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A COLORED VERSION OF TVERBERG'S THEOREM

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ABSTRACT

The main result of this paper is that given n red, n white, and n green points in the plane, it is possible to form n vertex–disjoint triangles $\Delta_1, \ldots, \Delta_n$ in such a way that Δ_i has one red, one white, and one green vertex for every $i = 1, \ldots, n$ and the intersection of these triangles is nonempty.

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Introduction

Let n, d, r with $n \geq (d+1)r$ be positive integers and consider a finite set $\mathcal{P}_n$ of n distinct points in R^d which are divided into $d+1$ subsets $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$, called colors, each of cardinality at least r. We say that $\mathcal{P}_n$ is r–properly colored. If $p_1, \ldots, p_{d+1}$ are points of $\mathcal{P}_n$ then we say that $\{p_1, \ldots, p_{d+1}\}$ and the simplex (possibly degenerate) $\text{conv}(p_1, \ldots, p_{d+1})$ which they determine is *multicolored* if, after suitable relabelling, $p_i \in \mathcal{C}_i$, $i = 1, \ldots, d+1$.

One of the best known elementary results in convex sets is Radon's theorem [1]:

Radon's Theorem. *Any* $d+2$ *points in* E^d *can be divided into two subsets* X, Y *with* $\text{conv}\, X \cap \text{conv}\, Y \neq \phi$.

The famous extension of Radon's theorem due to Tverberg [2] is:

Tverberg's Theorem. *Any* $r(d+1)-d$ *points in* E^d *can be divided into* r *disjoint sets* $X_1, \ldots, X_r$ *with* $\bigcap_{i=1}^{r} \text{conv}\, X_i \neq \phi$.

Recently, studies of the well–known k–set problem [3], [4], [8] have aroused considerable interest in the possible existence of a colored version of Tverberg's theorem. The results of this paper will, in particular, yield the bound $O(n^{3-1/27})$ on the number of possible ways a set of n points in E^3 can be divided in half by a hyperplane. This is an improvement over $O(n^{3-1/64})$ given in [4]. However, by a different method, the better bound $O(n^{3-1/3+\epsilon})$ has been obtained recently [8].

The Colored Tverberg Problem

Determine the least value $N(r,d)$ such that if $n \geq N(r,d)$ and $\mathcal{P}_n$ is an r–properly colored subset of E^d then there exists r disjoint multicolored subsets of $\mathcal{P}_n$,

$$\left\{p_{1,j}, \ldots, p_{(d+1),j}\right\}_{j=1}^{r}, \text{ say,}$$

such that

$$\bigcap_{j=1}^{r} \operatorname{conv}\{p_{1,j}, \ldots, p_{(d+1),j}\} \neq \phi .$$

For obvious reasons, we call the special case $r = 2$ the colored Radon problem.

Almost nothing is known about this problem. In [4] it is shown that $N(3,2) \leq 7$ but for $d \geq 3$, $r \geq 3$ it is not known that any finite $N(r,d)$ exists.

We make the conjecture that $N(r,d) = r(d+1)$. We shall prove it for $d = 1, 2$. The colored Radon Theorem $N(2,d) = 2(d+1)$ has been proved by many people independently and we will reproduce the proof due to Lovász [6] here.

Theorem. *For positive integers* r *and* d

(i) $N(r,1) = 2r$,

(ii) $N(r,2) = 3r$,

(iii) $N(2,d) = 2(d+1)$.

Note. If we have a set $\mathcal{P}$ in E^d which is r–properly colored, we shall say that $\mathcal{P}$ is r–divisible if there exist r disjoint multicolored subsets $\{p_{1,j}, \ldots, p_{d+1,j}\}_{j=1}^{r}$ with

$$\bigcap_{j=1}^{r} \operatorname{conv}\{p_{1,j}, \ldots, p_{d+1,j}\} \neq \phi .$$

We mention further that (ii) of the theorem has been proved (independently) J. Jaromczyk and G. Swiatek [7].

Proof of the Theorem.

