doubt on the multiplicity result in Theorem 2.1. However, RW argue that discounting imposes a cost on having a relationship because the formation and the termination of matches are random. But staying with one's current partner should not be costly. Thus, they consider a voluntary matching model with an endogenous choice of partner and demonstrate the existence of a large number of (non-competitive) equilibria even for the case in which $\delta < 1$.

Theorem 4.2. *(See RW) If $S = 1$ and the seller can choose in each period the buyer with whom he wishes to bargain then for each buyer b and any price $\frac{1}{1+\delta} \leq p \leq 1$ there exists a subgame perfect equilibrium in which b receives the good at the price equal to either p or $\frac{\delta p}{2-\delta}$, according to whether the seller or the buyer b is the proposer in their first encounter.*

Thus the indeterminacy and non-competitive outcomes are present in the model with discounting as well, irrespective of the number of buyers. But the strategies needed to implement the above equilibria for any $p < 1$ turn out to be unnecessarily complex. To establish the above result, for any price $\frac{1}{1+\delta} \leq p \leq 1$, RW construct the following subgame perfect equilibrium strategy profile. The seller s always offers p and agrees to accept $\frac{\delta p}{2-\delta}$ or more. A buyer always offers $\frac{\delta p}{2-\delta}$ and accepts p or less. In the first period s picks buyer b and in the case of disagreement s continues with the same buyer only if the same buyer did not deviate. If a buyer deviates at any period from the above strategy the seller discontinues the bargaining with him and picks a new buyer.

The above strategy, clearly, results in an agreement between the seller and buyer b at price p in the first period. But then why should the seller choose a strategy that involves selecting different partners depending on the previous history of moves? Consider a simpler strategy for the seller that *always chooses buyer b*, always offers p

[13] No equivalent result is known for the case of more than one seller.

to b and agrees to an offer if and only if the offer is $\frac{\delta p}{2-\delta}$. Clearly, if all buyers follow the above strategies, this simple strategy results in the same payoff as before but with less complexity.

In this section I extend the result of the previous section by showing that with complexity costs the only sequential (perfect Bayesian) equilibrium of the above game with endogenous choice of partners is also the competitive outcome. However, as was mentioned before, I obtain this result by using the stronger rs-complexity (both r-complexity and s-complexity) criterion.

The notation in this section is the same as in the previous section. In particular, e, h and d refer respectively to the history of actions in a period, histories of finite number of periods and the partial history of actions within a period. The definition of strategy in this section is the same as that in the case of random matching case except that here, with an endogenous choice of partners, a seller has to choose a partner at the beginning of each period. Formally, I represent the beginning of each period at which the seller has to choose a buyer by the null set Φ and define a strategy for seller s by a function

$$f_s : H^{\infty} \times (D_i \cup \Phi) \to C \cup \mathcal{B}$$

where $f_s(h, d) \in C_s(d)$ and $f_s(h, \Phi) \in \mathcal{B}$ for any $h \in H^{\infty}$ and for any $d \in D_s$.[14]

Similarly, the automaton representing a seller's strategy $M_s = \{Q_s, q_s^1, T, \lambda_s, \mu_s\}$ has the same structure as before except that the output function of any seller is now defined by

$$\lambda_s : Q_s \times (D_s \cup \Phi) \to C \cup \mathcal{B}$$

where $\lambda_s(q_i, d) \in C_s(d)$ and $\lambda_s(q_s, \Phi) \in \mathcal{B}$ for any $q_s \in Q_s$ and for any $d \in D_s$.

In the previous section with random matching, r-complexity (measuring the complexity of responses during a period) was sufficient to select uniquely the Walrasian outcome. In this section, we have an additional element of complexity of behaviour - the complexity of the sellers' decisions at the beginning of each period (at the null set Φ). I need to strengthen the definition of complexity to capture the complexity of conditioning the choice of the buyer at any period on the history of the game prior to that period.

One way of capturing the complexity of sellers' behaviour at the beginning of a period is to strengthen r-complexity definition to allow for responses to the null set Φ. Thus, in addition to r-complexity, I could require the complexity criterion to rank strategies (machines) for seller s according to the following complexity criterion.

Definition 15. *Machine $M_i' = \left\{Q_i', q_i^{1\prime}, T, \lambda_i', \mu_i'\right\}$ is more null-complex than another machine $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$, denoted by $M_i' \succ^n M_i$, if the machines M_i and M_i' are otherwise identical except that the choice of M_i to at the beginning of each period is simpler than that of M_i'. Formally, $M_i' \succ^n M_i$ if $Q_i = Q_i'$, $q_i^1 = q_i^{1\prime}$, $\mu_i = \mu_i'$, $\lambda_i'(q_i, d) =$*

[14]Clearly, the strategy of a buyer is defined in the same way as in the previous section.

$\lambda_i(q_i, d)$ *for all* d *and for all* q_i, *and there exists a set of states* $\overline{Q}_i \subset Q_i (= Q_i')$ *such that*

$$\left.\begin{array}{ll} \lambda_i'(q_i,\Phi) = \lambda_i(q_i,\Phi) & \text{if } q_i \notin \overline{Q_i} \\ \lambda_i(q_i,\Phi) = \lambda_i(q_i',\Phi) & \forall q_i, q_i' \in \overline{Q}_i, \\ \lambda_i'(q_i,\Phi) \neq \lambda_i'(q_i',\Phi) & \text{for some } q_i, q_i' \in \overline{Q}_i \\ \lambda_i'(q_i,\Phi) \neq \lambda_i'(q_i',\Phi) & \forall q_i \in Q_i/\overline{Q}_i \text{ and } \forall q_i' \in \overline{Q}_i \end{array}\right\} \quad (4.1)$$

However, it turns out that r-complexity together with n-complexity are not sufficient to select the competitive outcome. (I have a counter-example demonstrating the existence of a non-competitive outcome with this stronger complexity criterion[15] for the no discounting case.)

Another candidate for measuring the complexity of seller's choice of partners at the beginning of each period is the number of states of the machine. Clearly, a seller's machine needs to have as many states as the number of possible partners he chooses in the game. Putting it differently, if two machines for seller s are otherwise identical except that the first chooses fewer partners than the second, then the second machine must have more states than the first. I shall demonstrate below that s-complexity (a measure the number of induced rules at the beginning of each period) together with r-complexity (a measure of the complexity within a period) are sufficient to give us the selection result.

The main result of this section is stated for the no discounting case since this appears to be most amenable to indeterminacy type results.

Theorem 4.3. *Suppose the seller can choose in each period the buyer with whom he wishes to bargain (voluntary matching) and* $\delta = 1$. *Then, consider any PBECrs strategy profile* $f = \{f_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$. *If each strategy* f_i *is finite then* $\pi_s(f) = 1$ *for all* s, $\pi_b(f) = 0$ *for all* b, *the unique induced price is the competitive price of 1 and each* f_i *is stationary.*[16]

The proof of the above Theorem can be found in Appendix B for the case of the single seller. The result can be established for more than one seller by applying an induction argument on the set of sellers as in the proof of Theorem 2.3 in Appendix A.

