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AN ALTERNATIVE THEORY OF FIRM AND INDUSTRY DYNAMICS

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ABSTRACT

This paper provides a model of firm and industry dynamics that allows for entry, exit and firm–specific uncertainty generating variability in the fortunes of firms. It focuses on the impact of uncertainty arising from investment in research and exploration–type processes. It analyses the behavior of individual firms exploring profit opportunities in an evolving marketplace and derives optimal policies, including exit, in this environment. Then it adds an entry process and aggregates the optimal behavior of all firms, including potential entrants, into a rational expectations, Markov perfect industry equilibrium, and proves ergodicity of the equilibrium process. Numerical examples are used to illustrate the more detailed characteristics of the stochastic process generating industry structures that result from this equilibrium.

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I. INTRODUCTION

A salient feature of firm level data is the great variability in the fate of firms over time. Notable manifestations of this variability include the significant amount of simultaneous entry and exit, of simultaneous firm level job creation and destruction, and of variability in growth rates, found in the analysis of firm and establishment level panel data sets. These indications of differences in outcome paths among firms persist even after one controls for the entry date, location, and "industry" of the firm, and therefore for time, location, and industry specific differences in economic environments. Moreover they tend to be associated with a remarkable degree of heterogeneity among firms in the same industry in both the levels and the movements over time in the variables that we typically want to analyze (shares in industry output, investment, productivity, etc.)[2]. We provide a model of industry behavior which, because it incorporates idiosyncratic, or firm specific, sources of uncertainty, can generate the variability in the fortunes of firms observed in this data.

Although the data alone provide sufficient reason for developing a model capable of describing behavior in a world with idiosyncratic uncertainty, there is a policy, as well as a descriptive, need for such models. In a world where firms differ, policy and environmental changes are likely to have different impacts, and lead to different responses, in different firms. Since these responses are frequently nonlinear functions of the changing variable (entry and exit reflecting an extreme nonlinearity), any analysis of their effects, even if only an analysis of their aggregate impacts (say on industry supply or on productivity), requires both the underlying distribution of firms by the source of response heterogeneity,

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[2]Partly due to increased data availability, there has been a resurgence in the analysis of firm level panels over the last decade. For more detail on the empirical results that emanated from these studies, and the basis for our brief summary of some of their results, see Evans (1984, 1987), Dunne, Roberts, and Samuelson (1988), Pakes and Ericson (1989), Davis and Haltwinger (1991), and the literature cited in those articles. These articles also contain references to the extensive empirical literature on the nature, extent, and implications of the variation in output paths among firms.

and the (equilibrium) response of that distribution to the underlying policy or environmental change.[3] Of course, policy issues are often more directly related to the heterogeneity in the distribution of responses per se, as in, for example, the analysis of the effects of a policy or an environmental change on job turnover, on market structure, or on default probabilities. In these situations the whole focus is on characteristics of the distribution of the response heterogeneity, and hence the need for a structural model that allows for idiosyncratic uncertainty becomes even more obvious.

There is, of course, more than one source of idiosyncratic uncertainty that firms react to, and different sources are likely to be more relevant to analyzing behavior in some industries than in others. Though our model can accommodate other sources of uncertainty as well, we focus attention in this paper on the impact of the uncertainties generated by the random outcomes of research and exploration–like processes, and are thus primarily interested in providing a framework for analyzing behavior in industries where those processes are important. We need not, however, be too specific about the nature of competition in the spot market for current output, so the model ought to be applicable for a range of industries in which the outcomes of research and exploratory type processes are important.

The industry model is based upon a stochastic model of the growth of an enterprise through the active exploration of its economic environment. The active component is modeled as an investment process that is directed at the accumulation of a capital stock which improves the current profits that the enterprise is able to earn. The outcomes of the investment process are, however, far from certain. It may generate a sequence of events

[3]See Geweke (1985), and Pakes and McGuire (1992) for related discussions and numerical examples. The importance of explicitly accounting for heterogeneity in reponse patterns when analyzing the aggregate impacts of changes comes out clearly in the recent empirical work that uses disaggregate data to study such changes. See, for example, Thomas' (1990) study of the impact of FDA regulations on the rate of innovation in the pharmaceutical industry, or Olley and Pakes' (1992) study of the impact of deregulation on aggregate productivity growth in the telecommunications equipment industry.

that leads to phenomenal riches, or it may produce little of economic value. More precisely, it may produce less than the efforts of competitors (both inside and outside the industry), yielding a deterioration in the profitability of the enterprise and, quite possibly, leading to a situation in which it is optimal to abandon the whole undertaking. This endogenizes exit behavior, and provides a natural way of accounting for selection in the process of determining the evolution of the industry. The model is closed by showing the existence of a Markov perfect Nash equilibrium in the investment, entry, and exit decisions of each firm. Firms maximize their present discounted value given expectations about the evolution of their competition, and at equilibrium those expectations are fully consistent with the process generated by the optimal decisions of all firms within or entering the industry. Thus we show the existence of a rational expectations equilibrium with heterogeneous agents subject to idiosyncratic shocks.

There are several predecessors to our model in the literature. The earliest is the class of traditional investment models, including those dealing with investment in R&D and innovative activities (see, for example, Brock, 1972; Lucas, 1978; Abel and Blanchard, 1983). These models are similar to ours in that investment is the activity leading to growth. However, they typically postulate a deterministic relationship between investment efforts on the one hand, and the accumulation of capital on the other. They therefore do not attempt to deal with diversity in outcome paths conditional on the variables which determine the incentives to invest (a phenomena which is natural in a model, such as ours, with stochastic accumulation). More similar in spirit in this respect to our model are models of investment and stopping in research and exploration processes (see the review in Kamien and Schwartz, 1982; Roberts and Weitzman, 1981; Grossman and Shapiro, 1986). These, however, are models of the behavior of single firms, and hence do not attempt to analyze the impact of the interactions among competitors on both the descriptive and the policy questions of interest.

Models more similar to ours in the latter respect are the game–theoretic models of investment in R&D in patent races (see Dasgupta and Stiglitz, 1980; Reinganum, 1982; and the literature cited in those articles). The game–theoretic models, however, do not consider ongoing ventures; the competition terminates with a single prize.[4] Thus there is no notion of an evolving market for the output of the industry, or of continual change in industry structure. The ongoing nature of the interactions generated by outcomes of investment processes is stressed in the evolutionary model of Nelson and Winter (1982), a model similar in spirit to the one to be presented here. Indeed, the major difference between the Nelson–Winter model and ours is that ours is an optimizing model; i.e. in our model investment, entry, and exit decisions are made to maximize the entrepreneurs perceptions of the firm's expected discounted value of future net cash flows.

There are at least two other classes of models of industry dynamics that allow for heterogeneity among firms and idiosyncratic uncertainty, but both focus on a different source of idiosyncratic uncertainty than the one we concentrate on. Moreover there are variants of each that are truly closed in the sense that not only is the optimal behavior of individual firms determined, but an equilibrium is defined for the industry and shown to exist. In both cases the equilibrium allows for entry and exit, at least until a limiting stationary state is reached.

The earliest of the two consists of what might be called 'passive learning' models (Lippmann and Rummelt, 1982; Jovanovic, 1982). These models capture the idiosyncratic uncertainty in each undertaking by endowing it with an initially unknown, time–invariant parameter which determines the distribution of its profits thereafter. The (endogenous) dynamics of firm behavior are generated by a Bayesian learning process. That is, past profit realizations contain information which enables increasingly accurate predictions about future profitability. This generates an evolutionary selection process, with simultaneous entry and exit, and some very strong predictions regarding the nature of the

[4]A recent exception is P.K. Dutta, S. Lach, A. Rustichini (1991).

stochastic process generating sales among survivors. Pakes and Ericson (1989) compare the properties of this passive learning model to those of the model of active exploration presented here, and consider some simple ways of distinguishing when one or the other might be relevant.[5]

More recent developments include a class of dynamic models that emphasize the sunk cost nature of initial investments whose relative profitabilities change over time in response to the outcomes of some exogenous process. In this class of models, Dixit (1989a, 1989b) focuses on exchange rate variability, while Lambson (1989, 1992) focuses on variability in relative factor prices. In these models the realizations of the process generating the uncertainty are common to all firms, though their effects may differ. Though we could extend our model to incorporate (at least limited forms) of the uncertainties associated with either unknown initial conditions, or with a time–specific individual invariant process (as might the other authors be able to extend their models to incorporate the effects of stochastic accumulation), we pay little attention to the implications of those types of uncertainties in this paper; implications on which the other papers specifically focus.[6]

The next section begins with an overview of the behavioral assumptions in our model. We then formulate the decision problem faced by both incumbent firms and by potential entrants to the industry, and define an industry equilibrium. The equilibrium we work with is Markov Perfect Nash (in the sense of Maskin and Tirole, 1988a,1988b) in

[5]These models were a natural dynamic extension of the static models of industry equilibrium with heterogeneity among agents; see Lucas (1978), Khilstrom and Laffont (1978), and the summary in Brock and Evans (1985). Hopenhayn (1992) provides a hybrid model in which perfectly competitive firms are subject to exogenous productivity shocks, but do not engage in Bayesian learning as they know the distribution of those shocks.

[6]Dixit (1989b) considers only neutral efficiency differences among firms, i.e. if firms are differentially productive, then their relative productivities do not change at different realizations of the forcing process. In Lambson's (1989) model, firms choose technologies on the basis of current factor prices, and those choices may well turn out to be inefficient in the future. Then relative efficiencies are a function of the realization of the time–invariant process. For a discussion of the importance of initial, and of recurring, sunk costs and their relationship to the evolution of industry structure in several industries, see Sutton (1991).

investment strategies. It thus naturally allows for state variables that both differentiate among firms and are serially correlated; both are features of a model which are often considered to be necessary if the model is to be used to structure subsequent data analysis.

In section III we derive our general results characterizing the optimal investment, exit and entry decisions of firms either actively or potentially in the market, and showing the existence of a rational expectations, Markov perfect dynamic market equilibrium for the industry. We further characterize that equilibrium as an ergodic stochastic process, and discuss the general implications of that result for interpreting the observed dynamics of industry equilibria.

The model is general enough to encompass a number of detailed models of competition, and the answers to many questions of interest depend on the details of the functional forms which determine the nature of the relationships within it. As a result we have, elsewhere, developed a computational algorithm which computes and characterizes the equilibria associated with the different functional forms that can be fed into our model (see Pakes and McGuire, 1992). Section IV uses this algorithm to compute the equilibria from a Cournot–Nash, homogeneous product, version of our model in which firms are differentiated with respect to their efficiency of production; efficiencies which evolve with the outcomes of a research and exploration process and with the outcomes of an aggregate process which shifts the costs of factors of production to the industry. It then provides a brief comparison of these results to the results from the differentiated product version of our model used as the example in Pakes and McGuire (1992); a version in which firm's are differentiated by the quality of the product they produce, a quality which evolves with the outcomes of both a research and exploration process and with the value of products produced outside of the industry which vie for consumers' expenditures. The paper concludes, in Section V, with a discussion of potential extensions, focusing primarily (though not entirely) on steps that would allow us to make more intensive use of the model in interpreting data. Finally, the proofs of our results are gathered in a technical appendix.

II. AN INDUSTRY MODEL.

A. Overview.

The active force in our model is an entrepreneur or firm exploring a speculative idea, a perceived profit opportunity in some industry.[7] To learn the true value of this idea or opportunity, an entrepreneur must invest to enter the industry and then in developing and, possibly, in exploiting that idea or opportunity. Investment to enter the industry is a sunk cost, perhaps partially recoverable if there is some scrap value realizable on exit. The quantity of investment, together with parameters describing the evolution of the market and the competition, determine the probability distribution of outcomes from the exploratory activities of an active firm in each period.

Favorable outcomes from its own investment activity tend to move the firm towards "better" states; states in which its idea can be embodied in a good or service which is likely to be marketed more profitably. Favorable outcomes of direct competitors, or advances in alternatives to the industry's products, tend to move the firm toward less profitable states. Indeed, a firm whose investment activity generated a string of relatively unsuccessful outcomes may well find itself in a situation in which its idea is not perceived to be worth developing further, so that the enterprise is best liquidated and its salvageable resources committed to an alternate use. The model, therefore, generates exit as a natural outcome of an evolutionary process.

The opportunity and technology provided by this industry are open to all, so that the only distinction among firms is their achieved state of "success" [index of efficiency], $\omega \in \Omega \subset \mathbb{Z}$, in exploiting this opportunity.[8] The state, ω, of each firm within the industry is measured relative to an outside alternative. The outside alternative reflects the strength

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[7]We do not explore the nature of the firm in this model, but take it to be a unitary maximizing agent.

