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A STRATEGIC MARKET GAME WITH SECURED LENDING

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Abstract

We study stationary Markov equilibria for strategic, competitive games, in a market-economy model with one non-durable commodity, fiat money, borrowing/lending through a central bank or a money market, and a continuum of agents. These use fiat money in order to offset random fluctuations in their endowments of the commodity, are not allowed to borrow more than they can pay back (secured lending), and maximize expected discounted utility from consumption of the commodity. Their aggregate optimal actions determine dynamically prices and/or interest rates for borrowing and lending, in each period of play. In equilibrium, random fluctuations in endowment- and wealth-levels offset each other, and prices and interest rates remain constant.

As in our related recent work, KSS (1994), we study in detail the individual agents' dynamic optimization problems, and the invariant measures for the associated, optimally controlled Markov chains. By appropriate aggregation, these individual problems lead to the construction of stationary Markov competitive equilibrium for the economy as a whole.

Several examples are studied in detail, fairly general existence theorems are established, and open questions are indicated for further research.

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Contents

1 Introduction

In a previous paper (KSS, 1994), we constructed a stationary Markov equilibrium for an infinite horizon stochastic game with a continuum of players which modeled a simple economy with one nondurable commodity and a constant money supply. The players received random endowments during each period and had to decide how much of their current wealth to spend on consumption and how much to save for the future. Each agent sought to maximize the expected value of his or her total discounted utility.

Here we study a generalization in which agents are allowed to borrow and lend (or deposit) money during each period. Two simple lending mechanisms for borrowing and lending will be considered.

1.1 A Central or Outside Bank

We adopt a very simple model for a central bank which sets an interest rate for borrowers and another for depositors, and seeks only to balance the books by taking in as much money as it pays out. The bank is the only source of loanable funds in this model.

There are other interesting models for banking which we do not treat here, such as mutual banks, merchant banks, and stockholder-owned commercial banks.

We hope to consider some of these models in future work.

1.2 A Money Market

In this model agents offer money for loans and bid for loans in a money market. An interest rate is formed endogenously by dividing the aggregate bids for loans by the aggregate of the funds offered for loans.

This Cournot-style mechanism has been employed previously both in oligopoly theory and the study of strategic market games. A much more complicated model, in the manner of Bertrand and Edgeworth, would have each agent specify the quantity he or she wishes to borrow or lend, together with an interest rate above or below which the agent will leave the market. (See Dubey (1982) for a one-period game of this variety.)

There is a difficulty which arises in strategic market games when agents are allowed to borrow. If someone borrows money today with a promise to repay it at a later date, it is possible that the system may evolve to a state where the promise cannot be fulfilled. In other words, agents "can go bankrupt." We will sidestep this difficulty by restricting our attention to "secured lending" models, where agents are permitted to borrow only up to the lower bound of their income. We plan to consider models with active bankruptcy in a future paper.

1.3 Preview

In Section 2 we give a careful definition of the two models we will study and of what we mean by "stationary equilibrium." The dynamics for each model are fully specified, but we leave open the interesting question of how the models behave when they are not in equilibrium.

If either game is in equilibrium, then each individual agent faces a one- person game that is equivalent to a *discounted dynamic programming problem* for which the price of the commodity and the interest rates are fixed parameters. This one-person game is introduced in Section 3. It is pointed out that there always exists an optimal stationary strategy π^α for an agent α playing in the one-person game and that the stochastic process S_0^α, S_1^α, ..., corresponding to the player's wealth during successive periods, is a Markov chain when the strategy π^α is used.

Two properties of the Markov chains $\{S_n^\alpha\}$ are the key to the construction of stationary equilibria for the many-person games in Section 4. Indeed, if we assume that each agent's Markov chain has a stationary distribution μ_α with finite mean, and that the total interest paid by borrowers equals that received by lenders, then we are able to give an explicit construction of an equilibrium in terms of the μ_α.

Section 5 is devoted to a number of simple examples where the one-person games can be solved analytically and then used to construct a stationary equilibrium for the many-person game. An example is also presented where no such equilibrium exists for the money market model.

In Section 6 we return to treat the one-person game in more detail; we determine the structure of the optimal stationary strategies and give sufficient conditions for the corresponding Markov chains to have stationary distributions with finite means.

Most of Section 7 is devoted to proving the existence of stationary equilibrium for a modified form of the money market game in which there is "government intervention" to keep interest rates bounded.

2 Formulation of the Games with a Continuum of Agents

In this section we will define both the game with an outside bank and the game with a money market. Most of the notation is the same in both games.

Let $I = [0, 1]$ and let ϕ be a nonatomic probability measure on the Borel σ–field $\mathcal{B}(I)$. The set I is an index set for the collection of agents and ϕ represents the "spatial" distribution on this collection. Each agent $\alpha \in I$ has a utility function $u^\alpha : [0, \infty) \to [0, \infty)$ with $u^\alpha(0) = 0$ which is nondecreasing, concave, and has a finite right-hand derivative at the origin.

In every period $n \geq 1$ each agent α receives a random endowment $Y_n^\alpha(w)$ in units of a nondurable *commodity*. For each α the random variables Y_1^α, Y_2^α, ... are assumed to be nonnegative, integrable and independent, with common law λ^α. However, the *total endowment* of the commodity

$$Q = \int Y_n^\alpha(w)\phi(d\alpha) > 0 \qquad (2.1)$$

is taken to be nonrandom and constant from period to period. A technique of Feldman and Gilles (1985) gives a simple construction of jointly measurable variables $Y_n^\alpha(w) = Y_n(\alpha, w)$, $(\alpha, w) \in I \times \Omega$, $n \in \mathbb{N}$, which are IID for fixed α but aggregate to a constant as in (2.1).

At the beginning of every period $n \geq 0$ each agent α holds an amount $S_n^\alpha(w)$ in *fiat money*. Agents must decide in every period how much money to borrow or lend, and then how much to bid in the commodity market. Since the rules are slightly different in our two models, we will consider each in turn.

The random variables Y_n^α, S_n^α (and others, which are introduced below) are all defined on a given probability space $(\Omega, \mathcal{F}, P)$.

2.1 The Game with an Outside Bank

The bank sets two interest rates which remain fixed throughout the game: $r_1 = 1+\rho_1$ is the rate paid by borrowers and $r_2 = 1+\rho_2$ is the rate paid to depositors. We assume that $0 < r_2 \leq r_1 \leq 1/\beta$ where $\beta \in (0, 1)$ is a discount factor.

At the beginning of the nth period of play ($n \geq 1$), the price of the commodity is $p_{n-1}(w)$ and the money held by the bank is $m_{n-1}(w)$. Each agent α enters with $S_{n-1}^\alpha(w)$ in fiat money and with information represented by a σ–field $\mathcal{F}_{n-1}^\alpha$. (The σ–field $\mathcal{F}_{n-1}^\alpha$ measures past prices $\{p_k, k = 0, ..., n-1\}$ as well as past wealths, endowments, and actions $\{S_0^\alpha, S_k^\alpha, Y_k^\alpha, b_k^\alpha,\ k = 1, ..., n-1\}$; it may or may not measure corresponding quantities for other agents.) Based on this information, each agent bids a certain amount $b_n^\alpha(w) \in [0,\ S_{n-1}^\alpha(w) + k^\alpha]$ of fiat money for the commodity in the nth period.

The constant $k^\alpha \geq 0$ is the upper bound on loans to agent α. If $b_n^\alpha(w) > S_{n-1}^\alpha(w)$, agent α *borrows* $b_n^\alpha(w) - S_{n-1}^\alpha(w)$ from the bank. (We assume the bank has sufficient funds to make all requested loans.) If $b_n^\alpha(w) < S_{n-1}^\alpha(w)$, agent α deposits $S_{n-1}^\alpha(w) - b_n^\alpha(w)$ in the bank.

The total bid in period n is

$$B_n(w) := \int b_n^\alpha(w) \phi(d\alpha)\ . \tag{2.2}$$

We assume $b_n^\alpha(w) = b_n(\alpha, w)$ is jointly measurable in (α, w) so that $B_n(w)$ is a well-defined random variable.

The new price for period n is formed as

$$p_n(w) := B_n(w)/Q\ . \tag{2.3}$$

Each agent α then receives his or her bid's worth $x_n^\alpha(w) := \frac{b_n^\alpha(w)}{p_n(w)}$ of commodity, and consumes it immediately, thereby receiving a payoff in utility of

$$u^\alpha(x_n^\alpha(w))$$

in period n. Agent α's total payoff for the game is thus

$$\Sigma_{n=0}^\infty \beta^n u^\alpha(x_{n+1}^\alpha(w)).$$

The crucial assumption for the secured lending model is that

$$p_n(w)Y_n^\alpha(w) \geq r_1 k^\alpha, \quad \forall w \in \Omega,\ \alpha \in I,\ n \in \mathbb{N}. \tag{2.4}$$

Here $p_n(w)Y_n^\alpha(w)$ is agent α's endowment's worth in fiat money, and r_1k^α is α's maximum possible debt. Thus this assumption guarantees that every agent will be able to pay his or her debts and bankruptcy will not occur. (In fact, (2.4) depends on the endogenous price p_n which could, in principle, be formed so as to violate (2.4). However, we will study equilibria for which (2.4) holds when p_n always equals the equilibrium price p.)

The equation for the dynamics of the wealth process S_n^α of agent α is

$$S_n^\alpha = \left\{\begin{matrix} r_1(S_{n-1}^\alpha - b_n^\alpha) + p_nY_n^\alpha\ , & \text{if } S_{n-1}^\alpha \leq b_n^\alpha \\ r_2(S_{n-1}^\alpha - b_n^\alpha) + p_nY_n^\alpha\ , & \text{if } S_{n-1}^\alpha > b_n^\alpha \end{matrix}\right\} = g(S_{n-1}^\alpha - b_n^\alpha) + p_nY_n^\alpha$$

where

$$g(x) = \begin{cases} r_1 x\ , & \text{if } x \leq 0, \\ r_2 x\ , & \text{if } x \geq 0. \end{cases} \tag{2.5}$$

Notice that g is concave because of our assumption that $r_2 \leq r_1$.

It is a feature of our models that wealth in the form of fiat money is neither created nor destroyed. Thus, the total amount of wealth $W = W_n$ in period n should be the same for all n. Let us verify that $W_n = W_{n-1}$ as a check on the dynamics.

Write

$$W_n = m_n + \int S_n^\alpha \phi(d\alpha)$$

where m_n is the money in the bank at the end of period n. Now

$$\begin{aligned} m_n &= m_{n-1} + (\text{deposits}) - (\text{loans}) + \begin{pmatrix}\text{repayments} \\ \text{by borrowers}\end{pmatrix} - \begin{pmatrix}\text{payments} \\ \text{to depositors}\end{pmatrix} \\ &= m_{n-1} + \int_{S_{n-1}^\alpha \geq b_n^\alpha} (S_{n-1}^\alpha - b_n^\alpha)\phi(d\alpha) - \int_{S_{n-1}^\alpha < b_n^\alpha} (b_n^\alpha - S_{n-1}^\alpha)\phi(d\alpha) \\ &\quad + r_1 \int_{S_{n-1}^\alpha < b_n^\alpha} (b_n^\alpha - S_{n-1}^\alpha)\phi(d\alpha) - r_2 \int_{S_{n-1}^\alpha \geq b_n^\alpha} (S_{n-1}^\alpha - b_n^\alpha)\phi(d\alpha) \\ &= m_{n-1} + \int (S_{n-1}^\alpha - b_n^\alpha)\phi(d\alpha) - r_1 \int_{S_{n-1}^\alpha < b_n^\alpha} (S_{n-1}^\alpha - b_n^\alpha)\phi(d\alpha) \\ &\quad - r_2 \int_{S_{n-1}^\alpha \geq b_n^\alpha} (S_{n-1}^\alpha - b_n^\alpha)\phi(d\alpha). \end{aligned}$$

Now use equation (2.5) for the dynamics to get, in conjunction with (2.1), (2.3):

$$\begin{aligned} W_n &= m_n + \int S_n^\alpha \phi(d\alpha) \\ &= m_n + r_1 \int_{S_{n-1}^\alpha \leq b_n^\alpha} (S_{n-1}^\alpha - b_n^\alpha)\phi(d\alpha) + r_2 \int_{S_{n-1}^\alpha > b_n^\alpha} (S_{n-1}^\alpha - b_n^\alpha)\phi(d\alpha) + p_n \int Y_n^\alpha \phi(d\alpha) \\ &= m_{n-1} + \int (S_{n-1}^\alpha - b_n^\alpha)\phi(d\alpha) + p_n Q = m_{n-1} + \int S_{n-1}^\alpha \phi(d\alpha) - B_n + B_n = W_{n-1}. \end{aligned}$$

A *strategy* π^α for agent $\alpha \in I$ specifies the bids b_n^α for all n. As was mentioned above, the bids b_n^α are $\mathcal{F}_{n-1}^\alpha$–measurable, where $\mathcal{F}_{n-1}^\alpha$ is a σ–field that represents the information accumulated by agent α up to the beginning of period n. Of course, every b_n^α is measurable with respect to

$$\mathcal{F}_{n-1} = \bigvee_\alpha \mathcal{F}_{n-1}^\alpha,$$

the smallest σ–field containing all the $\mathcal{F}_{n-1}^\alpha$.

A collection $\Pi = \{\pi^\alpha,\ \alpha \in I\}$ of strategies is considered *admissible* for our game if, for every n, the bids $b_n(\alpha, w) = b_n^\alpha(w)$ are jointly measurable with respect to the product σ–field

$$\mathcal{G}_{n-1} = \mathcal{B}(I) \times \mathcal{F}_{n-1}.$$

(As Dubey and Shapley (1992) have pointed out, such a measurability assumption is somewhat artificial in a noncooperative game, but necessary in order to avoid a host of technicalities. Furthermore, mild assumptions do lead to admissible families in equilibrium.) We will consider only admissible Π in the sequel.

An admissible collection $\Pi = \{\pi^\alpha\}$, together with an initial distribution of wealth and the dynamics explained above, determines the distributions of all the random variables we have introduced. In particular, the expected payoff

$$e^\alpha = E \sum_{n=0}^{\infty} \beta^n u(x_{n+1}^\alpha)$$

is determined for each agent $\alpha \in I$, and we have a well-defined stochastic game.

Let

$$\nu_0(A, w) := \int_I 1_A(S_0^\alpha(w))\phi(d\alpha), \tag{2.6}$$

for $A \in \mathcal{B}([0, \infty))$, be the *initial distribution of wealth* in the form of fiat money. An admissible collection $\Pi = \{\pi_\alpha\}$, together with ν_0, determines the sequence of random measures

$$\nu_n(A, w) := \int_I 1_A(S_n^\alpha(w))\phi(d\alpha), \quad n \in \mathbb{N} \tag{2.7}$$

corresponding to the distribution of wealth in future time periods $n = 1, 2, ...$

At any time n, the total wealth held by all the agents is the quantity

$$\widetilde{W}_n(w) = \int_I S_n^\alpha(w)\phi(d\alpha) = \int_0^\infty s\nu_n(ds, w).$$

Obviously, $\widetilde{W}_n(w)$ cannot exceed the money supply W and the difference $W - \widetilde{W}_n(w) =: m_n(w)$ is the money held by the bank.

A strategy π^α for agent α is called *stationary* if it specifies bids of the form

$$b_n^\alpha(w) = c^\alpha(S_{n-1}^\alpha(w),\, p_{n-1}(w)), \quad n \in \mathbb{N}, \quad w \in \Omega$$

where $c^\alpha : [0,\infty) \times [0,\infty) \to [0,\infty)$ is measurable and $0 \le c^\alpha(s,p) \le s + k^\alpha$ for all (s,p). Such a strategy requires an initial price p_0 to be specified prior to period 1, so that b_1^α will be defined.

