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AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

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GLOBAL ANALYSIS AND ECONOMICS VI

GEOMETRIC ANALYSIS OF PARETO OPTIMA AND PRICE EQUILIBRIA UNDER CLASSICAL HYPOTHESES

Steve Smale

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GEOMETRIC ANALYSIS OF PARETO OPTIMA

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by

Steve Smale

The main goal is to understand the structure, especially local, of the set of Pareto Optima and price equilibria of classical economic systems in a differentiable setting. We use very little of previous papers in this series, but do use calculus of several variables in a systematic way.

One result obtained is the structure of a submanifold on the set Θ of Pareto Optima. The same result is found for the set of price equilibria Λ . Then the Fundamental Theorem of Welfare Economics is given in a strong form.

Theorem. The map φ from $\Lambda \to \Theta$ which assigns to a price equilibrium, the corresponding allocation defines a diffeomorphism from the set of all price equilibria to the set of all Optimal Allocations.

A diffeomorphism is a differentiable function between manifolds with a differentiable inverse. In particular φ is one-to-one and onto; thus for example an optimal allocation has a unique supporting price system, and this association is smooth over all optimal allocations.

A local analysis of Θ and Λ is made, which is used in work in progress on dynamic processes in economics.

The above results are obtained in models with production, but we have stopped short of analysis on the boundaries of the consumption sets. The analysis is an "interior analysis," as far as consumption sets go and we stay away from singularities of the production submanifolds.

In Section 2 our attention is devoted to the Walrasian price equilibria (emphasis on an initial endowment). A little study is made of conditions for such an equilibrium to be catastropic in the sense it is discontinuous in the parameters of the economy. Under these conditions the parameters of the economy are taken to be the endowment allocations.

Some of the results here were proved under different conditions in [9] and [10]. In this paper, however, we don't make any genericity hypotheses or use transversality theory, at least in any explicit way. Convexity conditions are used instead. On the other hand the kind of arguments used would seem to make the convexity conditions more a matter of a convenience than a matter of principle.

Conversations with Gerard Debreu and David Fried have been very helpful in working these ideas out.

Section 1

We review our setting of a pure exchange economy. Preferences of the i^{th} consumer are supposed to be represented by a C^2 utility function $u_i : P \to R$ where commodity space P is taken as

$$P = \{(x^1, \ldots, x^\ell) \in R^\ell | x^i > 0 \text{ each } i\} .$$

Let $S = \{y \in R^\ell | \Sigma(y^i)^2 = 1\}$ and $S_+ = S \cap P$. Define $g_i : P \to S$ by

$$g_i(x) = \frac{\text{grad } u_i(x)}{\|\text{grad } u_i(x)\|} ,$$

so that $g_i(x)$ is the unit normal to the indifference surface of the preference relation at x. Then we suppose throughout that

(1) $g_i(x) \in S_+$, all $x \in P$ (differentiable monotonicity).

If $V_x = \{v \in R^\ell | v \cdot g_i(x) = 0\}$, then the derivative, $Dg_i(x) : R^\ell \to V_x$ restricts to V_x to map $V_x \to V_x$ as a linear map, $\gamma_i(x)$. It is easily seen that $\gamma_i(x)$ is a symmetric linear map. We further suppose

(2) $\gamma_i(x)$ has only negative eigenvalues (differentiable convexity).

Here (2) is the same as the condition [9] that $D^2u_i(x)$ on $\operatorname{Ker} Du_i(x)$ is negative definite. It is also equivalent to convexity together with Debreu's hypothesis of positive Gaussian Curvature [5]. In particular (2) implies that $u_i^{-1}(c,\infty)$ is a strictly convex set for every real number c.

The space of states of this pure exchange economy with m agents is then

$$W = \{x \in (P)^m | x = (x_1, \ldots, x_m), x_i \in P, \Sigma x_i = s\}$$

where $s \in P$ is the fixed vector of total resources. Then $x \in W$ is Pareto optimal (or simply optimal) if there is no $y \in W$ with $u_i(y_i) \geq u_i(x_i)$ all i, strict inequality one i. Also x is called a strict Pareto optimum if $u_i(y_i) \geq u_i(x_i)$ all i implies that $y = x$.

As in [9] we let θ be the set of states $x = (x_1, \ldots, x_m) \in W$ which satisfy the first order condition: $\theta = \{x \in W | g_i(x_i)$ doesn't depend on $i\}$. For completeness we give a proof of the following well-known fact:

Proposition 1. The set of Pareto optimal points coincides with θ. Also so does the set of strict Pareto optimal points.

Proof. Strict Pareto optimal implies Pareto optimal and Pareto optimal implies the first order condition. See for example [10]. Now suppose $x = (x_1, \ldots, x_m) \in W$ satisfies the first order condition, with say $g_i(x_i) = p \in S_+$. Consider a state $y = (y_1, \ldots, y_m) \in W$ with $u_i(y_i) \geq u_i(x_i)$ all i. Now let Π be the orthogonal projection of R^ℓ onto the oriented line through p. By the differentiable convexity hypothesis it follows that $\Pi(y_i) \geq \Pi(x_i)$ each i with strict inequality in case $y_i \neq x_i$. But both $x, y \in W$ so $\Sigma x_i = \Sigma y_i$ and thus $\Sigma \Pi(x_i) = \Sigma \Pi(y_i)$. Therefore $x = y$ and we have that x is strictly Pareto optimal. This proves Proposition 1.

Remark. Of course the above proof works with milder convexity hypotheses on the u_i.

Corollary. The map $u : W \to R^m$ defined by $u(x)]_i = u_i(x_i)$ restricted to θ is one-to-one.