[8]$\mathbb{Z}$ is the ordered set of all integers; $\mathbb{Z}_+$ is the set of non–negative integers.

of competition outside the industry and it changes over time as a result of autonomous factors which shift the demand and/or the cost parameters of all firms producing in the industry (detailed examples are given below). Therefore higher ω indicates that the firm is in a stronger (more profitable) position relative to both other firms in, and competition from outside, the industry. There is a set of states, $\Omega^e \subset \Omega$, at which new firms may enter the industry after making a sufficient investment. We denote the industry structure at any point of time by $s = \{s_\omega\}_{\omega \in \mathbb{Z}} \in \mathbb{Z}_+^\infty$; so that s provides the number of firms at each possible ω state. The couple, (ω,s), will determine the state of the firm. That is, we will assume that (ω,s) determines the entire distribution of the firm's current and future profits, and hence the firm's viability as an enterprise.

This state changes as a result of the outcomes of the firm's own investment and development efforts, the outcomes of the efforts of other firms operating in the same market, and with changes in the overall market environment, i.e. in demand, input costs, and science and technology, in which it is embedded. The firm's own level of investment, denoted by $x_t \in \mathbb{R}_+$, is chosen to maximize the expected present discounted value of profits as a function of all information available at t. We assume this information to include the history of all past states and of the firm's own past investment decisions, i.e. $\{(\omega_{t'},s_{t'}),x_{t'}\}_{t'<t}$; the current state, (ω_t,s_t); and the probability laws governing the evolution of that state over time including the law governing the impact of the firm's own investment on that evolutionary pattern. Of course, those probabilities are determined, in part, by the investment decisions of all firms in or entering the industry. We assume that the firm never observes the investments of its competitors, even though the observable outcomes of those investments will determine the state of the firm's environment and hence its profits in the future. Thus we have in mind a model for a "diffuse" industry; an industry in which firms are not directly aware of the investment strategies of their actual and potential competitors, but are affected by the successful outcomes of those strategies. Note that since the investments of a firm's competitors are not observed, firms cannot

make decisions based on them.

Precisely how the state (ω,s) affects the present and likely future payoffs to the firm depends on the nature of competition and thus the associated type of market equilibrium achieved in each period. That will determine the "strength" of the competition faced, and hence which states are "better" for the firm. For our theoretical results we do not need to be too precise about the nature of the equilibrium in the spot market for current output. We will require that it generate a complete preorder, $\succeq$, over s, which unambiguously defines the strength of the competition. That is, we assume that, no matter what the firm's ω, current profits are (weakly) decreasing in s in the sense of $\succeq$. Also, conditional on any s, current profits are (weakly) increasing in (the natural linear order of) ω. Hence many models of the interaction among firms (including price taking "competitive" models) abide by our assumptions.

To illustrate, we have developed and numerically analyzed two examples. The first is a differentiated product model where spot market equilibrium is Nash in prices (for a discussion of such models see Caplin and Nalebuff, 1991, and the literature cited there). There ω indexes the difference in the distribution of consumer utilities obtained from consuming the firm's product and the utility obtained by spending all of the consumer's income on alternatives outside the industry (see Pakes and McGuire, 1992). The second example is a homogeneous product model in which equilibrium is Nash in quantities and firms differ in their production efficiencies. Here ω indexes the difference between the firm's index of production efficiency and an index of factor cost. The second example is presented, and briefly compared to the first, in Section IV of this paper.

The dynamics of the model are generated by the stochastic outcomes of the firms' own investments and the outcome of an exogenous process reflecting improvements made by competition outside the industry. Outcomes of this exogenous stochastic process generate a correlated non–positive stochastic shift in all the firms' ω's, reflecting increases in the quality of goods outside the industry that vie for the consumer's dollar (and/or increases in labor costs). It is, therefore, a source of continuous dynamic competitive pressure that forces all firms in the industry to struggle to maintain profits and survive. Note that this exogenous process can induce a positive correlation in the profits of different firms in the same industry, a phenomenon we often observe in data (Without the exogenous process, any outcome which leads to an increase in profits for one firm would necessarily reduce the profits of its rivals). Also, it is assumed that the outcomes of the exogenous process generating increases in the knowledge stock outside the industry are embodied in the new generations of potential entrants to this industry; otherwise entry would eventually die out, and with it the industry. That is, the new generation of entrants brings with it knowledge which was not available to previous generations.

Entry takes place at some $\omega^0 \in \Omega^e$, a set of states that in varying degrees reflects relevant developments/advances outside the industry. A new entrant incurs a sunk cost of entry, x^e, and then takes a full period to set up the specific fixed capital with which it enters. The precise state of entry depends on the "quality" or "efficiency" of the entering firm, i.e. on how "good" its idea or innovation is relative to the achieved standards of the industry. This we assume to be unknowable ex–ante; an idea must be tried, and time, money and effort invested, before competitiveness can be precisely known. Hence there is only a common knowledge distribution, $P(\omega^0)$, over the potential entry states, Ω^e, indicating the uncertainty of both entrants and incumbents as to the competitiveness of potential entrants.[9]

Exit from the industry takes place when firms realize that the expected present discounted value of remaining in the industry is less than the opportunity cost of remaining, i.e. less than the resale/scrap value of the assets currently employed there. This naturally occurs in "low" ω–states; just how low depends on the strength of competition in

[9]We assume Ω^e is bounded above, implicitly restricting the amount of progress that can be made in the area/niche of this industry while remaining outside the industry, and will show that there is a lower bound such that such that rational entry would never occur below that ω.

the industry (i.e. on s). Indeed, without a continuing series of "successes" in investment, the negative drift imposed by outside competition and new entrants will *inevitably* drive incumbents to exit. Thus the industry is in a continual state of flux, in a sense that we make precise below, with entry, exit, growth, decline, and reversals of fortune continually occurring.

B. The Assumptions.

The opportunity presented to each firm by the industry is defined by the following set of model primitives, which are common knowledge to all actual and potential participants.

$$\{A(\omega,s),\ p(\omega'\,|\,\omega,\cdot),\ q_{\omega}(\hat{s}'\,|\,s),\ [m(s),\ P(\omega^0),\ \{x^e_m\}_{m=1}^{\infty}],\ \phi,\ c(\omega),\ \beta\}_{(\omega,s)\in\Omega\times S}.$$

We first describe these objects, and then present the formal assumptions required for our general model.

The state space is $\Omega\times S \subset \mathbb{Z}\times\mathbb{Z}^{\infty}_{+}$, where S is a set of counting measures on $\mathbb{Z}$. The structure of the industry, that is s, the list which counts the number of firms in each state ω, is just such a measure. The function $A(\omega,s)$ gives the payoff or profits of a firm from its current production and sales activities. It is a reduced form, reflecting the equilibrium of the industry spot market, and its detailed characteristics can vary from example to example. $p(\omega'\,|\,\omega,x)$ is the given firm's probability transition function which depends on the level of its investment activity, $x \in \mathbb{R}_{+}$. It gives the probability of shifting into state ω', conditional on being in state ω and investing amount x. $q_{\omega}(\hat{s}'\,|\,s)$ provides the firm's beliefs about the transition probabilities for the other firms in, or entering, the industry, given that the firm under consideration is in state ω. Here $\hat{s} \equiv s-e_{\omega}$, where e_{ω} is a vector with one in the ω–th place and zero elsewhere, so that $\hat{s}$ is a measure providing the location of the firm's competitors. Thus next period's industry structure will be $s' = \hat{s}' + e_{\omega'}$, where $\hat{s}'$ includes any new entrants and ω' is the new state achieved by the firm in question.

The triple $[m(s),P(\omega^0), \{x_m^e\}_{m=1}^{\infty}]$ characterizes the conditions of entry into the industry. The number of entrants stimulated by any structure (state of competition), s, is given by the function m(s). The initial investment required to begin the process of entry is x_m^e, which may depend on the number of firms entering. Finally, the state, ω^0, at which a new firm enters the industry is determined by the probability distribution $P(\cdot)$ with support [supp(P)] Ω^e.

The parameter ϕ gives the opportunity cost of being in the industry; it is the amount recoverable on exit. The function $c(\omega)$ gives the unit cost of activity level x, so that investment activity costs $c(\omega)\cdot x$, and net revenues or profits are given by:

$$R(\omega,s;x) = A(\omega,s) - c(\omega)\cdot x. \tag{1}$$

Finally, β is the common discount factor of all the agents in the model.

We use the following assumptions for our general results.

A.0) $\omega \in \Omega \subset \mathbb{Z}$; $s \in S \subset \mathbb{Z}_+^{\infty}$, with $\succeq$ a complete preorder on S.

A.1) $\beta \in (0,1)$; $\phi \in \mathbb{R}$.

A.2) $\forall \omega$, $c(\omega) \in [\underline{c},\infty)$, $\underline{c} > 0$.

A.3) $\forall s \in S$, $\lim_{\omega\to\infty} A(\omega,s) = \bar{A} \leq \infty$ and $\lim_{\omega\to-\infty} A(\omega,s) < (1-\beta)\phi$. $A(\cdot)$ is non–decreasing in ω for all s, and is non–increasing in s, ordered by $\succeq$, for each ω. Finally, $\forall\ \omega$, $\forall\ s \in \hat{S}_n(\omega)$, $A(\omega,s) \leq (1-\beta)\phi + o(1/n)$, where $\hat{S}_n(\omega) \equiv \{s \in S \mid \sum_{\omega' \geq \omega} s_{\omega'} \geq n\}$.

A.4) $\forall\omega\in\Omega$, $\forall x\geq 0$, $p(\cdot\mid\omega,x)$ is formed from the convolution of two distributions with finite connected support: $\pi(\cdot\mid\omega,x)$ with $\text{supp}(\pi) = \{\omega' \mid \omega'=\omega+\tau,\ \tau=0,\dots,k_1\}$; $p_0 = \{p_\eta\}_{-k_2}^{0}$ with $\text{supp}(p_0) \subset \{\omega' \mid \omega'=\omega+\eta,\ \eta=-k_2,\dots,0\}$. $\pi(\cdot\mid\omega,x)$ is stochastically increasing, continuous in x, $\frac{\partial\pi}{\partial x}(\omega\mid\omega,x) < 0$,

$\frac{\partial\pi}{\partial x}(\omega'\,|\,\omega,x) > 0$ and concave at each $\omega' \in \{\omega+1, \dots, \omega+n\}$, and $\pi(\omega'\,|\,\omega,0) = \begin{cases} 1 & \text{if } \omega' = \omega \\ 0 & \text{otherwise} \end{cases}$

A.5) m(s) firms enter in each period, $m{:}S \longrightarrow \mathbb{Z}_+$. Each entrant pays $x^e_m > \beta\phi$, nondecreasing in the number of entrants, m. The entry process is completed at the beginning of the succeeding period, when each entrant becomes an incumbent at some state $\omega^0 \in \Omega^e \subset \Omega$ with probability $P(\omega^0) = \sum_{\eta=-k_2}^{0} p_\eta \cdot \pi^e(\omega^0-\eta)$. Ω^e is a compact connected set.

A.6) There exists a regular Markov transition kernel, $Q{:}\mathbb{Z}^\infty_+ \times \mathbb{Z}^\infty_+ \longrightarrow [0,1]$, i.e.:

$$\forall\, B \subset S,\ \forall\, s \in S,\ \sum_{s' \in B} Q(s'\,|\,s) = \text{Prob}\{s_{t+1} \in B\,|\,s_t = s\},$$

with range $\mathcal{S}(s) \equiv \{s'\,|\,Q(s'\,|\,s) > 0\} \neq \emptyset$, such that the functions $q_\omega(\hat{s}'\,|\,s) \equiv \Sigma_\eta\, q_\omega(\hat{s}'\,|\,s,\eta)p_0(\eta)$ are the consistent marginal transition probabilities derived from it for $\hat{s} = s - e_\omega$. The stochastic kernels Q and q_ω have the Feller property, i.e. each maps the space of continuous functions on S, C(S), into itself.

A.7) a. There exists a constant $M < \infty$, such that, for all $s \in S$, $m(s) \leq M$.

b. The set of potential feasible industry structures, $S \subset \mathbb{Z}^\infty_+$, is compact.

(A.3) gives the consequences of spot market competition. Whatever the structural model that lies behind $A(\omega,s)$, we require it to have the property that if we increase the number of competitors with ω's at least as large as the firm's own ω then, eventually, the firm's profits will fall to less than $(1-\beta)\phi$, the annuity value of the recoverable assets obtained by the firm when it exits. Similarly we require that no matter the competition inside the industry, there is sufficient competition from outside that a firm whose ω drops low enough will eventually find its profits to be less than $(1-\beta)\phi$.