A very brief sketch of the proof of Theorem 4.3 for the case of the single seller (Appendix B) is as follows. Fix a PBECrs profile f and let M be the PBECrs machine

[15] This stronger definition of complexity (r-complexity together with n-complexity) is used in Chatterjee and Sabourian (1999).

[16] As in the proof of Theorem 2.3, the finiteness of f_i is only needed because complexity costs enters the players' preferences lexicographically in the definition of PBEC-rs. If postive fixed cost is assumed, as in Definition 10, I do not need to assume that each f_i is finite; Theorem 4.3 holds without this assumption because the positive cost of each state implies that in equilibrium players use finite machines (strategies).

profile that implements f. For any b, let

$$z(b) = \max_h \pi_b(\langle f \mid h \rangle).$$

First, it is shown that $z(b)$ is the same for all b (this is because the seller selects a buyer at each period). Denote $z(b)$ by z. Next consider the set of histories

$$H(b) \equiv \{h \in H^\infty \mid \pi_b(\langle f \mid h \rangle) = z\}.$$

The proof of the Theorem considers two separate cases:

(i) For all b and for all $h_b \in H(b)$ the probability that the seller s chooses another buyer $b' \neq b$ after h_b is zero (I call this property α in Appendix B).

(ii) There exists b, $h_b \in H(b)$ and another buyer $b' \neq b$ such that the probability that the seller s chooses b' after h_b is positive.

Case (ii) is like the random matching model (there is a positive probability of choosing another buyer) and the proof that f results in a payoff of 1 for the seller is similar to that of Theorem 2.3 in Section 3.

When f satisfies case (i), I show that if $\pi_s(f) < 1$ then it is possible to construct another machine for the seller that generates the same payoff as the equilibrium machine M_s and economizes on s-complexity. This is done by first showing that if $\pi_s(f) < 1$ then s selects at least two buyers b and b' with a positive probability.. (Otherwise, there is a buyer that is never chosen on the equilibrium path and thus by r-complexity he will accept any positive price; this contradicts $\pi_s(f) < 1$).

Next, let q_s^b and $q_s^{b'}$ be the states of the equilibrium machine of s after any histories $h_b \in H(b)$ and $h_{b'} \in H(b')$ respectively. Since after any history $h_i \in H(i)$, $i = b, b'$, the seller selects only buyer i, it can be shown that s obtains the minimum continuation payoff of $1 - z$ any time the equilibrium machine of s is in state q_s^i. Now consider another machine M'_s for s that is otherwise identical to the equilibrium machine of s except that the two states q_s^b and $q_s^{b'}$ are replaced by a single absorbing state[17] q' that always chooses buyer b, always offers $1 - z(b)$ and always accepts an offer if and only if the offer is not less than $1 - z$. Using r-complexity, it is also shown that for any b the equilibrium machine M_b either *always* accepts an offer of $1 - z$ or *always* offers $1 - z$. Therefore, $(M'_s, M_b, M_{-s,b})$ result in an immediate agreement at a price $1 - z$ after any history that changes the state of the machine M'_s to q'. Thus M'_s induces the same payoff as the equilibrium machine M_s and has less states than the latter. But this results in a contradiction.

5. Concluding Remarks

Finally, I would like to conclude this paper with some remarks and conjectures on the various ways of extending and expanding the results of this paper.

[17]A state is absorbing if once in it the state of the machine in the next period remains the same for all possible inputs.

5.1. Equal number of buyers and sellers

The selection result in this paper shows that those on the short side of the market (the sellers in the model presented) receive all the surplus generated by exchange in any equilibrium with complexity costs. What if the number of buyers equals the number of sellers? In this case, complexity considerations do not select among the set of possible equilibrium prices. But notice that this is consistent with the competitive outcome; when $B = S$, any price between 0 and 1 is a competitive price.

5.2. Complexity criterion and alternative machine specification

R-complexity used to obtain the competitive outcome in the random matching model is a very weak concept. In the voluntary matching model, I use r-complexity together with s-complexity. Clearly, this division between the two notions of complexity reflects the machine specification I have adopted in this paper. It is possible that with a different machine specification (e.g. states of the machines changing within a period) one may be able to establish the selection results of this paper with a different notion of complexity.

5.3. Equilibrium concept

In this paper, the equilibrium concept adopted is PBECl. Any profile of strategies $f = (f_i)_{i \in \mathcal{B} \cup \mathcal{S}}$ is PBECl if it is both a perfect Bayesian equilibrium and is such that for all i

$$\text{if some strategy } f_i' \text{ is a best responses to } f_{-i} \text{ then } f_i' \succeq^l f_i$$

As I mentioned before, PBECl imposes the notion of credibility directly on the set of NECl profiles. Another way of ensuring that NECl strategy profiles are credible is to allow strategies (machines) to tremble and consider the limit of Nash equilibrium with trembles and l-complexity as the trembles become small. This approach is adopted by Chatterjee and Sabourian (1999, 2000). My conjecture is that the results of this paper remain valid with this alternative formulation of credibility, irrespective of the order in which complexity costs and trembles enter the limiting arguments.

There are several different ways of weakening PBECll concept. First, one could consider the following weaker equilibrium concept (Kalai and Neme (1992) use a similar notion of equilibrium).

A strategy profile f is weakly PBECl if it is both a sequential equilibrium and is such that for all i

$$\text{if some strategy } f_i' \text{ is s.t. } \langle f_i' \mid h, d\rangle \text{ is a best responses to } \langle f_{-i} \mid h, d\rangle \text{ for all } (h, d) \text{ then } f_i' \succeq f_i$$

This is a weaker concept than PBECll because complexity costs enter lexicographically among strategies that are best responses at *every* information sets. Here, I also conjecture that the results obtained in this paper remains valid if the weakly PBECll equilibrium concept is adopted.[18]

Another way of weakening the equilibrium concept in this paper is to use solution concepts based on the notion of strict dominance rather than NECl or PBECll, which are based on the idea of Nash equilibrium.[19] For example, the concept of NECl strategy profile could be replaced by the following solution concepts.

(i) A strategy f_i is strictly dominated with complexity costs l (denoted by SDl) if for all f_i' and for all f_{-i}

$$\begin{aligned}&\text{either } \pi_i(f_i, f_{-i}) > \pi_i(f_i', f_{-i}) \\ &\text{or } \pi_i(f_i, f_{-i}) = \pi_i(f_i', f_{-i}) \text{ and } f_i' \succ^l f_i\end{aligned}$$

(ii) Iterated strict dominant solution with complexity costs l (denoted by ISDl) is defined as the set of strategies that survive the process of iteratively deleting, at each round, every SDl strategies.

I could also replace PBECl criterion with some thing like Pearce's(1984) extensive form rationalizability together with $l-$complexity (or with strategies that survive iterated conditional dominance together with $l-$complexity; see Fudenberg and Tirole (1991) section 4.6 for the definition of conditional dominance). It is my conjecture that one may obtain the selection results in this paper with these weaker solution concepts.[20]

5.4. Richer models of trade

RW's model considered in this paper is very simple. It is my conjecture that the results of this paper hold if one introduces a different matching/bargaining arrangement into RW's model. A more interesting issue would be to consider complexity costs in richer models of exchange than that considered by RW. For example, one could address the issues considered in this paper with a heterogeneous set of buyers and sellers and/or models when trade decision is not restricted to a single unit of a good. Or one could look at exchange economy with many goods where agents trade their endowments sequentially. (For example, Gale 1986, 2000.) It is an open question whether complexity costs allow one to select the competitive outcomes among the set of equilibria in these richer models of exchange as well.