The following was stated in [10], but no proof was given.

Theorem. The set of Pareto optimal points θ is an $(m-1)$-dimensional submanifold in W.

Our proof relates to the set Λ of price equilibria and the "Fundamental Theorem of Welfare Economics." Towards this end define a space of states $\mathcal{S} = (P)^m \times S_+$ and

$$\Lambda = \{(x,p) \in \mathcal{S} \mid g_i(x_i) = p, \ \Sigma x_i = s\}.$$

Proposition 2. Λ is a submanifold of $\mathcal{S}$ of dimension $m-1$.

The proof of Proposition 2 contains the major part of the argument of this section and proceeds as follows. The following Lemmas 1 and 2 are easy consequences of the calculus, and the implicit function theorem.

Lemma 1. $\Lambda_0 = \{(x,p) \in \mathcal{S} \mid \Sigma x_i = s\}$ is a submanifold of $\mathcal{S}$ of codimension (i.e. $\dim \mathcal{S} - \dim \Lambda_0$) ℓ with tangent space at $(x,p) \in \Lambda_0$ given by

$$T_{x,p}(\Lambda_0) = \{(\bar{x},\bar{p}) \in (R^\ell)^m \times p^\perp \mid \Sigma \bar{x}_i = 0\}$$

where $p^\perp = \{v \in R^\ell \mid v \cdot p = 0\}$.

Lemma 2. For each $i = 1, \ldots, m$, $\Lambda_i = \{(x,p) \in \mathcal{S} \mid g_i(x_i) = p\}$ is a submanifold of $\mathcal{S}$ of codimension ℓ-1 and at $(x,p) \in \Lambda_i$ its tangent space is

$$T_{x,p}(\Lambda_i) = \{(\bar{x},\bar{p}) \in (R^\ell)^m \times p^\perp \mid Dg_i(x_i)(\bar{x}_i) = \bar{p}\} .$$

Recall in Lemma 2, that $Dg_i(x_i) : R^\ell \to p^\perp$ is the derivative (as a linear transformation) of $g_i : P \to S_+$ at x_i , and $T_p(S_+) = p^\perp$.

Note that Λ is the intersection $\Lambda = \bigcap_{i=0}^m \Lambda_i$. Thus for the proof of Proposition 2 it is sufficient to show that the Λ_i have normal intersection at a given $(x,p) \in \Lambda$. This means that the linear subspaces $T_{x,p}(\Lambda_i)$ intersect normally or that

$$\text{dimension } \bigcap_{i=0}^m T_{x,p}(\Lambda_i) = m-1 .$$

For the moment let $T = \bigcap_{i=0}^m T_{x,p}(\Lambda_i)$. Then

$$T = \{(\bar{x},\bar{p}) \in (R^\ell)^m \times p^\perp \mid \Sigma x_i = 0,\ Dg_i(x_i)\bar{x}_i = \bar{p}\} .$$

Further define

$$\Delta = \{\beta \in R^m | \beta = (\beta_1, \ldots, \beta_m), \Sigma \beta_i = 0\}$$

and $\varphi : T \to \Delta$ by $\varphi(\bar{x}) = (p\cdot\bar{x}_1, \ldots, p\cdot\bar{x}_m)$. Note that since $\Sigma\, p\cdot\bar{x}_i = p\cdot\Sigma\,\bar{x}_i = 0$, φ is well-defined.

From what we have said, Proposition 2 is a consequence of the following lemma.

Lemma 3. The map $\varphi : T \to \Delta$ is a linear isomorphism.

Proof. Let $\beta \in \Delta$. We will show that there is a unique $\bar{x} \in T$ such that $\varphi(\bar{x}) = \beta$. For each i , $\bar{x}_i$ can be written uniquely in the form $\bar{x}_i = \bar{x}_i' + \bar{x}_i''$, with $p\cdot\bar{x}_i' = 0$ and $\bar{x}_i'' \in \text{Ker}\, Dg_i(x_i)$ (remember that $g_i(x_i) = p$, and by our differentiable convexity hypothesis on the preferences, $Dg_i(x_i)$ restricted to $p^\perp$ is an isomorphism; thus $\text{Ker}\, Dg_i(x_i)$ and $p^\perp$ provide a direct sum decomposition of R^ℓ).

Towards solving $\varphi(\bar{x}) = \beta$ for $\bar{x}$, let $\bar{x}_i''$ satisfy $p\cdot\bar{x}_i'' = \beta_i$. Then $p\cdot\Sigma\,\bar{x}_i'' = 0$; so that $\Sigma\, x_i'' \in p^\perp$.

Since $\gamma_i = Dg_i(x_i)$ restricted to $p^\perp$ is symmetric with negative eigenvalues for each i , so is γ_i^{-1} and $\Sigma\,\gamma_i^{-1}$ as well. Thus there exists a unique $\bar{p} \in p^\perp$ satisfying $\Sigma\,\gamma_i^{-1}\bar{p} + \Sigma\,\bar{x}_i'' = 0$. Let $\bar{x}_i' = \gamma_i^{-1}\bar{p}$. Then $\Sigma\,\bar{x}_i = 0$, $Dg_i(x_i)\bar{x}_i = \bar{p}$, $\bar{x} \in T$ and $\varphi(\bar{x}) = \beta$. This finishes the proof of Lemma 3, and Proposition 2.

We now write $T = T_{x,p}(\Lambda)$ as the tangent space of the manifold Λ at x .

Proposition 3. (Fundamental Theorem of Welfare Economics) The set θ is a submanifold of W, and the map of Λ into W which sends (x,p) into x is diffeomorphism α_1 of Λ onto θ.