(A.4) implies that $\omega' = \omega + \tau + \eta$, where the realization of τ is determined by the outcome of the firm's research expenditures and has a distribution given by $\pi(\cdot\,|\,\omega,x)$, while the realization of η is determined by the outcome of the process determining the value of the outside alternative and has a distribution given by p_0. Consequently $p\{\omega'=z|\omega,x,\eta\} \equiv \pi(z-\eta|\omega,x)$ and $p\{\omega'=z|\omega,x\} \equiv \Sigma_{\eta'}\pi(z-\eta'|\omega,x)p_{\eta'}$. Similarly the distribution of both entering states (in A.5) and of the likely locations of ones competitors (in A.6) are also obtained by first conditioning on η. Note that though a firm can stop investing (choose $x=0$), if $x=0$ the firm's ω cannot improve, and will, in fact, stochastically decay with negative realizations of η. The assumptions on the derivatives of $\pi(\cdot)$ are only used to insure the uniqueness of the firms choice of level of investment. Provided $\pi(\cdot)$ is everywhere continuous and stochastically increasing in x, we could use any other condition guaranteeing that uniqueness.

(A.5) describes the entry process, incorporating the impact of the negative drift on firms engaged in the process of entry. It is essentially a free entry assumption that captures our idea of relevant competition in an industry of the sort modeled here.[10] It also indicates that the real sunk cost of entry is $x^e_\cdot - \beta\phi$ as any entrant could recover ϕ next period by immediately exiting after becoming an incumbent. The last two assumptions are auxiliary in the sense that they are used to restrict agents' perceptions, and then are shown to be natural consequences of an equilibrium given those perceptions. Indeed, in equilibrium, $Q(\cdot|\cdot)$ will be completely determined by $p(\cdot|\cdot,\cdot)$, i.e. $\pi(\cdot)$, p_0, and $P(\cdot)$, and the optimal investment decisions of all firms active or entering the industry. So will $m(s)$ and, given any initial industry structure, the set of industry structures that could possible be realized in any future period.

C. The Incumbent's Decision.

We can now formulate an incumbent firm's decision problem. The incumbent

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[10]There are alot of possible entry assumptions that could be inserted here without affecting the general nature of the theoretical results. We did not delve futher into this both because the free entry assumption seemed the natural place to start, and because so little is known about empirically relevant alternatives.

makes decisions to maximize the expected present value of net cash flows. At any time t, in any state (ω_t, s_t), it must decide to continue or to exit the industry, and if it continues in operation, it must decide how much to invest. It thus solves[11]

$$(2)\ W_t(\omega_t, s_t) \equiv \max\left\{ \sup_{\{x_\tau, \chi_\tau\}_{\tau=t}^{\infty}} E_t\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} R(\omega_\tau, s_\tau; x_\tau)\chi_\tau + (\chi_{\tau-1} - \chi_\tau)\phi \,\Big|\, (\omega, s)\right], \phi \right\},$$

where χ_τ is the continuation decision $[\chi_\tau = 1 \Longrightarrow \text{continue};\ \chi_\tau = 0 \Longrightarrow \text{exit}]$, and $x_\tau \geq 0$ is the amount to be invested, in period τ. Clearly, $\chi_\tau = 0$ implies that, for all $\sigma \geq \tau$, $\chi_\sigma = x_\sigma = 0$. For any given $\{x_\tau, \chi_\tau\}$, the distribution used to form the expectation in (2) can be derived from the firm's perception of the Markov transition kernel for its competitors, $\{q_\omega(\hat{s}' \mid s)\}$, and the controlled Markov process governing the evolution of the firm's own state, $\{p(\omega' \mid \omega, x)\}$.[12]

In any state, the incumbent firm compares the expected present discounted value of remaining in the industry, assuming optimal future decisions, to the opportunity cost of remaining, ϕ. If the latter is larger, it exits, foregoing $R(\omega,s;0)$ and all potential future earnings in the industry. If not, it invests $x \geq 0$, receives $R(\omega,s;x)$, and retains the option of further activity in the industry starting in a new state (ω', s') at the beginning of the next period.

This formulation has an inherently stationary Markovian structure. That is, the current state, (ω_t, s_t), and the current decisions, x_t and χ_t, are sufficient to completely determine its dynamics, i.e. the evolution to the next state, (ω_{t+1}, s_{t+1}). This implies that the optimal investment strategy, if it exists, can be chosen from the class of stationary Markov strategies, vastly simplifying its analysis.[13] Thus we are justified in writing $x(\omega,s)$

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[11]See Chapter 9 of Stokey, Lucas, and Prescott (1989) for more detail on setting up related intertemporal optimization problems.

[12]This distribution can be explicitly written using the Chapman–Kolmogorov equation. See Doob (1953), p. 88.

[13]This is a standard result of the literature on optimization in a Markovian environment. See, for example, Dynkin and Yushkevich (1975), p. 148, or Stokey, Lucas, Prescott (1989), Chapter 9.1.

and $\chi(\omega,s)$; that is, both the investment and shutdown decisions are stationary functions of only the current state (ω,s).

This together with boundedness implies that if a solution exists to the entrepreneur's problem it must satisfy the Bellman equation

$$(3)\ V(\omega,s) = \max\left[\sup_{x\geq 0}\left\{R(\omega,s;x)+\beta\cdot\sum_{\eta'}\sum_{s'}\sum_{\omega'} V(\omega',s')p(\omega'\,|\,\omega,x,\eta')q_{\omega}(\hat{s}'\,|\,s,\eta')p_{\eta'}\right\}, \phi \right]$$

as can readily be seen by substitution. In any state the optimal policy thus involves first choosing a level of investment that maximizes the expression in braces on the r.h.s. of (3). This requires selecting an investment level equalizing current marginal costs with the marginal change in the expected present value of the states that might be realized next period. When the expected future value generated by optimal investment is less than or equal to the opportunity cost of the entire enterprise, ϕ, then the optimal decision is to liquidate the enterprise. For the model to be well formulated we need to show that (2) has a solution so that $V(\omega,s)$, $\chi(\omega,s)$, $x(\omega,s)$ are well defined.

D. The Entrant's Decision

An entrant faces a similar optimization problem, with the added uncertainty as to where he will be, once in the industry. Entry decisions are taken at the beginning of each period, and the process of entry takes a full period (A.5). Thus firms deciding to enter in period t become incumbents at the beginning of period $t+1$. Attempted entry is always successful upon payment of the sunk cost, x^e_m, which depends on the number of firms, $\mathbf{m}$, entering at t. As an incumbent at some ω^0, the new firm at $t+1$ invests (or exits) to solve (3), i.e. to generate the maximal value, $V(\omega^0,s_{t+1})$, where s_{t+1} includes all entrants from the preceding period.

Though always possible, entry into the industry is not always desirable. Any potential entrant must evaluate the expected value of optimal behavior in the industry, labeled $V^e(s,m)$, relative to the cost of entry, x^e_m, both of which depend on the number of new firms entering in that period. Note that this is an expectation over all the states $\omega^0 \in \Omega^e$ at which the firm might enter, and is the same ex–ante for all potential entrants.

Assumptions (A.5) and (A.6) imply that

$$(4)\ V^e(s,m) \equiv \beta \cdot \sum_{\eta'} \sum_{s'} \sum_{\hat{\omega}_m} \sum_{\omega^0} V(\omega^0, s' + e_{\omega^0} + \hat{\omega}_m) \cdot \pi^e(\omega^0 - \eta') \cdot \prod_{j=1}^{m-1} \pi^e(\omega_j^0 - \eta') \cdot q^0(s' \mid s, \eta') \cdot p_{\eta'}$$

$$\geq \phi,$$

where $\hat{\omega}_m \equiv \sum_{j=1}^{m-1} e_{\omega_j^0}$, and $q^0(\cdot \mid \cdot)$ is the marginal of $Q(\cdot \mid \cdot)$ for incumbents only.[14] The given firm enters at ω^0 with probability $P(\omega^0)$. The other $m-1$ entrants come in each at their own ω_j^0 according to the same probability distribution, adding the vector of entrants, $\hat{\omega}_m$, to the old incumbents new structure s'.

If $V^e(s,m) \leq x_m^e$ for all $m \geq 1$, then no entry can optimally take place: the expected value of being in the industry at some ω^0 cannot justify the sunk cost of even one entrant. We assume that the ex–ante identical firms enter sequentially until the expected value of entry falls sufficiently to render further entry unprofitable. That occurs when $V^e(s,m+1) - x_{m+1}^e \leq 0 < V^e(s,m) - x_m^e$, so that m is the number of new firms that rationally enter that period. Formally, the number of entrants into any industry structure s is thus the (single–valued) function:

$$(5) \qquad m(s) \equiv \begin{cases} 0 & \text{if } V^e(s,m) \leq x_m^e \quad \text{for all } m \geq 1 \\ \min\{m \in \mathbb{Z}_+ \mid V^e(s,m) > x_m^e,\ V^e(s,m+1) \leq x_{m+1}^e\} & \text{otherwise} \end{cases}$$

This is the primary consequence of rational optimization on the part of potential entrants into this industry; given an existing structure s, $m(s)$ new firms find the industry potentially profitable. For the model to be well formulated we need to show that $m(s)$ is well defined and finite for all $s \in S$ as assumed in (A.7.a).

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[14] $q^0(\cdot)$ is given by a multinomial distribution from the $|s|$ independent transitions with probabilities $p(\cdot \mid \cdot,\cdot)$, ignoring the entrants induced by the structure s. See the Remark in Section III.

E. The Equilibrium.

We study the dynamic equilibrium of the industry; spot market equilibrium is subsumed in the reduced form profit function, $A(\omega,s)$. The dynamic equilibrium arises from the competitive interaction of independent firms both within and entering the industry. All firms know the structure of the industry, s, their place in it, ω, and the likely impact of their own investment on their future values of ω. Firms also have perceptions, embodied in $q_{\omega}(\cdot)$, about how the structure of the industry, and hence the states of its competitors, will change. However, the actual process driving the change in that structure is a consequence of the unobserved decisions of others both in and entering the industry. The industry is said to be in dynamic equilibrium when the process generating the change in industry structure is accurately reflected in the perceptions of each of the firms entering or active in the industry. Thus the equilibrium is one of "rational expectations."

In equilibrium incumbents act so as to maximize their value (3) of being in the industry, conditional on the true distribution of future states generated by the optimal behavior of all incumbents and potential entrants, exiting when that value is less than the value of alternative uses of their resources, ϕ. New competitors enter until the value of entry (4), also calculated conditional on this true distribution, is less than the sunk cost of entry. That optimal behavior is characterized in the next section where it is understood that the distribution of future states that is used in making decisions is this rational expectations equilibrium distribution.

Formally, we define an equilibrium for this industry as the 6–tuple,

$$\left[\{V(\omega,s),\ x(\omega,s),\ \chi(\omega,s),\ Q(s'\,|\,s),\ m(s)\}_{(\omega,s)\in\Sigma},\ s^0\right], \tag{6}$$

with $\Sigma \equiv \Omega \times S$ and $\Omega \equiv (0, \ldots, K)$, $K < \infty$, such that

6.a) $\forall\ (\omega,s) \in \Sigma$, $V(\omega,s)$ satisfies (3):

$$V(\omega,s) = \max\Big[R(\omega,s;x(\omega,s)) + \beta\cdot\Big\{\sum_{\eta'}\sum_{\omega'}\sum_{\hat{s}'} V(\omega',\hat{s}'+e_{\omega'})\times$$
$$\times q_{\omega}(\hat{s}'|s,\eta')p\big[\omega'|\omega,x(\omega,s),\eta'\big]p_{\eta'}\Big\}, \Phi\Big]$$

6.b) $\forall\,(\omega,s) \in \Sigma$, $x(\omega,s)$ and $\chi(\omega,s)$ solve (3) and satisfy:

$$\Big\{-c(\omega) + \beta\cdot\sum_{\eta'}\sum_{\omega'} \hat{V}(\omega'|\omega,s,\eta')\cdot p_x(\omega'|\omega,s,\eta')\cdot p_{\eta'}\Big\}\cdot x(\omega,s) = 0,$$

$$\{V(\omega,s) - \Phi\}\cdot[\chi(\omega,s) - 1] = 0,$$

where $\hat{V}(\omega'|\omega,s,\eta') \equiv \sum_{\hat{s}'} V(\omega',\hat{s}'+e_{\omega'})q_{\omega}(\hat{s}'|s,\eta')$;