(i) $N(r,1) = 2r$. This we can do by induction. Trivially $N(1,1) = 2$. Now assume $N(r,1) = 2r$ for some $r \geq 1$. Let $\mathcal{P}_{2(r+1)}$ be an $(r+1)$–properly colored set of $2(r+1)$ points on the real line. Let $\inf \mathcal{P} = A$ and we suppose that A is colored 1. Let B be the largest point of $\mathcal{P}$ which is colored 2. The removal of A and B from $\mathcal{P}_{2(r+1)}$ yields a r properly colored subset which we can divide into r multicolored intervals with a common point of intersection which can be chosen in the interval $[A,B]$. The inclusion of the multicolored interval $[A,B]$ yields the required $r+1$ multicolored intervals.

(ii) $N(r,2) = 3r$. We adopt the Tverberg approach of taking points $P, P_2, \ldots, P_{3r}$ and $Q, P_2, \ldots, P_{3r}$ in algebraically independent positions. Assuming that the set $P, P_2, \ldots, P_{3r}$ is r–divisible we shall prove that the set $Q, P_2, \ldots, P_{3r}$ is r–divisible. Since there certainly are positions for $P, P_2, \ldots, P_{3r}$ which are r–divisible, (ii) will be established if we can prove the above result.

In fact it will be convenient to prove the stronger result that when the points are in algebraically independent positions then the interiors of the r multicolored triangles contain a common point of intersection. As in Tverberg's approach we consider the set $(1-t)P + tQ, P_2, \ldots, P_{3r}$, $0 < t < 1$, and consider the set T of those t in $[0,1]$ for which $(1-t)P + tQ, P_2, \ldots, P_{3r}$ is r–divisible. T is a non–empty, since $0 \in T$, closed set and let t_0 be the maximum of T. We show that $t_0 = 1$ (and the result follows) by showing that if $t_0 < 1$ then there exists $t > t_0$ with $t \in T$. Now suppose that $t_0 < 1$ and consider the situation at t_0.

Since we are unable to continue using the subdivision of

$$\{(1-t)P + tQ, P_2, \ldots, P_{3r}\}$$

used at t_0 one of two possibilities must have occurred:

(i) Two of the multicolored triangles used at t_0 will intersect in a degenerate way, i.e. if the triangles are T_1, T_2, then T_1 and T_2 are weakly separated by a line ℓ and a vertex of T_2 will lie on an edge of T_1. All other triangles will contain this vertex of T_2 in their interior.

(ii) Three of the multicolored triangles used at t_0 will intersect in a single point 0 say which lies in the relative interiors of their edges. All other triangles will contain 0 in their interior.

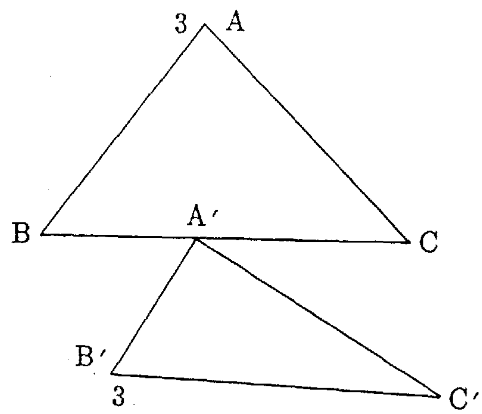
We fist consider possibility (i).

Let T_1 have vertices A, B, C and T_2 have vertices A', B', C' where A' is the point $(1-t_0)P + t_0Q$. If ℓ^+, ℓ^- are the two half planes determined by ℓ we suppose that T_1 lies in ℓ^+ and T_2 lies in ℓ^-. We suppose that A' lies in the edge BC and as t increases from t_0, A' moves to a position A_t' in the interior of ℓ^- and hence the triangles ABC, $A_t'B'C'$ do not intersect. Another possibility is that B lies on the edge $A'C'$ but the arguments for this possibility are similar and will therefore be omitted.