[18] In contrast to Abreu and Rubinstein's (1988) selection results with NECs in 2-player repeated games, Kalai and Neme (1992) demonstrate a Folk Theorem type result for the weakly PBECs strategies in the repeated prisoner's dilemma.

[19] The equilibrium concepts based on the strict dominance criterion are more attrative than those based on Nash equilibrium because it is easier to justify them in terms of either rationality arguments or in terms of evolutionary/learning stories.

[20] In fact, in the proof of the two selection results of this paper (Theorem 2.3 and 4.3), there are many results (Lemmas) characterising properties of NECr strategy profiles. It is reasonably easy to show that these characterisation results apply equally well to the set of ISDl strategy profiles.

5.5. Complexity and the properties of bargaining games

Chatterjee and Sabourian (1999,2000) and Sabourian (1999) also use complexity costs to select (uniquely) among the large number of equilibria in n-person complete information alternating bargaining game and in 2-person one-sided incomplete information bargaining game, respectively. In particular, these papers try to provide a justification for stationary equilibria in these class of dynamic games. Complexity costs, however, do not always select a unique equilibrium or provide a justification for stationary/Markov strategies in dynamic games (for example repeated games; see Abreu and Rubinstein (1988) and Bloise (1998)). This paper, together with Chatterjee and Sabourian (1999, 2000) and Sabourian (1999) demonstrate that non-stationary equilibria of the dynamic models involving bargaining are not always robust to the introduction of complexity considerations. Bargaining games have the following two properties:

(i) the (last) responder can always end the game by accepting an offer;

(ii) the payoffs the players receive depend on the value and the time of the final agreement and not on the history of play up to the final agreement.

These two features give complexity considerations a role in selecting among a large number of equilibria in these class of dynamic games.

6. Appendix A: Proof of Theorem 2.3 with an arbitrary number of sellers

The proof for the case of more than one seller is by induction on the number of sellers S in the market. In section 3, it was shown that the result holds for the case of $S = 1$. To complete the proof of Theorem 2.3 with an arbitrary number of sellers, I need to show that if any PBECr with less than S sellers results in a payoff of 1 for each seller then any PBECr profile with S sellers also results in a payoff of 1 for each seller. This is done by repeating some of the arguments for the case of $S = 1$.

I shall now provide a brief sketch of the induction argument that follows. The first four Lemmas in this appendix establish that if Theorem 2.3 holds when the number of sellers is less than S then in a market with exactly S sellers and for any PBECr strategy profile that results in a payoff of less than 1 for some of the S sellers, we can not have an agreement at a price of 1 in any match between a seller and a buyer after any history. This implies (Lemma 12) that if Theorem 2.3 holds when the number of sellers is less than S then for any PBECr profile f in a market with exactly S sellers the following holds

$$\text{if } \pi_s(f) < 1 \text{ for some } s \text{ then } \pi_s(\langle f \mid h \rangle) < 1 \text{ for all } s. \tag{6.1}$$

Finally, the proof of Lemma 14 demonstrates that if Theorem 2.3 holds when the number of sellers is less than S, then for any PBECr profile with S sellers there exits a history such that the continuation payoff of some seller is one. (Intuitively, this is because at some point a pair of a buyer and a seller will leave the market and by assumption the remaining sellers will receive a continuation payoff of 1.) But this contradicts (6.1) unless the PBECr strategy profile results in a payoff of 1 for each of the S sellers. Thus if Theorem 2.3 holds when the number of the sellers is less than S it also holds when there are exactly S sellers.

Now I turn to the formal proof of the above.

Lemma 8. *Consider any NECr strategy profile f in a market with exactly S sellers then if $\pi_s(f) = 1$ for some seller s; then*

either there exists a buyer b such that after every history if s and b are matched with s as the proposer then they will agree on a price of 1 ($\exists\, b$ such that $f_s(h, s, b) = 1$ and $f_b(h, s, b, 1) = A$ for all h)

or there exists a buyer b such that after every history if s and b are matched with b as the proposer then they will agree on a price of 1 ($\exists\, b$ such that $f_b(h, b, s) = 1$ and $f_s(h, b, s, 1) = A$ for all h).

Proof. Since $\pi_s(f) = 1$ for some seller s it follows that

(i) either $f_s(h, s, b) = 1$ and $f_b(h, s, b, 1) = A$ for some h and for some buyer b

(ii) or $f_b(h, b, s) = 1$ and $f_s(h, b, s, 1) = A$ for some h and for some buyer b.

Now consider each of the possibilities in turn. In the first case, by Lemma 1, $f_b(h, s, b, 1) = A$ for some h implies that

$$f_b(h, s, b, 1) = A \text{ for all } h \tag{6.2}$$

But this implies that

$$f_s(h, s, b) = 1 \text{ for all } h. \tag{6.3}$$

(Otherwise, $f_s(h, s, b) = p \neq 1$ for some h. But then s could obtain a payoff of 1 and reduce r-complexity by following a strategy that is otherwise identical to f_s except it always offers 1 to b. Such a strategy guarantees a payoff of 1, given that other players are following f_{-s}, because if the ordered pair (s, b) does not occur then the payoff to s would be the same as that s obtains if it follows f_s, namely $\pi_s(f) = 1$, and if the ordered match (s, b) does occur after some history then, by condition (6.2) s and b will agree on a price of 1.)

Now consider the second possibility in which there exists b and h such that $f_b(h, b, s) = 1$ and $f_s(h, b, s, 1) = A$. By Lemma 1 this implies that

$$f_s(h, b, s, 1) = A \text{ for all } h \tag{6.4}$$

But then

$$f_b(h, b, s) = 1 \text{ for all } h. \tag{6.5}$$

(If $f_b(h,b,s) = p \neq 1$ for some h then b could economize on r-complexity and obtain at least the same payoff, given f_{-b}, as he would obtain with strategy f_b by following another strategy f'_b that is otherwise identical to f_b except that $f'_b(h,b,s) = p$ for all h such that $f_b(h,b,s) = 1$. Since, by condition (6.4), s always accepts an offer of 1 by b, it follows that $\pi_b(f'_b, f_{-b}) \geq \pi_b(f_b, f_{-b})$.)