6.c) $\forall\,(s',s) \in S\times S$, $\qquad Q(s'|s) \equiv \sum_{\eta} p_{\eta}\cdot Q_{\eta}(s'|s)$, with

$$Q_{\eta}(s'|s) \equiv \sum_{Y\in\mathcal{Y}(s'|s)} \prod_{j=0}^{K} m_{\eta}(y_{0j},\dots,y_{Kj}|s_j)\cdot m^{e}_{\eta}(y_{0j},\dots,y_{Kj}|m(s)),$$

where $Y \equiv [y_{ij}] \in \mathbb{Z}_{+}^{(K+1)^2}$, y_{ij} is the number of firms shifting to s'_i from s_j, $\mathcal{Y}(s'|s) \equiv \{Y\in\mathbb{Z}_{+}^{(K+1)^2} |\ Y\cdot e = s',\ e\cdot Y = (m(s),s)\}$, $m_{\eta}(y_j|s_j)$ is the multinomial probability of $y_j = (y_{0j},\dots,y_{Kj})$ firms out of s_j going to the states $i=0,\dots,K$, conditional on η, and $m^{e}_{\eta}(y_j|m(s))$ is the same for the $m(s)$ new entrants;[15]

6.d) $\forall\, s \in S$, equation (5) determines the number of entrants, $m(s)$: $\forall\, t$, $m_t >$ > 0 if and only if $x^e_1 \le V^e(s_t,1)$ [defined in (4)], where $m_t =$ $= m(s_t) \equiv \min\{\, m\in\mathbb{Z}_{+} \mid x^e_m \le V^e(s_t,m),\ V^e(s_t,m+1) < x^e_{m+1}\,\}$;

6.e) There is an exogenously given initial state, $s^0 \in S$. □

<u>Remark</u>: Note that we are assuming in this definition that the number of states can be bounded above and below; that is indeed proved in Proposition 1 below. The optimal policy, $\{x(\omega,s),\chi(\omega,s)\}$ and (A.4) together define Markov transition probabilities from

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[15] Y is a matrix summarizing one way that the vector s' might have been generated, and $\mathcal{Y}(s'|s)$ is the set of all feasible such matrices. A row i of Y shows the numbers of firms moving into state i, while a column of Y, y_j, shows the allocation of firms in state j among next period's states, $i = 0, \dots, K$. The first column shows the allocation of new entrants among the states within $\Omega^e \subset [0,K] \subset \mathbb{Z}_{+}$.

each active state ℓ, to each feasible state j, conditional on each possible value of η as $\pi(j-\eta\,|\,\ell,x(\ell,s)) \equiv p_{j\ell}(\eta,s)$. For every s, equilibrium defines a matrix of transition probabilities for incumbents as $P(s) \equiv \sum_{\eta'} p_{\eta'} P(\eta',s)$ where $P(\eta,s) \equiv [\, p_{j\ell}(\eta,s) \,]_{j,\ell=0}^{K}$. These transition probabilities, together with the distribution of incumbents along the rows of this matrix and the entry rule, determine $Q_\eta(s'\,|\,s)$ in (6.c). To actually compute $Q_\eta(s'\,|\,s)$ we note that the multinomial theorem implies that the s_j firms in state $\omega = j$ allocate themselves among the K+1 possible states, relabeled $\{0, \dots, K\}$, with conditional probabilities, conditional on progress outside the industry (on η) given by

$$(7) \qquad m_\eta(y_j\,|\,s_j) \equiv \left\{\frac{s_j!}{(y_{0,j})!\cdot(y_{1,j})!\cdot \;\cdots\; \cdot(y_{K,j})!}\right\}\cdot \prod_{\ell=0}^{K}\left[p_{j\ell}(\eta)\right]^{y_{j\ell}},$$

A similar expression gives the distribution of the m(s) entrants over the states in $\Omega^e \subset \{0, \dots, K\}$, with conditional probabilities $p_{j0}(\eta)$ given by $\pi^e(\omega^0-\eta)$. Thus $Q_\eta(s'\,|\,s)$, as defined in (6.c), is the probability that optimal investment strategies will generate a shift in the structure from s to s′ conditional on the outside competition making a positive advance of η (implying that all incumbents and entrants in this industry will drift downward by as much as η if their investment efforts fail to yield a counteracting advance). It follows that $Q(s'\,|\,s)$ is the unconditional probability that $s_{t+1}=s'$ when $s_t=s$. Finally, $q_\omega(\hat{s}'\,|\,s,\eta)$ is just the marginal distribution over the competing firms, conditional on η, or

$$(8) \qquad q_\omega(\hat{s}'\,|\,s,\eta) = \sum_{\omega'} Q_\eta(\hat{s}'+e_{\omega'}\,|\,s),$$

as can be seen by direct substitution. □

This is a "rational expectations" equilibrium, as all firms optimize with respect to a given distribution of future states (industry structures), $Q(\cdot\,|\,s)$, and their optimal decisions generate industry transitions with precisely the distribution that they used in their optimizing calculation (6.c). That is, $Q(\cdot\,|\,s)$, the transition probability function determining the evolution of the industry structure, is derived by aggregating the incumbent firms transition probabilities, $p(\omega' | \omega, x(\omega,s))$, where $x(\omega,s)$ is the optimal investment strategy, with the distribution of the m new entrants, $\prod_{j=1}^{m} P(\omega_j^0)$. The dependence of current market returns, $A(\omega,s)$, on structure s (A.3) insures that the spot market for current output clears. Investment, exit (6.b), and entry (6.d) are all optimal given the individual state and industry structure, and that state and structure evolve according to the anticipated distribution, which in turn is generated by that optimizing behavior.

The equilibrium defined above is also a Nash equilibrium in investment strategies defined for all (ω,s)–nodes in the game tree. By assumption, firms take the distribution of outcomes of others' decisions as fixed, thereby choosing their exit and investment decisions independently of others in the industry. That is, firms make decisions on the basis of the *outcomes* of their potential and actual competitors activities; they do not react to, or perhaps even know, the extent of their competitors investments in those activities. As the optimal strategies and transition probabilities are functions of the payoff relevant states the equilibrium is a Markov Perfect Nash Equilibrium in the sense of Maskin and Tirole (1988a and b). Agents optimize with respect to all payoff relevant state variables, $(\omega,s) \in \Sigma$, solving dynamic programming problems that are interdependent only through those variables. Their investment strategies, $x(\omega,s)$, remain optimal at every state, regardless of how that state was reached, against the optimal decisions of all other agents.[16]

At the heart of this dynamic equilibrium is a (time homogeneous) Markov process, $(S, Q(\cdot | \cdot), s^0)$, on the space of industry structures (counting measures of firms in the industry), S, defined by Q, a transition kernel determining the distribution of s_{t+1} conditional on all alternative possible values of s_t, and by s^0, the initial state (see Section III.C below). A realization of this process is a unique sequence $\{s_t\}_{t=0}^{\infty}$ where s_0

[16]This is an immediate consequence of the dynamic programming formulation and the consistency of all firms problems at equilibrium.

$= s^0$ and s_t is a realization from the distribution $Q(\cdot|s_{t-1})$. Associated with each such realization of this process are the sequences: $\{m_t\}$, the optimal entry process derived from (6.d); $\{\underline{\underline{\omega}}_t\}$, the highest exit states defined by $\underline{\underline{\omega}}_t = \max\{\ \omega|\ \chi(\omega,s_t)=0\ \}$; and $\{f_t\}$, the number of firms that exit in period t, $f_t \equiv \sum_{\omega=0}^{\underline{\underline{\omega}}_t} s_{\omega,t}$. The notion of equilibrium guarantees that the distribution of these sequences is generated by the optimal investment strategies of both incumbents and potential entrants and that the spot market for current output always equilibrates.

All decisions within a period are understood to be taken simultaneously, based on common knowledge of the industry structure , s_t, the number of entrants that this structure will call forth, $m_t = m(s_t)$, the exit states that the structure generates, $\{\omega|\ \omega\leq\underline{\underline{\omega}}_t(s_t)\}$, and the distribution of future states that will arise from that structure, $Q(\cdot|s_t)$. While m_t new firms are entering, incumbents either rationally exit $[\chi(\omega,s_t)=0]$ or invest $x(\omega,s_t) \geq 0$ generating their transition probabilities which at equilibrium will collectively, when combined with the distribution of new entrants, precisely coincide with those given by the common knowledge distribution $Q(s'|s_t)$. This yields the new industry structure at the beginning of the next period in which again entry, exit, and investment decisions will be made. We emphasize that in equilibrium all decisions are made optimally by firms that are fully cognizant of the structure of the industry and the distribution of its future evolution. To close the model we need to show that these decisions can be consistently taken, i.e. that such a stochastic dynamic equilibrium exists.

III. RESULTS.

A. Characterizing Optimal Agent Behavior.

The primary agent in this model is an incumbent firm; the first result characterizes its behavior. It begins by showing that an optimal solution exists to the decision problem (2), giving well defined investment and exit decisions and a well defined value to the firm

(3), and then characterizes the optimal policies.[17] Entrants are distinguished only in the initial period of their entry; thereafter they are incumbents. In our simple formulation the only question that needs to be answered about their behavior is how many find it profitable to sink x_m^e in order to enter the industry. Our second result shows that it is finite in any period and indeed will be zero if competition within the industry is sufficiently strong. In combination the results of these propositions imply that the state space S is compact, as assumed (A.7.b) for some of the results characterizing incumbent behavior. These results also allow us to show the consistency of our assumptions about the industry structure transition probabilities (A.6), setting the stage for a proof of existence of our rational expectations equilibrium in the next subsection.

<u>Proposition 1</u>: Consider the firm's decision problem (2). Under assumptions (A.0) through (A.7):

a) There exist: (i) a unique $V(\omega,s)$, $V:\mathbb{Z}\times\mathbb{Z}_+^\infty \to \mathbb{R}_+$, monotonic increasing in ω, uniformly bounded, and satisfying (3); (ii) an $\bar{x} < \infty$ and a unique optimal investment policy (function), $x(\omega,s)$, $x:\mathbb{Z}\times\mathbb{Z}_+^\infty \to \mathbb{R}_+$, with $x(\omega,s) \leq \bar{x}$; and (iii) an optimal termination policy $\chi(\omega,s)$, $\chi:\mathbb{Z}\times\mathbb{Z}_+^\infty \to \{0,1\}$; solving (2) [or (3)] for $\forall\,(\omega,s) \in \mathbb{Z}\times\mathbb{Z}_+^\infty$.

b) There exist two finite boundaries in $\mathbb{Z}\times\mathbb{Z}_+^\infty$, $\underline{\omega}(s)$ and $\bar{\omega}(s)$, such that $x(\omega,s) = 0$ if $(\omega,s) \in C \equiv C_\ell \cup C_u$, where $C_\ell \equiv \{(\omega,s)\,|\,\omega < \underline{\omega}(s)\}$ and $C_u \equiv \{(\omega,s)\,|\,\omega > \bar{\omega}(s)\}$, and there exists a finite lower bound $\underline{\underline{\omega}}(s) \in \mathbb{Z}$ such that $\chi(\omega,s) = 0$ if and only if $(\omega,s) \in \{(\omega,s)\,|\,\omega \leq \underline{\underline{\omega}}(s)\} = L$. Further, $\inf_s \underline{\underline{\omega}}(s) > -\infty$, and $\sup_s \bar{\omega}(s) < \infty$.

c) There exists a random variable, $T:\mathbb{Z}\times\mathbb{Z}_+^\infty \to \mathbb{Z}_+$, $T(\omega_0,s_0) = \inf\{t \geq 0\,|\,(\omega_0,s_0) = (\omega_0,s_0)$ and $(\omega_t,s_t) \in L\}$, associating each initial state, (ω_0,s_0), with the first time, t, such that $\chi_t \equiv \chi(\omega_t,s_t) = 0$, where (ω_t,s_t) is the state achieved in period t

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[17]See, for example, D. Blackwell (1965) and E.V. Denardo (1967). The now standard textbook discussion is in Stokey, Lucas, and Prescott (1989).

under the optimal policy $\{x(\omega,s),\chi(\omega,s)\}$. $T(\omega_0,s_0) < \infty$, a.s. and is stochastically increasing in ω.