Definition 2.1. We say that an admissible collection $\widetilde{\Pi} = \{\widetilde{\pi}^\alpha,\ \alpha \in I\}$ of stationary strategies results in a *stationary Markov equilibrium* (p,μ), where $p \in (0,\infty)$ and μ is a probability measure on $\mathcal{B}([0,\infty))$, if with $p_0 = p$ and $\nu_0 = \mu$,

(i) we have $p_n = p$, $\nu_n = \mu$, $\forall n \in \mathbb{N}$, when every agent α plays the strategy $\widetilde{\pi}^\alpha$, and

(ii) for every $\beta \in I$, $\widetilde{\pi}^\beta$ is optimal among all strategies for agent β when every other agent α, $\alpha \neq \beta$, plays $\widetilde{\pi}^\alpha$.

2.2 The Game with a Money Market and an Endogenous Rate of Interest

The notation for agents $\alpha \in I$, and prices p_n, wealth processes S_n^α, and utility functions u^α remains the same. There is no longer an outside bank, but instead there is a *money market* where each agent α can borrow or lend in each period.

Here is how the money market works. Each agent $\alpha \in I$ has a wealth of $S_{n-1}^\alpha(w)$ at the beginning of period n. If agent α chooses to borrow, then he or she bids $i_n^\alpha(w) \in [0,\ k^\alpha]$ in IOU notes, where $k^\alpha \ge 0$ is an upper bound on IOU notes allowed to agent α. If agent α chooses to lend, then he or she offers an amount $\ell_n^\alpha(w) \in [0,\ S_{n-1}^\alpha(w)]$ in fiat money. Let

$$I_n(w) := \int i_n^\alpha(w)\phi(d\alpha)$$

be the total of the IOU notes bid in period n and let

$$L_n(w) := \int \ell_n^\alpha(w)\phi(d\alpha)$$

be the total money offered for lending. (The functions $i_n(\alpha,w) = i_n^\alpha(w)$, $\ell_n(\alpha,w) = \ell_n^\alpha(w)$ are assumed to be jointly measurable in α and w, with $i_n^\alpha(w)\ell_n^\alpha(w) \equiv 0$.)

If $I_n(w)$ and $L_n(w)$ are both positive, then an interest rate is formed as

$$r_n(w) = 1 + \rho_n(w) := \frac{I_n(w)}{L_n(w)}. \tag{2.8}$$

A borrower α, who has bid $i_n^\alpha(w)$ in IOU notes, obtains a loan of $i_n^\alpha(w)/r_n(w)$ in fiat money, and bids

$$b_n^\alpha(w) = S_{n-1}^\alpha(w) + i_n^\alpha(w)/r_n(w)$$

in the commodity market. A lender α, who has offered $\ell_n^\alpha(w)$ in the loan market, bids

$$b_n^\alpha(w) = S_{n-1}^\alpha(w) - \ell_n^\alpha(w)$$

units of fiat money in the commodity market. Notice that $b_n^\alpha(w) > S_{n-1}^\alpha(w)$ if and only if α is a borrower and $b_n^\alpha(w) < S_{n-1}^\alpha(w)$ if and only if α is a lender.

If either $I_n(w)$ or $L_n(w)$ equals zero, the money market is closed and we assume that each agent α bids his or her entire fortune

$$b_n^\alpha(w) = S_{n-1}^\alpha(w)$$

in the commodity market. (This assumption of no hoarding is quite arbitrary. It is not a crucial assumption because we want to construct an equilibrium with an active loan market where neither $I_n(w)$ nor $L_n(w)$ is zero.) In order that the interest rate always be defined, we set

$$r_n(w) = 1 + \rho_n(w) := 1 \tag{2.9'}$$

in this case.

The price of the commodity is formed as before (recall (2.3)):

$$p_n(w) := \frac{B_n(w)}{Q} = \frac{\int b_n^\alpha(w)\phi(d\alpha)}{Q} \ . \tag{2.3'}$$

However, there is no bank and all of the money is bid. That is, when the money market is active

$$\begin{aligned} \int b_n^\alpha(w)\phi(d\alpha) &= \int S_{n-1}^\alpha(w)\phi(d\alpha) + \frac{1}{r_n}\int i_n^\alpha(w)\phi(d\alpha) - \int \ell_n^\alpha(w)\phi(d\alpha) \qquad (2.10) \\ &= \int S_{n-1}^\alpha(w)\phi(d\alpha) = W, \end{aligned}$$

where W is the supply of fiat money, which again is held constant from period to period; see (2.14) and (2.15) below. The same holds true when the money market is inactive. Thus *the price is constant*

$$p_n(w) \equiv p := W/Q.$$

Each agent α receives as before $x_n^\alpha(w) = b_n^\alpha(w)/p(w)$ units of the commodity, and an endowment of $pY_n^\alpha(w)$ in fiat money. Agent α thus enters the next period with wealth

$$S_n^\alpha = r_n(S_{n-1}^\alpha - b_n^\alpha) + pY_n^\alpha, \tag{2.11}$$

and this equation for the dynamics can be rewritten as

$$S_n^\alpha = \begin{cases} -i_n^\alpha + pY_n^\alpha \ , & \text{if} \quad \alpha \text{ is a borrower,} \\ r_n\ell_n^\alpha + pY_n^\alpha \ , & \text{if} \quad \alpha \text{ is a lender,} \\ pY_n^\alpha \ , & \text{if} \quad \alpha \text{ is neither.} \end{cases} \tag{2.12}$$

The assumption of secured lending now takes the form

$$pY_n^\alpha(w) \geq k^\alpha. \tag{2.13}$$

This insures that no bankruptcy occurs since we have already assumed that $i_n^\alpha \leq k^\alpha$. (Unlike the situation with an outside bank in (2.4), our assumption of secured lending in the money market is not affected by an endogenous price.)

Let us check that our dynamics preserve the money supply as they did with an outside bank. The total wealth W_n in period n is now given by $W_n(w) = \int S_n^\alpha(w)\phi(d\alpha)$ since all the wealth is held by the agents. If the money market is active, then, by (2.10),

$$\begin{aligned} W_n(w) &= -\int i_n^\alpha(w)\phi(d\alpha) + r_n(w)\int \ell_n^\alpha(w)\phi(d\alpha) + p\int Y_n^\alpha(w)\phi(d\alpha) \qquad (2.14) \\ &= -I_n(w) + \frac{I_n(w)}{L_n(w)} \cdot L_n(w) + \frac{W}{Q} \cdot Q = W. \end{aligned}$$

If the money market is closed, then

$$W_n(w) = p\int Y_n^\alpha(w)\phi(d\alpha) = \frac{W}{Q} \cdot Q = W. \qquad (2.15)$$

Thus, the price- and interest-formation rules (2.3′) (2.9), and (2.9′) guarantee that the money supply is conserved in both cases.

A *strategy* π^α for an agent α specifies the IOU bids i_n^α and loan offers ℓ_n^α for all n. (For each α and n, at most one of the quantities i_n^α, ℓ_n^α is strictly positive.) As in the case of an outside bank, i_n^α and ℓ_n^α are $\mathcal{F}_{n-1}^\alpha$–measurable, where $\mathcal{F}_{n-1}^\alpha$ represents the information accumulated by agent α up to the beginning of period n. For all α, i_n^α and ℓ_n^α are measurable with respect to

$$\mathcal{F}_{n-1} = \bigvee_\alpha \mathcal{F}_{n-1}^\alpha.$$

A collection $\Pi = \{\pi^\alpha,\ \alpha \in I\}$ is now called *admissible* if, for every n, the IOU bids $i_n(\alpha, w) = i_n^\alpha(w)$ and loan offers $\ell_n(\alpha, w) = \ell_n^\alpha(w)$ are jointly measurable with respect to

$$\mathcal{G}_{n-1} = \mathcal{B}(I) \times \mathcal{F}_{n-1}.$$

As with an outside bank, an admissible collection $\Pi = \{\pi^\alpha\}$ and an initial distribution of wealth ν_0 as in (2.7) determine the distributions of all the random variables we have defined, along with future wealth distributions $\{\nu_n,\ n \geq 1\}$ as in (2.8) and expected payoffs $\{e^\alpha\}$ as in (2.6), for all agents. Notice that for every n the interest rate $r_n(w)$ is $\mathcal{F}_{n-1}$–measurable and that the commodity bids $b_n(\alpha, w) = b_n^\alpha(w)$ are $\mathcal{G}_{n-1}$–measurable, as are the wealth processes $S_{n-1}(\alpha, w) = S_{n-1}^\alpha(w)$.

For the money market game, a *stationary strategy* π^α specifies IOU bids and loan offers in the form

$$\begin{aligned} i_n^\alpha(w) &= i^\alpha(S_{n-1}^\alpha(w),\ r_{n-1}(w)) \\ \ell_n^\alpha(w) &= \ell^\alpha(S_{n-1}^\alpha(w),\ r_{n-1}(w)) \end{aligned}$$

where i^α and ℓ^α are measurable mappings from $[0,\infty) \times [0,\infty)$ to $[0,\infty]$ such that $0 \leq i^\alpha(s,r) \leq k^\alpha$, $0 \leq \ell^\alpha(s,r) \leq s$, and $i^\alpha(s,r)\ell^\alpha(s,r) = 0$ for all (s,r). Such a strategy requires the specification of an initial interest rate r_0 for the definition of i_1^α and ℓ_1^α.

Definition 2.2. An admissible collection $\widetilde{\Pi} = \{\widetilde{\pi}_\alpha,\ \alpha \in I\}$ of stationary strategies is said to result in a *stationary Markov equilibrium* (r, μ), where $r \in (0, \infty)$ and μ is a measure on $\mathcal{B}([0, \infty))$, if, with $r_0 = r$ and $\nu_0 = \mu$,

(i) we have $r_n = r$, $\nu_n = \mu$, $\forall n \in \mathbb{N}$, when every agent α plays the strategy $\widetilde{\pi}^\alpha$,

(ii) $\forall \beta \in I$, $\widetilde{\pi}^\beta$ is optimal for agent β among all strategies for agent β when every agent α, $\alpha \neq \beta$, plays $\widetilde{\pi}^\alpha$.

3 Formulation of the One-Person Game as a Dynamic Programming Problem

Consider a single agent playing in either the game with an outside bank or with a money market. If the many-person game is in a stationary equilibrium in the sense of Definition 2.1 or 2.2, then this agent faces a discounted dynamic programming problem in which the interest rates r_1 and r_2 and the price p are fixed parameters. Of course, $r_1 = r_2$ in the case of the money market.

The key to our construction of stationary equilibria for the many-person games is in the analysis of this one-person dynamic programming problem. The same analysis works for both the model with an outside bank and the money market model.

Here are the basic ingredients for the dynamic programming problem:

(3.1) The *state space* $\mathcal{S} = [0, \infty)$. A state $s \in \mathcal{S}$ represents the wealth of the agent in fiat money.

(3.2) The *utility function* $u : \mathcal{S} \to \mathcal{S}$ is concave, nondecreasing, $u(0) = 0$, and u has a finite derivative from the right at 0. We write $u'(0) = u'_+(0)$.

(3.3) The *action sets* $B(s) = [0,\, s+k]$ for $s \in \mathcal{S}$ where $k > 0$. We interpret the agent's choice of an action $b \in B(s)$ as a decision to purchase b/p units of the commodity, where $p \in (0, \infty)$ is the price of the commodity.

(3.4) The *reward function* $r(s, b) = u(b/p)$ reflects our interpretation that the agent receives utility $u(b/p)$ from the consumption of his or her commodity purchase. We introduce the function r only to make the connection to dynamic programming and will not use it further.

(3.5) The *law of motion* determines the distribution $q(\cdot|s, b)$ of the next state S_1 for an agent at state s who selects action b by the rule

$$S_1 = g(s - b) + pY$$

where Y is a nonnegative, integrable random variable with a given distribution λ, and $g(\cdot)$ is given by (2.6). The positive constants $r_1 = 1 + \rho_1$, $r_2 = 1 + \rho_2$ are interpreted as the interest rates for borrowing and lending, respectively. The assumption $pY \geq r_1 k$ guarantees that loans can always be paid back.

(3.6) The *discount factor* $\beta \in (0, 1)$.

We assume as in Section 2.1 that $0 < r_2 \leq r_1 \leq 1/\beta$.

A player begins at some state s_0 and selects a *plan* $\pi = (\pi_1, \pi_2, \dots)$ where π_n makes a measurable choice of the action b_n on day n as a function of the sequence $(s_0, b_1, s_1, \dots, b_{n-1}, s_{n-1})$ of previous states and actions. The plan π is *stationary* if it is of the form $b_n = c(s_{n-1})$, for all n, and some measurable function $c : \mathcal{S} \to [0, \infty)$ such that $c(s) \in B(s)$. We will call such a c a *consumption function.*

A plan π, together with the law of motion, determines the distribution of the stochastic process $s_0, b_1, s_1, b_2, \dots$ of states and actions. The *return function* of a plan π is defined as

$$I(\pi)(s) := E^{\pi}_{s_0=s}\left[\sum_{n=0}^{\infty} \beta^n u(b_{n+1})\right], \; s \in \mathcal{S}. \tag{2.7}$$

The *optimal return or value function* is defined by

$$V(s) := \sup_{\pi} I(\pi)(s), \; s \in \mathcal{S}. \tag{2.8}$$

A plan π is *optimal* if $V = I(\pi)$.

If u is bounded, then our problem is a discounted dynamic programming problem as defined by Blackwell (1965). It follows from our assumptions (3.2) about u that u is dominated from above by the linear function $f(x) = u'(0)x$. This is sufficient, as it was in KSS (1994), for many of Blackwell's results.

In particular, V satisfies the *Bellman equation*

$$V(s) = \sup_{0 \leq b \leq s+k} [u(b/p) + \beta E V(g(s-b) + pY)]. \tag{2.9}$$

We introduce the operator T defined for Borel functions $\psi : [0, \infty) \to [0, \infty)$ by

$$(T\psi)(s) = \sup_{0 \leq b \leq s+k} [u(b/p) + \beta E \psi(g(s-b) + pY)]. \tag{2.10}$$

Then (3.9) can be written in the form

$$TV = V.$$

Here is the well-known characterization of optimal stationary plans given by Blackwell (1965). The extension to our situation is straightforward and we omit the proof.

Theorem 3.1. *For a stationary plan π corresponding to the consumption function c, the following conditions are equivalent:*

(a) $I(\pi) = V$.

(b) $V(s) = u(c(s)/p) + \beta E V(g(s - c(s)) + pY), \; s \in \mathcal{S}$.

(c) $T(I(\pi)) = I(\pi)$.

Under our assumptions the value function V is continuous (and also concave and nondecreasing). This can be shown directly, or by using the methods of Chapter I in Bellman (1957). Hence, for every s, the maximum on the right-hand side of (3.9) is attained at some $b = c(s)$. By the theorem above, c determines an optimal stationary plan π.

Corollary 3.2. *There exists an optimal stationary plan.*

In general, the optimal stationary plan need not be unique, as is illustrated by examples in Jayawardene (1993). It is unique when u is strictly concave. Much more detailed information about the optimal plan will be given in Section 6.