Thus it follows from the above two possibilities that either there exists a buyer b such that conditions (6.2) and (6.3) hold or there exists a buyer b such that conditions (6.4) and (6.5) hold. ■

Lemma 9. *Suppose that all PBECr strategy profiles in markets with less than $S > 1$ sellers result in a payoff of 1 for each seller. Then for any PBECr strategy profile f in a market with exactly S sellers the following holds:*

$$\text{if } \pi_{s'}(f) < 1 \text{ for some seller } s' \text{ then } \pi_s(f) < 1 \text{ for all sellers } s$$

Proof. Suppose not; then $\pi_{s'}(f) < 1$ and $\pi_s(f) = 1$ for some s' and for some s. Now consider a strategy $f'_{s'}$ for s' such that

$$\begin{array}{ll} f'_{s'}(h,d) = f_{s'}(h,d) & \forall\ h \text{ such that there are less than } S \text{ sellers} \\ f'_{s'}(h,s',b) = 1 & \forall\ b \text{ and } \forall\ h \text{ such that there are } S \text{ sellers} \\ f'_{s'}(h,b,s',p) = R & \forall\ b,\ \forall\ h \text{ and } \forall\ p \text{ such that there are } S \text{ sellers} \end{array}$$

Since $\pi_s(f) = 1$, by the previous Lemma, we have

$$\begin{array}{l} \text{either } \exists b \text{ such that } f_s(h,s,b) = 1 \text{ and } f_b(h,s,b,1) = A \text{ for all } h \\ \text{or } \exists b \text{ such that } f_b(h,b,s) = 1 \text{ and } f_s(h,b,s,1) = A \text{ for all } h \end{array} \tag{6.6}$$

Therefore, it follows from (6.6) that if the players choose the profile $(f'_{s'}, f_{-s'})$ then with probability one some seller is going to reach an agreement with a buyer in finite time. (Otherwise, by the law of large numbers, seller s will meet each buyer b both as a proposer and as a responder; but then by (6.6) an agreement will be reached.)

Given that $(f'_{s'}, f_{-s'})$ results in an agreement in finite time, there are two possible set of outcome paths.

Case A: s' reaches an agreement with a buyer no later than any other seller reaches an agreement. In this case given the definition of $f_{s'}$ the agreement must involve a price of 1.

Case B: some seller $s'' \neq s'$ reaches an agreement with a buyer and leaves the market in some finite time (before s' reaches an agreement). Since $(f'_{s'}, f_{-s'})$ is identical to $(f_{s'}, f_{-s'})$, when there are less than S sellers in the market, and since $(f_{s'}, f_{-s'})$ constitutes a PBECr, it follows from the supposition of the induction argument that if $(f'_{s'}, f_{-s'})$ is implemented then all the remaining sellers will receive a payoff of 1 after s'' leaves the market.

Therefore, in both cases $(f'_{s'}, f_{-s'})$ results in a payoff of 1 for player s'. Since $(f_{s'}, f_{-s'})$ is a PBECr, it follows that $\pi_{s'}(f_{s'}, f_{-s'}) \geq \pi_{s'}(f'_{s'}, f_{-s'}) = 1$. But this contradicts the assumption that $\pi_{s'}(f_{s'}, f_{-s'}) < 1$. ■

Lemma 10. *Suppose that all PBECr strategy profiles in markets with less than $S > 1$ sellers result in a payoff of 1 for each seller. Then for any PBECr strategy profile f in a market with exactly S sellers the following holds: if $\pi_{s'}(f) < 1$ for some seller s' then we have $f_b(h, s, b, 1) = R$ for all h, for all b and for all s.*

Proof. Suppose not; then $f_b(h, s, b, 1) = A$ for some b, for some s and for some h. By Lemma 1 this implies that

$$f_b(h, s, b, 1) = A \text{ for all } h \tag{6.7}$$

Now consider any strategy f'_s for s such that

$$\begin{array}{ll} f'_s(h, d) = f_s(h, d) & \forall\ h \text{ such that there are less than } S \text{ sellers} \\ f'_s(h, s, b) = 1 & \forall\ b \text{ and } \forall\ h \text{ such that there are } S \text{ sellers} \\ f'_s(h, b, s, p) = R & \forall\ b,\ \forall\ h \text{ and } \forall\ p \text{ such that there are } S \text{ sellers} \end{array}$$

Clearly, if the strategy profile (f'_s, f_{-s}) is chosen there are two possible set of outcomes.

Case A: some seller $s' \neq s$ reaches an agreement with a buyer and leaves the market in some finite time (before s reaches an agreement). Since when there are less than S sellers in the market (f'_s, f_{-s}) is identical to (f_s, f_{-s}) and since (f_s, f_{-s}) constitutes a PBECr, it follows from the supposition of the induction argument that if (f'_s, f_{-s}) is implemented then all the remaining sellers, including s, will receive a payoff of 1 after s' leaves the market.

Case B: there does not exists a seller $s' \neq s$ reaches an agreement with a buyer and leaves the market before s reaches an agreement (This case can includes outcomes in which no player leaves the market.) But this implies, conditional on no agent leaving the market before s, that the match given by the ordered pair (s, b) occurs with probability 1. But, if (s, b) occurs it follows from the definition of f'_s and condition (6.7) that f'_s can guarantee s a payoff of 1. Therefore, if f'_s is chosen the expected payoff to s (conditional on no agent leaving the market before s) is 1.

Since in both cases the expected payoff to s if (f'_s, f_{-s}) is chosen is 1 and the strategy profile f is a PBECr it follows that $\pi_s(f) = 1$. But given the previous Lemma, this contradicts the hypothesis that $\pi_{s'}(f) < 1$. ■

Lemma 11. *Suppose that all PBECr strategy profiles in markets with less than $S > 1$ sellers result in a payoff of 1 for each seller. Then for any PBECr strategy profile f in a market with exactly S sellers the following holds: if $\pi_{s'}(f) < 1$ for some seller s', then we have for all s, for all b and for all h*

$$\textit{either} \qquad f_b(h, b, s) \neq 1 \qquad \textit{or} \qquad f_s(h, b, s, 1) = R.$$

Proof. Suppose not; then there exists s, b and h such that $f_b(h, b, s) = 1$ and $f_s(h, b, s, 1) = A$. By Lemma 1 this implies that

$$f_s(h, b, s, 1) = A \text{ for all } h \tag{6.8}$$

Also since f results in a payoff less than 1 for s', by Lemma 9, it follows that

$$\pi_s(f) < 1 \tag{6.9}$$

The next step is to show that there exists h' such that $f_b(h', b, s) = p' \neq 1$. Suppose not; then s could always obtain a payoff of 1 by choosing strategy f'_s such that for any $(h, d) \in H^\infty \times D_s$

$$f'_s(h,d) = \begin{cases} 1 & \text{if } h \text{ s.t. there are } S \text{ sellers \& } d = (s, b') \text{ for some } b' \\ A & \text{if } h \text{ is s.t. there are } S \text{ sellers \& } d = (b', s, 1) \text{ for some } b' \\ R & \text{if } h \text{ is s.t. there are } S \text{ sellers, \& } d = (b', s, p) \text{ for some } b' \text{ \& } p < 1 \\ f_s(h,d) & \text{otherwise.} \end{cases}$$

This is because if the strategy profile (f'_s, f_{-s}) is chosen, as in the proof of the previous Lemma, there are two possible set of outcomes.

Case A: some seller $s' \neq s$ is going to reach an agreement with a buyer and leaves the market in some finite time before s reaches an agreement. Since (f'_s, f_{-s}) is identical to (f_s, f_{-s}), when there are less than S sellers in the market, and (f_s, f_{-s}) constitutes a PBECr, it follows from the supposition of the induction argument that if (f'_s, f_{-s}) is implemented then all the remaining sellers, including s, will receive a payoff of 1 after s' leaves the market.