PROOF: See Appendix. □

An incumbent firm in state ω facing an industry structure s has an expected present discounted value of $V(\omega,s)$. When $V(\omega,s) = \phi$ it will optimally exit the industry. This is the case at all (ω,s) with $\omega \leq \underline{\underline{\omega}}(s)$. Hence we will never observe a firm with an efficiency less than $\underline{\underline{\omega}} = \min\{\, \underline{\underline{\omega}}(s) \mid s \in S \,\}$. When $V(\omega,s) > \phi$, the firm pursues an optimal investment policy, $x(\omega,s) \in [0,\bar{x}]$, earning a current cash flow of $R(\omega,s) = A(\omega,s) - c(\omega)x(\omega,s)$. Part b) of the proposition proves the existence of boundaries $\underline{\omega}(s)$, and $\bar{\omega}(s)$, such that $x(\underline{\omega}(s)-\tau,s) = x(\bar{\omega}(s)+\tau,s) = 0$, for all $\tau \geq 1$. Since ω cannot increase in value without some investment (A.4), and the distribution of increments to ω has finite support, an immediate consequence of this optimal behavior is that we will never find a firm at ω–states higher than $\bar{\bar{\omega}} = \max\{\bar{\omega}(s)+k_1 \mid s \in S\}$. It follows that, together, (A.4), (A.7), and Proposition 1 imply that the relevant set of states is the compact, connected interval $\{\underline{\underline{\omega}}, \dots ,\bar{\bar{\omega}}\} \subset \mathbb{Z}$. Hence the compactness of Ω in our definition of equilibrium (6) is satisfied and, by relabeling, we can set $\Omega = \{0,1, \dots ,K\}$. The space of admissible structures, then, is surely no greater than (K+1)–dimensional: $S \subset \mathbb{Z}_+^{K+1} \subset \mathbb{Z}_+^{\infty}$.

The results in (a) to (c) provide a fairly detailed characterization of incumbent behavior. Part (a) guarantees that incumbent behavior is well defined and shows that the valuation of optimal behavior satisfies the natural monotonicity property in ω; greater success gives a higher value. Part (b) gives two types of "coasting" states, $\mathcal{C}_u$ and $\mathcal{C}_\ell$ in which the firm neither invests in, nor exits from, the industry. Coasting in "successful" states, $\mathcal{C}_u$, reflects the optimal response to a situation in which the expected marginal gain to further advance is outweighed by the marginal cost of further investment, $c(\omega)$. Recall that the return to investing is an increase in the probability of transiting to higher ω. The value of these increments is given by the "slope" of the value function. Since the value function is bounded that slope must eventually become less than the marginal cost of (even zero) investment. There are also states in which $A(\omega,s)$ is *low*, $x(\omega,s)$ goes to zero, and yet the firm does not leave the industry. Indeed the firm can choose to stay in the industry even in situations where it is optimal to shut down current production (possibly incurring a fixed cost for mothballing its plant). In these cases fixed costs are incurred, and exit values are foregone, because of the likelihood that an improved future condition $(s_{t+1} \prec s_t)$ will lead to a situation where it pays to produce and invest again.

There is, however, a limit to such lower coasting. When $(\omega,s) \in C$, $E(\Delta\omega | \omega,s) <$ < 0 as $x(\omega,s) = 0$, and hence ω drifts lower with probability $\sum_{\eta<0} p_\eta$ (A.4). This will reduce the value of the enterprise, $V(\omega',s')$, unless there is a countervailing shift in s so that $s' \prec s$. Indeed, without a random "improvement" in s, parts (b) and (c) insure that the firm will enter a true "liquidation state," $(\omega,s) \in L$, where $V(\omega,s) = \phi$ indicates the optimality of exit from the industry. That optimal liquidation ultimately occurs in finite time with probability one, despite the possibility of exogenous improvements, is the principal content of part (c).

Proposition 1 characterizes behavior in an industry in which active exploration and learning through investment is required for survival. We know that eventually all firms will die, but the life cycle of the firm can include a variety of different types of activity, including periods of active struggle and learning $[x_t > 0]$, with its successes $[\omega_{t+1} > \omega_t]$ and failures $[\omega_{t+1} \leq \omega_t]$, periods of coasting on the successful outcomes of past efforts wherein no exploratory investment takes place but profits are derived from previous development, and, possibly, periods of coasting wherein a firm does not earn any profits and its current prospects do not warrant further investment, but there is some probability that the market will "improve" $[s_{t+1} \prec s_t]$, and that probability is enough to deter the firm from exit. Due to outside competition $[p_0]$ and entry $[\omega_m]$ the state is inexorably moving in a direction unfavorable to the firm. Only through active investment $[x > 0]$

can the firm hope to counteract this pressure. Yet, despite its best efforts, the firm must eventually succumb and liquidate, even though phenomenal profits may have been earned in the interval between birth and death. This situation is schematically illustrated, along with several possible sample paths for a typical firm, in Figure 1.

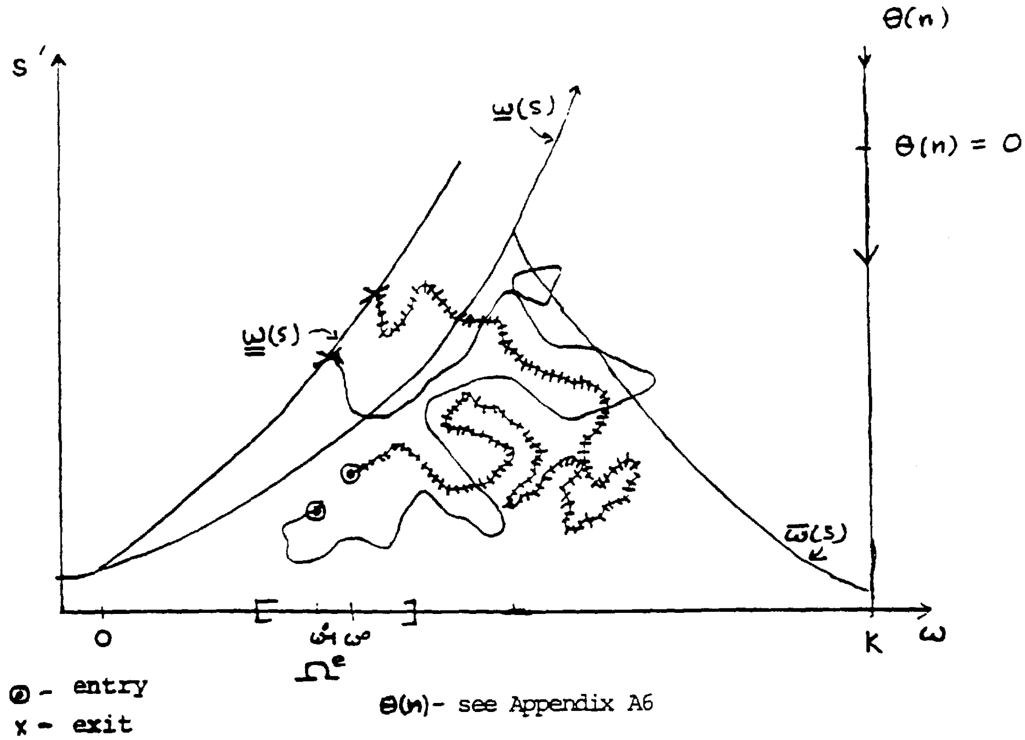
Despite the finite life of firms, it might be possible for entry rates to generate an industry that can grow without bound. That is, it might be possible for either a countable set of firms to decide to enter at some s $[V(\omega,s) > x^e_m,\ \forall\ m\geq 1]$, or for there to be a steady excess of entrants over exitors which would cause $|s|$ to grow without bound, thus violating (A.7) and much of Proposition 1. We note that if $V(\cdot)$ were isotone to $\succeq$ on S [i.e. $\forall\omega,\ s_1 \succ s_2 \Longrightarrow V(\omega,s_1) \prec V(\omega,s_2)$], then new entrants would increase s' driving $V(\cdot,s')$, and hence $V^e(s,m)$, down, eventually choking off entry. Unfortunately, the subtleties generated by the interactions among agents (particularly in entry deterrence), imply that it is not in general true that $V(\cdot)$ is isotone in s, so that one cannot use this fact to stop entry (or induce exit) as the number of firms in the industry grows.

To bound the number of firms in the industry, we provide a direct proof of the fact that $V(\omega,s)$ can be made arbitrarily small by increasing the number of active firms in the industry. This will imply that $m(s)$ is finite for all $s \in \mathbb{Z}^{K+1}_+$ and that there exists an $N < \infty$ such that

$$(9) \qquad S \equiv \left\{ s\in\mathbb{Z}^{K+1}_+ \;\middle|\; |s| \equiv \sum_{\omega=0}^{K} s_\omega \leq N \right\},$$

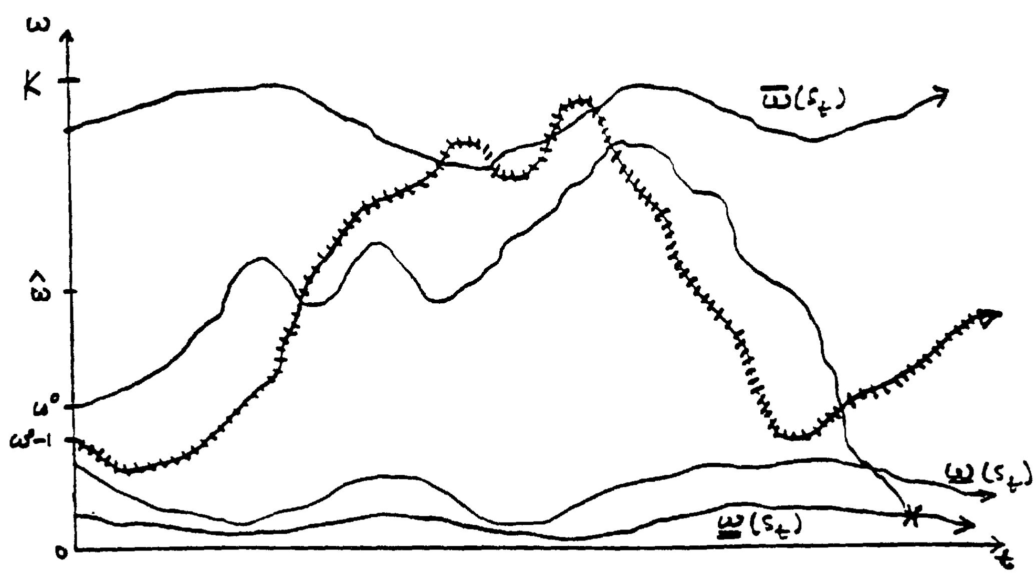
i.e. S is compact. Hence (A.7) is justified and w.l.o.g. we can normalize the full state space to $\Sigma = \{(\omega,s) \in \mathbb{Z}\times\mathbb{Z}^{K+1}_+ |\ \omega \in \Omega \subset \mathbb{Z}_+,\ s \in S \subset \mathbb{Z}^{K+1}_+ \}$, where $\Omega = \{0, \dots ,K\}$ and s counts the finite number of firms in each such possible state. This is a key step in showing the existence of a competitive equilibrium in this model.

To prove this we fix ω and an arbitrary structure s and consider a sequence of industry structures that increases the number of firms at ω, i.e. $\{s_n(\omega)\}_{n=1}^{\infty}$, where $s_n(\omega) = s + n\cdot e_\omega$. The following proposition shows that no matter which ω and s we a)



State Space Dynamics

b)



Time Dynamics

Figure 1.

fix, as n increases, the value to being in the industry at that ω falls to the exit value. Consequently enough entry will, in fact, choke off further entry, and there can never be more than a finite number of firms at any ω.

PROPOSITION 2: Let $s_n(\omega) \equiv s + n \cdot e_\omega$. Under Assumptions (A.0) to (A.6), for all $\omega \in \Omega$, and all $s \in \mathbb{Z}_+^\infty$: $\lim_{n \to \infty} V(\omega, s_n(\omega)) = \phi$, i.e. $\forall \epsilon > 0 \ \exists n_\epsilon$ such that $n \geq n_\epsilon$ implies $V(\omega, s_n(\omega)) < \phi + \epsilon$.

COROLLARY 2.1: There exists an $M < \infty$ such that, $\forall\, m \geq M$, $V^e(s,m) \leq x_m^e$, $\forall\, s \in S$.

COROLLARY 2.2: There exists an $N < \infty$ such that $V^e(1,s) < x_1^e$, i.e. $m(s) = 0$, for all $s \in \hat{S}_n(1)$ with $n \geq N$.

PROOF: See Appendix. □

This is a quite strong result. It shows first that the number of entrants is to any structure is surely finite: (A.7.a) holds. Second, it shows that there can be no rational entry if there are sufficiently many firms in the industry. That is, with a sufficiently large number, N, of firms in the industry, regardless of their individual states of success, $V^e(\cdot, s)$ will be so small that no new firms will enter. Indeed, we show that for some finite yet sufficiently large N, $V(\omega, s) = \phi$ [Lemma 1 in Appendix]. Thus exit must take place, while none can enter. This insures a compact state space for the industry structure; provided that the initial industry structure has no more than N firms, there will never be more than N firms in the industry: (A.7.b) holds. Note also that the fact that S is finite makes it possible to compute equilibria for our model (see Section IV below).