Let π be an optimal stationary plan corresponding to the function c. We will sometimes write $c(s) = c(s, \theta)$ where θ is some subset of the parameters (r_1, r_2, p). For example, when we are contemplating the many-person game with an *outside bank*, we sometimes write $c(s) = c(s, p)$ to emphasize the dependence on the endogenous price p. If we plan an application to the game with a *money market*, we may write $c(s) = c(s, r)$ since it is the interest rate which can change in that game.

If an agent plays according to c in the *money market game* at equilibrium with interest rate r, we can write

$$i(s) = \left\{ \begin{matrix} r(c(s,r) - s) & , & \text{if} \quad c(s,r) > s \\ 0 & , & \text{otherwise} \end{matrix} \right\}$$

for the agent's IOU bid, and

$$\ell(s) = \left\{ \begin{matrix} s - c(s,r) & , & \text{if} \quad s \geq c(s,r) \\ 0 & , & \text{otherwise} \end{matrix} \right\}$$

for the agent's loan offer. Thus the choice of c is equivalent to specifying i and ℓ.

Suppose now that a player begins at $S_0 = s$ and plays according to the stationary plan π associated with the function c. Then the process S_0, S_1, ... of successive states is a *Markov chain* with transitions given by

$$S_{n+1} = g(S_n - c(S_n)) + pY_{n+1}, \; n \in \mathbb{N}_0 \tag{2.11}$$

where Y_1, Y_2, ... are IID with distribution λ. A basic assumption in the next section will be that *this chain has a stationary distribution with finite mean.* Sufficient conditions for this assumption to be satisfied will be given in Section 6.

4 The Construction of Stationary Markov Equilibrium

In this section we show how to construct a stationary Markov equilibrium for the game with an outside bank and also for the game with a money market. The construction depends on two basic assumptions: 1. Each agent uses a stationary plan, which is optimal for the one-person game and for which the associated Markov chain has a stationary distribution with finite mean. 2. The "books balance," in the sense that interest paid by borrowers equals that received by lenders.

The construction is much simpler when all the agents are *homogeneous*, in the sense that they all have the same utility function $u^\alpha = u$ and income distribution $\lambda^\alpha = \lambda$ for all $\alpha \in I$. We will consider first an outside bank with homogeneous agents, and then a money market with homogeneous agents. We will then go on to treat each model with countably many types of agents.

4.1 The Game with an Outside Bank and Homogeneous Agents

Fix a price $p \in (0, \infty)$ and interest rates $r_1 = 1 + \rho_1$, $r_2 = 1 + \rho_2$ with $0 < r_2 \leq r_1 \leq 1/\beta$.

Here is the form of our two basic assumptions for this case.

Assumption 4.1. Assume that the one-person problem of Section 3 has an optimal stationary strategy π corresponding to $c(s) = c(s, p)$ and that the Markov chain of (3.11) has an invariant distribution $\mu = \mu(\cdot, p)$ such that $\int s\mu(ds) < \infty$.

Assumption 4.2. Under the stationary wealth distribution μ, the bank "balances its books" in the sense that

$$\rho_2 \int_{s > c(s)} (s - c(s))\mu(ds) = \rho_1 \int_{s \leq c(s)} (c(s) - s)\mu(ds). \tag{4.1}$$

This equality simply says that under the wealth distribution μ, if all the agents play according to π, then the interest paid by the bank to depositors is the same as that paid by borrowers to the bank.

Lemma 4.3. $p = \frac{\int c(s,p)\mu(ds)}{EY}$.

Proof. Let S_0, S_1, ... be the Markov chain of (3.11), and write (3.11) for $n = 1$ as

$$S_1 = \begin{cases} (1 + \rho_1)(S_0 - c(S_0)) + pY_1 & , \ \text{if } S_0 \leq c(S_0) \\ (1 + \rho_2)(S_0 - c(S_0)) + pY_1 & , \ \text{if } S_0 > c(S_0). \end{cases} \tag{4.2}$$

Suppose S_0 has the (stationary) distribution μ, and so S_1 has distribution μ also. Take expectations in (4.2) and use the fact that $ES_0 = ES_1$ to get

$$\begin{aligned} 0 &= -\int c(s)\mu(ds) + \rho_1 \int_{s \leq c(s)} (s - c(s))\mu(ds) + \rho_2 \int_{s > c(s)} (s - c(s))\mu(ds) + pEY \\ &= -\int c(s)\mu(ds) + pEY. \end{aligned}$$

The final equality is by Assumption 4.2. ∎

Theorem 4.4. *Under Assumptions 4.1 and 4.2, the family $\Pi = \{\pi^\alpha, \alpha \in I\}$ with $\pi^\alpha = \pi$ for all $\alpha \in I$, results in a stationary Markov equilibrium (p, μ) for which the fixed interest rates are r_1, r_2.*

Proof. As in KSS (1994), use the technique of Feldman and Gilles (1985) to construct the income variables $Y_n(\alpha, w) = Y_n^\alpha(w)$ so that

(i) for every given $\alpha \in I$: $Y_1(\alpha, \cdot)$, $Y_2(\alpha, \cdot)$, $\cdots$ are IID with distribution λ,
(ii) for every given $w \in \Omega$: $Y_1(\cdot, w)$, $Y_2(\cdot, w)$, $\cdots$ are IID with distribution λ.
Then the Markov chain

$$S_n(\alpha, w) = g(S_{n-1}(\alpha, w) - c(S_{n-1}(\alpha, w), p)) + pY_n(\alpha, w)$$

has the same dynamics for each fixed α and each fixed w.

By Assumption 4.1, μ is a stationary distribution for the chain $S_n(\alpha, \cdot)$ for each α, and consequently is also a stationary distribution for the chain $S_n(\cdot, w)$ for each w, under the assumption that the endogenous prices $p_n(w)$ remain equal to p.

Assume that $p_0 = p$ and that the initial wealth distribution $\nu_0 = \mu$. Take expectations in (2.1), to obtain $EY = Q$; then by Lemma 4.3

$$\begin{aligned} p_1(w) = \frac{B_1(w)}{Q} &= \frac{\int c(S_0^\alpha(w), p)\phi(d\alpha)}{Q} \\ &= \frac{\int c(s,p)\mu(ds)}{EY} = p. \end{aligned}$$

Since μ is an invariant distribution for the chain, we also have $\nu_1 = \mu_1$.

By induction, $p_n = p$ and $\nu_n = \mu$ for all n.

We have verified clause (i) of Definition 2.1. Clause (ii) follows from the optimality of π in the one-person game, together with the fact that a change of strategy by a single player cannot affect the value of the price. ■

4.2 The Game with a Money Market and Homogeneous Agents

As in the previous section, we assume that $u^\alpha = u$ and $\lambda^\alpha = \lambda$ for all α. Fix the price $p = W/Q$ and a single interest rate $r = 1 + \rho > 0$.

Our two basic assumptions are now as follows:

Assumption 4.5. There is an optimal strategy π for the one-person problem corresponding to $c(s) = c(s, r)$ such that the Markov chain of (3.11) has an invariant distribution $\mu = \mu(\cdot, r)$ with $\int s\mu(ds) < \infty$.

Assumption 4.6. Under the wealth distribution μ, the amount borrowed equals the amount offered for lending, and both quantities are positive, i.e.,

$$\int_{s \le c(s)} (c(s) - s)\mu(ds) = \int_{s > c(s)} (s - c(s))\mu(ds) \neq 0. \tag{4.3}$$

The equality can be rewritten in this simpler way:

$$\int s\mu(ds) = \int c(s)\mu(ds). \tag{4.4}$$

Theorem 4.7. *Under Assumptions 4.5 and 4.6, the family $\Pi = \{\pi^\alpha, \alpha \in I\}$, with $\pi^\alpha = \pi$ as in Assumption 4.5 for all $\alpha \in I$, results in a stationary Markov equilibrium (r, μ) for which the fixed price is p.*

Proof. Construct the variable $Y_n(\alpha, w) = Y_n^\alpha(w)$ exactly as in the proof of Theorem 4.4. Then the Markov chain

$$S_n(\alpha, w) = r(S_{n-1}(\alpha, w) - c(S_{n-1}(\alpha, w), r)) + pY_n(\alpha, w)$$

has the same dynamics for each fixed α and each fixed w.

By Assumption 4.5, μ is a stationary distribution for the chain $S_n(\alpha, \cdot)$ for each α and therefore is also a stationary distribution for the chain $S_n(\cdot, w)$ for each w, if the endogenous interest rates $r_n(w)$ remain equal to the fixed value r.

To check this, assume that $r_0 = r$ and that $\nu_0 = \mu$. Then

$$\begin{aligned} r_1(w) &= \frac{\int i_1^\alpha(w)\phi(d\alpha)}{\int \ell_1^\alpha(w)\phi(d\alpha)} \\ &= \frac{\int_{c(s)>s} i(s)\mu(ds)}{\int_{c(s)\le s} \ell(s)\mu(ds)} \\ &= \frac{r\int_{c(s)>s}(c(s)-s)\mu(ds)}{\int_{c(s)\le s}(s-c(s))\mu(ds)} = r, \end{aligned}$$

where the last line is by (4.3) of Assumption 4.6. Since μ is invariant for the chain, we also have $\nu_1 = \mu$.

By induction, $r_n = r$ and $\nu_n = \mu$ for all n, which verifies (i) of Definition 2.2. Clause (ii) follows from the optimality of π in the one-person game and the fact that a single player cannot affect the interest rate. ■

4.3 An Outside Bank and Countably Many Types of Agents

In this section the notation and much of the argument resembles Section 7.6 of KSS (1994).

Suppose that the space of agents I is measurably partitioned into a finite or countably infinite collection of types $\{I_k\}$. Agents of the same type k are assumed to have the same utility function u_k and income distribution λ_k. Assume also that, for each k, $w_k := \phi(I_k) > 0$. Thus, in particular, there are uncountably many agents of each type.

Fix a price $p \in (0, \infty)$ and interest rates $r_1 = 1 + \rho_1$, $r_2 = 1 + \rho_2$ with $0 < r_1 \le r_2 \le 1/\beta$.

Here are our familiar assumptions slightly reformulated.

Assumption 4.8. Assume that, for each type k of agent, the one-person game with the given parameters has an optimal stationary strategy π_k corresponding to $c_k(s) = c_k(s; p)$ and that the associated Markov chain has a stationary distribution $\mu_k = \mu_k(\cdot, p)$ with finite mean.

The invariance property of μ_k can be expressed symbolically as

$$\mu_k(A, p) = \int_0^\infty \lambda_k\left(\frac{A - g(s - c_k(s, p))}{p}\right) \mu_k(ds, p) \tag{4.5}$$

for all $A \in \mathcal{B}([0,\infty))$. (Cf. $(7.4)'$ of KSS (1994).)

As in KSS (1994), we aggregate these stationary distributions to form

$$\overline{\mu}(A,p) := \Sigma_k w_k \mu_k(A,p) = \int_I \mu^\alpha(A,p)\phi(d\alpha) \tag{4.6}$$

for $A \in \mathcal{B}([0,\infty))$ where we are using the notation $\mu^\alpha = \mu_k$ for all $\alpha \in I_k$ and all k. It is this aggregate measure $\overline{\mu}$ which now plays the role of a stationary wealth distribution.

Assumption 4.9. Assume that the bank balances its books in the sense that

$$\rho_2 \Sigma_k w_k \int_{s>c_k(s,p)} (s - c_k(s,p))\mu_k(ds,p) = \rho_1 \Sigma_k w_k \int_{s\leq c_k(s,p)} (c_k(s,p) - s)\mu_k(ds,p).$$

The left-hand side of this equality represents the total interest paid to depositors of all types and the right-hand side is the total interest paid by borrowers of all types. Observe that the books need not balance for a given type of agent considered in isolation from other types. It could happen that one type of agent is cash-poor, but has sufficient income to finance loans from a second type of cash-rich agents.

Lemma 4.10. *We have*

$$\begin{aligned} p &= \frac{1}{Q}\Sigma_k w_k \int_{I_k} c_k(s,p)\mu_k(ds,p) \\ &= \frac{1}{Q}\int_I c^\alpha(s,p)\overline{\mu}(ds,p), \end{aligned}$$

where $c^\alpha = c_k$ *for* $\alpha \in I_k$.

Proof. The second equality is by definition of $\overline{\mu}$ in (4.6). To prove the first equality, take expectations in equation (2.5) for the dynamics of an agent α. Assume $n = 1$ and that S_0^α has the invariant distribution $\mu^\alpha = \mu^\alpha(\cdot,p)$ and $b_1^\alpha = c^\alpha(S_0^\alpha) = c^\alpha(S_0^\alpha, p)$. This gives

$$\begin{aligned} ES_1^\alpha &= ES_0^\alpha - \int c^\alpha(s)\mu^\alpha(ds) + \rho_1 \int_{s\leq c^\alpha(s)} (s - c^\alpha(s))\mu^\alpha(ds) \\ &\quad + \rho_2 \int_{s>c^\alpha(s)} (s - c^\alpha(s))\mu^\alpha(ds) + pEY^\alpha. \end{aligned}$$

Since $ES_1^\alpha = ES_0^\alpha$, this can be written as

$$\int c^\alpha(s)\mu^\alpha(ds) = -\rho_1 \int_{s\leq c^\alpha(s)} (c^\alpha(s) - s)\mu^\alpha(ds) + \rho_2 \int_{s>c^\alpha(s)} (s - c^\alpha(s))\mu^\alpha(ds) + pEY^\alpha.$$

Now integrate with respect to α, using Assumption 4.9 and remembering that $c^\alpha = c_k$, $\mu^\alpha = \mu_k$ for $\alpha \in I_k$, to get

$$\Sigma_k w_k \int c_k(s)\mu_k(ds) \;=\; p\int_I \int_\Omega Y^\alpha(w)P(dw)\phi(d\alpha)$$

$$= p\int_\Omega \int_I Y^\alpha(w)\phi(d\alpha)P(dw)$$
$$= p\int_\Omega QP(dw) = pQ.$$

The next to last equality is by our standing assumption that the total endowment is the nonrandom quantity Q. ■

Theorem 4.11. *Under Assumptions* 4.8 *and* 4.9*, the family of strategies* $\Pi = \{\pi^\alpha\}$ *with* $\pi^\alpha = \pi_k$ *for all* $\alpha \in I_k$ *and all* k*, results in a stationary Markov equilibrium* $(p, \overline{\mu})$ *for which the fixed interest rates are* r_1*,* r_2*.*

Proof. Use the Feldman–Gilles construction to obtain for each k and all $n = 1, 2, ...$ functions ${}^kY_1^\alpha$, ${}^kY_2^\alpha$, ... to represent the daily endowments for an agent $\alpha \in I_k$.

For each k, introduce the random measures

$${}^k\nu_n(A, w) := \frac{1}{w_k}\phi\{\alpha \in I_k \,:\, S_n^\alpha(w) \in A\}$$

for $A \in \mathcal{B}([0,\infty))$ and $n = 0, 1, ...$ The measure ${}^k\nu_n$ corresponds to the distribution of wealth during period n among agents of type k. The measure ν_n representing the distribution of wealth among all agents can be written

$$\nu_n(A, w) = \Sigma_k \, {}^k\nu_n(A, w)w_k.$$

To verify clause (i) of Definition (2.1), assume that $p_0(w) = p$ and ${}^k\nu_0(\cdot, w) = \mu_k(\cdot, p)$. (So, in particular, $\nu_0(\cdot, w) = \overline{\mu}(\cdot, p)$.) The proof that $p_1(w) = p$ and ${}^k\nu_1(\cdot, w) = \mu_k(\cdot, p)$ (and so $\nu_1(\cdot, w) = \overline{\mu}(\cdot, p)$) is almost exactly the same as the proof of Theorem 7.7 in KSS (1994). Just use Lemma 4.10 for the calculation (7.8)′ in KSS (1994), and use (4.5) to replace (7.4′) in the final calculation.