Case B: there does not exist a seller $s' \neq s$ who reaches an agreement with a buyer and leaves the market before s leaves the market. (This case can include outcomes in which no player leaves the market.) But this implies that there are three possibilities: some buyer accepts the offer of 1 by s, some buyer $b' \neq b$ makes an offer of 1 or by the law of large numbers s will eventually be matched with b and will receive an offer of 1 (by assumption) from b. Clearly, in all the three cases s receives a payoff of 1.

Since (f'_s, f_{-s}) result in a payoff of 1 in both above cases and the strategy profile f is a PBECr it follows that $\pi_s(f) = 1$. But this contradicts (6.9). Therefore, there exists h' such that $f_b(h', b, s) = p' \neq 1$.

Now consider a strategy f'_b such that

$$\begin{array}{ll} f'_b(h', d) = p' & \text{for all } (h, d) \text{ such that } f_b(h, d) = 1 \\ f'_b(h, d) = f_b(h, d) & \text{otherwise} \end{array}$$

Clearly, the only difference between f_b and f'_b is that in some instance f_b proposes an offer of 1 (and by (6.8) this is accepted) and f'_b does not; thus f'_b induces as much payoff as f_b and it is also less r-complex than f_s according to Definition 6. But, by Remark 3, this is a contradiction. ■

Lemma 12. *Suppose that all PBECr strategy profiles in markets with less than $S > 1$ sellers result in a payoff of 1 for each seller. Then for any PBECr strategy profile f in a market with exactly S sellers the following holds: if $\pi_{s'}(f) < 1$ for some seller s' then for all s, for all b and for all h we have that $\pi_s(f \mid h, s, b) < 1$, $\pi_s(f \mid h, b, s) < 1$ and thus $\pi_s(f \mid h) < 1$.*

Proof. This follows from no buyer accepts an offer of 1 (Lemma 10) and from the ordered pair (b, s) never reaching an agreement at a price of 1 (Lemma 11). ■

Lemma 13. *For any perfect Bayesian equilibrium strategy profile f in a market with exactly S sellers the following holds: there exists a history h, a move by nature η (a match $q(.)$, and a choice of a proposer $m_q(s)$ for each match between s and $q(s)$) such that some s and some buyer b reach an agreement after history h and after η.*

Proof. Suppose not; then for all h, for all η and for all s no agreement is reached. Therefore, $\pi_i(f) = 0$ for every player i. Now consider any s, any b any price offer $p > 0$ by b to s at $t = 1$. Clearly, s's optimal response to any such p is to accept (otherwise, s will receive zero by the supposition). But then b could obtain a positive payoff by offering any price $0 < p < 1$. But this contradicts $\pi_b(f)$=0. ■

Lemma 14. *Suppose that all PBECr strategy profiles in markets with less than $S > 1$ sellers result in a payoff of 1 for each seller. Then any PBECr strategy profile f in a market with exactly S sellers results in a payoff of 1 for each seller.*

Proof. Suppose not; then there exists a PBECr strategy profile f in a market with exactly S sellers that results in a payoff of less than 1 for some seller. Thus, by the Lemma 12 we have that

$$\pi_s(f \mid h) < 1 \qquad \text{for all } s \text{ and for all } h \tag{6.10}$$

By the previous Lemma, there exists a h', some move by Nature η and some s' and some buyer b' such that s' and b' reach an agreement after h and η. Now consider any player $s \neq s'$ and a new strategy f'_s for s such that for all h and for all d

$$f'_s(h, d) = \begin{cases} 1 & \text{if } h = h' \text{ and if } d = (s, b) \text{ for some } b \\ A & \text{if } h = h' \text{ and if } d = (b, s, 1) \text{ for some } b \\ R & \text{if } h = h' \text{ and if } d = (b, s, p) \text{ for some } b \text{ and for some } p < 1 \\ f_s(h', d) & \text{if } h \neq h' \end{cases}$$

Now let e be the outcome of the period after h', given that Nature has chosen η and given that the players follow the strategy profile (f'_s, f_{-s}). Then if s plays the game according to f'_s after history $h'' \equiv (h', e)$ then he will receive a payoff of 1. This is because if e results in an agreement between s and some buyer b then it follows from the definition of f'_s that the agreement will be at price 1. Also, since there are at most $S-1$ sellers in the market after h'', it follows from the supposition and from the definition of f'_s (namely that $f'_s(h,d) = f_s(h,d)$ at every h that follows h''), that all those sellers remaining in the market after h'' have a continuation payoff of 1. Therefore, in either case

$$\pi_s(f'_s, f_{-s} \mid h'') = 1.$$

But since f is a perfect Bayesian equilibrium it follows that

$$\pi_s(f \mid h'') \geq \pi_s(f'_s, f_{-s} \mid h'').$$

But the last two conditions contradict condition (6.10). ■

Now note that, by induction on the number of sellers, it follows from the last Lemma together with the proof of Theorem 2.3 for the case of a single seller (in section 3) that for any PBECr profile strategy f in a market with S sellers $\pi_s(f) = 1$ for every s and thus $\pi_b(f) = 0$ for every b. This implies that the unique induced price is 1 and, by the same arguments as that in the last paragraph of section 3, f is stationary.

7. Appendix B: Proof of Theorem 4.3 for the case of a single seller s

Lemmas 1, 2, 3, 4, 5 also hold for the voluntary matching model and the proofs of these Lemmas with in this case with deterministic endogenous matching arrangement are similar to those found in section 3. In fact, the proofs of 1, 4 and 5 are identical to those in section 3 and I will not repeat the arguments. Here, I shall only provide the proofs for Lemmas 2 and 3 when the trading arrangement is voluntary.

Proof of Lemma 2 for the voluntary matching model: Suppose not; then $f_b(h,s,b,1) = A$ for some b and for some h. By Lemma 1 this implies that

$$f_b(h,s,b,1) = A \text{ for all } h \tag{7.1}$$

Now consider any strategy f'_s for s that always chooses player b and proposes 1 and rejects all offers. Since, with probability 1, s will have the opportunity to make a proposal to player b in finite time, it follows from (7.1) that f'_s can guarantee s a payoff of 1; but this is a contradiction. ■

Proof of Lemma 3 for the voluntary matching model: Suppose not; then there exists b and h such that $f_b(h,b,s) = 1$ and $f_s(h,b,s,1) = A$. Therefore, by Lemma 1

$$f_s(h,b,s,1) = A \text{ for all } h.$$

Also, since f results in a payoff less than 1 there exists h' such that $f_b(h',b,s) = p' \neq 1$ (otherwise s could always obtain 1 by always choosing b, always making an offer of 1 and only accepting an offer of 1; following this strategy results eventually in b making an offer of 1 to s). Now consider a strategy f_b' such that

$$\begin{array}{ll} f_b'(h',d) = p' & \text{for all } (h,d) \text{ such that } f_b(h,d) = 1 \\ f_b'(h,d) = f_b(h,d) & \text{otherwise} \end{array}$$

Clearly, f_b' induces as much payoff as f_b and is less r-complex than f_b according to Definition 6. But, by Remark 3, this is a contradiction. ■

Now, as in Section 3, for any f, let

$$m_s^b(f) = \min_{h \in H^\infty} \pi_s(\langle f \mid \langle h, s, b \rangle)$$

$$m_b^b(f) = \min_{h \in H^\infty} \pi_s(\langle f \mid h, b, s \rangle)$$

$$\overline{b}(f) \equiv \text{ arg min}_b \ (m_b^b(f) + m_s^b(f)) \tag{7.2}$$

$$z(b;f) = \max_{h \in H^\infty} \pi_b(\langle f \mid h \rangle)$$

$$H(b;f) = \{h \in H^\infty \,| \pi_b(\langle f \mid h \rangle) = z(b;f)\}$$

Note also that if f is finite then $z(b;f)$ and $m_i^b(f)$ are well defined for $i = b, s$, and $H(b;f)$ is not empty.