B. EXISTENCE OF EQUILIBRIUM.

We can now prove the existence of a rational expectations equilibrium for this model of active exploration and learning through investment. This closes the model by showing that the assumptions on the industry structure and its evolution used to determine optimal behavior are in fact consistent with that behavior. To do so, we show that given $Q(\cdot|s)$, as defined in (6.c), the optimal decisions of incumbents solving (3) and entrants (satisfying equation 4) generate transition probabilities which aggregate to form $Q(\cdot|s)$ as defined in (6.c). This requires a fixed point argument that is presented in the proof (in the Appendix) of Theorem 1. In essence, it involves showing that investment, entry and exit decisions depend continuously on the distribution of future states which in turn depends continuously on those decisions. The continuous compound function maps a compact, convex space of probability distributions into itself, and hence has a fixed point: a rationally anticipated Markov transition function $Q(\cdot|s)$ as defined in (6.c).

Before presenting a formal statement of our existence theorem, we note that assumptions (A.6) and (A.7) need no longer be imposed; they are now a consequence of the more basic assumptions, and our definitions of equilibrium transitions and entry decisions (6.c–d). (A.7) was shown to hold in the corollaries to Proposition 2, while (A.6) follows from the following proposition.

PROPOSITION 3: Under assumptions (A.0) – (A.5), assumption (A.6) holds with $Q(\cdot|\cdot)$ defined using (6.c) and (7), when $q_\omega(\hat{s}'|s)$ is defined by equation (8).

PROOF: See Appendix. □

Thus the last two assumptions in Proposition 1 were made merely to facilitate analysis of a single firm in the industry: they are natural consequences of our equilibrium formulation.

We can now state the result that closes this model of active exploration and learning through investment.

THEOREM 1: Under Assumptions (A.0) – (A.5) there exists an equilibrium (6), satisfying conditions (6.a–e).

PROOF: See Appendix. □

This theorem shows both existence of equilibrium and that the preceding results for a firm in the industry are valid at equilibrium. Due to the autonomous structure of the model the equilibrium is characterized by stationary valuation of states, stationary optimal investment strategies, and stationary Markov transition probabilities. Yet the sequence of states for any firm, and, indeed, the sequence of (almost surely finite) structures for the entire industry, are truly random realizations from an underlying stochastic structure. This structure is determined by the precise nature of the underlying parametric forms of the model, and by our equilibrium conditions. We now turn to its analysis.

C. EQUILIBRIUM DYNAMICS.

This dynamic equilibrium is characterized by a remarkable degree of flux. Active firms are truly heterogeneous, distinguished by their "state of success," ω, and have truly idiosyncratic outcomes to even identical investment decisions. Multiple rank reversals (according to any criteria of interest, eg. sales, profitability) are possible during the life of any collection of firms (cohort), as well as simultaneous entry and exit $[\underline{\underline{\omega}}(s) < \omega^0 \in \Omega^e]$. All firms die almost surely in finite time, yet new firms continually enter to try their skill and luck in the evolving industry. Thus the structure of the industry can change dramatically over time, though we know that it must remain everywhere finite [Corollary 2.2]. In view of this continual change, the question arises as to whether there is any useful characterization of the "average" structure of the industry and its relation to the industry's long run evolution.

Among the things that we would like to know are whether the industry structure settles down into some recurrent pattern and, if so, the characteristics of that pattern. For example, does the industry survive forever, or might it fade away as fewer and fewer firms enter while old firms exit one after another? If the industry does survive, is there a sense in which we can speak of a long–run average number of firms, or structure, for the industry? What determines these and other characteristics of the process defined by the industry equilibrium, and how do they change in response to perturbations of various environmental and policy parameters? This section proves a result which lies at the heart of our ability to answer these questions: we prove the ergodicity of the stochastic process defined by the industry equilibrium.[18] Some direct implications of this ergodicity will be noted outright, but many of the more interesting questions, questions about the nature of the ergodic distribution for example, will depend on detailed characteristics of the actual functional forms that go into our model. We begin to explore some of these in the examples of Section IV. But they cannot be seriously addressed until a more detailed study of appropriate functional forms has been made, a topic on which we are currently working.

Before turning to a formal analysis, we would like to emphasize two points on its relevance. First, one of the advantages of an explicit dynamic model such as ours is that it allows us to study the distribution of the entire sequence of structures that the industry passes through and not just some notion of a limit structure. Our focus here on long run averages stems from the fact that, at least in the absence of a specific empirical example with a particular value for s_0, if one wants to investigate the effect of a policy or environmental change on the (distribution of the possible) structures of the industry, a natural place to start is to investigate the effects of these changes on the time–average of the structures the industry will pass through. This leads us immediately to the question of whether there is a time average, in particular one that is independent of initial conditions, to which all sequences converge.

Second, for these limiting results to be appropriate, our behavioral assumptions

[18]Here we use ergodicity in the wide sense: a stochastic process is said to be ergodic if it converges to a stationary ergodic process. See Halmos (1956) or Friedman (1970) for the definition and discussion of ergodicity.

might have to provide an adequate approximation to those prevalent in the industry over very long time periods. One could, for example, imagine industries whose investment patterns approximate our noncooperative Nash assumptions in, say, the early period of their life cycle but then switch over to patterns that are better approximated by more complex models of interfirm interaction involving, say, some form of collusion, after certain structures or states are reached.

Formally, the evolution of the equilibrium structure of the industry, s_t, is given by

$$s_t = (I[\{\omega > \underline{\underline{\omega}}(s_{t-1})\}] \cdot s_{t-1})' + \omega_{m(s_{t-1})} \tag{10}$$

where $I[\{\omega > \underline{\underline{\omega}}(s_{t-1})\}]$ is a diagonal matrix whose diagonal elements are either unity [if $\omega > \underline{\underline{\omega}}(s_{t-1})$] or zero (otherwise), $\omega_{m(s_{t-1})}$ is the realization of the counting measure giving the location of firms paying their entry fee in t–1, and 'prime' indicates a realization from the distribution $q^0(\cdot|\cdot)$ [see (4)].[19] Here equilibrium transition probabilities, entry, and exit are defined in (6.b–d). Proposition 2 implies that the state space for this stochastic process, S, is compact, and so has a finite number of elements. Let $Q(s,s')$ be the stationary transition matrix of the equilibrium transition probability function $Q(s'|s)$ defined in (6.c). Then $s \equiv \{s_t\}_{t=0}^{\infty}$ is a Markov process with stationary transitions given by the $|S|\times|S|$–matrix Q and with distribution [sample path probabilities]

$$P_{s^0}\{s_t = \bar{s}_t \text{ for } t = 0, \dots, n\} = e_{s^0} \cdot \prod_{t=0}^{n-1} Q(\bar{s}_t, \bar{s}_{t+1})$$

for a specific path $\bar{s} = (\bar{s}_1, \bar{s}_2, \dots)$ when the process begins in state s^0. Similarly P_ν is the distribution of this Markov process when the initial state has probability ν_s of being in state, i.e. having structure, s. Therefore, the distribution of industry structures evolving from an initial s^0 after n periods can be written

$$\mu_n(s^0)\ [\mu_n(\nu)] = e_{s^0} Q^n\ [\nu \cdot Q^n] \in \Delta^S, \tag{11}$$

---

[19] $q^0(\cdot)$ is given by a multinomial distribution from the $|s|$ independent transitions with probabilities $p(\cdot|\cdot,\cdot)$, ignoring the entrants induced by the structure s. See the Remark in Section III.

where Q^n is the n–th iterate (power) of Q and Δ^S is the $(|S|-1)$–dimensional simplex. That is, $\mu_n(\nu)$ is an $|S|$–vector whose elements $\mu_{n,s}$ give the probability that the structure of the industry, with initial distribution ν, is in state s after n periods.

This notation enables us to formulate our principal result on industry equilibrium dynamics. It is that the evolution of the industry is ergodic; that is, the stochastic process defined by the industry equilibrium possesses a unique limiting distribution of states (of industry structures). Thus, though the sequence of industry structures generated by the equilibrium process remains truly stochastic, never settling down to any limiting structure, the time average of these sequences will, regardless of the distribution we start with, converge to a unique ergodic distribution with probability one.

<u>Theorem 2</u>: Under Assumptions (A.0) through (A.5) at equilibrium (6):

a) The stochastic process $s = \{s_t\}_{t=0}^{\infty} \in (S^{\infty}, \otimes \mathscr{S})$ with initial state s^0 is Markov with stationary transitions $Q(s,s')$ and distribution P_{s^0}, where $\mathscr{S}$ is the σ–field of all subsets of S.

b) The state space, S, contains a unique, positive recurrent communicating class $R \subset S$.

c) There exists a unique, invariant probability measure, μ^*, on S such that

$$\mu^*_s = [\mathrm{m}Q(s,s)]^{-1} \text{ for } s \in R, \text{ and } \mu^*_s = 0 \text{ for } s \in S \backslash R,$$

where $\mathrm{m}Q(s,s')$ is the P_s–expectation of the time of first reaching state s'.

d) $\forall\, s \in S,\ \mu_n(s) \xrightarrow[n\to\infty]{} \mu^*$.

<u>Corollary</u>: P_{μ^*} is the distribution of a stationary, ergodic Markov process with transition Q, i.e. $\mu^* Q = \mu^*$.

<u>Proof</u>: See Appendix. □

Ergodicity of the equilibrium process generating industry structures has a number of implications. First, it implies that the industry structure evolves in a nondegenerate though increasingly regular way over time, so that there never is a "limit" structure of the industry. Indeed, all viable industry structures, that is all structures in the recurrent class $R \subset S$, are realized infinitely often. Thus, just as there is continual flux in the relative position of firms in the industry, there is continual change in the industry structures that those firms comprise. A given industry structure generates investment, exit, and entry decisions as optimal responses to the valuation of the opportunity presented by the industry. The idiosyncratic outcomes of these investment decisions, together with the evolution of the state of competitors from outside the industry, determine the structure of the industry at the beginning of the next period, a structure that is only probabilistically related to the structure which generated it. Though all firms eventually die, entrants replenish the population of active firms, and hence the industry of this model lives forever, eventually going through all the epochs determined by its recurrent states and its transition kernel.

Another consequence of ergodicity is that, after some time, a certain stochastic regularity will appear in the evolution of the industry. If the initial structure is transient, $s^0 \in S\backslash R$, then a finite (a.s.) time will be spent shifting to some recurrent structure, $s \in R$. Thereafter, the portion of time spent in any state $\underline{s} \in R$ will approach the invariant probability of that state, $\mu^*_{\underline{s}}$:

(12) $$\lim_{T\to\infty} \frac{1}{T} \sum I_{\underline{s}} = \mu^*_{\underline{s}}.$$

Thus the structure of the industry, s_t, while shifting randomly in response to the idiosyncratic outcomes of optimal decisions by firms, will spend more time near "natural" states, with a "natural" number of incumbents, entrants and exits. What is "natural" will depend on the values of the underlying parameters of the industry,

$$\theta \equiv \left\{ A(\cdot), c(\cdot), \phi, \beta, \hat{\omega}, \pi(\cdot), p_0, P, x^e_m, \Omega^e \right\},$$

and will be reflected in the mass of the invariant measure over the set of recurrent structures. Thus, over time, structures that are natural or normal for this industry will reveal themselves as more likely by their more frequent occurrence: time averages will approximate state averages, i.e. the ergodic distribution, μ^*.

A final consequence of ergodicity is that the influence of any initial situation systematically fades, becoming irrelevant for the future evolution of the industry. As Theorem 2.d indicates, the actual distribution over industry structures, μ_n, evolving from any initial structure, s^0, (or distribution over structures, ν), converges to the unique invariant distribution, μ^*, hence losing any information that it contained about the initial condition of the industry. Indeed, a strong Markov property [Freedman (1983), Chapter 1.3] holds in this class of models; the future is independent of the past conditional on any measurable (Markov time) event. Thus two possible histories for the industry with different initial conditions (structures), once they intersect in any state, as they must with probability one, have identical distributions over future sample paths conditional on that intersection. Further, this property should be inherited by any firm or subset of firms within the industry. This ergodic characteristic of the model differentiates it from other stochastic dynamic equilibrium models currently in the literature (see, for eg., Jovanovic 1982) and allows us to build a simple nonparametric test for the empirical validity the model presented here (see Pakes and Ericson, 1989).