The proof of clause (ii) of Definition (2.1) appeals as usual to the optimality of each π_k in the corresponding one-person game and to the fact that no single agent can affect the price. ■

4.4 A Money Market and Countably Many Types of Agents

As in the previous section the space of agents I is partitioned into a finite or countable collection of types $\{I_k\}$. The same notation u_k, λ_k, and $w_k = \phi(I_k)$ is used for the utility function, income distribution, and proportion of agents of type k, respectively.

As in Section 4.2 we fix a price $p = W/Q$ and a single interest rate $r = 1 + \rho > 0$.

Here is the last variation on our two basic assumptions.

Assumption 4.12. Assume that, for each type k of agent, the one-person game with the given parameter has an optimal stationary strategy π_k corresponding to $c_k(s) = c_k(s, r)$ and that the associated Markov chain has a stationary distribution $\mu_k = \mu_k(\cdot, r)$ with finite mean.

Recall from Section 3 that we can express the IOU bid $i^{(k)}(s)$ and loan offer $\ell^{(k)}(s)$ for an agent $\alpha \in I_k$ who plays π_k by

$$\begin{aligned} i^\alpha(s) = i^{(k)}(s) &= \begin{cases} r(c_k(s) - s) & \text{, if } c_k(s) > s \\ 0 & \text{, if not,} \end{cases} \\ \ell^\alpha(s) = \ell^{(k)}(s) &= \begin{cases} s - c_k(s) & \text{, if } c_k(s) < s \\ 0 & \text{, if not.} \end{cases} \end{aligned}$$

Just as in (4.5) the invariance property of μ_k can be written as

$$\mu_k(A, r) = \int_0^\infty \lambda_k\Big(\frac{A - g(s - c_k(s,r))}{p}\Big)\,\mu_k(ds, r) \tag{4.7}$$

for all $A \in \mathcal{B}([0,\infty))$. As in the previous section we aggregate the stationary distributions of the various types to form

$$\begin{aligned} \overline{\mu}(A, r) &:= \sum_k w_k \mu_k(A, r) \\ &= \int_I \mu^\alpha(A, r)\phi(d\alpha) \end{aligned} \tag{4.8}$$

for all $A \in \mathcal{B}([0,\infty))$ where we set $\mu^\alpha = \mu_k$ for $\alpha \in I_k$.

Assumption 4.13. Under this aggregate wealth distribution we assume that the total amount lent equals the total offered for lending and that both quantities are positive; i.e.

$$\sum_k w_k \int_{s \le c_k(s)} (c_k(s) - s)\mu_k(ds) = \sum_k w_k \int_{s > c_k(s)} (s - c_k(s))\mu_k(ds) \neq 0. \tag{4.9}$$

Theorem 4.14. *Under assumptions* (4.12) *and* (4.13), *the family of strategies* $\Pi = \{\pi^\alpha\}$ *with* $\pi^\alpha = \pi_k$ *for* $\alpha \in I_k$, *results in a stationary competitive equilibrium* $(r, \overline{\mu})$ *for which the fixed price is* p.

Proof. Construct the variables ${}^kY_n^\alpha(w)$ and define the measure ${}^k\nu_n$ exactly as in the proof of Theorem 4.11. Define $r_0(w) = r$ and ${}^k\nu_0(\cdot, w) = \mu_k(\cdot, r)$ for all k so that in particular $\nu_0(\cdot, w) = \overline{\mu}(\cdot, r)$. To see that $r_1(w) = r$, calculate thus:

$$\begin{aligned} r_1(w) &= \frac{\int i_1^\alpha(w)\phi(d\alpha}{\int \ell_1^\alpha(w)\phi(d\alpha)} \\ &= \frac{\int i^\alpha(S_0^\alpha(w))\phi(d\alpha)}{\int \ell^\alpha(S_0^\alpha(w))\phi(d\alpha)} \\ &= \frac{\sum_k w_k \int i^{(k)}(s)\mu_k(ds)}{\sum_k w_k \int \ell^{(k)}(s)\mu_k(ds)} \\ &= \frac{r\sum_r w_k \int_{s \le c_k(s)} (c_k(s) - s)\mu_k(ds)}{\sum_k w_k \int_{s > c_k(s)} (s - c_k(s))\mu_k(ds)} \\ &= r, \end{aligned}$$

where the final equality is by Assumption 4.13. The proof that ${}^{k}\nu_1(\cdot, w) = \mu_k(\cdot, r)$ for all k (and so $\nu_1(\cdot, w) = \overline{\mu}(\cdot, r)$) uses (4.7) and then follows the proof of Theorem 7.7 in KSS (1994).

Clause (i) of Definition 4.1 now follows from induction. The proof of clause (ii) is by now familiar. ■

Remark 4.14. The results of this and the previous section can be extended to the case of uncountably many agents, as in Remark 7.8 of KSS (1994).

4.5 The Substitution of a Bank for a Money Market and Vice-Versa

Suppose the game with a money market (as in Section 4.2 or 4.4) satisfies the basic assumptions and therefore has a stationary Markov equilibrium (r, μ) for a given price p. Then the money market can be replaced by a bank in the sense that the corresponding game with an outside bank has the equilibrium (p, μ) for the fixed rates $r_1 = r_2 = r$. This is because the basic assumptions for an outside bank are also satisfied.

Conversely, if the game with an outside bank satisfies the basic assumptions leading to a stationary Markov equilibrium (p, μ) for which the fixed rates are $r_1 = r_2 = r \neq 1$, then the bank can be replaced by a money market. However, if $r_1 = r_2 = r = 1$, it may not be possible to replace the bank by a money market. In the first example of the next section, a situation arises in which every agent is a borrower. This causes no trouble for a bank if all the agents pay back the amounts they borrow, but a money market is impossible since there are no lenders. The second example also shows a bank may work when a money market fails.

5 Some Examples

A method for constructing a stationary Markov equilibrium was presented in the previous section. Now we will apply the method in some simple examples. To do so, we have, in each example, to find the optimal plan π for a class of one-person dynamic programming problems and then check that π is optimal by showing that its return function $I(\pi)$ satisfies the Bellman equation: $T(I(\pi)) = I(\pi)$ as in Theorem 3.1. In the examples of this section, we will omit the straightforward, tedious verifications of the Bellman equation. The details are given for one of the examples in an appendix.

The examples have been selected to illustrate various possibilities. In Example 5.1 there is a trivial equilibrium with an outside bank, but no stationary money market equilibrium except for the somewhat unstable situation where $r = 1/\beta$ and agents are indifferent between borrowing and lending. Example 5.2 provides another situation in which it is easy to find a stationary equilibrium with an outside bank but delicate to do so with a money market. The difficulty with the money market is largely due to the fact that in Example 5.2 the utility function saturates at a finite level. In Example 5.3 we consider a nonsaturating utility and find a money market solution. Example 5.4 has an outside bank solution where the bank charges a higher interest rate to borrowers than it pays to depositors. In Example 5.5, we construct a stationary equilibrium for a game with two types of agents. In all of the other examples, agents are assumed to be homogeneous. The last example treats a nonstochastic situation where an analytic solution turns out to be quite easy.

Example 5.1. *A Linear Utility Function.* Suppose that for all $\alpha \in I$ and $x \geq 0$, $u^\alpha(x) = u(x) = x$. Assume also that the income variables Y_n^α have the same distribution λ and write Y for a generic variable with this distribution.

Consider first the one-person game with fixed parameters p and $r_1 = r_2 = r = 1+\rho$ where $p > 0$, $0 < r < 1/\beta$ and $pY \geq rk$. The unique optimal plan π is to borrow up to the limit k and spend everything. Thus π corresponds to the consumption function $c(s) = s+k$ for all s. The Markov chain of (3.11) satisfies

$$\begin{aligned} S_{n+1} &= r(S_n - (S_n + k)) + pY_{n+1} \\ &= -rk + pY_{n+1}\ , \quad n \geq 0. \end{aligned}$$

Obviously, the stationary distribution μ of this chain is the distribution of $-rk+pY$. Let $Q = I(\pi)$ be the return function for π. Then

$$\begin{aligned} Q(s) &= E^\pi_{s_0=s}\left[\sum_{n=0}^{\infty} \beta^n u(c(S_n))\right] \\ &= s + k + \sum_{n=1}^{\infty} \beta^n(-rk + pEY_{n+1} + k) \\ &= s + k + \frac{\beta}{1-\beta}(pEY - \rho k). \end{aligned}$$

It is easy to see that Q satisfies the Bellman equation

$$Q(s) = \max_{0 \leq b \leq s+k} [b + \beta EQ(r(s-b) + pY)],$$

which establishes that π is optimal (recall Theorem 3.1).

Turn now to the many-person game with an outside bank. If $r \neq 1$, then there is no stationary Markov equilibrium. The bank cannot balance its books because every agent will borrow k in each period and pay back rk resulting in a decrease, if $r > 1$, or an increase, if $r < 1$, of the total wealth held by the agents. However, there is an equilibrium with $r = 1$. In this case, every agent borrows k and pays back k so that the books balance. (Assumption 4.2 holds trivially since $\rho_1 = \rho_2 = 0$.)

In the money market game, there is no stationary Markov equilibrium with $r < 1/\beta$ because no funds are offered for lending. When $r = 1/\beta$, it is possible to construct an equilibrium based on the fact that agents are indifferent between lending and borrowing for consumption.

Example 5.2. *A Piecewise-Linear Utility Function with Saturation.* Assume that each agent $\alpha \in I$ has the utility function

$$u^\alpha(x) = u(x) = \begin{cases} x & , \quad 0 \leq x \leq 1, \\ 1 & , \quad x > 1. \end{cases}$$

Assume also that the income variables, represented by Y, have the distribution

$$P[Y = 1/2] = 1 - \gamma, \quad P[Y = 3/2] = \gamma$$

with $0 < \gamma < 1$, so that $EY = \frac{2\gamma+1}{2}$.

Consider the one-person game with parameters $p = 1$, $r_1 = r_2 = r = 1$, and $k = 1/2$. Observe that $pY = Y \geq 1/2 = rk$. The optimal plan π corresponds to the consumption function

$$c(s) = \begin{cases} s + 1/2 & , \quad 0 \leq s \leq 1/2, \\ 1 & , \quad s > 1/2. \end{cases}$$

Thus an agent, with cash $s < 1$, borrows up to $1/2$ or just enough to reach 1; an agent, with cash $s \geq 1$, lends (or deposits) the excess $s-1$. (The proof of the corresponding result for the model without a loan market is given in great generality in the appendix of KSS (1994).) The Markov chain of (3.11) takes the form

$$S_{n+1} = \begin{cases} Y_{n+1} - 1/2 & , \quad 0 \leq S_n \leq 1/2, \\ S_n - 1 + Y_{n+1} & , \quad S_n > 1/2. \end{cases}$$

For $S_n > 1/2$, the chain behaves like a random walk with mean drift

$$-1 + EY = \gamma - 1/2.$$

Thus the chain will not have a stationary distribution for $\gamma \geq 1/2$. Assume now that $0 < \gamma < 1/2$. Then the chain has a unique stationary distribution μ concentrated on the set $\{n/2 : n = 0, 1, ...\}$ as follows:

$$\mu(0) = \delta(1 - \gamma), \quad \mu(1/2) = \delta\gamma,$$
$$\mu(n/2) = \delta \left(\frac{\gamma}{1 - \gamma} \right)^{n-1} \quad \text{for} \quad n \geq 2,$$

where $\delta := (1 - 2\gamma)/(1 - \gamma)$. Clearly, μ has a finite mean; in fact, $\int s\mu(ds) = \gamma \frac{3-4\gamma}{2(1-2\gamma}$ and $\int c(s)\mu(ds) = \gamma + \frac{1}{2} = EY$, in accordance with Lemma 4.3.

It follows from Theorem 4.4 that the price $p = 1$ and wealth distribution μ form a stationary Markov equilibrium for the many-person game with an outside bank. We have already established Assumption 4.1, and Assumption 4.2 is trivial since $\rho_1 = \rho_2 = 0$.

We would like to apply Theorem 4.7 to see that the rate $r = 1$ and distribution μ form a stationary Markov equilibrium for the money market game. Assumption 4.5 is satisfied, but Assumption 4.6, which says that the amount offered for lending must equal that lent, here becomes the equation

$$\sum_{n=3}^{\infty} (n/2 - 1)\mu(n/2) = \tfrac{1}{2}(\mu(0) + \mu(1/2)).$$

Straightforward algebra shows that the only value of γ for which the equation holds is $\gamma = \frac{1}{3}$.

Example 5.3. *A Money Market Equilibrium for a Piecewise-Linear Utility Function without Saturation.* Assume that each agent α has utility function

$$u^\alpha(x) = u(x) = \begin{cases} x & , \quad 0 \le x \le 2\frac{1}{2}, \\ 2\frac{1}{2} + \eta(x - 2\frac{1}{2}) & , \quad x > 2\frac{1}{2}, \end{cases}$$

where $0 < \eta < 1$. Suppose that the income variables, represented by Y, satisfy

$$P[Y = 1] = 1/2 = P[Y = 4],$$

and that we are in the one-person game with parameters $p = 1$, $r_1 = r_2 = r = 2$, and $k = 1/2$. Then $pY = Y \ge 1 = rk$. Let π be the stationary plan that corresponds to the consumption function

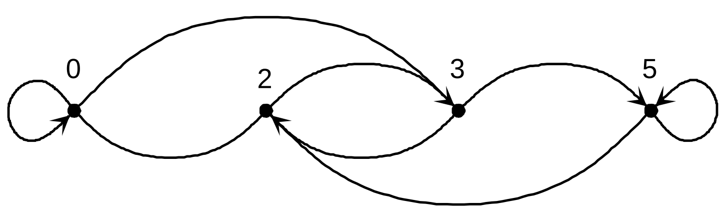
$$c(s) = \begin{cases} s + \frac{1}{2} & , \quad 0 \le s \le 2 \\ 2\frac{1}{2} & , \quad 2 \le s \le 3 \\ s - \frac{1}{2} & , \quad s \ge 3. \end{cases} \tag{5.1}$$

Notice that under this plan an agent with less than $2\frac{1}{2}$ units of fiat money borrows the maximum or just enough to reach $2\frac{1}{2}$, while an agent with more than $2\frac{1}{2}$ units will lend (or deposit) the excess up to a maximum of $\frac{1}{2}$. It is shown in an appendix that π is optimal when

$$1 \ge \beta(1 + \eta) \ge \eta \ge \beta^2(1 + \eta) + \beta\eta.$$

For example, π is optimal if $\beta = 1/4$ and $\eta = 1/3$.

An agent who plays π will reach a wealth $s \in \{0, 2, 3, 5\}$ after at most 2 steps and then follow the finite chain below.



All transitions are with probability 1/2 and the unique stationary distribution is the uniform distribution on $\{0, 2, 3, 5\}$. Thus one-half of the agents (those at 0 and 2) are borrowing 1/2 and paying back 1; the other half are lending (or depositing) 1/2 and getting back 1. The books obviously balance, so there is an equilibrium with a money market or with an outside bank by Theorems 4.7 and 4.4, respectively. Note that $EY = \int c(s)\mu(ds) = \int s\mu(ds) = 2\frac{1}{2}$, which is consistent with $p = 1$ in Lemma 4.3.