One form of the Fundamental Theorem of Welfare Economics, e.g. [1], [3], [8] asserts under more general conditions that the map $\alpha_1 : \Lambda \to \theta$ above is onto. In other words every optimal allocation is supported by a price system. Of course Proposition 3 implies Theorem 2. See also Section 3 for the case with production.

The proof of Proposition 3 uses Proposition 2 and the "first order theory." More precisely, define $\alpha : W \times S_+ \to W$ by $\alpha(x,p) = x$. If $(x,p) \in \Lambda \subset W \times S_+ \subset \mathcal{A}$, $\alpha(x,p) \in \theta$. Let $\alpha_1 : \Lambda \to \theta$ be the restriction of α. Define $\beta : W \to W \times S_+$ by $\beta(x) = (x, g_1(x_1))$; so $\beta(x) \in \Lambda$ if $x \in \theta$. Let $\beta_1 : \theta \to \Lambda$ be the restriction of β. Then on Λ, $\beta_1 \circ \alpha_1$ is the identity. Also β is an imbedding (see below). Therefore one can conclude that on Λ, α_1 is an imbedding. The rest follows.

For $x \in W$ define K_x to be the kernel of the derivative $Du(x) : T_x(W) \to R^m$, of $u : W \to R^m$. Thus K_x is a linear subspace of $T_x(W) = \{\bar{x} \in (R^\ell)^m \mid \Sigma \bar{x}_i = 0, \bar{x}_i \in R^\ell\}$.

<u>Proposition 4</u>. K_x is a transversal to θ or $K_x \cap T_x(\theta) = 0$ in $T_x(W)$.

This proposition implies that $T_x(W)$ has as a direct sum decomposition $K_x \oplus T_x(\theta)$. For the proof note that if $\bar{x} \in K_x$ then $\bar{x}_i \in \operatorname{Ker} Du_i(x_i)$, or $\bar{x}_i \cdot g_i(x_i) = 0$. Thus in the proof Lemma 3, $\bar{x}_i'' = 0$, all i. Then $\bar{x}_i' = 0$, $\bar{x}_i = 0$, and $\bar{x} = 0$. Proposition 4 is proved.

An <u>imbedding</u> of a manifold is a C^1 map which is one-to-one, and the derivative is one-to-one at each point.

<u>Corollary</u>. The map $u : W \to R^m$ restricted to θ is an imbedding.

The corollary, besides Proposition 4 uses the corollary to Proposition 1.

Define a submanifold $\overline{K}_x$ (an affine one) of W by considering K_x to be contained in W with its origin at x . In the Edgeworth Box (Figure 1), $\overline{K}_x$ is the tangent line to the indifference curves at $x \in \theta$, lying naturally in the box. Formally $\overline{K}_x = \{y \in W | y = x + \overline{x}, \overline{x} \in K_x\}$.

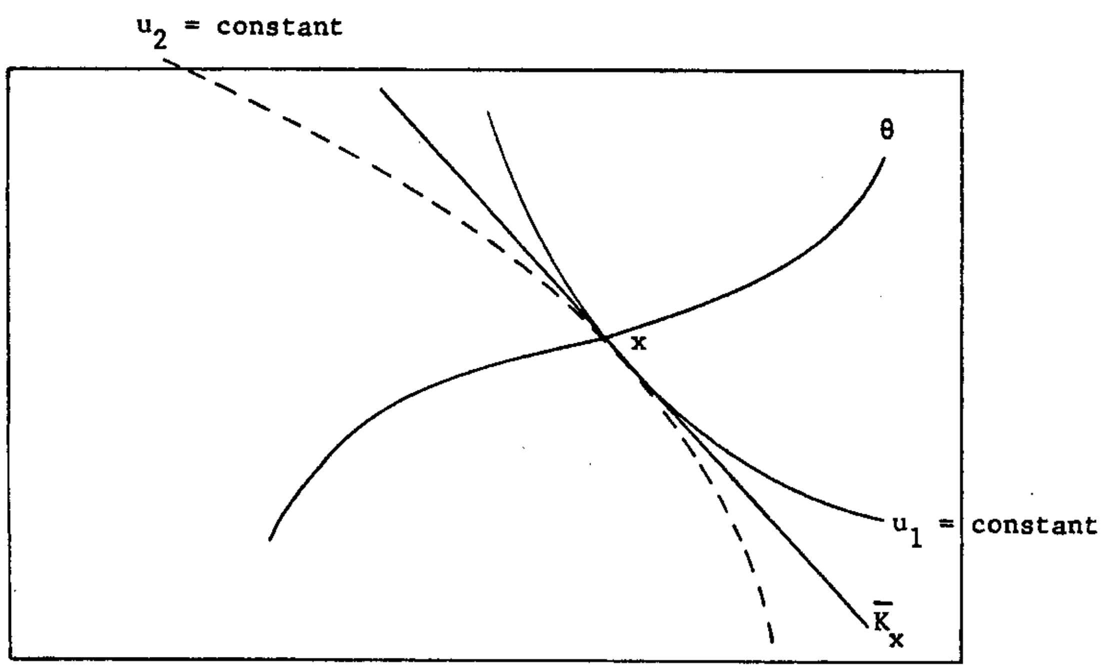

FIGURE 1

Then from Proposition 4, $\overline{K}_x$ and θ intersect transversally in W at the point x .

<u>Proposition 5</u>. For each x in θ , $\overline{K}_x \cap \theta = x$. Also there is a neighborhood $N(\theta)$ of θ in W with the property that if $r \in N(\theta)$, there is a unique $x \in \theta$ such that $r \in \overline{K}_x$.