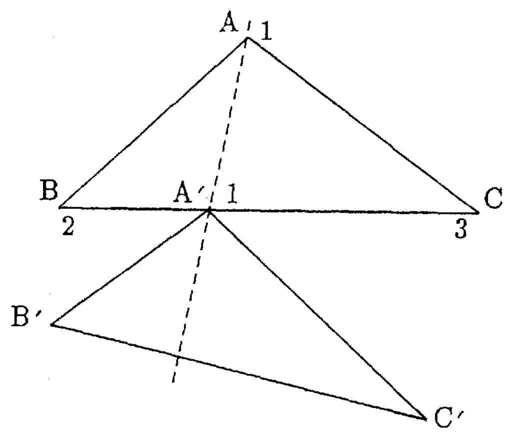
What we shall show is that it is possible, as A' moves slightly to A_t', to rearrange the six points A, B, C, A_t', B', C' into two multicolored triangles whose interiors meet within any given neighborhood of A' (of B if B lies in the edge $A'C'$) by varying the distance between t and t_0 accordingly. This ensures that for $t > t_0$ and t close to t_0, the r multicolored triangles (the two newly distributed triangles and the $r-2$ remaining triangles in the r–division at t_0) have a common point in their interiors.

<u>Case 1</u>. *In the line ℓ the three points A', B, C do not have distinct colors.*

Let us suppose that the color 3 is not amongst the colors of A', B, C. Then A has color 3 and we suppose that B' has color 3. Then $AA_t'C'$, $B'BC$ are the required triangles.

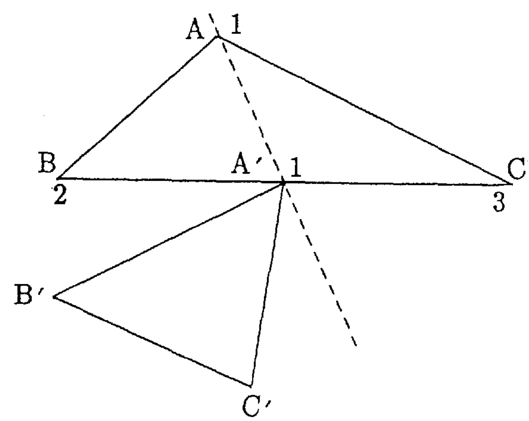


<u>Case 2</u>. *In the line* ℓ *the three points* A', B, C *have the distinct colors* 1, 2, 3 *respectively.*



If the line through AA' meets the interval $(B'C')$ then the triangles $AB'C'$, A'_tBC will do.

Otherwise suppose that $B'C'$ lies on the same side of the line AA' as does B.



If B' is colored 2 then $B'AC$, $BC'A'$ will do. If B' is colored 3 then $B'BA'$, $AC'C$ will do. So, if $t_0 < 1$, (i) cannot arise.

We now consider the possibility (ii):

There are three multicolored triangles T_1, T_2, T_3, with the point $(1 - t_0)P + t_0Q$ in T_1, whose intersection is a single point 0 say belonging to the relative interiors of the sides of T_1, T_2, T_3. Further, if $T_1(t)$ is the multicolored triangle with $(1 - t_0)P + t_0Q$ replaced by $(1-t)P + tQ$, an increase from t_0 to t means that $T_1(t)$, T_2, T_3 no longer have a common point of intersection.

We consider the nine vertices of T_1, T_2, T_3, three colored 1, three colored 2, and three colored 3 which we try to rearrange as the vertices of three multicolored triangles whose intersection still contains 0 but also contains an interior point. Thus when $(1 - t_0)P + t_0Q$ is moved to $(1-t)P + tQ$, $t > t_0$ but $t - t_0$ small, the rearranged triangles still have a non–empty intersection. In fact we will try to rearrange two of the three triangles so that one contains 0 in its interior and the other contains 0 on its boundary. We may not always succeed but we gain information about the arrangement of points.

The triangles T_1, T_2, T_3 have three edges AB, DE, GH, one each respectively, passing through 0, with third vertices C, F, I respectively. We regard the nine vertices as arranged circularly around 0 with each edge AB, DE, GH carrying a normal direction to indicate the halfplane containing the third vertex. Of course, the intersection of the three halfplanes is precisely 0.

Consider two of these edges AB, DE. Two of these vertices say B, E will be given the same color, say 3. Consider first the case when A, D have different colors say 1, 2. Figure 1 indicates the three different possible arrangements.