Any strategy profile f defines a probability distribution on the set of outcome paths in this game. From this, one can compute the probability of any finite history $h \in H^\infty$, given that the players choose a given strategy profile f. I shall denote such a probability by $\theta(h;f)$. Also, with some abuse of notation, let

$\theta(h,d;f) =$ probability of $(h,d) \in H^\infty \times D$ given that the players choose strategy profile f

$\theta(h,b;f) =$ probability of (h,b) given that the players choose strategy profile f

where (h,b) refers to history h followed by the seller choosing b as the partner in the next period. Finally, for any strategy profile f, I denote the set of histories that occur with a positive probability and the probability that s chooses a buyer b for the first time after history h by $\Omega(f)$ and $\beta(h,b;f)$, respectively. Thus,

$$\Omega(f) = \{h \in H^\infty \,|\theta(h;f) > 0\}\}$$

$$\beta(h,b,f) = \sum_{h' \in \Sigma^b} \theta(h,h',b;f)$$

where

$$\Sigma^b = \left\{h = (e^1,...,e^t) \in H^\infty \,|e^\tau \text{ does not involve a match between } s \text{ and } b \text{ for all } \tau \leq t\right\}$$

Henceforth, I fix a strategy profile f and refer to $m_s^b(f)$, $m_b^b(f)$, $\overline{b}(f)$, $z(b;f)$, $H(b,f)$, $\theta(h,d;f)$, $\Omega(f)$ and $\beta(h,b;f)$ by m_s^b, m_b^b, $\overline{b}$, $z(b)$, $H(b)$, $\theta(h,d)$, Ω and $\beta(h,b)$ respectively.

Lemma 15. *Suppose $S = 1$. Then for any finite subgame perfect equilibrium strategy profile f we have $z(b) = z(b')$ for all b and b'.*

Proof. Suppose not; then

$$z(b') > z(b) + \epsilon$$

for some b, for some b' and for some $\epsilon > 0$.

Consider any $h_{b'} \in H(b')$. Since $\pi_b(\langle f \mid h_{b'}\rangle) = z(b')$ it follows that

$$\pi_s(\langle f \mid h_{b'}\rangle) \leq 1 - z(b') < 1 - z(b) - \epsilon \tag{7.3}$$

Now consider a strategy f'_s for s that always chooses buyer b, rejects all offers and always makes the proposal $1-z(b)-\epsilon$. Clearly, b always accepts the proposal $1-z(b)-\epsilon$. Therefore (f'_s, f_{-s}) guarantees a payoff of $1 - z(b) - \epsilon$ after history $h_{b'}$. Since f is a subgame perfect equilibrium we have $\pi_s(\langle f \mid h_{b'}\rangle) \geq \pi_s(\langle f'_s, f_{-s} \mid h_{b'}\rangle) \geq 1 - z(b) - \epsilon$. But this contradicts condition (7.3). ■

Since $z(b) = z(b')$ for all b and b' henceforth, for any strategy profile f, I shall refer to $z(b)$ by z.

Lemma 16. *Suppose $S = 1$. Then for any finite subgame perfect equilibrium strategy profile f with voluntary matching we have $m_{\bar{b}}^{\bar{b}} \geq m_s^{\bar{b}}$.*

Lemma 16 is a restatement of Lemma 6 for the voluntary matching model. The steps of the proofs of the two Lemmas are identical and therefore I will omit stating the proof of Lemma 16.

Definition 16. *A strategy profile f is said to satisfy property α if for all b, for all $h_b \in H(b)$ and for all $b' \neq b$ the probability that s chooses b' after h_b is zero. Formally, f satisfies property α if for all b, for all $h_b \in H(b)$*

$$\beta(h_b, b') = 0 \text{ for all } b' \neq b$$

Lemma 17. *Suppose $S = 1$. Then for any finite PBECr strategy profile f that does not satisfy property α we have $\pi_s(f) = 1$.*

Proof. Suppose not; then $\pi_s(f) < 1$. Now let

$$\bar{\epsilon} = \min_{b \in \mathcal{B},\, h \in H^\infty} \pi_b(\langle f \mid h, b, s\rangle) \tag{7.4}$$

Since f is finite $\bar{\epsilon}$ is well defined. Also, by $\pi_s(f) < 1$ and Lemma 5, we have $\bar{\epsilon} > 0$.

Now since f does not satisfy property α there exists b, $h_b \in H(b)$ and $b' \neq b$ such that

$$\beta(h_b, b') > 0.$$

By the definition of $z(b)$ and Lemma 15 we have that $z = z(b) = \pi_{\overline{b}}(\langle f \mid h_b \rangle)$. Therefore, since the seller's minimum continuation payoff is at least $1/2(m_s^{\overline{b}} + m_{\overline{b}}^{\overline{b}})$, we can write an upper bound on z as follows

$$\begin{aligned}&z \leq 1 - 1/2(m_s^{\overline{b}} + m_{\overline{b}}^{\overline{b}}) - \\ &\textstyle\sum_{h \in \Sigma^{b'}} \theta(h_b, h, b')[1/2(\pi_{b'}(\langle f \mid h_b, h, s, b' \rangle) + 1/2\pi_{b'}(\langle f \mid h_b, h, b', s \rangle]\end{aligned} \tag{7.5}$$

(The third terms on the RHS of the last inequality is simply the sum of the expected continuation payoff of $b' \neq b$ after history h_b.)

Therefore, it follows from (7.4)

$$z \leq 1 - 1/2(m_s^{\overline{b}} + m_{\overline{b}}^{\overline{b}}) - \frac{\overline{\epsilon}}{2} \sum_{h \in \Sigma^{b'}} \theta(h_b, h, b') \tag{7.6}$$

Thus it follows from (7.6) and from the definition of $\beta(h_b, b')$ that

$$z \leq 1 - 1/2(m_s^{\overline{b}} + m_{\overline{b}}^{\overline{b}}) - \frac{\overline{\epsilon}\beta(h_b, b')}{2} \tag{7.7}$$

Now, by the same argument as that which follows (3.7) in the proof of Lemma 7, I now show that

$$z \geq 1 - m_s^{\overline{b}} \tag{7.8}$$

To show this, suppose otherwise; then

$$m_s^{\overline{b}} < 1 - z - \epsilon \text{ for some } \epsilon > 0. \tag{7.9}$$

Now consider any history h and suppose that s makes a price offer of $(1 - z(\overline{b}) - \epsilon)$ to $\overline{b}$ after $(h, s, \overline{b})$. Since $z = z(\overline{b})$ is the maximum continuation payoff of $\overline{b}$, it follows that this offer will be accepted by $\overline{b}$. Thus $m_s^{\overline{b}} \geq 1 - z(\overline{b}) - \epsilon$. But this contradicts condition (7.9). Therefore, condition (7.8) holds.