<u>Proof</u>. Let $y \in \overline{K}_x \cap \theta$. Since $y \in \overline{K}_x$, $(y_i - x_i) \cdot g_i(x_i) = \dot{0}$, and $u_i(y_i) < u_i(x_i)$, using our convexity hypothesis. Therefore $y \notin \theta$.

The second part follows from a simple version of the tubular neighborhood theorem of differential topology. See [6].

Corollary. For each endowment allocation r in $N(\theta)$ there exists a unique Walras price equilibrium.

The proof follows from Proposition 5 and the observation that for $x \in \theta$, $p = g_1(x_1)$, (x,p) is a Walras price equilibrium for the endowment allocation r if and only if $r \in \bar{K}_x$.

Compare Balasko [2].

Remark. Suppose that each $u_i : P \to R$ satisfies the boundary condition $u_i^{-1}(c,\infty)$ is closed in R^ℓ each c. Then it follows from the existence theory (see Debreu [4]) that $\bigcup_{x \in \theta} \bar{K}_x = W$.

Remark. One can say some things about $u : W \to R^m$ from the point of view of singularities of maps (see [6]). If x in W is not in θ, then the derivative $Du(x)$ of u at x is non-singular (i.e. onto) or x is a regular point of u. This is a consequence, for example, of the rank proposition in [10]. Furthermore at each $x \in \theta$, u is a fold, the simplest kind of singularity. This follows from the fact that the second intrinsic derivative $\Sigma \lambda_i D^2 u_i(x_i)$ is a non-degenerate form on the kernel of $Du(x)$ (see [10] and [6]).

Section 2

We begin by defining a space of Walras equilibria with fixed total resources, $s \in P$. Keeping the situation of the previous section as to conditions on the preference relations and notations, let $Q_s = \{r \in (P)^m | \Sigma r_i = s\}$, $\mathcal{S}_s = W \times S_+$. $\mathcal{S}_s$ constitutes a space of states of the pure exchange economy; Q_s will be the space of parameters of the economy. An element of Q_s is an endowment allocation (or "endowment reallocation") keeping the total resources always the same. The methods used in the sequel permit one to make other choices for the parameter space.

Define

$$\Sigma = \{(r,x,p) \in Q_s \times \mathcal{S}_s \mid p\cdot x_i = p\cdot r_i,\ g_i(x_i) = p,\ i = 1, \ldots, m\} .$$

Then $(r,x,p) \in \Sigma$ means exactly that (x,p) is a Walrasian price equilibrium relative to the endowments given by $r = (r_1, \ldots, r_m)$.

Proposition 1. Σ is a submanifold of $Q_s \times \mathcal{S}_s$ with $\dim \Sigma = \dim Q_s$ and the tangent space $T_{r,x,p}(\Sigma)$ of Σ at (r,x,p) is given as the set of $(\bar{r},\bar{x},\bar{p}) \in (R^{\ell})^m \times (R^{\ell})^m \times p^{\perp}$ which satisfy $\Sigma\, \bar{r}_i = 0$, $\Sigma\, \bar{x}_i = 0$

$$\bar{p}\cdot(r_i - x_i) + p\cdot(\bar{r}_i - \bar{x}_i) = 0 ,\quad i = 1, \ldots, m$$

(the m^{th} equation here is redundant)

$$Dg_i(x_i)(\bar{x}_i) = \bar{p} ,\quad i = 1, \ldots, m$$

Proof. Let $\varphi_i : Q_s \times \mathcal{S}_s \to R$ be defined by $\varphi_i(r,x,p) = p\cdot x_i - p\cdot r_i$. Then it is easily checked that φ_i is regular (has no critical points) and so $\varphi_i^{-1}(0)$ is a submanifold. Similar reasoning with the other equations reduces the proof to checking that all the submanifolds defined by each of the equations have transversal intersection (compare to the proof of Proposition 2 of Section 1). Thus the proof is reduced to showing that the following system of linear equations in $(\bar{r},\bar{x},\bar{p}) \in (R^{\ell})^m \times (R^{\ell})^m \times p^{\perp}$ has at most an $(m-1)\ell$ dimensional space of solutions for given $(r,x,p) \in Q_s \times \mathcal{S}_s$:

$$\Sigma \bar{r}_i = 0 , \quad \Sigma \bar{x}_i = 0$$

$$\bar{p}\cdot(x_i - r_i) + p\cdot(\bar{x}_i - \bar{r}_i) = 0$$

$$Dg_i(x_i)\bar{x}_i = \bar{p}$$

To prove this define a linear map

$$\varphi : (R^{\ell})^m \times (R^{\ell})^m \times p^{\perp} \to R^{\ell} \times R^{\ell} + (R)^{m-1} \times (p^{\perp})^m$$

by sending $(\bar{r},\bar{x},\bar{p})$ into

$$(\Sigma \bar{r}_i, \Sigma \bar{x}_i, \bar{p}\cdot(x_i - r_i) + p\cdot(\bar{x}_i - \bar{r}_i)]_{i=1}^{m-1}, Dg_i(x_i)(\bar{x}_i) - \bar{p}]_{i=1}^{m}) .$$

If φ can be shown to be surjective (onto), a simple counting of dimensions gives us what we need. Thus let $\alpha_1 \in R^{\ell}$, $\alpha_2 \in R^{\ell}$, $\beta_i \in R$, $i = 1, \ldots, m-1$, and $\delta_i \in p^{\perp}$, $i = 1, \ldots, m$ be given. Solving the following system of equations in $(\bar{r},\bar{x},\bar{p})$ will finish the proof.