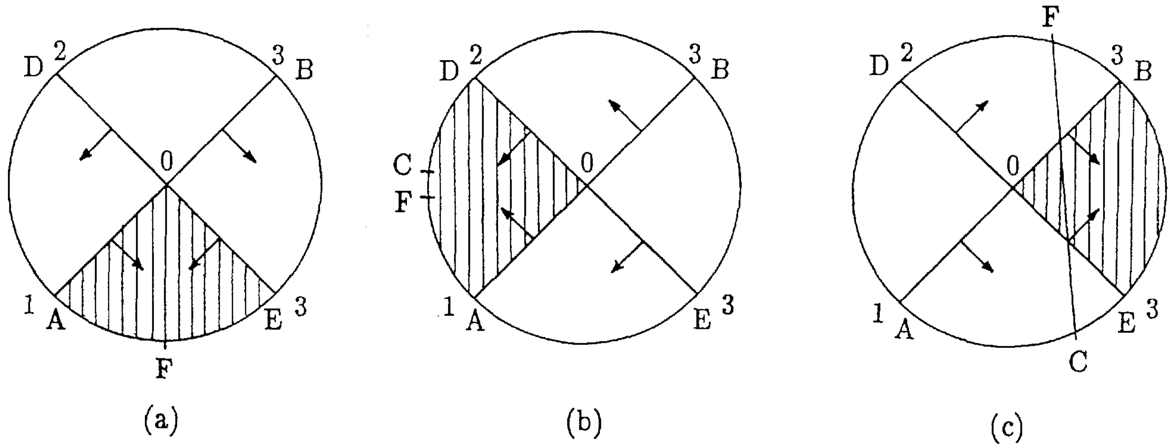


FIGURE 1

(a) If $F \in \widehat{AD}$ (the circular arc between A and D taken clockwise) and the segment FC does not meet the sector D0B then FCB , ADE are the required triangles. If FC meets D0B then FEC , ABD are the required triangles.

Consequently in case (a) we may suppose that F lies in $\widehat{EA}$ the common arc of intersection of the triangles T_1 and T_2.

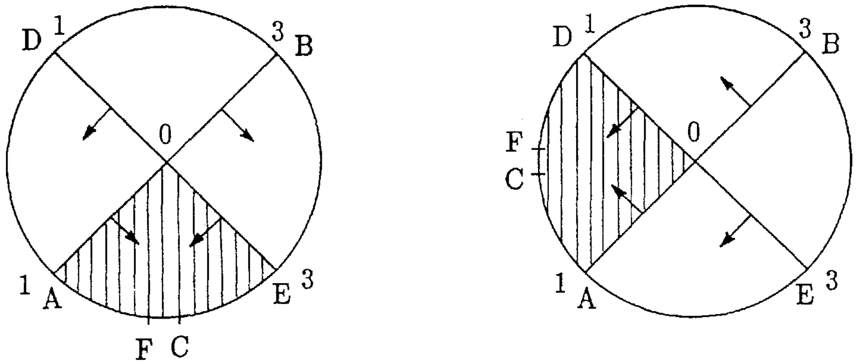
(b) If C lies in $\widehat{DB}$ and the segment FC does not meet the sector B0E then FCE , ABD are the required triangles. If FC meets the sector B0E (and C lies in $\widehat{DB}$), then ACE , FDB are the required triangles. Consequently we may suppose that C lies in $\widehat{AD}$. If F lies in $\widehat{EA}$ then BCF and AED are the required triangles. So F also lies in $\widehat{AD}$.

Consequently, in case (b), we may suppose that both C and F lie in AD , the common arc of intersection of the triangles T_1 and T_2.

(c) If the segment FC does not meet the sector B0E then ADB and CFE are the required triangles.

Consequently, in case (c), we may suppose that the segment FC meets the sector B0E, the common sector of intersection of the triangles T_1 and T_2.

Now suppose that A, D have the same color 1 say. Figure 2 indicates the two possible arrangements.



(d) (e)

FIGURE 2

(d) If F is in $\widehat{AD}$ then we may change the region of intersection from E0A to D0B by interchanging F and C i.e. using the triangles ABF, DEC. The intersection $T_3 \cap ABF \cap DEC$ contains 0 and an interior point, as required. So we may suppose that F and C lie in $\widehat{EA}$ the arc of intersection of the triangle T_1 and T_2.