But (7.8), together with condition (7.7), imply that

$$m_s^{\overline{b}} \geq 1/2(m_s^{\overline{b}} + m_{\overline{b}}^{\overline{b}}) + \frac{\overline{\epsilon}\beta(h_b, b', M)}{2}$$

Therefore, it follows from Lemma 16 that

$$m_s^{\overline{b}} \geq m_s^{\overline{b}} + \frac{\overline{\epsilon}\beta(h_b, b', M)}{2}$$

But since $\overline{\epsilon} > 0$ and $\beta(h_b, b') > 0$ this is a contradiction. Therefore, $\pi_s(f) = 1$. ■

Lemma 18. *Consider any subgame perfect equilibrium strategy profile f. Then $\pi_s(\langle f \mid \langle h \rangle) \geq 1 - z$ for all h.*

Proof. Suppose not; then there exists h such that $\pi_s(\langle f \mid \langle h \rangle) < 1 - z - \epsilon$ for some $\epsilon > 0$. Now consider a strategy $f_s^{'}$ that always chooses the same buyer, always offers $1-z-\epsilon$ and rejects all offers. Since every16 buyer always accepts any offer below $1-z$ it follows that $\pi_s(f_s^{'}, f_{-s} \mid \langle h \rangle) = 1 - z - \epsilon > \pi_s(f \mid \langle h \rangle)$. But this is a contradiction. ■

Lemma 19. *For any NECr strategy profile f we have*

$$f_s(h, b, s, 1-z) = f_s(h', b, s, 1-z) \text{ for all } b, \ h \text{ and } h' \in H^\infty$$

Proof. Suppose not; then there exists b, h and $h' \in H^\infty$ such that $f_s(h, b, s, 1-z) = A$ and $f_s(h', b, s, 1-z) = R$. Now consider another strategy f_s' that is defined by

$$\begin{array}{ll} f_s'(h, d) = R & \text{if } d = (b, s, 1-z) \\ f_s'(h, d) = f_s(h, d) & \text{otherwise} \end{array}$$

Clearly, f_s is more r-complex than f_s' according to Definition 6. Thus, by Remark 3, to obtain a contradiction I need to show that $\pi_s(f_s, f_{-s}) \leq \pi_s(f_s', f_{-s})$. But note that f_s and f_s' differ only on the set $\overline{H} \equiv \{(h, b, s, 1-z) \mid f_s(h, b, s, 1-z) = A\}$. Since, after any h, $f_s^{'}$ always rejects $(b, s, 1-z)$, it follows from the previous lemma that

$$\pi_s(f_s^{'}, f_{-s} \mid h, b, s, 1-z) \geq 1 - z.$$

Moreover, by the definition of $\overline{H}$, we have

$$\pi_s(f_s, f_{-s} \mid h, b, s, 1-z) = 1 - z \text{ for any } (h, b, s, 1-z) \in \overline{H}.$$

Therefore, $\pi_s(f_s, f_{-s}) \leq \pi_s(f_s', f_{-s})$. But this results in a contradiction. ■

Lemma 20. *For any NECr strategy profile f and for any b we have*

$$f_b(h, s, b, 1-z) = f_b(h', s, b, 1-z) \text{ for all } h \text{ and } h' \in H^\infty$$

Proof. Suppose not; there exists b, h and h' such that $f_b(h, s, b, 1-z) \neq f_b(h', s, b, 1-z)$. Now consider $f_b' \in F_b$ that is otherwise identical to f_b except that $f_b'(h, s, b, 1-z) = A$. Clearly, f_b' is less r-complex than f_b according to Definition 6. Moreover, after every history f_b' induces at least the same payoff as f_b (this is because f_b by rejecting $1-z$ can guarantee at most a payoff of z). But, by Remark 3, this is a contradiction. ■

Lemma 21. *For any NECr profile f and for any b*

$$\text{either for all } h \text{ we have } f_b(h, s, b, 1-z) = A \tag{7.10}$$

$$\text{or for all } h \text{ we have } f_b(h, b, s) = 1-z \tag{7.11}$$

Proof. Since for any b we have $\pi_b(\langle f \mid h\rangle) = z$ for some h and $\pi_b(\langle f \mid h\rangle) \leq z$ for all h, it follows that for any b

either (i) there exists h' such that $f_s(h', s, b) = 1 - z$ and $f_b(h', s, b, 1 - z) = A$
or (ii) there exists h' such that $f_b(h', b, s) = 1 - z$ and $f_s(h, b, s, 1 - z) = A$

If (i) then condition (7.10) follows from Lemma 20. If (ii) then it follows from Lemma 19 that $f_s(h, b, s, 1 - z) = A$ for all h. But then after any (h, b, s) buyer b can obtain a payoff of z by offering $1 - z$. Since z is the maximum payoff that b can obtain after any history it follows that b always offers $1 - z$ (otherwise b could always economize on r-complexity according to Definition 6 and obtain the same maximum payoff of $1 - z$). ■

Now I need to define some further notation. For any such $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$ and for any state of the machine $q \in Q_i$, let $M_i(q) = \{Q_i, q, T, \lambda_i, \mu_i\}$. Thus $M_i(q)$ is otherwise identical to the machine M_i except that the initial state of $M_i(q)$ is q whereas the initial state of M_i is q_i^1.

Also, denote the machine induced by $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$ after a history h by $\langle M_i \mid h\rangle$. Thus $\langle M_i \mid h\rangle = M(\mu_i(q_i^1, h))$, where, as in Section 2 condition (2.1), with some abuse of the notation, $\mu_i(q_i^1, h)$ denotes the state of machine M_i after any history h. Similarly, denote the profile of machines induced by $M = (M_i, M_{-i})$ after a history h by $\langle M \mid h\rangle$.