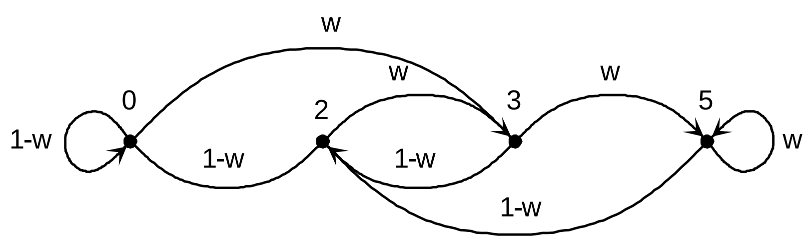
Example 5.4. *An Outside Bank Which Sets Two Different Interest Rates.* Assume that each agent has the same utility function u as in Example 5.3. Suppose the income variables have distribution given by

$$P[Y = 1] = 1 - w \ , \quad P[Y = 4] = w \ ,$$

for some $0 < w < 1$. Consider the one-person game with parameters fixed at $p = 1$, $r_1 = 3$, $r_2 = 2$, and $k = 1/3$. Then $pY \geq 1 = r_1 k$. Let π be the stationary plan corresponding to

$$c(s) = \begin{cases} s + \frac{1}{3} & , \quad 0 \leq s \leq 2\frac{1}{6}, \\ 2\frac{1}{2} & , \quad 2\frac{1}{6} \leq s \leq 3, \\ s - 2\frac{1}{2} & , \quad s \geq 3. \end{cases}$$

As in Example 5.3, it is easy to see that an agent, who follows this plan, will after no more than two days have a fortune $s \in \{0, 2, 3, 5\}$ and then follow the chain below.



The stationary distribution of this chain is the measure

$$\mu(0) = (1-w)^2, \quad \mu(2) = \mu(3) = w(1-w), \quad \mu(5) = w^2.$$

Agents with wealth 0 or 2 will borrow 1/3 and pay back $r_1 \times \frac{1}{3} = 1$ to the bank for an aggregated net gain to the bank of

$$\tfrac{2}{3}(\mu(0) + \mu(2)) = \tfrac{2}{3}(1-w).$$

Agents with wealth 3 or 5 will deposit 1/2 and get back $r_2 \times \frac{1}{2} = 1$. The aggregated net loss to the bank is

$$\tfrac{1}{2}(\mu(3) + \mu(5)) = \tfrac{1}{2}w.$$

The bank will balance its books if and only if

$$\tfrac{2}{3}(1-w) = \tfrac{1}{2}w;$$

that is,

$$w = \tfrac{4}{7}.$$

Fix $w = 4/7$. It can then be shown that the plan π is optimal for the one-person game when the parameters β and η satisfy the conditions: $0 < \beta \leq 1/3$, $7\eta = 6\beta + 8\beta\eta$. (The proof consists of checking the Bellman equation and is similar to that given in the appendix for Example 5.3.) Thus, for these values of the parameters, we have a stationary Markov equilibrium for the game with an outside bank by Theorem 4.4. There can be no money market equilibrium with $r_1 \neq r_2$.

Example 5.5. *A Money Market with Two Types of Agents.* Suppose that the space of agents I is partitioned into two sets I_1 and I_2. For agents $\alpha \in I_1$, assume that

$$u^\alpha(x) = u_1(x) = \begin{cases} x & , \quad 0 \leq x \leq 2\frac{1}{2}, \\ 2\frac{1}{2} + \eta(x - 2\frac{1}{2}) & , \quad x > 2\frac{1}{2} \end{cases}$$

and that the income variables Y_n^α are distributed like the generic variable Y_1 where

$$P[Y_1 = 1] = \tfrac{1}{4}, \quad P[Y_1 = 4] = \tfrac{3}{4}.$$

These agents of type 1 are much like the agents in the two preceding examples. Assume that $\phi(I_1) = 2/3$; i.e., two-thirds of all agents are of type 1.

For agents $\alpha \in I_2$, assume that $u^\alpha(x) = u_2(x) = x$ for all x as in Example 5.1 and that income variables Y_n^α are like Y_2 where

$$P[Y_2 = 1] = \tfrac{1}{2} = P[Y_2 = 3].$$

Note that $\phi(I_2) = 1 - \phi(I_1) = \frac{1}{3}$.

Consider now the one-person game with $r = 2$ and $p = 1$ first for a player of type 2 with loan limit $k_2 = 1/2$. Assume that $\beta < 1/2$ so that $\beta r < 1$. Then the optimal plan π_2, as explained in Example 5.1, is to borrow up to the limit and spend everything. This results in a Markov chain with stationary distribution μ_2 equal to the distribution of $-rk_2 + pY = -1 + Y$. So

$$\mu_2(0) = \mu_2(2) = \tfrac{1}{2}.$$

Notice that all players of type 2 are borrowers so that no money market would be possible with them alone.

Consider next the one-period game for a player of type 1 with $r = 2$, $p = 1$ as before with loan limit $k_1 = 1/2$. It can be shown that the stationary plan π_1 with consumption function c as in (5.1) is optimal for β and η satisfying

$$4 \geq 2\beta(1 + 3\eta) \geq 4\eta \geq \beta^2(1 + 3\eta) + 6\beta\eta.$$

(The proof is similar to that presented for Example 5.3 in the appendix.) We assume in what follows that these inequalities are satisfied. The Markov chain corresponding to this strategy eventually reaches the set $\{0, 2, 3, 5\}$ where it has transitions like those of the chain in Example 5.4 with $w = 3/4$. The stationary distribution μ_1 is given by

$$\mu_1(0) = 1/16, \quad \mu_1(2) = \mu_1(3) = 3/16, \quad \mu_1(5) = 9/16.$$

Let $\overline{\mu} = \frac{2}{3}\mu_1 + \frac{1}{3}\mu_2$ be the aggregated measure as in (4.3).

To check that the books balance in the many-person game, observe first that agents with wealth 0 or 2, whether of type 1 or type 2, borrow 1/2. The total amount borrowed is thus

$$\tfrac{1}{2}(\overline{\mu}(0) + \overline{\mu}(2)) = \tfrac{1}{4}.$$

Agents with wealth 3 or 5 are necessarily of type 1 and lend 1/2. The total amount lent is

$$\tfrac{1}{2}(\overline{\mu}(3) + \overline{\mu}(5)) = \tfrac{1}{4}.$$

This verifies Assumption 4.13. It now follows from Theorem 4.14 that the family of strategies $\Pi = \{\pi^\alpha\}_{\alpha \in I}$, where $\pi^\alpha = \pi_1$ for $\alpha \in I_1$ and $\pi^\alpha = \pi_2$ for $\alpha \in I_2$, results in a stationary Markov equilibrium for the money market game.

Example 5.6. *A Nonstochastic Money Market.* Assume that every agent α has the same utility function u which is smooth, increasing and strictly concave. Assume that all of the income variables Y_α^n are equal to a positive constant y.

Consider the one-person game with $r = 1/\beta$, $p = 1$, and $k = y$. The unique optimal plan for this game is the stationary plan π corresponding to the consumption function

$$c(s) = (1-\beta)s + \beta y.$$

To see that this is so, observe first that

$$\begin{aligned} s &= \beta^{-1}(s - c(s)) + y \\ &= r(s - c(s)) + y \end{aligned}$$

and consequently the Markov chain resulting from π is the constant sequence $S_n = s$ for all s. Thus the reward function is

$$\begin{aligned} Q(s) &= u(c(s)) + \beta u(c(s)) + \beta^2 u(c(s)) + \cdots \\ &= \frac{u(c(s))}{1-\beta} = \frac{u((1-\beta)s + \beta y)}{1-\beta}. \end{aligned}$$

It is easy to see that Q satisfies the Bellman equation

$$Q(s) = \max_{0 \le a \le s+y} [u(a) + \beta Q(r(s-a)+y)]$$

and so π is optimal by Theorem 3.1. The strict convexity of u implies that π is the unique optimal plan.

Now consider the many-person game with a money market. Suppose every agent plays π and that the initial distribution of wealth is a nondegenerate measure μ on $\mathcal{B}([0,\infty))$ which has mean y. Any such μ is invariant for the Markov chain $\{S_n\}$ determined by c because

$$S_{n+1} = \beta^{-1}(S_n - c(S_n)) + y = S_n \ .$$

To see that the books balance, notice that $c(s) > s$ (respectively, $c(s) < s$) if and only if $s < y$ ($s > y$). Thus, the balance equation (4.3) of Assumption 4.6 can be written as

$$\int_0^y (c(s) - s)\mu(ds) = \int_y^\infty (s - c(s))\mu(ds). \tag{5.2}$$

This holds because, by assumption,

$$\int_0^\infty s\mu(ds) = y$$

and, by definition of c,

$$\int_0^\infty c(s)\mu(ds) = \int_0^\infty \{(1-\beta)s + \beta y\}\mu(ds) = (1-\beta)y + \beta y = y.$$

The assumption that μ does not degenerate to a point mass guarantees that the quantities in (5.2) are not zero. Theorem 4.7 now applies to show that $r = 1/\beta$ and μ form a stationary Markov equilibrium for every nondegenerate μ with mean y.

6 The One-Person Game in More Detail

We now return to the one-person game for a more detailed study of the value function, the structure of optimal stationary strategies, and properties of the corresponding Markov chains. Some of these results are of interest for their own sake. However, our main concerns are to give sufficient conditions for the first basic assumption used in Section 4 for the construction of a stationary Markov equilibrium and to lay the foundations for the existence proof to be presented in Section 7.

In order to obtain sharper results and to simplify our analysis, we will assume in this section:

(A1) The utility function u is strictly concave, strictly increasing, and twice continuously differentiable. Also, $0 < r_2 \leq r_1 < 1/\beta$, $P[Y \geq kr_1] = 1$.

All of the assumptions of Section 3 also remain in force. Here is our main result for the one-person game (cf. Theorem 4.1 of KSS (1994)).

Theorem 6.1.

(a) *The value function V is concave, strictly increasing, and continuously differentiable.*

(b) *There is a unique optimal stationary plan π corresponding to a continuous consumption function $c : [0, \infty) \to [0, \infty)$ such that $0 \leq c(s) \leq s+k$. Furthermore, the functions $c(s)$ and $s - c(s)$ are nondecreasing.*

(c) $V'(s) = u'(c(s))$ for $s \geq 0$.

(d) *There exist s^*, t^* with $0 \leq s^* \leq t^* \leq \infty$ such that $c(s) > s$ for $0 \leq s < s^*$, $c(s) = s$ for $s^* \leq s \leq t^*$, and $c(s) < s$ for $s > t^*$. Furthermore, $s^* < t^*$ if and only if $r_1 > r_2$. Indeed*

$$\begin{aligned} s^* &= I(\beta r_1 E V'(Y)) \\ t^* &= I(\beta r_2 E V'(Y)) \end{aligned}$$

where I is the inverse function for u'. ■

As we have before, we continue to simplify notation by writing, for example, $u'(0)$ for the right derivative $u'_+(0)$ of u at 0.

It seems likely that part (a) of the theorem could be strengthened to say that V is strictly concave, but our proof does not show this.

Our proof of Theorem 6.1, like that of Theorem 4.1 in KSS (1994), will rely on a careful study of a basic recursion which uses the operator T of (3.10).

6.1 The Basic Recursion

Define

$$v(s) = v_w(s) = Tw(s) = \sup_{0 \le b \le s+k} [u(b/p) + \beta E w(g(s-b) + pY)] \tag{6.1}$$

where $w : [0, \infty) \to [0, \infty)$ satisfies the following assumptions:

(A2) w is nondecreasing, concave, and has a continuous derivative on $[0, \infty)$.

(A3) $w'(0) < u'(0)$.

In this section, we will show that the properties assumed for w also hold for Tw. Moreover, it is a standard result of dynamic programming that the value function V is the limit of $T^n u$. So we will be able to deduce properties of V in the next section.

Proposition 6.2.

(a) *The function* $v = Tw$ *has all the properties assumed for* w *in* (A2) *and* (A3).

(b) *For each* $s \ge 0$*, there is a unique action* $c_w(s) \in [0, s+k]$ *that achieves the supremum in* (6.1)*. In particular,*

$$(Tw)(s) = u(c_w(s)/p) + \beta E w(g(s - c_w(s)) + pY).$$

(c) *The functions* $c_w(s)$ *and* $s - c_w(s)$ *are nondecreasing. Hence,* c_w *is continuous.*

(d) $(Tw)'(s) = u'(c_w(s))$.

(e) *There exist* s_w*,* t_w *with* $0 < s_w \le t_w \le +\infty$ *such that*

$$\begin{aligned} c_w(s) &> s \text{ for } 0 \le s < s_w, \\ &= s \text{ for } s_w \le s \le t_w, \\ &< s \text{ for } t_w < s. \end{aligned}$$

Indeed, $s_w = I(\beta r_1 E w'(Y))$ *and* $t_w = I(\beta r_2 E w'(Y))$ *where* I *is the inverse of* u'*.*

Proof. For a slight simplification of notation, we assume without loss of generality that $p = 1$.

To prove (b), we introduce the function

$$\psi_{s,w}(b) = \psi_s(b) = u(b) + \beta E w(g(s-b) + Y)$$

for $0 \le b \le s+k$. It follows from the concavity of g and w and the strict concavity of u that ψ_s is also strictly concave. This is enough to establish (b).

To verify (e), notice that

$$\psi'_s(b) = \left\{ \begin{array}{lll} u'(b) - \beta r_2 \cdot E w'(r_2(s-b) + Y) & ; & 0 \le b < s \\ u'(b) - \beta r_1 \cdot E w'(r_1(s-b) + Y) & ; & s < b \le s+k \end{array} \right\}.$$

Therefore,

$$c_w(s) > s \text{ iff } \psi'_s(s+) > 0 \text{ iff } u'(s) > \beta r_1 E w'(Y) \text{ iff } s < s_w,$$

and similarly

$$c_w(s) < s \text{ iff } \psi'_s(s-) < 0 \text{ iff } u'(s) < \beta r_2 E w'(Y) \text{ iff } s > t_w.$$

Also $c_w(0) > 0$ and so $s_w > 0$ because $\psi'_0(0+) = u'(0) - \beta r_1 E w'(Y) > u'(0) - w'(0) > 0$.

We will first prove (c) and (d) for functions w that are C^2; i.e. functions with two continuous derivatives. To simplify notation, we temporarily write c for c_w. Let us consider four cases.

Case 1. $0 \le s < (q-k)^+$ where $q = I(\beta r_1 E w'(Y - r_1 k))$.
For s in this interval, $\psi'_s((s+k)^-) > 0$ and so $c(s) = s + k$. Obviously (c) holds.

Case 2. $(q-k)^+ \le s \le s_w$. Here $c(s)$ satisfies $\psi'_s(c(s)) = 0$ or $u'(c(s)) = \beta r_1 E w'(r_1(s - c(s)) + Y)$. By the Implicit Function Theorem, c is C^1 and we can differentiate to get

$$c'(s) u''(c(s)) = \beta r_1^2 (1 - c'(s)) E w''(r_1(s - c(s)) + Y).$$

Hence,

$$0 \le c'(s) = \frac{\beta r_1^2 E w''(r_1(s - c(s)) + Y)}{u''(c(s)) + \beta r_1^2 E w''(r_1(s - c(s)) + Y)} \le 1.$$

In particular, both $c(s)$ and $s - c(s)$ are nondecreasing; and we have

$$(Tw)(s) = u(c(s)) + \beta E w(r_1(s - c(s)) + Y).$$

Case 3. $s_w \le s \le t_w$. Here $c(s) = s$ and (c) is obvious; we have $(Tw)(s) = u(s) + \beta E w(Y)$.