(1) $\Sigma \bar{r}_i = \alpha_1$, (1') $\Sigma \bar{x}_i = \alpha_2$

(2) $\bar{p}\cdot(x_i - r_i) + p\cdot(\bar{x}_i - \bar{r}_i) = \beta_i$, $i = 1, \ldots, m-1$

(3) $Dg_i(x_i)(\bar{x}_i) - \bar{p} = \delta_i$, $i = 1, \ldots, m$.

Towards finding the solution, write $\alpha_2 = \alpha_2' + \alpha_2''$, with $\alpha_2'\cdot p = 0$, $\alpha_2'' \in \operatorname{Ker} Dg_1(x_1)$. As in Section 1, let $\gamma_i = Dg_i(x_i)\big|_{p^{\perp}}$, $\gamma_i : p^{\perp} \to p^{\perp}$. Choose $\bar{p}$ so that $(\Sigma \gamma_i^{-1})\bar{p} + \Sigma \gamma_i^{-1}\delta_i = \alpha_2'$ and $\bar{x}_i' = \gamma_i^{-1}(\bar{p} + \delta_i)$. Then $\Sigma \bar{x}_i' = \alpha_2'$. Let $\bar{x}_1'' = \alpha_2''$, $\bar{x}_1 = \bar{x}_1' + \bar{x}_1''$, $\bar{x}_i = \bar{x}_i'$ for $i > 1$. Then (1') and (3) are satisfied.

Now one can easily choose $\bar{r}_i$, $i = 1, \ldots, m-1$ so that (2) is satisfied and finally $\bar{r}_m$ satisfying (1). The proof of Proposition 1 is finished.

In a similar fashion one can define

$$\Sigma^* = \{(r,x,p) \in Q \times \mathcal{S} \mid \Sigma x_i = \Sigma r_i,\ p\cdot x_i = p\cdot r_i,\ g_i(x_i) = p\} .$$

The following is proved in the same way as Proposition 1.

Proposition 2. Σ^* is a submanifold of $Q \times \mathcal{S}$ with $\dim \Sigma^* = \dim Q$, and the tangent space $T_{r,x,p}(\Sigma^*)$ of Σ^* at (r,x,p) is given as the set of $(\bar{r},\bar{x},\bar{p}) \in (R^{\ell}) \times (R^{\ell})^m \times p^{\perp}$ which satisfy:

$$\Sigma \bar{r}_i = \Sigma \bar{x}_i$$

$$p\cdot(\bar{r}_i - \bar{x}_i) + \bar{p}\cdot(r_i - x_i) = 0 , \quad i = 1, \ldots, m$$

(the m^{th} equation here is redundant)

$$Dg_i(x_i)(\bar{x}_i) = \bar{p} , \quad i = 1, \ldots, m .$$

Corollary. (Debreu [4]) Except for a set of initial endowments of Q of measure 0, the number of price equilibria is discrete. Furthermore with Debreu's boundary condition (see Remark at the end of Section 1), one has the exceptional set closed and finite replaces discrete.

For the proof of the Corollary, one simply applies Sard's theorem to the projection $Q \times \mathcal{S} \to Q$ restricted to Σ^* as in [9].

Returning to the situation of Proposition 1 one may think of the following figure.

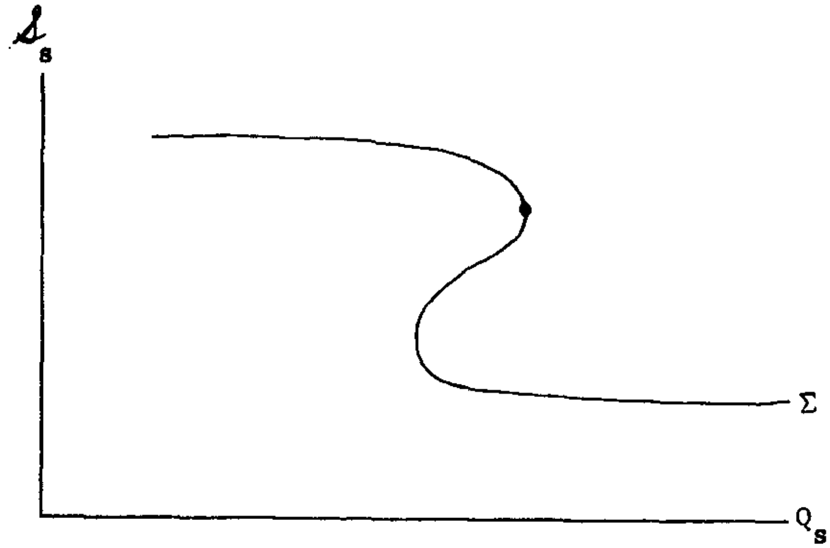

FIGURE 2

Let $\Pi_{Q_s} : Q_s \times \mathcal{S}_s \to Q_s$ be the projection and $\Pi : \Sigma \to Q_s$, its restriction to Σ. We consider the question: for what $(r,x,p) \in \Sigma$ is Π catastrophic? In other words, find a condition on (r,x,p) such that the linear map $D\Pi(r,x,p)$ be singular as a map from one linear space to another of the same dimension. For non-catastrophic (or regular) $(r,x,p) \in \Sigma$ one knows by the implicit function theorem that the price equilibrium (x,p) varies continuously with the parameters of the economy r. On the other hand if (r,x,p) is catastrophic then a small change in the parameter r of the economy could produce a large jump in prices. In Figure 2 the big dot on Σ is an example of a catastrophic point.

The following gives an analytic criterion for $(r,x,p) \in \Sigma$ to be catastrophic. Remember $(r,x,p) \in \Sigma$ if and only if (x,p) is a price equilibrium for the endowments given by r.