(e) If C lies in $\widehat{DB}$ and F lies in $\widehat{EA}$ we may change the region of intersection A0D to B0E by using the triangles ABF, CED. The intersection $T_3 \cap ABF \cap DEC$ contains 0 and an interior point, as required.

If C lies in $\widehat{DB}$ and F lies in $\widehat{AD}$ we may change the region of intersection A0D to D0B by using the triangles ABF, CED. The intersection $T_3 \cap ABF \cap DEC$ contains 0 and an interior point, as required. So we may suppose that C lies in $\widehat{AD}$.

So we may suppose that F and C lie in $\widehat{AD}$ the arc of intersection of the triangles T_1 and T_2.

So in the cases (a), (b), (d), (e) (at least) one of the points F and C lies in the sector of intersection.

Now consider the three diameters AB , DE , GH as in Figure 3.

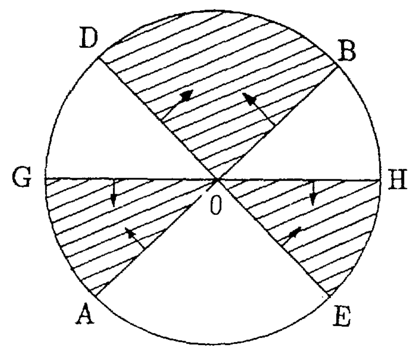


FIGURE 3

There will be three regions of pairwise intersections of the triangles T_1 , T_2 , T_3 determined by the arcs $\widehat{DB}$, $\widehat{HE}$, $\widehat{AG}$. Consider the pairs EA , GD , BH . Suppose E and A receive the same color 1 say. Then, by (b) and (e) we see that the arc $\widehat{DB}$ contains both points C and F . The other two regions of pairwise intersection will contain at least one point of C , F , I , and hence an obvious contradiction, unless one of the pairwise intersections, corresponding to AB , GH say falls into case (c). Thus A , G are labelled with the same color 1 and B , H receive different colors, necessarily 2, 3 (say B receives color 2). Now either D has color 2 and (a) applies or D has color 3 and (d) applies. In both cases $\widehat{HE}$ contains I . Consequently C , F and I lie in the arc $\widehat{DE}$ and hence the chord CI does not meet the (interior of) the sector A0G as required by (c) applied to AB , GH .

So we may suppose that none of the pairs EA , GD , BH receive the same color.

Suppose that two of the diameter pairs say AB, DE are similarly colored. Say A, D colored 1 and B, E colored 3. Then, by (d), both C and F lie in $\widehat{DB}$. Unless case (c) arises amongst the other two sets of diameter pairs an immediate contradiction arises since $\widehat{HE}$ and $\widehat{AG}$ will both contain at least one point of C, F and I. So suppose that DE, HG fall into case (e) i.e. H is colored 3. But then B and H have the same color, contradiction.

So we may suppose that none of the diameters are similarly colored. Now only cases (a) and (c) can arise. Let us suppose that case (a) arises for the diameters AB, DE colored 1, 3, 2, 3 respectively. Then G is colored 1 and H is colored 2. Consequently C lies in arc $\widehat{DB}$ and as the pair DE, GH also falls into case (a), I lies in arc $\widehat{HE}$. The pair AB, GH falls into case (c) and so the chord CI must intersect the interior of the sector $A0G$ which contradicts C, I lying in $\widehat{DE}$.

Finally, we suppose that only case (c) arises. Let A, B, D, E be colored 2, 3, 3, 1 respectively. Then H is colored 1 and G is colored 2. The triangle CFI meets the interior of each of the sectors $D0B$, $H0I$, $A0G$ and so contains 0 in its interior. Consequently CFI, AHD, BEG are the required triangles.

This completes the proof that if $t_0 < 1$, (ii) cannot arise and hence completes the proof of (ii) of the theorem.