Case 4. $t_w \le s$. An argument similar to that for Case 2 again proves (c).

Assertion (d) is obvious in Cases 1 and 3. For Cases 2 and 4, differentiate the equation in (b) to get, for example in Case 2;

$$\begin{aligned}(Tw)'(s) &= c'(s) u'(c(s)) + \beta r_1 (1 - c'(s)) E w'(r_1(s - c(s)) + Y) \\ &= \beta r_1 E w'(r_1(s - c(s)) + Y) = u'(c(s)).\end{aligned}$$

There is no trouble at the endpoints of the various intervals because, as is easily checked, the right and left derivatives always agree. The proof of (c) and (d) is now complete for C^2 functions w.

Consider now a $w(\cdot)$ satisfying (A2) and (A3) that is, perhaps, only C^1. Then there exist C^2 functions $w_n(\cdot)$, $n \in \mathbb{N}$, satisfying the same assumptions and such that $w_n(\cdot)$ converges up to $w(\cdot)$ and $w_n'(\cdot)$ converges down to $w'(\cdot)$ on $[0,\infty)$. (For example, extend $w(\cdot)$ to be C^1 and satisfy (A2) on the whole real line and take $w_n(s) = Ew(s - Z_n)$, where the Z_n are positive random variables that have smooth densities and converge down to zero almost surely.)

Write $c_n(\cdot)$, for $c_{w_n}(\cdot)$, and $c(\cdot)$ for $c_w(\cdot)$. Then $w_n'(\cdot) \geq w_{n+1}'(\cdot) \geq w'(\cdot)$ so that $\psi'_{s,w_n}(\cdot) \leq \psi'_{s,w_{n+1}}(\cdot) \leq \psi'_{s,w}(\cdot)$ and $c_n(\cdot) \leq c_{n+1}(\cdot)$. Define

$$\widetilde{c}(s) := \lim_n \uparrow c_n(s), \quad s \in \mathcal{S}.$$

Now $c_n(s)$ and $s - c_n(s)$ are nondecreasing in s for every n by (c) applied to the C^2 function w_n. Hence, $\widetilde{c}(s)$ and $s - \widetilde{c}(s)$ are nondecreasing also, and so $\widetilde{c}(s)$ is continuous.

Lemma 6.3. *For every s,*

(i) $(Tw_n)(s) \uparrow (Tw)(s)$,

(ii) $c_n(s) \uparrow c(s)$ *(i.e. $\widetilde{c}(s) = c_w(s)$).*

Proof. (i) $(Tw)(s) \geq (Tw_n)(s) = u(c_n(s)) + \beta E w_n(g(s - c_n(s)) + Y) \uparrow u(c(s)) + \beta E w_n(g(s - c(s)) + Y) = (Tw)(s)$. The second inequality holds because c_n achieves the supremum in the definition of Tw_n; the two equalities are instances of (b).

(ii) By (i), we have $(Tw)(s) = \lim_n[u(c_n(s)) + \beta E w_n(g(s - c_n(s)) + Y] = u(\widetilde{c}(s)) + \beta E w(g(s - \widetilde{c}(s)) + Y)$. Thus $\widetilde{c}(\cdot) = c_w(\cdot)$, by (b). ■

Now we can complete the proofs of (c) and (d) in Proposition 6.2. Property (c) is immediate from part (ii) of the lemma. For (d), use part (i) and (d) applied to the w_n to get

$$\begin{aligned}(Tw)(s) - (Tw)(0) &= \lim_n [(Tw_n)(s) - (Tw_n)(0)] \\ &= \lim_n \int_0^s u'(c_n(t))dt = \int_0^s u'(c(t))dt.\end{aligned}$$

Differentiate to get (d).

It only remains for us to check property (a). That Tw is nondecreasing and C^1 follows from (c) and (d). To see that Tw is concave, observe from (c), (d) that $(Tw)'(s) = u'(c_w(s))$ is decreasing. To see that $(Tw)'(0) < u'(0)$, use (d) and the fact from (e) that $c_w(0) > 0$.

The proof of Proposition 6.2 is now complete. ■

Remark 6.4. Suppose w is nondecreasing and concave, but not necessarily C^1. The function

$$\psi_s(b) = u(b) + \beta E w(g(s - b) + Y), \; 0 \leq b \leq s + k$$

is the sum of a strictly concave and a concave function. So $\psi_s(\cdot)$ is strictly concave and must therefore have its maximum at a unique point $c_w(s)$ in $[0,\, s + k]$ which satisfies the equation of Proposition 6.2(b).

6.2 Proof of Theorem 6.1

Let

$$V_0 := u, \quad V_{n+1} := TV_n$$

for $n = 0, 1, ...$ Then V_n is, for each n, the optimal reward function for the n–day dynamic programming problem. It is a standard (and straightforward) result that V_n converges up to V as $n \to \infty$.

By Proposition 6.2 on the basic recursion, every V_n satisfies assumptions (A2) and (A3). Thus V is clearly nondecreasing and concave. A simple direct argument shows that V_n and V are, in fact, strictly increasing.

By (3.9) and Remark 6.3 there is, for each s, a unique $c(s) \in [0, s+k]$ such that

$$V(s) = (TV)(s) = u(c(s)) + \beta EV(g(s - c(s)) + Y).$$

We are continuing to assume that $p = 1$.

By Theorem 3.1, the stationary plan π corresponding to c is the unique optimal plan.

Set $c_n = c_{V_n}$ in the notation of Proposition 6.2. Then, for $n \geq 1$,

$$V_n(s) = (TV_{n-1})(s) = u(c_n(s)) + \beta EV_{n-1}(g(s - c_n(s)) + Y).$$

It follows from Schäl (1975) that

$$c(s) = \lim_n c_n(s).$$

By Proposition 6.2, $c_n(s)$ and $s - c_n(s)$ are nondecreasing for each n.

Hence, the same holds for $c(s)$ and $s - c(s)$. In particular, c is continuous.

Now apply Proposition 6.2(d) to get $V(s) - V(0) = \lim_n[V_n(0) - V_n(0)] = \lim_n \int_0^s u'(c_n(x))dx = \int_0^s u'(c(x))dx$. Hence,

$$V'(s) = u'(c(s)) \ .$$

We have verified conditions (a), (b), and (c) of Theorem 6.1 and also that V satisfies assumption (A2). To verify condition (d), consider the function $\psi_s(b) = u(b) + \beta EV(g(s-b) + Y)$. Notice that

$$\begin{aligned} \psi'_s(0) &= u'(0) - \beta r_1 EV'(Y) \\ &\geq u'(0) - \beta r_1 V'(0) = u'(0) - \beta r_1 u'(c(0)) \\ &\geq (1 - \beta r_1)u'(0) > 0. \end{aligned}$$

Hence, $c(0) > 0$ and $V'(0) = u'(c(0)) < u'(0)$. So V' satisfies (A3) as well as (A2). Now apply Proposition 6.2(e) with $w = V$ to get condition (d) of Theorem 6.1. This completes the proof.

Here is an additional property of the optimal consumption function c. The proof is essentially the same as that of Theorem 4.3 in KSS (1994).

Theorem 6.5. *Under the hypotheses of Theorem* 6.1*, we have* $\lim_{s\to\infty} c(s) = \infty$.

6.3 The Solution as a Function of the Parameters

Let $\theta = (r_1, r_2, p)$ be the vector composed of the two interest rates and the price of the commodity. Also, write $V_\theta(s)$ and $c_\theta(s)$ for the value function and optimal consumption function for the one-person game with parameters corresponding to θ. In this section we will use results of Langen (1981) to see that both $V_\theta(s)$ and $c_\theta(s)$ are continuous functions of θ. (We already know from Theorem 6.1 that these functions are continuous in s for fixed θ.)

We will assume that θ varies over the set

$$D = \{(r_1, r_2, p) : 0 < r_2 \leq r_1 < 1/\beta, \, 0 < p < \infty\}.$$

The discount factor $\beta \in (0, 1)$ will be held constant. The bound on borrowing $k = k(\theta)$ is assumed to be a continuous function of θ.

Proposition 6.6. *Suppose $\theta_n \to \theta$ as $n \to \infty$ where $\theta, \theta_1, \theta_2, \ldots$ lie in $\mathcal{S}$. Then $V_{\theta_n} \to V_\theta$ and $c_{\theta_n} \to c_\theta$ uniformly on compact subsets of S.*

Proof. The convergence of V_{θ_n} to V_θ follows from a slight extension of a nice result of Langen (Theorem 5.1, 1981). The result applies directly when the utility function u is bounded. For the general case, where u satisfies (A1), it is straightforward to show that, for $s \geq 0$, $\theta \in D$, and $m = 1, 2, \ldots$,

$$|V_\theta(s) - V_{\theta,m}(s)| \leq (\beta r_1)^m \ell(s)$$

where $V_{\theta,m}(s)$ is the m–day optimal reward function and $\ell(\cdot)$ is a linear function of s. This inequality can be substituted for Langen's inequality (5.1), to see that V_{θ_n} converges to V_θ pointwise. The uniform convergence on compact sets follows from the continuity and monotonicity of these functions.

Consider now the convergence of the c_{θ_n}. By Theorem 6.1(b) and Theorem 3.1(b), for each s, $c_{\theta_n}(s)$ is the unique point in $[0, s + k(\theta_n)]$ such that

$$V_{\theta_n}(c_{\theta_n}(s)) = u(c_{\theta_n}(s)/p_n) + \beta E V_{\theta_n}(g_n(s - c_{\theta_n}(s)) + p_n Y).$$

Here $\theta_n = (r_1^{(n)}, r_2^{(n)}, p_n)$ and

$$g_n(x) = \begin{cases} r_1^{(n)} x & \text{if } \; x \leq 0, \\ r_2^{(n)} x & \text{if } \; x > 0. \end{cases}$$

Let $\{b_k\}$ be a subsequence of the $\{c_{\theta_n}(s)\}$ that converges to some $b \in [0, s + k(\theta)]$. We can use the convergence property of the V_{θ_n} to conclude that

$$V_\theta(b) = u(b/p) + \beta E V_\theta(g(s - b) + pY).$$

(For the convergence of the expectations, observe that $V_{\theta_n}(g_n(s - c_{\theta_n}(s)) + p_n Y) \leq u'(0)(\overline{r}s + \overline{p}Y)$ where $\overline{r}$ is an upper bound on the $r_1^{(n)}$ and $\overline{p}$ is an upper bound on the p_n. Thus Lebesgue's Dominated Convergence Theorem applies.)

Moreover, the unique b that satisfies this equation is $b = c_\theta(s)$. So we conclude that $c_{\theta_n}(s) \to c_\theta(s)$. The uniform convergence on compact sets again follows from the continuity and monotonicity of these functions. ■

We will now use the continuity properties just established to derive two technical results needed in the sequel.

Corollary 6.7. *The infimum of $c_\theta(s)$, taken over all $s \geq 0$ and θ belonging to a fixed compact subset of D, is strictly positive.*

Proof. By Theorem 6.1, $c_\theta(s)$ is nondecreasing in s and $c_\theta(0) > 0$. Hence

$$\inf_{\theta, s} c_\theta(s) = \inf_\theta c_\theta(0).$$

The second infimum is positive because $\theta \to c_\theta(0)$ is continuous by Proposition 6.6, and θ ranges over a compact set ■

Corollary 6.8. *Suppose $\inf_s u'(s) > 0$. Then the supremum of $r_2(s - c_\theta(s))$, taken over all $s \geq 0$ and $\theta = (r_1, r_2, p)$ belonging to a fixed compact subset K of D, is finite.*

Proof. Set $\alpha := \inf_s u'(s)$. Then, if $\theta \in K$ and $c_\theta(s) < s$,

$$\begin{aligned} \alpha < u'(c_\theta(s)) &= \beta r_2 E V'_\theta(r_2(s - c_\theta(s)) + pY) \leq \beta r_2 V'_\theta(r_2(s - c_\theta(s)) \\ &\leq \beta r^* u'(c_\theta(r_2(s - c_\theta(s))) \leq \beta r^* u'(0), \end{aligned}$$

where $r^* = \sup\{r_2 : (r_1, r_2, p) \in K\}$. By assumption (A1) and the compactness of K, $\beta r^* < 1$. Hence, $\alpha < \frac{\alpha}{\beta r^*} < u'(0)$ and with $I = (u')^{-1}$, we have

$$c_\theta(r_2(s - c_\theta(s))) \leq I\left(\frac{\alpha}{\beta r^*}\right).$$

Let

$$\eta(\theta) := \sup\left\{s \geq 0 : c_\theta(s) \leq I\left(\frac{\alpha}{\beta r^*}\right)\right\}.$$

Then $\eta(\theta)$ is finite for each θ since $c_\theta(s) \to \infty$ by Theorem 6.4. It is straightforward using Proposition 6.6 to check that η is upper semi-continuous. Hence, η achieves its finite supremum, say η^*, on the compact set K. It follows that $r_2(s - c_\theta(s)) \leq \eta^* < \infty$ for $\theta \in K$. ■

6.4 Existence of a Stationary Distribution with Finite Mean

Recall that the construction of a stationary Markov equilibrium in Section 4 was based on two assumptions. We will now give some simple, but fairly general, sufficient conditions for the first assumption: namely that the agent has an optimal stationary plan for which the corresponding Markov chain has a stationary distribution with a finite mean. The second assumption, that the "books balance," is much more delicate and we have not been able to find satisfactory conditions for it.

Fix the parameters r_1, r_2, p as in Assumption (A1), and let $c(\cdot)$ be the optimal consumption function given by Theorem 6.1. Consider the associated Markov chain $\{S_n\}$ with dynamics given by (3.11).

Proposition 6.9. *If* $\inf_s u'(s) > 0$*, then the Markov chain* $\{S_n\}$ *has a stationary distribution with finite mean.*

Proof. Apply Corollary 6.7 to the special case where the compact set K is the singleton $\{(r_1, r_2, p)\}$, to obtain a constant η^* such that

$$g(s) \le r_2(s - c(s)) \le \eta^* < \infty \tag{6.2}$$

for all $s \ge 0$. (The first inequality is trivial.) Let L be the operator associated with the chain $\{S_n\}$; i.e., if π is the distribution of S_0, then $L\pi$ is the distribution of $S_1 = g(S_0 - c(S_0)) + pY$ in the notation of (2.6). Next, define

$$\Lambda := \{\pi : \pi \text{ is a probability measure on } \mathcal{B}([0,\infty)) \text{ and } \int s\pi(ds) \le \eta^* + pEY\}. \tag{6.3}$$

It is easy to check that (i) Λ is a compact, convex subset of the space of probability measures on $\mathcal{B}([0,\infty))$ with the usual weak topology and, using (6.2), that (ii) L maps Λ continuously into itself. The Schauder–Tychonoff Theorem (cf. Dunford and Schwartz (1964)) applies to yield a fixed point of L. ■

Suppose the hypothesis of Proposition 6.9 does not hold, so that $\inf_s u'(s) = \lim_{s\to\infty} u'(s) = 0$. Then it is no longer true in general that the chain $\{S_n\}$ has a stationary distribution. The argument for Proposition 6.9 fails because $s - c(s)$ is no longer bounded and, if $r_2 > 1$, then $r_2(s - c(s))$ tends to $+\infty$ as s approaches $+\infty$. However, there is an existence result when $r_1 = r_2 = 1$.