<u>Proposition 3</u>. The point $(r,x,p) \in \Sigma$ is catastrophic if and only if the linear map $h_{r,x,p} : T_x(\theta) \to \Delta$ is an isomorphism where $h_{r,x,p} = h^*_{r,x,p} + \omega_x$ with

$$h^*_{r,x,p}(\bar{x}) = (Dg_1(x_1)(\bar{x}_1)\cdot(x_1 - r_1), \ldots, Dg_m(x_m)(\bar{x}_m)\cdot(x_m - r_m))$$

and

$$\varphi_x(\bar{x}) = (p\cdot\bar{x}_1, \ldots, p\cdot\bar{x}_m) .$$

The map φ_x was studied in Lemma 3 in Section 1.

<u>Proof of Proposition 3</u>. First we examine analytically what it means for a point (r,x,p) in Σ to be catastrophic. Fixing (r,x,p) in Σ, we have in Proposition 1 equations on $(\bar{r},\bar{x},\bar{p})$, describing $T_{r,x,p}(\Sigma)$. Then from the definitions, (r,x,p) is non-catastrophic if and only if given any $\bar{r}$ with $\Sigma\,\bar{r}_i = 0$, there exist $(\bar{x},\bar{p})$ with $(\bar{r},\bar{x},\bar{p}) \in T_{r,x,p}(\Sigma)$. In other words, can the equations of Proposition 1 always be solved for $(\bar{x},\bar{p})$? Referring to Proposition 3 now, if $\bar{x} \in T_x(\theta)$, then $\Sigma\,\bar{x}_i = 0$ and $Dg_i(x_i)\bar{x}_i$ is independent of i, and say is $\bar{p}$.

Thus the condition for (r,x,p) to be non-catastrophic amounts to solving for $\bar{x}$, the equations,

$$\bar{p}\cdot(x_i - r_i) + p\cdot\bar{x}_i = p\cdot\bar{r}_i , \quad i = 1, \ldots, m$$

and $\Sigma\, p\cdot\bar{r}_i = 0$.

The equivalence with the condition of Proposition 3 can now be seen. Proposition 3 is proved.

The following gives an affirmative answer to a conjecture Debreu made to me in connection with his recent work on the rate of convergence of the core.

Corollary. If the endowment allocation r is a regular economy then the map $\psi : \theta \to \Delta$ defined by $x \to (g_1(x_1)\cdot(x_1 - r_1), \ldots, g_m(x_m)\cdot(x_m - r_m))$ is a diffeomorphism for x in a neighborhood of a Walrasian price equilibrium (relative to r).

The proof of the corollary is obtained by simply differentiating the map ψ, applying Proposition 3 and then the implicit function theorem.

Remark. Proposition 3 yields some perspective on what can cause catastrophic jumps in prices in the framework of general equilibrium theory. By Lemma 3 of Section 1, φ_x is always an isomorphism; thus it is the effect of $h^*_{r,x,p}$, which causes jumps. If $\|h^*_{r,x,p}\|$ is small then there are no catastrophes. This term $h^*_{r,x,p}$ can get big for two reasons. One is that $x_i - r_i$ becomes large or that r gets far from θ (compare Proposition 5 of Section 1). The other is that the curvature of the indifference surfaces becomes large from the expression $Dg_i(x_i)$. One should keep in mind that all of this is an interior analysis. Every consumer owns at least a little of each commodity.

Section 3

Our model of an economy with production goes as follows. To each of m consumers is associated a consumption set X_i, an open set in R^ℓ, $i = 1, \ldots, m$. We suppose $m \geq 1$. On each X_i is supposed a preference relation which is represented by a C^2 utility function $u_i : X_i \to R$. Define $g_i : X_i \to S_+$ by $g_i(x) = [\text{grad } u_i(x)]/[\|\text{grad } u_i(x)\|]$ for each x in X_i. We suppose throughout that g_i satisfies the differentiable monotonicity and convexity hypotheses of Section 1.

It is also supposed that there are n producers, and to each is associated a technology which is represented by a closed submanifold Y_α in R^ℓ, $\alpha = 1, \ldots, n$. Note that we are not making the very restrictive hypothesis that Y_α be a hypersurface.

For our economy with production, but no prices explicitly yet, define the space τ of attainable states by

$$\tau = \{(x,y) \in X \times Y \mid \Sigma x_i = \Sigma y_\alpha + s\} .$$

Here X and Y are Cartesian products $X = \Pi_i X_i$, $Y = \Pi_\alpha Y_\alpha$, $x_i \in X_i$, $y_\alpha \in Y_\alpha$ and $s \in R^\ell$ denotes the endowed resources of the economy. The defining condition of τ of course just relates the total consumption to the total production.

<u>Proposition 1</u>. τ is a submanifold of $X \times Y$.

<u>Proof</u>. One needs to check that in $X \times (R^\ell)^m$, the submanifolds defined by the Y_α and the condition $\Sigma x_i = \Sigma y_\alpha + s$ all intersect in general position. Thus fix $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, $x_i \in X_i$ and $y_\alpha \in Y_\alpha$ with $\Sigma x_i = \Sigma y_\alpha + s$. Then it is to be shown that the set of $(\bar{x},\bar{y})$ with $\bar{x}_i \in R^\ell$, $\bar{y}_\alpha \in T_{y_\alpha}(Y_\alpha)$ and $\Sigma \bar{x}_i = \Sigma \bar{y}_\alpha$ has dimension equal to $\ell m + \Sigma \dim Y_\alpha - \ell$. For this, let $\bar{y}_\alpha \in T_{y_\alpha}(Y_\alpha)$, $\alpha = 1, \ldots, n$ be given. Then the $\bar{x}_i$ satisfy the single vector equation $\Sigma \bar{x}_i = \Sigma \bar{y}_\alpha$ with exactly $\ell m - \ell$ degrees of freedom. This very easy proposition is proved.