Remark. It is not possible to carry through the argument in E^3 as we have done in E^2. Notice that in E^2, when the intersection of (say) two multicolored simplices S_1 and S_2 became a single point 0 it was possible to rearrange the vertices of S_1 and S_2 so as to form two other multicolored simplices T_1 and T_2 with 0 in their intersection. We give an example of two tetrahedra $S_1 = \text{conv}\{A, B, C, D\}$ and $S_2 = \text{conv}\{A', B', C', D'\}$ where this is not possible. Let the points A, B, C, D, A', B', C', D' be colored 1, 2, 3, 4, 3, 4, 1, 2, respectively, and let

$$A = (1,0,0)\ ,\ B = (-1,0,0)\ ,\ C, D \text{ close to } (1,1,1)\ ,$$
$$A' = (0,-1,0)\ ,\ B' = (0,1,0)\ ,\ C, D \text{ close to } (1,1,-1)\ .$$

Then $S_1 \cap S_2$ is the origin 0. Assume T_1 and T_2 are two multicolored tetrahedra with vertices from A, B, C, D, A', B', C', D' and $0 \in T_1 \cap T_2$. As $0 \notin \operatorname{conv}\{A, C, D, B', C', D'\}$, A' and B must be in different tetrahedra, $A' \in T_1$ and $B \in T_2$, say. Then $A \in T_2$ since $0 \notin \operatorname{conv}\{B, C, D, B', C', D'\}$, and similarly $B' \in T_1$. But now the only way to have all colors in T_1 and T_2 is to have $T_1 = \operatorname{conv}\{A', B', C', D'\}$ and $T_2 = \operatorname{conv}\{A, B, C, D\}$.

(iii) $\underline{N(2,d) = 2(d+1)}$. In E^{d+1} consider the cross–polytope X with vertices, $\pm e_i$, $i = 1, \ldots, d+1$, where $e_1, \ldots, e_{d+1}$ are the unit coordinate vectors. Let $\mathcal{P} = \{1, \ldots, d+1; 1', \ldots, (d+1)'\}$ be a 2–properly colored set in E^d of $2(d+1)$ points such that points i and i' are colored i, $i = 1, \ldots, d+1$. We define

$$\sigma(e_i) = i\ ,\ \sigma(-e_i) = i'\ ,\ i = 1, \ldots, d+1\ .$$

We can extend σ to a continuous map of ∂X into E^d by taking

$$\sigma(x) = \sum_{i=1}^{d+1} \lambda_i \sigma(v_i) \quad \text{where } x = \sum_{i=1}^{d+1} \lambda_i v_i \in \partial X$$

$\lambda_i \geq 0$, $\sum_{i=1}^{d+1} \lambda_i = 1$, $i = 1, \ldots, d+1$, v_i are vertices of X.

By the Borsuk–Ulam theorem [5] there exists x and $-x$ in ∂X with $\sigma(x) = \sigma(-x)$. If $\{v_i\}_{i=1}^{d+1}$ are the vertices in the facet of X containing x then $\{-v_i\}_{i=1}^{d+1}$ are the vertices in the facet containing $-x$ and $\{\sigma(v_i)\}_{i=1}^{d+1}$, $\{\sigma(-v_i)\}_{i=1}^{d+1}$ are the vertices of two multicolored d–simplices which intersect in the point $\sigma(x)$.

REFERENCES

[1] J. Radon, "Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten," Math. Ann., 83 (1921), 113–115.

[2] H. Tverberg, "A generalisation of Radon's Theorem," J. London Math. Soc., 41 (1966), 123–128.

[3] P. Erdős, L. Lovász, A. Simmons and E. G. Strauss, Dissection graphs of planar point sets, in A Survey of Combinatorial Theory (J. N. Shrivastava et al., eds.), North–Holland, Amsterdam, 1973, 139–149.

[4] I. Bárány, Z. Füredi and L. Lovász, On the number of halving planes, in "Proc. 5th Ann. Sympos. Comput. Geom., 1989," 140–144 and to appear in Combinatorica.

[5] K. Borsuk, Drei Sätze über die n–dimensionale euklidische Sphäre, Fundam. Math., 20 (1933), 177–190.

[6] L. Lovász, private communication (1989).

[7] J. Jaromczyk, G. Swiatek, private communication (1989).

[8] B. Chazelle, H. Edelsbrunner, L. J. Giubas, M. Sharir, "Points and triangles in the plane and halving planes in space" (manuscript).