Proposition 6.10. *Assume* $\inf_s u'(s) = 0$*,* $r_1 = r_2 = 1$*, and* $EY^2 < \infty$*. Then the Markov chain* $\{S_n\}$ *has a stationary distribution with finite mean.*

Proof. This follows from Theorems 1 and 2 of Tweedie (1988). First, observe that $\{S_n\}$ is a weak Feller process because the functions g and c appearing in the transition formula (3.11) are continuous. Then check Tweedie's Condition F with, in his notation, $g(s) = s$, $\varepsilon = 1$, and $A = [0, \widetilde{s}]$ where $c(\widetilde{s}) > pEY + 1$. Such an $\widetilde{s}$ exists by Theorem 6.5 and, for $s \ge \widetilde{s}$, $E[S_{n+1}|S_n = s] = s - c(s) + pEY \le s - 1$.

On the other hand, for $0 \le s \le \widetilde{s}$:

$$E[S_{n+1}|S_n = s] = s - c(s) + pEY \le \widetilde{s} - c(\widetilde{s}) + pEY \le \widetilde{s} - 1.$$

By Tweedie's Theorem 2, $\{S_n\}$ has a stationary distribution μ such that $0 < \mu(A) \le 1$.

To see that μ has finite mean, apply Tweedie's Theorem 1(iii) with $f(s) = s$. Take $g(s) = s^2$ in Conditions M1 and F2 and let $A = [0, \widetilde{t}]$ where $\widetilde{t}$ is the maximum of $\widetilde{s}$ and $p^2 \mathrm{Var}\, Y + 1$. For $s > \widetilde{t}$,

$$E[S_{n+1}^2 | S_n = s] = p^2 \mathrm{Var}\, Y + E[S_{n+1} | S_n = s]^2 \leq p^2 \mathrm{Var}\, Y + (s-1)^2 < s^2 - s.$$

This implies condition M1. Condition F2 is easy to check. ■

7 The Existence of Stationary Markov Equilibrium

The existence of stationary equilibrium has already been established in Section 4 under two basic assumptions. The first assumption, that each agent's optimally controlled Markov chain has a stationary distribution with finite mean, follows from natural assumptions about the model as was shown in Section 6.4. The second basic assumption, that the books balance, is trivially satisfied by an outside bank that sets $\rho_1 = \rho_2 = 0$. However, the second assumption is, in general (i.e., for non-zero ρ_1, ρ_2), much more delicate.

We will give a brief discussion of the model with an outside bank. Then we will introduce a modified version of the *money market game*, for which a fairly general existence theorem can be proved.

7.1 The Model with an Outside Bank

Consider again the model with an outside bank and countably many types of agents as in Section 4.3. Assume that each type of agent has a utility function which satisfies assumption (A1) of Section 6. Further, assume that for each type k of agent with $\inf_s (u_k)'(s) = 0$, the income variable Y^k has a finite second moment.

Fix $r_1 = r_2 = 1$ or, equivalently $\rho_1 = \rho_2 = 0$. Also, fix a price $p \in (0, \infty)$ and assume that $pY^\alpha \geq k^\alpha$ for each agent α.

By Theorem 6.1, each agent α of each type k has a unique optimal stationary plan $\pi^\alpha = \pi_k$. Furthermore, by Propositions 6.9 and 6.10, the associated Markov chain has a stationary distribution $\mu^\alpha = \mu_k$ with finite mean. Form the aggregated wealth distribution $\overline{\mu}$ as in (4.3).

Theorem 7.1. *The family of strategies $\{\pi^\alpha\}$ results in a stationary Markov equilibrium $(p, \overline{\mu})$ with interest rates $r_1 = r_2 = 1$.*

Proof. This will follow from Theorem 4.11 once Assumptions 4.8 and 4.9 are verified. Assumption 4.8 is immediate from Propositions 6.9 and 6.10. Assumption 4.9 is a triviality since, by hypothesis $\rho_1 = \rho_2 = 0$. ■

It is natural to ask whether there always exists a stationary equilibrium with positive interest rates ρ_1 and ρ_2. This seems unlikely when the agents are homogeneous and $\inf_s u'(s) = 0$ as in Example 5.2.

Question 7.2. Consider the model with an outside bank and homogeneous agents with utility function u such that $\inf_s u'(s) > 0$. Does there always exist a stationary Markov equilibrium with ρ_1 and ρ_2 positive?

We suspect that the answer is yes, but that the equilibrium may involve active bankruptcy. We hope to return to this question in a subsequent paper which allows for bankruptcy.

7.2 The Model with a Money Market

The object of the remaining sections is to establish the existence of a stationary Markov equilibrium for a modified version of the money market game with homogeneous agents.

Consider first the game as originally formulated. In Example 5.2 we saw that there need not exist a stationary Markov equilibrium. We suspect that this is a common occurrence when the utility remains bounded.

Question 7.3. Does there always exist a stationary Markov equilibrium for the money market game with homogeneous agents and a utility function u such that $\inf_s u'(s) > 0$?

We do not know the answer to this question, but will give an affirmative answer for a modified game, wherein the endogenous interest rates are controlled by "government intervention" so that they are bounded away from $1/\beta$.

7.3 A Game with a Regulated Money Market

For simplicity, we assume that the agents are homogeneous with utility function u and generic income variable Y such that $Y \geq kQ/W$; here $Q = EY$ is the quantity of the commodity produced each period, $W = W_0$ is the amount of fiat money initially held by the agents, and $k > 0$. Let ε and δ be small positive numbers such that

$$0 < \varepsilon \leq \min\{k/2,\, \beta^{-1} - 1\},\ 0 < \delta < \min\{W, \varepsilon\}. \tag{7.1}$$

Our new game will be quite similar to the money market game formulated in Section 2.2. One difference is that every agent is required in every period to offer at least ε for lending and to bid at least δ in IOU notes.

Indeed, suppose that, at the beginning of each period n, each agent α has a wealth $S^\alpha_{n-1}(w) \geq \varepsilon$ and is required to offer $\ell^\alpha_n(w) \in [\varepsilon,\, S^\alpha_{n-1}(w)]$ for lending and $i^\alpha_n(w) \in [\delta,\, k-\varepsilon]$ in IOU notes. To simplify the bookkeeping, we assume that at most one of the quantities $\ell^\alpha_n(w)$ and $i^\alpha_n(w)$ can exceed its minimum value. As usual, we assume that $\ell^\alpha_n(w)$ and $i^\alpha_n(w)$ are jointly measurable in (α, w). The total amounts offered for lending and in IOU notes, respectively, are

$$L_n(w) := \int \ell^\alpha_n(w)\phi(d\alpha), \quad I_n(w) := \int i^\alpha_n(w)\phi(d\alpha). \tag{7.2}$$

Observe that

$$\varepsilon \leq L_n(w) \leq \int S^\alpha_{n-1}(w)\phi(d\alpha), \quad \delta \leq I_n(w) \leq k - \varepsilon.$$

Thus the ratio

$$\widetilde{r}_n(w) := \frac{I_n(w)}{L_n(w)} \tag{7.3}$$

is well-defined. Set

$$r_\varepsilon := \frac{1}{\beta} - \varepsilon. \tag{7.4}$$

If $\widetilde{r}_n(w) \leq r_\varepsilon$, the *interest rate* $r_n(w) = 1 + \rho_n(w)$ for period n is taken to be $\widetilde{r}_n(w)$. However, if $\widetilde{r}_n(w) > r_\varepsilon$, the government (or gamemaster) offers additional funds $G_n(w)$ for lending so that the interest rate for period n is

$$r_n(w) = \frac{I_n(w)}{L_n(w) + G_n(w)} = r_\varepsilon.$$

Thus

$$r_n(w) := \widetilde{r}_n(w) \wedge r_\varepsilon = \frac{I_n(w)}{L_n(w)} \wedge r_\varepsilon. \tag{7.5}$$

In order that G_n be always defined, we set $G_n(w) = 0$ if $\widetilde{r}_n(w) \leq r_\varepsilon$. Then

$$0 \leq G_n(w) := \left(\frac{I_n(w)}{r_\varepsilon} - L_n(w)\right) \vee 0 \leq \frac{k-\varepsilon}{r_\varepsilon} - \varepsilon \leq \frac{k}{r_\varepsilon}. \tag{7.6}$$

With the definition (7.6), the equality

$$r_n(w) = \frac{I_n(w)}{L_n(w) + G_n(w)} \tag{7.7}$$

is always valid.

Once the interest rate $r_n(w)$ is formed, each agent α obtains a loan of $i^\alpha_n(w)/r_n(w)$ and bids

$$b^\alpha_n(w) = S^\alpha_{n-1}(w) - \ell^\alpha_n(w) + i^\alpha_n(w)/r_n(w) \tag{7.8}$$

in the commodity market. (As in the unregulated model, no hoarding is allowed.)

When loans are repaid, the government receives a profit of $\rho_n(w)G_n(w)$, where $r_n(w) = 1 + \rho_n(w)$. However, the government is required to spend its profit in the next period. Thus the price of the commodity in period n is formed as

$$\begin{aligned} p_n(w) &= \frac{\int b^\alpha_n(w)\phi(d\alpha) + \rho_{n-1}(w)G_{n-1}(w)}{Q} \\ &= \frac{W_{n-1}(w) + G_n(w) + \rho_{n-1}(w)G_{n-1}(w)}{Q}, \end{aligned} \tag{7.9}$$

by virtue of (7.6)–(7.8); here

$$W_{n-1}(w) = \int S^\alpha_{n-1}(w)\phi(d\alpha)$$

is the total amount of money held by agents at the beginning of period n, and we set $W_0 = W$ and $\rho_0 G_0 = 0$. Thus, at the end of the nth period, agent α has wealth

$$S_n^\alpha = r_n(S_{n-1}^\alpha - b_n^\alpha) + p_n Y_n^\alpha = r_n \ell_n^\alpha - i_n^\alpha + p_n Y_n^\alpha \tag{7.10}$$

in fiat money.

In this model, the total wealth levels W_n can fluctuate, but the quantities $W_n + \rho_n G_n$ remain constant. To see this, let $n \geq 1$ and calculate:

$$\begin{aligned} W_n + \rho_n G_n &= \int S_n^\alpha \phi(d\alpha) + (r_n - 1)G_n \\ &= \int (r_n \ell_n^\alpha - i_n^\alpha + p_n Y_n^\alpha)\phi(d\alpha) + (r_n - 1)G_n \\ &= \left(\frac{I_n}{L_n + G_n} L_n - I_n + \frac{W_{n-1} + G_n + \rho_{n-1}G_{n-1}}{Q} Q\right) + \left(\frac{I_n}{L_n + G_n} - 1\right)G_n \\ &= W_{n-1} + \rho_{n-1}G_{n-1}. \end{aligned} \tag{7.11}$$

Thus

$$W_n + \rho_n G_n = W_0 = W \tag{7.12}$$

for all $n \geq 1$, and the formula for the price $p_n(w)$ of (7.9) can now be written in the simpler form

$$p_n(w) = \frac{W + G_n(w)}{Q}. \tag{7.13}$$

It follows that p_n is bounded because G_n is bounded:

$$p_* := \frac{W}{Q} \leq p_n(w) \leq \frac{W + k/r_\varepsilon}{Q} := p^*. \tag{7.14}$$

Since $i_n^\alpha(w) \leq k - \varepsilon$, $p_n(w) \geq W/Q$, and $Y_n^\alpha(w) \geq kQ/W$, we have $S_n^\alpha(w) \geq -i_n^\alpha(w) + p_n(w)Y_n^\alpha(w) \geq \varepsilon$ as assumed.

By construction, the interest rate r_n is bounded above by r_ε; cf. (7.5). To get a positive lower bound, observe from (7.2), (7.12) that

$$L_n(w) \leq \int S_{n-1}^\alpha(w)\phi(d\alpha) = W_{n-1}(w) = W - \rho_{n-1}(w)G_{n-1}(w) \leq W.$$

(The quantity $\rho_{n-1}G_{n-1}$ is nonnegative, since $G_{n-1} > 0$ implies $r_{n-1} = r_\varepsilon$ and $\rho_{n-1} = r_\varepsilon - 1 > 0$.) Consequently,

$$\widetilde{r}_n(w) = \frac{I_n(w)}{L_n(w)} \geq \frac{\delta}{W}$$

and, by the definition (7.5) of r_n and (7.1), (7.2):

$$r_* := \frac{\delta}{W} \leq r_n(w) \leq r_\varepsilon. \tag{7.15}$$

As in our other games, each agent α receives $x_n^\alpha = b_n^\alpha / p_n$ units of the commodity and $u(x_n^\alpha)$ in utility in the nth period. The agent's objective is to maximize the expectation of total discounted reward, namely

$$E \sum_{n=0}^{\infty} \beta^n u(x_n^\alpha).$$

Strategies and admissible collections of strategies are defined as in Section 2. A stationary strategy π^α will now specify IOU bids and loan offers in the form

$$\begin{aligned} i_n^\alpha(w) &= i^\alpha(S_{n-1}^\alpha(w),\, r_{n-1}(w),\, p_{n-1}(w)) \\ \ell_n^\alpha(w) &= \ell^\alpha(S_{n-1}^\alpha(w),\, r_{n-1}(w),\, p_{n-1}(w)), \end{aligned}$$

where i^α and ℓ^α are measurable functions such that $\delta \le i^\alpha(s,r,p) \le k - \varepsilon$, $\varepsilon \le \ell^\alpha(s,r,p) \le s$, and either $i^\alpha(s,r,p) = \delta$ or $\ell^\alpha(s,r,p) = \varepsilon$.

Definition 7.4. An admissible collection $\Pi = \{\pi^\alpha,\, \alpha \in I\}$ of stationary strategies results in a *stationary Markov equilibrium* (r,p,μ) where $0 < r < \infty$, $0 < p < \infty$, and μ is a probability measure on $\mathcal{B}([0,\infty))$ if, with $r_0 = r$, $p_0 = p$, and $\nu_0 = \mu$, we have

(i) $r_n = r$, $p_n = p$, and $\nu_n = \mu$ for all $n \ge 1$,

(ii) each strategy π^β is optimal for agent β when every other agent α ($\alpha \ne \beta$) plays π^α.

Theorem 7.5. *For δ sufficiently small, there is a stationary Markov equilibrium for the regulated money market with homogeneous agents and a utility function u that satisfies assumption* (9.1) *and* $\inf_s u'(s) > 0$.

The rest of this section is devoted to the proof of Theorem 7.5. The method is to use the Schauder–Tychnoff fixed point theorem in a way analogous to that of Whitt (1975). The proof will also use results for the one-person game from Section 6. We begin with a look at how the one-person game is related to the regulated money market game.

Consider an agent with wealth $s \ge \varepsilon$ who is playing in the regulated money market game. The agent could elect to bid $i = \delta$ in IOU notes and to lend $\ell \in [\varepsilon, s]$, or elect to lend $\ell = \varepsilon$ and bid $i \in [\delta,\, k - \varepsilon]$. The resulting commodity bid

$$b = s - \ell + i/r,$$

for a given r, would take values in the interval $[\delta/r,\, s + k_r]$ where

$$k_r = \frac{k - \varepsilon}{r} - \varepsilon.$$

Suppose the agent believes the game to be in equilibrium at price p and interest rate r. Then the agent would prefer to make the optimal bid $c_\theta(s)$ in the one-person game with parameters $\theta = (r,p)$ and $k(\theta) = k_r$. (In the notation of Section 6.3, $\theta = (r,r,p)$

which we abbreviate here to (r,p).) If $c_\theta(s) < \delta/r$, then the bid $c_\theta(s)$ would not be available. However, θ takes values in the compact set

$$K = [r_*,\, r_\varepsilon] \times [p_*,\, p^*] \tag{7.16}$$

(recall (7.14), (7.15)). Thus, by Corollary 6.7,

$$c_* := \inf_{\theta, s} c_\theta(s) > 0.$$

We now assume that $\delta \le c_* r_*$ so that $c_\theta(s) \ge \delta/r_* \ge \delta/r$, and the bid $c_\theta(s)$ is available to the agent. (This small technicality could be avoided by treating a more general one-person game in Section 6.)