Define functions $u_i : \tau \to R$, $i = 1, \ldots, m$ by $u_i(x,y) = u_i(x_i)$, where $u_i(x_i)$ is the value of the individual utility at x_i. There should be no serious confusion using the u_i for two slightly different meanings. One may now define the notion of admissable curve and infinitesimal Pareto set θ in τ as in [10]. That is $(x,y) \in \tau$ belongs to θ if and only if there is no curve $\varphi : (-1,1) \to \tau$ with $\varphi(0) = (x,y)$ and $\frac{d}{dt}(u_i\varphi(t)) > 0$ all i, t.

Proposition 2. (First order) A point (x,y) of τ is infinitesimal Pareto (i.e. $(x,y) \in \theta$) if and only if

(a) $g_i(x_i)$ is some constant vector say p in S_+, and

(b) $p \in N_{y_\alpha}(Y_\alpha)$ (i.e. this vector is normal to Y_α at y_α).

The proof uses the first order condition, e.g. [10]. Thus $(x,y) \in \theta$ if and only if there exist $\lambda_i \geq 0$, $i = 1, \ldots, m$, not all zero such that $\Sigma \lambda_i Du_i(x,y)(\bar{x},\bar{y}) = 0$ all $(\bar{x},\bar{y}) \in T_{x,y}(\tau)$. Supposing $(x,y) \in \theta$, in this equation, one can take $\bar{y} = 0$, so that $\Sigma \bar{x}_i = 0$. Then as in [10], λ_i grad $u_i(x_i)$ is independent of i, and one has (a). For (b) fix i and α and take $\bar{x}_i = \bar{y}_\alpha$ with all the other components of $\bar{x}$ and $\bar{y}$ zero. This leads to $g_i(x_i)\cdot\bar{y}_\alpha = 0$ or $p\cdot\bar{y}_\alpha = 0$ for all $\bar{y}_\alpha \in T_{y_\alpha}(Y_\alpha)$. The converse is similar.

Before proceeding any further we introduce a convexity type of hypothesis on the technology submanifolds Y_α.

Hypothesis on Y_α. For each $p \in S_+$ the real valued map $f_p : Y_\alpha \to R$ which sends y in Y_α to $p\cdot y$ has exactly one critical point and that critical point is a non-degenerate maximum.

We assume this in all that follows.

Suppose that for $p \in S_+$, y^* is the maximum given in the hypothesis. Then $Df_p(y^*) = 0$ and $D^2f_p(y^*)$ is a negative definite symmetric bilinear form on the tangent space $T_{y^*}(Y_\alpha)$ of Y_α at y^*.

One can write $D^2f_p(y^*) = p\cdot H_{y^*}$ where H_{y^*} is the second fundamental form of the submanifold Y_α at y^* .

Let us look at this geometric invariant of the technology a bit. One has defined for any $y \in Y_\alpha$, the second fundamental form H_y as for example in [7]. This form H_y is a symmetric bilinear form defined on the tangent space $T_y(Y_\alpha)$ with value in the normal space $N_y(Y_\alpha)$ of all vectors orthogonal to $T_y(Y_\alpha)$. One can think of H_y as simply the second derivative of the inclusion $Y_\alpha \to R^\ell$ with values projected into $T_y(Y_\alpha)$. For $p \in N_y(Y_\alpha)$, we write $p \cdot H_y$ as this real valued form on $T_y(Y_\alpha)$.

Our hypothesis above has the following consequence. For $y \in Y_\alpha$, let $\Pi_y : R^\ell \to T_y(Y_\alpha)$ be the orthogonal projection and if $p \in S_+ \cap N_y(Y_\alpha)$, define $Q_y : p^\perp \to p^\perp$ by $Q_y(v) \cdot w = p \cdot H_y(\Pi_y v, w)$. Then Q_y is symmetric, and from the hypothesis on Y_α , Q_y will have non-positive eigenvalues.

Proposition 3. The Pareto optimal points in the space of attainable states τ of our economy with production coincide with the points of θ , and in fact the points in θ are strict Pareto optimal points. The map $u : \tau \to R^m$ whose coordinates are u_i , restricted to θ is one-to-one.

Proof. Let $(x^*, y^*) \in \theta$ and $(x,y) \in \tau$, with $u_i(x,y) \geq u_i(x^*, y^*)$ all i . We wish to show that $(x,y) = (x^*, y^*)$. Let $p = g_1(x_1^*)$ (keeping in mind Proposition 2) and let Π be the projection of R^ℓ onto the oriented line through p .

Since $u_i(x_i) \geq u_i(x_i^*)$, it follows from our convexity condition on the u_i and Proposition 2 that $\Sigma \, \Pi x_i \geq \Sigma \, \Pi(x_i^*)$ and strict inequality if $x \neq x^*$. From $\Sigma x_i = \Sigma y_\alpha + s$, $\Sigma x_i^* = \Sigma y_\alpha^* + s$ it follows that $\Sigma \, \Pi y_\alpha \geq \Sigma \, \Pi y_\alpha^*$. But by Proposition 2 and the hypothesis on Y_α , this is impossible unless $y_\alpha = y_\alpha^*$, each α . The rest follows.

Proposition 4. (Fundamental Theorem of Welfare Economics, simple form) Given $(x^*, y^*) \in \theta \subset \tau$, there is some $p^* \in S_+$ (which is unique) with the properties

(a) x_i^* maximizes utility u_i on the budget set $\{x_i \in X_i | p^* \cdot x_i \leq p^* \cdot x_i^*\}$

and (b) y_α^* maximizes profit $p^* \cdot y_\alpha$ on Y_α.