Theorem 2's general characterization of the stochastic process generating the (equilibrium) dynamics of industry structures gives rise to several more detailed questions. Some revolve around the typical configurations of firms generated by the model, and how these configurations are likely to evolve over time (their likely sample paths). For example, does the unique stationary ergodic distribution, to which the time–average of the industry structures eventually converges (see 2.d), possess a large number of small firms or a small number of large firms? At a more detailed level, are the industry structures of the recurrent class "similar," so that one can think of the industry's structure "settling down" after some finite number of periods. Or does this recurrent class contain very diverse structures, so that no matter how long the time period elapsed since the "startup" of the industry we will still observe the industry structure undergoing distinct evolutionary patterns? To the extent that the recurrent class contains quite divergent industry structures, do the sample paths through these structures typically cycle, and if so, with what periodicity, or are there Poisson–type events that cause relatively quick and sharp discontinuities in the industry structure? Which structures of the recurrent class generate large amounts of simultaneous entry and exit, and which generate periods of high investment? Finally, and perhaps most importantly, how long will it generally take before the industry's structure enters the recurrent class, and through what type of sample paths does an industry typically pass before its recurrent pattern becomes evident?

We have begun to explore such questions, and how their answers change in different policy or environmental settings, in some numerical examples. Some answers appear highly sensitive to precise functional forms (even to parameter values), while others seem more robust to these detailed assumptions. We turn to one such example now, and compare it to others that we have computed elsewhere.

IV. AN EXAMPLE.

As a more detailed example we consider a homogeneous product market having producers with different, but constant, marginal costs. Marginal costs, say θ_{ω}, are determined by the multiple of a firm specific efficiency index and a common factor price index. That is if $s\tau$ and $s\eta$ are the logarithm of the factor price index and of the firm's efficiency index respectively, then $\omega \equiv s\tau - s\eta$ and $\theta_{\omega} = \exp(-\omega)$. Firms' R&D investments are directed at improving their efficiency of production (increasing their $s\tau$). Factor prices $(s\eta)$ are a nondecreasing stochastic process generating a correlated negative drift in the state of the firms in the industry. It is to overcome this drift as well as to undercut competitors in the industry that firms invest.

The spot market equilibrium in this market is assumed to be Nash in quantities.

Consequently market shares and profits (gross of fixed costs) are inversely related to marginal cost, and will be increasing in ω. More formally, letting q_i be firm i's output, $Q = \sum q_i$, and f be the fixed cost of production, the profits of our classic Cournot oligopolists are given by

$$\pi_i = p(Q)q_i - \theta_i q_i - f$$

where

$$p(Q) = D - Q.$$

It is straightforward to show that the unique Nash equilibrium for this problem gives quantities and price as

$$q_i^* = \max\{0, p^* - \theta_i\}, \quad \text{and} \quad p^* = \frac{1}{n^*+1}\left[D + \sum_{i=1}^{n^*} \theta_i\right]$$

where n^* is the number of firms with $q^* > 0$. Current profits can therefore be written as

$$A(\omega,s) = [p^*(s) - \theta_\omega]^2 - f_\omega,$$

where

$$p^*(s) = \frac{1}{n^*+1}\left[D + \sum_{\omega \geq \omega^*} s_\omega \cdot \theta_\omega\right], \quad \text{and} \quad \omega^* = \min\{\omega \mid q_\omega > 0\}.$$

We note that this current profit function is, in many senses, an extreme alternative to the profit function used in the other example of our model which has been numerically analyzed (see Pakes and McGuire, 1992). The latter example considers a differentiated product industry in which all firms have the *same* (constant) marginal costs but are differentiated by the quality of the product they produce; a quality which increases with successful research activity. In that example the spot market equilibrium was assumed to be Nash in prices. We come back to a comparison of these two examples below.

To complete the specification we need to provide our assumptions on the costs and effects of investment, as well as our entry rule. These are

$$\theta_\omega = \gamma e^{-\omega}, \quad \text{with} \quad \pi(\omega' \mid \omega, x) = \pi(\omega' - \omega \mid x) \equiv \pi(\tau - \eta \mid x),$$

$$\pi(\tau \mid x) = \begin{cases} ax/(1+ax) & \text{that } \tau=1 \\ 1/1+ax) & \text{that } \tau=0 \end{cases}, \quad p_0 = \begin{cases} 1-\delta & \text{that } \eta=0 \\ \delta & \text{that } \eta=-1 \end{cases}, \quad c_\omega = c,$$

$$\Omega^e = \{\omega^0 - 1, \omega^0\}, \quad x_1^e = x^e \text{ and } x_m^e = \infty \text{ for } m > 1, \quad \text{and} \quad \omega^e = \omega^0 + \eta.$$

Transitions in ω are determined by the difference between the increment in efficiency of production generated by the outcomes of the firm's own research activity (τ), and the increment in the factor price index (η). τ can either increase by 1 or stay the same. The probability of τ increasing is an increasing function of investment, and the cost of a unit of investment is independent of ω. η also either increases by one or stays the same, but here the probabilities are given by an exogenous process. In each period there is at most one entrant who pays a set up cost of x^e and enters in the following year at state ω^0 if the cost of production has not increased in the interim, and at ω^0-1 if it did (if η for the period was -1). Note that by assuming that the period of time at which we actually observe new data points is larger than the decision period of the model, this specification could allow for both many entrants, and for richer conditional distributions for the changes in ω, per data period (while still maintaining the computational advantages available when there are single step transitions).

It is easy to see that this specification (together with an appropriate choice for β and ϕ) satisfies all of (A.0) through (A.5), and hence that all of the above results hold. Only (A.3) perhaps requires checking for

$$A(\omega,s) = \max\left\{\left[\frac{D + \sum_{k\geq\omega^*} s_k \cdot \gamma e^{-k}}{|s^*| + 1} - \gamma e^{-\omega}\right]^2 - f,\ -f\right\},$$

where $|s^*| = \sum_{k\geq\omega^*} s_k$. Clearly $\bar{A} = (D+\gamma)^2$, and $A(\omega,s)$ is increasing in ω and decreasing in s with the natural vector preorder. In particular, $A(\omega,s) \downarrow -f$ as s_k increases at any $k \geq \omega$, or as ω falls for any s. Note that if $A(\omega,s) = -f$ then marginal cost is greater than price and the firm is not active in the spot market. The same firm can, however, still be a participant in the industry. That is plants will be mothballed without being dismantled if there is sufficient hope that the environment will improve to the extent that it will pay to bring the plant back on line in the future.

As this example satisfies our basic assumptions, all of the results of Section III hold and we have a well defined dynamic equilibrium that generates an ergodic Markov process in industry structures. Of course the purpose of constraining ourselves to the parameterization in our example is to generate more detailed results then those contained in our general propositions. To obtain the more detailed results we substitute the specification given above into the computational algorithm developed in Pakes and McGuire (1992), an algorithm developed explicitly to calculate the equilibria generated by the model developed in this paper, initialize the various parameters, and let that algorithm calculate the policy functions for all (potential and active) firms. Pakes and McGuire (1992) also provide a set of auxiliary programs which help interpret the results. One subset of these programs generate descriptive statistics that describe the industry structures, and the welfare implications, of the Markov Perfect Nash (MPN) equilibrium. A second subset of these programs calculates the optimal policies for both a social planner and a multiplant monopolist (or equivalently a perfectly colluding cartel) faced with the same cost and demand primitives that generate the MPN equilibria, and then generate the descriptive statistics and welfare measurements that emanate from the equilibria obtained from these institutional environments. The colluder makes all decisions (investment, quantities marketed, entry, and exit) to maximize the expected discounted value of the total profits earned in the market. Similarly the planner maximizes consumer surplus (note that there is a question of whether there is a feasible set of institutional arrangements which could lead to an industry which follows either a colluder's or a planner's dictates).

Some of the results from these computations are listed in Table 1 (the footnote at the bottom of the table provides the precise values of the model's parameters that underlie these calculations). All descriptive statistics are obtained from simulation runs starting with one firm at the entry state [i.e. $s_0 = e_{\omega^0}$], and then using the computed policies to simulate from that point. Panel A and B provide descriptive statistics from a 10,000 period simulation run. Panel C provides the distribution of expected discounted values

Table 1: Simulated Quantities From a Homogeneous Product Model*

A. % of Simulated Periods with

	MPN	Colluder	Planner
1 firm active	27.9	92.4	98.3
2 firms active	70.8	7.6	1.7
3 firms active	1.2	0	0
4 firms active	.1	0	0
Entry and Exit	16.5	5.4	1.2
Entry or Exit	20.4	10.0	2.1

B. Average (standard deviation of)

	MPN	Colluder	Planner
price	1.79(.35)	2.22(.36)	**
total investment	1.05(.41)	.68(.29)	.84(.41)
entry	.19	.08	.02
number active	1.74(.48)	1.08(.27)	1.02(.13)

C. Welfare Runs (average and, in parenthesis, standard deviation of)

	MPN	Colluder	Planner
1. Discounted Consumer Benefits	27.4 (6.4)	6.6 (5.5)	**
2. Discounted Net Cash Flow	11.6 (5.4)	22.5 (8.5)	-----
3. Discounted Entry-Exit Fees	2.5 (1.0)	1.0 (1.0)	-----
4. Discounted Welfare	36.5 (11.9)	28.1 (14.1)	58.8

*All runs are based on the specification described in the text with the following parameter values $D = 4$. $f = .2$, $x^e = .4$, $w^o = 4$, $\emptyset = .2$, $c = 1$, $S = .7$, $a = 3$, $\beta = .925$. Panels A and B are obtained from a run which starts with one firm entering the industry, goes 10,000 periods, and then calculates the appropriate descriptive statistics. Panel C is obtained by doing 100 runs, each starting with one entrant and each lasting 100 periods. The appropriate discounted values are taken from each run, and then their averages and standard deviations across runs are computed.

**The welfare result for the planner can be read off the value function which is computed exactly. The planner sets price equal to the marginal cost of the minimum cost producer. This minimum marginal cost averaged .15 with a standard deviation of .39.

from 100 independent simulation runs of 100 periods each.

Panel A indicates that this is an industry which is most often a duopcly, though in a significant fraction (about a quarter) of the periods only one firm is active. Note that "monopoly" positions here are built up solely from successful past research; a firm which is efficient enough will deter entry. Of course, as noted in our theoretical results, even the most efficient of firms will eventually decay and be taken over by more successful competitors. Consequently it is not the same two or three firms that are active in all of the periods. Indeed this industry exhibits substantial entry and exit; there is entry in about 19% of the periods. Moreover entry and exit are positively correlated, a fact which is consistent with the time series evidence in many (though not all) industries (see Dunne Roberts and Samuelson, 1988), and which very clearly brings out the need for allowing for idiosyncratic sources of uncertainty.

Figure 2 provides a section of the optimal investment policy surface. The vertical axis gives the investment of a firm as a function of its own ω (ω_1) and the ω of a competitor (ω_2) when no other firms are active. From the figure it is clear that the firm starts investing at $\omega = 3$ or $\omega = 4$ depending on the value of its competitors ω. Thereafter investment seems to be an initially increasing and then decreasing function of the firm's own ω. It is worthwhile considering this investment pattern in somewhat more detail.

In a separate paper (Ericson and Pressman, 1989) we note that the investment function *must* be an initially increasing and then decreasing function of ω for a monopolist with functional forms for the primitives similar to the one used here. The intuition behind this result follows from the form of the value function. Recall that what investment does in our model is to increase the probability of increments to ω. Consequently investment will increase when increments to ω result in larger increments to the value function. Thus an initially convex and then concave value function will generate an initially increasing and then decreasing policy function. Indeed, the value function for all of our models is initially

Firm 1's Optimal Investment Policy

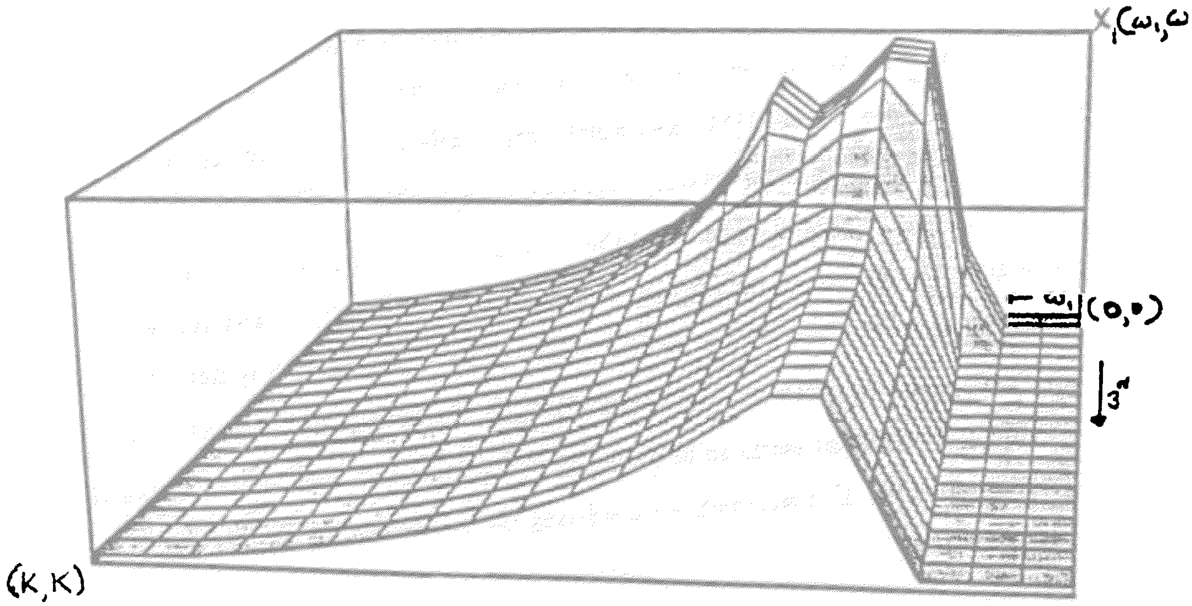


Figure 2.

convex, and eventually concave. This is a result of the fact that it is bounded from below and above (more intuitively, a product or a technique of production, must be developed somewhat before it can be used profitably thus generating the initial convexity, and a good enough product will eventually take over the whole market, thus generating the eventual concavity). In the case of a monopolist with a simple enough profit function we can show, in addition, that the value function has only one point of inflection. Once we allow for free entry and consider sections of the value function that hold competitors ω's fixed, then, as we explain presently, there need not be only one inflection point as we increase our own ω. However we do maintain the initial convexity and eventual concavity of the value function and, as we now explain,this tends to have very distinctive implications for the sample paths of firms.