Consider the function ψ which maps the current interest rate, price, and wealth distribution (r,p,μ) to the corresponding quantities $(r_1,\, p_1,\, \mu_1)$ for the next period, under the assumption that each agent plays according to the unique optimal stationary strategy for the one-person game with parameters $\theta = (r,p)$ and $k(\theta) = k_r$. We want to show that ψ has a fixed point. First we need a more explicit definition of ψ. We take the domain of ψ to be the set

$$\Delta = K \times M. \tag{7.17}$$

Here K is the compact set of (7.16), and M is the collection of all probability measures on $\mathcal{B}([0,\infty))$, which satisfy

$$\int s\mu(ds) \le W \tag{7.18}$$

and are stochastically smaller than

$$r_\varepsilon \eta^* + (k - \varepsilon)\left(\frac{r_\varepsilon}{r_*} - 1\right) + p^* Y, \tag{7.19}$$

where

$$\eta^* := \sup\{s - c_\theta(s) \,:\, s \ge 0,\, \theta \in K\}. \tag{7.20}$$

It follows from Corollary 6.8 that η^* is finite. Hence, M is tight and therefore compact in the weak-star topology. Thus Δ is also compact in the product topology. Note also, from (7.14), that the expectation of the random variable in (7.19) is $r_\varepsilon \eta^* + (k - \varepsilon)\left(\frac{r_\varepsilon}{r_*} - 1\right) + W + \frac{k}{r_\varepsilon} \ge W$, in accordance with (7.18).

For $\theta \in K$, write

$$c_\theta(s) = s - \ell_\theta(s) + i_\theta(s)/r \tag{7.21}$$

where $\ell_\theta(s)$ and $i_\theta(s)$ are the loan offer and IOU bid of an agent with wealth s. Only one of these can exceed its minimum; so either $i_\theta(s) = \delta$ and $\ell_\theta(s) = s - c_\theta(s) + \delta/r$, or else $\ell_\theta(s) = \varepsilon$ and $i_\theta(s) = r(c_\theta(s) - s + \varepsilon)$. Thus

$$\begin{aligned} \ell_\theta(s) &= (s - c_\theta(s) + \delta/r) \vee \varepsilon \in [\varepsilon, s], \\ i_\theta(s) &= r(c_\theta(s) - s + \varepsilon) \vee \delta \in [\delta, k - \varepsilon]. \end{aligned} \tag{7.22}$$

An agent who lends and borrows in accord with $\ell_\theta(\cdot)$ and $i_\theta(\cdot)$ will be playing optimally if the interest rate and price remain fixed at $\theta = (r, p)$.

Both $\ell_\theta(\cdot)$ and $i_\theta(\cdot)$ are continuous in θ, uniformly on compact sets because $c_\theta(\cdot)$ is (by Proposition 6.6). It follows that the totals of the IOU bids and loan offers given by the integrals

$$I \equiv I(r,p,\mu) := \int i_\theta(s)\mu(ds), \quad L \equiv L(r,p,\mu) := \int \ell_\theta(s)\mu(ds) \tag{7.23}$$

are continuous functions of (r, p, μ) on Δ. This can be shown directly, or by an application of Theorem 3.5 in Langen (1981). Likewise, the quantity

$$G \equiv G(r,p,\mu) := \left(\frac{I}{r_\varepsilon} - L\right) \vee 0, \tag{7.24}$$

which represents the "funds offered by the government," is a continuous function of (r, p, μ). It develops that the new interest rate

$$r_1 \equiv r_1(r,p,\mu) := \frac{I}{L+G}, \tag{7.25}$$

and the new price

$$p_1 \equiv p_1(r,p,\mu) := \frac{W+G}{Q}, \tag{7.26}$$

are also continuous in (r, p, μ). Consider now

$$\mu_1 \equiv \mu_1(r,p,\mu) := (\text{distribution of } S_1 = -i_\theta(S) + r_1\ell_\theta(S) + p_1 Y) \tag{7.27}$$

where the independent random variables S, Y have distributions μ, λ respectively.

Lemma 7.6. *The mapping $\psi : (r, p, \mu) \to (r_1,\, p_1,\, \mu_1)$ of (7.25)–(7.27) is continuous, and maps Δ into itself.*

Proof. The continuity properties of r_1, p_1, ℓ_θ and i_θ imply the continuity of $\mu_1(r, p, \mu)$, hence also that of the mapping ψ.

Now from (7.22), (7.23) and (7.18), we have

$$\delta \le I \le k - \varepsilon, \quad \varepsilon \le L \le \int s\mu(ds) \le W;$$

and by analogy with (7.14), (7.15) we deduce that $(r_1,\, p_1) \in K$. It remains to show $\mu_1 \in M$. From

$$S_1 = -i_\theta(S) + r_1\ell_\theta(S) + p_1 Y \tag{7.28}$$

and (7.23)–(7.26), we have

$$\begin{aligned} \int s\mu_1(ds) &= -\int i_\theta(s)\mu(ds) + r_1 \int \ell_\theta(s)\mu(ds) + p_1 Q \\ &= \frac{I}{L+G}L - I + W + G = W - G(r_1 - 1) \le W \end{aligned}$$

since, either $G = 0$, or $G > 0$ and $r_1 = r_\varepsilon = \frac{1}{\beta} - \varepsilon \geq 1$. This verifies (7.18) for μ_1; to verify (7.19), let

$$b_\theta(S) = S - \ell_\theta(S) + \frac{i_\theta(S)}{r_1}$$

denote the "bid for commodity," recall (7.21), and rewrite (7.28) as

$$\begin{aligned} S_1 &= r_1(S - b_\theta(S)) + p_1 Y = r_1(S - c_\theta(S)) + r_1(c_\theta(S) - b_\theta(S)) + p_1 Y \\ &= r_1(S - c_\theta(S)) + i_\theta(S)\left(\frac{r_1}{r} - 1\right) + p_1 Y \\ &\leq r_\varepsilon \eta^* + (k - \varepsilon)\left(\frac{r_\varepsilon}{r^*} - 1\right) + p^* Y \end{aligned}$$

from (7.20), (7.22) and $(r, p) \in K$, $(r_1, p_1) \in K$. ■

This completes the proof that the mapping $\psi : (r, p, \mu) \to (r_1, p_1, \mu_1)$ is continuous from Δ into Δ. By the Schauder–Tychonoff Theorem, ψ has a fixed point (r', p', μ').

Suppose every agent plays according to the stationary strategy corresponding to $c_{\theta'}(\cdot) = c_{(r',p')}(\cdot)$ and that the initial conditions are $r_0 = r'$, $p_0 = p'$, and $\nu_0 = \nu'$. Since (r', p', μ') is a fixed point of ψ it follows that $r_n = r'$, $p_n = p'$ and $\nu_n = \nu'$ for all $n \geq 1$, and also that every agent is playing optimally against the given strategies of the others.

The proof of the theorem is now complete.

Question 7.6. Is the stationary Markov equilibrium unique?

8 Acknowledgment

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9 Appendix: A Proof of Optimality

Let π be the stationary plan introduced in Example 5.3 with corresponding consumption function c as in (5.1). Assume that the parameters in the example satisfy the condition:

$$1 \geq \beta(1 + \eta) \geq \eta \geq \beta^2(1 + \eta) + \beta\eta. \tag{9.1}$$

Let $Q(s) = I(\pi)(s)$ be the expected return for an agent who starts at s and plays π. To show π is optimal, it suffices, by Theorem 3.1, to verify that Q satisfies the Bellman equation:

$$Q(s) = \max_{0 \leq b \leq s + 1/2} [u(b) + \beta E Q(2(s - b) + Y)]. \tag{9.2}$$

The first step is the calculation of Q using the identity

$$\begin{aligned} Q(s) &= u(c(s) + \beta E Q((2(s-c(s)) + Y) \\ &= u(c(s)) + \frac{\beta}{2}[Q(2(s-c(s))+1) + Q(2(s-c(s))+4)]. \end{aligned}$$

This identity together with the formula (5.1) defining c gives the following:

$$Q(s) = \begin{cases} s + \frac{1}{2} + \frac{\beta}{2}[Q(0) + Q(3)] & , \quad 0 \le s \le 2, \\ 2\frac{1}{2} + \frac{\beta}{2}[Q(2s-4) + Q(2s-1)] & , \quad 2 \le s \le 3, \\ 2\frac{1}{2} + \eta(s-3) + \frac{\beta}{2}[Q(2) + Q(5)] & , \quad s \ge 3. \end{cases}$$

In the middle equality above where $2 \le s \le 3$, we have $2s-4 \le 2$ and $2s-1 \ge 3$, so we can substitute from the first and last equalities to get

$$Q(s) = 2\tfrac{1}{2} + \tfrac{\beta}{2}\left\{\left[2s - 3\tfrac{1}{2} + \tfrac{\beta}{2}(Q(0)+Q(4)\right] + \left[2\tfrac{1}{2} + \eta(2s-4) + \tfrac{\beta}{2}(Q(2)+Q(5))\right]\right\}.$$

Differentiation gives

$$Q'(s) = \begin{cases} 1 & , \quad 0 < s < 2, \\ \beta + \beta\eta & , \quad 2 < s < 3, \\ \eta & , \quad s > 3. \end{cases}$$

Left and right derivatives can be obtained at the endpoints of the three intervals by continuity.

Observe that condition (9.1) implies that Q' is decreasing and Q is concave.

In order to check the Bellman equation (9.2), define, for each fixed s, the function

$$\psi(b) := \psi_s(b) := u(b) + \beta E Q(2(s-b)+Y) = u(b) + \tfrac{\beta}{2}[Q(2(s-b)+1) + Q(2(s-b)+4)]$$

for $0 \le b \le s + \frac{1}{2}$. The function ψ is concave because u and Q are concave. What we must show is that ψ attains its maximum at $b = c(s)$. We will consider three cases.

Case 1. $0 \le s \le 2$.

Here $c(s) = s + \frac{1}{2}$ and it suffices to show that $\psi'_-\left(s+\frac{1}{2}\right) \ge 0$ where ψ'_- denotes the left derivative of ψ. For $0 \le b \le s + \frac{1}{2}$, we have $b \le 2\frac{1}{2}$ and

$$\psi(b) = b + \tfrac{\beta}{2}[Q(2(s-b)+1) + Q(2(s-b)+4]).$$

Hence,

$$\psi'_-(b) = 1 - \beta[Q'_+(2(s-b)+1) + Q'_+(2(s-b)+4)]$$

where Q'_+ denotes the right derivative of Q. Set $b = s + \frac{1}{2}$ to get

$$\psi'_-\left(s+\tfrac{1}{2}\right) = 1 - \beta[Q'_+(0) + Q'_+(3)] = 1 - \beta[1+\eta] \ge 0$$

by condition (9.1).

Case 2. $2 < s \le 3$.

In this case, $c(s) = 2\frac{1}{2}$ and we need to show that $\psi'_-\left(2\frac{1}{2}\right) \geq 0 \geq \psi'_+\left(2\frac{1}{2}\right)$. To prove the first inequality, observe as in Case 1 that

$$\psi'_-\left(2\tfrac{1}{2}\right) = 1 - \beta[Q'_+(2s-4) + Q'_+(2s-1)] \geq 1 - \beta[Q'_+(0) + Q'_+(3)] \geq 0,$$

where the first inequality holds because Q'_+ is nonincreasing and the second by the argument in Case 1.

To prove that $\psi'_+\left(2\frac{1}{2}\right) \leq 0$, let $2\frac{1}{2} \leq b \leq s + \frac{1}{2}$ and write

$$\psi(b) = 2\tfrac{1}{2} + \eta\left(b - 2\tfrac{1}{2}\right) + \tfrac{\beta}{2}[Q(2(s-b)+1) + Q(2(s-b)+4)]. \qquad (9.3)$$

Hence,

$$\psi'_+(b) = \eta - \beta[Q'_-(2(s-b)+1) + Q'_-(2(s-b)+4)]$$

and, in particular,

$$\psi'_+\left(2\tfrac{1}{2}\right) = \eta - \beta[Q'_-(2s-4) + Q'_-(2s-1)] \leq \eta - \beta[Q'_-(2) + Q'_-(5)] = \eta - \beta(1+\eta) \leq 0$$

where the first inequality holds because Q'_- is nonincreasing and the second by (9.1).

Case 3. $s > 3$.

Here $c(s) = s - \frac{1}{2}$. So we need to show $\psi'_-\left(s - \frac{1}{2}\right) \geq 0 \geq \psi'_+\left(s - \frac{1}{2}\right)$. To prove the first inequality, let $2\frac{1}{2} \leq b \leq s + \frac{1}{2}$, write $\psi(b)$ as in (9.3) and take the left derivative at $b = s - \frac{1}{2}$ to get

$$\psi'_-\left(s - \tfrac{1}{2}\right) = \eta - \beta[Q'_+(2) + Q'_+(5)] = \eta - \beta[\beta + \beta\eta + \eta].$$

The final quantity is nonnegative by condition (9.1).

Take the right derivative of ψ at $b = s - \frac{1}{2}$ to get

$$\psi'_+\left(2 - \tfrac{1}{2}\right) = \eta - \beta[Q'_-(2) + Q'_-(5)] = \eta - \beta[1 + \eta].$$

This quantity is less than or equal to zero by (9.1).

The proof that π is optimal is now complete.

10 References

Bellman, R. (1957). *Dynamic Programming.* Princeton, NJ: Princeton University Press.

Blackwell, D. (1965). "Discounted Dynamic Programming," *Ann. Math. Stat.* **36**, 226–235.

Dubey, P. (1982). "Price–Quantity Strategic Market Games," *Econometrica* **50**(1), 111–126.

Dubey, P. and L. Shapley (1994). "Noncooperative General Exchange with a Continuum of Traders: Two Models," *Journal of Mathematical Economics* **23**(1): 253–293.

Dunford, N. and J. T. Schwartz (1964). *Linear Operators. Part I: General Theory.* New York: J. Wiley & Sons/Interscience.

Feldman, M. and C. Gilles (1985). "An Expository Note on Individual Risk Without Aggregate Uncertainty," *Journal of Economic Theory* **35**, 26–32.

Jayawardene, K. K. A. (1993). "Essays on Queueing Systems and Strategic Market Games," PhD. Thesis, Department of Operations Research, Yale University.

Karatzas, I., M. Shubik, and W. Sudderth (KSS) (1994). "Construction of Stationary Markov Equilibria in a Strategic Market Game," *Math. of Oper. Research* **19**, 975–1006.

Langen, H.–J. (1981). "Convergence of Dynamic Programming Models," *Math. of Oper. Research* **6**, 493–512.

Meyn, S. P. and R. L. Tweedie (1993). *Markov Chains and Stochastic Stability.* New York: Springer–Verlag.

Schäl, M. (1975). "Conditions for Optimality in Dynamic Programming, and for the Limit of n–Stage Optimal Policies to be Optimal," *Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete* **32**, 179–196.

Tweedie, R. L. (1988). "Invariant Measures for Markov Chains with No Irreducibility Assumptions," *Journal of Applied Probability* **25A**, 275–286.

Whitt, W. (1975). "Stationary Equilibria in an Economy with Money, Uncertainty, Infinitely-Many Time-Periods and a Continuum of Agents," unpublished manuscript, Yale University.