Conversely let (x^*, y^*, p^*) in $\tau \times S_+$ satisfy (a) and (b). Then $(x^*, y^*) \in \theta$ and is therefore Pareto optimal.

Remarks. One could equivalently say that $x^* \in X$ is an optimal allocation (for this economy) provided there is some $y^* \in Y$ with $(x^*, y^*) \in \tau$ and morevoer $(x^*, y^*) \in \theta$, and restate Proposition 4 in terms of optimal allocations in X.

Proof of Proposition 4. Let $(x^*, y^*) \in \theta$. Then choose $p^* = g_1(x_1)$. Then (a) is satisfied (Proposition 2 and properties of u_i). Also (b) is satisfied (Proposition 2 and hypothesis on Y_α).

The converse follows easily from Propositions 2 and 3.

Now we extend the preceeding framework a bit to consider all states of the above economy together with prices. Thus let $\mathcal{S} = X \times Y \times S_+$ and define

$$\Lambda = \{(x,y,p) \in \mathcal{S} | g_i(x_i) = p,\ p \in N_{y_\alpha}(Y_\alpha),\ \Sigma x_i = \Sigma y_\alpha + s\} .$$

It follows from Proposition 4, that elements of Λ can be thought of as price equilibria (or equilibria relative to a price system in the terminology of Debreu [3]). Note that an element of Λ does not depend on some endowment.

Proposition 5. The set Λ is a submanifold of $\mathcal{S}$ of dimension $m-1$. Furthermore the tangent space $T_{x,y,p}(\Lambda)$ is the set of $(\bar{x},\bar{y},\bar{p})$ in $T_{x,y,p}(\mathcal{S})$ which satisfy

$$\Sigma \bar{x}_i = \Sigma y_\alpha$$

$$Dg_i(x_i)\bar{x}_i = \bar{p}$$

$$Q_{y_\alpha}(\bar{p}) = \bar{y}_\alpha .$$

Lemma 1. $\Lambda_\alpha = \{(x,y,p) \in X \times Y \times S_+ | p \in N_{y_\alpha}(Y_\alpha)\}$. Then Λ_α is a submanifold of codimension $\ell-1$ with tangent space

$$T_{x,y,p}(\Lambda_\alpha) = \{\bar{x},\bar{y},\bar{p} | Q_{y_\alpha}(\bar{p}) = \bar{y}_\alpha\} .$$

Proof. The condition that $p \in N_{y_\alpha}(Y_\alpha)$ can be replaced by $p \cdot D\varphi_\alpha(y_\alpha) = 0$ where $\varphi_\alpha : Y_\alpha \to R^\ell$ is the inclusion and $p \cdot D\varphi_\alpha(y_\alpha)$ is the map $T_{y_\alpha}(Y_\alpha) \to R$ defined by $\bar{y}_\alpha \to p \cdot \bar{y}_\alpha$.

The rest of the proof of Lemma 1 proceeds by calculus.

The proof of Proposition 5 is now reduced (by arguments in Proposition 2 of Section 1) to showing that the following map is an isomorphism:

$$\psi : T_{x,y,p}(\Lambda) \to \Delta$$

$$(\bar{x},\bar{y},\bar{p}) \to (p \cdot \bar{x}_1, \ldots, p \cdot \bar{x}_m) .$$

Here $T_{x,y,p}(\Lambda)$ is defined as in Proposition 5 even though Λ has not yet been shown to be a manifold.

The proof of this is similar to the proof of Lemma 3 of Section 1 and we omit it.

<u>Theorem</u>. (Strong form of the Fundamental Theorem of Welfare Economics) The set of Pareto Optima θ is a submanifold of $\mathcal{S}$ ($\theta \subset \tau \subset \mathcal{S}$) of dimention m-1 , so is the set of price equilibria Λ , and the map $\varphi : \Lambda \to \theta$, induced by $(x,y,p) \to (x,y)$ is well-defined and a diffeomorphism.

The proof follows from Proposition 5 in the same way as Proposition 3 of Section 1 was proved.

Again one may define $K_{x,y}$ as the kernel of the map $Du(x,y) : T_{x,y}(\tau) \to R^m$ and one can prove as in Section 1 that for $(x,y) \in \theta$, the intersection of $K_{x,y}$ with $T_{x,y}(\theta)$ is zero.

REFERENCES

[1] Arrow, K. and F. Hahn. General Competitive Analysis. San Francisco: Holden-Day, 1971.

[2] Balasko, Y. "Some Results on Uniqueness and Stability in General Equilibrium Theory," reprint, Berkeley, 1974.

[3] Debreu, G. Theory of Value. New York: Wiley, 1959.

[4] ________. "Economies with a Finite Set of Equilibria," Econometrica, 38 (1970), 387-392.

[5] ________. "Smooth Preferences," Econometrica, 40 (1972), 603-616.

[6] Golubitsky, M. and V. Guilemin. Stable Mappings and Their Singularities. New York: Springer, 1973.

[7] Hermann, R. Differential Geometry and the Calculus of Variations. New York: Academic Press, 1968.

[8] Intrilligator, M. Mathematical Optimization and Economic Theory. Englewood Cliffs, N.J.: Prentice-Hall, 1971.

[9] Smale, S. "Analysis and Economics IIA, Extension of a Theorem of Debreu," Jour. of Math. Econ., 1 (1974), 1-14.

[10] _______. "Global Analysis and Economics III, Pareto Optima and Price Equilibria, Jour. of Math. Econ. (forthcoming).