Lemma 22. *Suppose f is a PBECrs. If $\pi_s(f) < 1$ then there are at least two buyers b and b' such that if players follow the strategy profile f then s is matched with both buyers with a positive probability.*

Proof. Suppose not; then given f, after every history that occurs with a positive probability s chooses some fixed b as his partner. Let $M = \{M_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$ be any NECrs machine profile that implements f, where $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$. Then it follows from the definition of NECrs that $\lambda_s(q) = b$ for all $q \in Q_i$ (otherwise, s could save the states that choose partners other than b and obtain the same payoff). Thus

$$f_s(h) = b \text{ for all } h$$

Now, consider any history (h, s, b', p) for any $p < 1$ and for any $b' \neq b$. Since f constitutes a subgame perfect equilibrium and s always chooses b as a partner, it follows that $f_{b'}(h, s, b', p) = A$ (otherwise, b' receive a zero payoff). Thus

$$\pi_s(\langle f \mid h, s, b'\rangle) = 1 \text{ for all } h \text{ and for all } b' \neq b. \tag{7.12}$$

(If $\pi_s(\langle f \mid h, s, b'\rangle) < 1$ for some h and for some $b' \neq b$, then s can always obtain a payoff greater than $\pi_s(\langle f \mid h, s, b'\rangle)$ after (h, s, b') by offering a price p such that $1 > p > \pi_s(\langle f \mid h, s, b'\rangle)$; by the previous argument this will be accepted by b'.) But condition (7.12) contradicts Lemma 4. ■

Lemma 23. *Consider any finite PBECrs strategy profile f that satisfies property α. Let $M = \{M_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$ be any NECrs machine profile that implements f, where $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$. Suppose that f is such that b is selected as a partner at some period with a positive probability ($\beta(\Phi, b) > 0$). Also for any $h_b \in H(b)$, let $q_s^b = \mu_s(q_s^1, h_b)$. Then*

$$\pi_b(\langle M \mid h \rangle) = z \qquad \text{for any } h \text{ such that } \mu_s(q_s^1, h) = q_s^b \tag{7.13}$$

Proof. By property α, we have that $\langle f_s \mid h_b \rangle$ (and thus $\langle M_s \mid h_b \rangle$) always chooses buyer b after any history h' such that $\theta(h'; \langle f \mid h_b \rangle) > 0$. Therefore, since $\pi_b(\langle M \mid h_b \rangle) = z$, it follows that $\langle M_b \mid h_b \rangle$ and $\langle M_s \mid h_b \rangle$ result in a payoff of z for player b irrespective of what machines (strategies) the other players adopt. Thus

$$\pi_b(\langle M_b \mid h_b \rangle, \langle M_s \mid h_b \rangle, M'_{-b,s}) = z \text{ for all } M'_{-b,s} \tag{7.14}$$

Note that by assumption $q_s^b = \mu_s(q_s^1, h_b)$. Therefore,

$$\langle M_s \mid h_b \rangle = M_s(q_s^b) \tag{7.15}$$

Also, it follows from (7.13) that

$$\langle M_s \mid h \rangle = M_s(q_s^b) \tag{7.16}$$

Conditions (7.14), (7.15) and (7.16) together imply

$$\begin{aligned} &\pi_b(\langle M_b \mid h_b \rangle, \langle M_{-b} \mid h \rangle) = \\ &\pi_b(\langle M_b \mid h_b \rangle, M_s(q_s^b), \langle M_{-b,s} \mid h \rangle) = \pi_b(\langle M_b \mid h_b \rangle, \langle M_s \mid h_b \rangle, \langle M_{-b,s} \mid h \rangle) = z \end{aligned} \tag{7.17}$$

But since f is a subgame perfect equilibrium, it follows from (7.17) that

$$\pi_b(\langle M \mid h \rangle) \geq \pi_b(\langle M_b \mid h_b \rangle, \langle M_{-b} \mid h \rangle) = z \tag{7.18}$$

By definition of z, the continuation payoff of b is less or equal to z. This, together with (7.18), imply that $\pi_b(\langle M \mid h \rangle) = z$. ■

Lemma 24. *For any finite PBECrs profile f that satisfies property α we have $\pi_s(f) = 1$.*

Proof. Suppose not; then $\pi_s(f) < 1$. Then by Lemma 22 there exist at least two buyers b and b' such that s is going to be matched with both b and b' with positive probability.

Let $M = \{M_i\}_{i \in \mathcal{B} \cup \mathcal{S}}$ be any NECrs machine profile that implements f, where $M_i = \{Q_i, q_i^1, \lambda_i, \mu_i\}$. Consider any $h_i \in H(i)$ for any $i = b, b'$. Let $q_s(i) = \mu_s(q_s^1, h_i)$. Now define another machine M'_s for s that is otherwise identical to M_s except that the pair of states $\overline{Q}_s \equiv \{q_s(b), q_s(b')\}$ is replaced by a single absorbing state q' that always chooses b, always offers $1-z$ and accepts a price offer if and only if the price offer is no less than $1-z$. Thus $M_s^{'}$ is defined by $\{Q_s/\overline{Q}_s, q^{1\prime}, \lambda_s^{'}, \mu_s^{'}\}$ where

$q^{1\prime} = q_s^1$ if $q_s^1 \notin \overline{Q}_s$ and $q^{1\prime} = q'$ otherwise
for all $q \in Q_s/\overline{Q}_s$ and for all $d \in D_s \cup \Phi$, $\lambda_s'(q,d) = \lambda_s(q,d)$
$\lambda_s'(q',\Phi) = b$, $\lambda_s'(q',s,b) = 1-z$ and $\lambda_s'(q',b,s,p) = A$ if and only if $p \geq 1-z$
$\forall e \in E$ and $\forall q \in Q_s/\overline{Q}_s$, $\mu_s'(q,e) = \mu_s(q,e)$ if $\mu_s(q,e) \notin \overline{Q}_s$ and $\mu_s'(q,d) = q^{'}$ otherwise
$\forall e \in E$, $\mu_s'(q',e) = q'$

Clearly, M_s is more s-complex than M_s'. Now I demonstrate a contradiction by showing that $\pi_s(M_s, M_{-s}) = \pi_s(M_s', M_{-s})$.

First, let

$$\overline{H}_s = \left\{ h \in H^\infty \,\middle|\, \mu_s(q_s^1, h) \in \overline{Q}_s \right\}$$

Now, note that by Lemma 23 we have

$$\text{if } h \in \overline{H}_s \text{ then } \pi_s(\langle M_s, M_{-s} \mid h \rangle) = 1-z \tag{7.19}$$

Also, from the definition of $M_s^{'}$ and Lemma 21 we have

$$\text{if } h \in \overline{H}_s \text{ then } \pi_s(\langle M_s' \mid h \rangle, \langle M_{-s} \mid h \rangle) = \pi_s(M_s'(q'), \langle M_{-s} \mid h \rangle) = 1-z. \tag{7.20}$$

The first equality in (7.20) follows from M_s' being in state q' after any h such that $\mu_s(q_s^1, h) \in \overline{Q}_s$. The second equality in (7.20) follows from $M_s'(q')$ always selecting b, always offering $1-z$ and always accepting an offer if the price offer is no less than $1-z$, and from Lemma 21 (b either always accepting $1-z$ or always offering $1-z$).

If, on the other hand , h such that if $h \notin \overline{H}_s$ then by the definition of M_s' the profiles (M_s, M_{-s}) and (M_s', M_{-s}) behave in exactly the same way at any period following such h. This, together with (7.19) and (7.20) imply that $\pi_s(\langle M_s', M_{-s} \mid h \rangle) = \pi_s(\langle M_s, M_{-s} \mid h \rangle)$ for all h. But this is a contradiction because $M_s^{'}$ has a fewer states than M_s and yields the same payoff as M_s. ■

Now Lemma 17 and 24 imply that for any PBECrs profile f we have $\pi_s(f) = 0$ for all s and thus $\pi_b(f) = 0$ for all b. This implies that the unique equilibrium price is one and, by the same arguments as that in the last paragraph of section 3, f is stationary.

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