In particular, it implies that new entrants begin with a relatively low level of investment. As a result most entrants will never actually overcome the negative drift imposed by advances of its competitors both inside and outside the industry, and die at early ages. This generates high mortality rates in an initial "learning" period, and a large fraction of entrants whose realized discounted value of returns from participating in the industry are negative. On the other hand the few new entrants who do get a good sequence of initial draws begin to increase their profits and invest more, thereby increasing the probability that they develop even further. Of course the successful firms will eventually pass over an inflection point of the value function, and decrease their investment, at which point there expected increment in ω will fall. However once their ω falls back to near the inflection point their investment will pick up again, so that an initially successful firm will tend to be productive for a long period of time. This, in turn, implies that both the lifetime and the realized value distributions from our model will tend to be very skewed. Detail on the simulated distribution of firm values and life spans for the differentiated products version of our model is given in Pakes and McGuire (1992). The distributions obtained from the homogeneous product version of our model described here had the same general characteristics, characteristics which are not too different from the results on life spans and value distributions reported in the empirical literature.[20]

Figure 2 also shows how the subtleties generated by the interactions among agents can destroy any simple generalizations on the form of the value function. Consider any one of the sections in which ω_2 is low $(\omega_2 \leq 5)$ and follow the investment pattern of the first firm as its ω increases. As before it is initially increasing until about $\omega = 5$, and then it decreases, but at $\omega = 8$ we see a surge of investment, which heads back down after $\omega = 9$. The reason for the increase in investment at $\omega = 8$ is to deter entry. It works out that a potential entrant finds it profitable to enter if there is one firm in the industry at $\omega = 7$, but not if there is one firm in the industry at $\omega =9$. This surge in investment destroys the simple characterization of the investment function that we would get if there were no potential competitors.

Coming back to the first panel of the table, it is clear that both the planner, and the colluder, tend to generate equilibria with less firms then does the Markov Perfect Nash solution. Indeed, given that the optimal policy for both the planner and the colluder is to have only one firm actually produce output in any period, the firm with the lowest θ_ω, it is somewhat surprising that either of these two institutional structures ever find it optimal to have more than one firm active. They do because it is sometimes optimal for them to run parallel R&D efforts (see Nelson, 1960), and then only use the most efficient production technique developed. Still the logic behind the fact that both the colluder and the planner have less entry and generate less investment (panel B) than does the MPN solution is clear enough; entry and investment decisions in the MPN solution depend on

[20]The literature on lifespan distribution is extensive; see Dunne, Roberts, and Samelson (1988), Pakes and Ericson (1989), and the literature cited in those articles. There is less literature on value distributions, but there is a substantial literature on both the distribution of sales, and persistance in the process generating the sales of different firms (there is also a smaller literature on the distribution of profits, and persistence in the process generating the profits of differnt firms). In addition to the literature cited above, see Evans (1987), Hall (1987), and Mueller (1986), and the literature referred to therein.

the expected incremental cash flow going to the entrant and to the investor, and some of this cash flow is taken away from the incumbents. Both the colluder and the planner internalize the losses to incumbents and hence invest less (Mankiw and Whinston, 1986). In this respect the results here are similar to those in the differentiated products example, however in that example the planner had distinctly more entry and investment than did the colluder. This because the planner took into account the increase in consumer surplus caused by the increase in the number of products marketed whereas the colluder did not (see Pakes and McGuire, 1992). There is no reason to have this difference between the planner and the colluder in the homogeneous product case. Indeed, for our particular values of the parameters the colluder generates more entrants, but the planner invests more after entry.

There were several other interesting aspects of the numerical results that were similar to those obtained from the differentiated products case. First, note that though the colluder generates an industry structure that looks much more like the planner than does the industry structure from the MPN solution, the welfare generated by the MPN solution is much higher (and hence closer to that generated by the planner). More generally we have consistently been surprised by the extent to which institutional structures which generate "similar market structures" (similar numbers of firms active, similar shares for the largest firms, similar entry and exit, etc.), can have very different welfare implications, and institutional arrangements which lead to very different market structures can generate very similar welfare results. Also we have found surprisingly high standard deviations for the welfare results from any given institutional structure. In this example, the average difference in total welfare between the MPN and the Colluder's solution is less than the standard deviation of the welfare results from either of them. This should make us wary about generalizing from case study attempts to compare different institutional arrangements; this is true even of case studies that have a "laboratory perfect" comparison to make (in the sense that the other primitives of the model are the same in the two institutional arrangements being compared).

The big difference between the welfare results in the homogeneous and differentiated product cases is that in the differentiated product example the welfare from the MPN solution was generally within 2 or 3 percent of the welfare that a planner could generate, even when equilibrium typically involved only two firms active, leaving little room for improvement over the "free market." In the homogeneous product case, at least one with parameters that typically generate only one or two firms that are active, the difference between the welfare generated by the planner and that generated by the MPN solution seems to be much more substantial (on the order of 40%). At least for welfare comparisons, it might just be important whether a differentiated or a homogeneous product model best suits the industry being studied.

V. CONCLUDING REMARKS.

We noted in the Introduction that models of firm and industry behavior that allowed for idiosyncratic, or firm specific, uncertainties and entry and exit were required in order to account for many of the phenomena exhibited in firm level data sets. These phenomena include: simultaneous entry and exit; strikingly different outcome paths from similar initial conditions, investment strategies, and exogenous events; and industry structures that never seem to remain stable. We also noted that the need for models which can account for such phenomena is not merely descriptive, but indeed lies at the heart of our ability to analyze many of the impacts of policy and environmental changes.

This paper has provided one possible model of firm and industry dynamics that, because it allows for idiosyncratic uncertainties and entry and exit, can account for these empirical phenomena. It focuses on the effects of the uncertainties generated by the random outcomes of exploratory investments, although it can incorporate other sources of uncertainty as well. Successful investment outcomes move the firm to states where its output can be marketed more profitably. The actual profit that the firm is able to earn in any period depends not only on its own level of development, but also on the levels of development of all other firms active in the industry, as well as on the state of outside competition. We assume that firms are expected discounted value maximizers, so that assumptions on the evolution of the environment outside the firm and on the stochastic impact of investment determine the firm's optimal investment decision in each state. We then show that there is a dynamic rational expectations, Markov perfect, Nash equilibrium in investment strategies. The firm dynamics generated by the equilibrium process can be described by a stochastic process (Markov chain) on possible individual states of success, and industry dynamics as an ergodic Markov process on the space of industry structures, i.e. counting measures providing the number of active firms at each possible level of development. The equilibrium generates almost surely a finite number of firms, and simultaneous entry and exit as rational responses to the opportunity presented by the industry.

The focus of this paper has been on the basic logic and implications of the model in a framework that is general enough to accommodate primitives that could be thought appropriate for a broad number of industries in which research and exploration processes are important. Even at this level of generality, however, the model is rich enough to both generate empirically testable implications (see Pakes and Ericson, 1989), and to suggest nonparametric procedures for correcting for selection (induced by entry and exit) and simultaneity (induced by endogenous input demands) problems when analyzing firm's responses to policy and environmental changes (see Olley and Pakes, 1992). On the other hand, many of the more detailed issues that one might want to analyze with the model depend on the finer properties of the primitives of our model, and are currently buried in the relationship between those primitives and the nature of the equilibrium process generating industry dynamics (these primitives include the profit and cost of capital functions, the entry and exit fees, the functions determining the impact of investment on transition probabilities, and the discount rate).

For both policy and descriptive purposes we will ultimately be interested in the relationship between each primitive and the recurrent class of industry structures, the ergodic distribution on that class, and the nature of the transition process into that class. This would enable us to analyze how a change in either a policy variable (such as an R&D tax credit, or a tariff) or in the external environment (such as a technical change that increased the effectiveness of external competition, or a shift in the structure of demand), affect the nature of the equilibrium process generating industry supply, productivity, shut downs, default probabilities, job creation and destruction at the firm level, etc.

There are at least three (related) ways of proceeding to the more detailed analysis required to unravel these relationships. In order of (what we believe to be) increasing difficulty, they are: simulation based on assumed functional forms and particular parameter values for all of the primitives (see section IV), comparative dynamics within parametric classes, and simulation based on estimated functional forms. As noted in various parts of the paper, we are pursuing all three of these in related research (with varying degrees of success to date).

We believe that there is also a need for more detailed theoretical analysis of several issues. Two come to mind as being particularly important. The first is integrating a more explicit analysis of the other sources of idiosyncratic uncertainty discussed in the introduction to the analysis. The second, and, for some industries, particularly "high–tech industries", we think more important, is to incorporate a "spillover" process so that the advances of a firm's competitors can have a positive impact on a firm's own ability to make advances.

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APPENDIX

PROPOSITION 1: Consider the firm's decision problem (2). Under assumptions (A.0) through (A.7):

a) There exists: (i) a unique $V(\omega,s)$, $V:\mathbb{Z}\times\mathbb{Z}_+^\infty \longrightarrow \mathbb{R}_+$, monotonic increasing in ω, uniformly bounded, and satisfying (3); (ii) an $\bar{x} < \infty$ and a unique optimal investment policy (function), $x(\omega,s)$, $x:\mathbb{Z}\times\mathbb{Z}_+^\infty \longrightarrow \mathbb{R}_+$, with $x(\omega,s) \leq \bar{x}$; and (iii) an optimal termination policy $\chi(\omega,s)$, $\chi:\mathbb{Z}\times\mathbb{Z}_+^\infty \longrightarrow \{0,1\}$; solving (2) [or (3)] for $\forall\ (\omega,s) \in \mathbb{Z}\times\mathbb{Z}_+^\infty$.

b) There exist two finite boundaries in $\mathbb{Z}\times\mathbb{Z}_+^\infty$, $\underline{\omega}(s)$ and $\bar{\omega}(s)$, such that $x(\omega,s) = 0$ if $(\omega,s) \in C \equiv C_\ell \cup C_u$, where $C_\ell \equiv \{(\omega,s)\,|\,\omega < \underline{\omega}(s)\}$ and $C_u \equiv \{(\omega,s)\,|\,\omega > \bar{\omega}(s)\}$, and there exists a finite lower bound $\underline{\underline{\omega}}(s) \in \mathbb{Z}$ such that $\chi(\omega,s) = 0$ if and only if $(\omega,s) \in \{(\omega,s)\,|\,\omega \leq \underline{\underline{\omega}}(s)\} = L$. Further, $\inf_s \underline{\underline{\omega}}(s) > -\infty$ and $\sup_s \bar{\omega}(s) < \infty$.

c) There exists a random variable, $T:\mathbb{Z}\times\mathbb{Z}_+^\infty \longrightarrow \mathbb{Z}_+$, $T(\omega_0,s_0) = \inf\{t \geq 0\,|\,(\omega_0,s_0) = (\omega_0,s_0)$ and $(\omega_t,s_t) \in L\}$, associating each initial state, (ω_0,s_0), with the first time, t, such that $\chi_t \equiv \chi(\omega_t,s_t) = 0$, where (ω_t,s_t) is the state achieved in period t under the optimal policy $\{[x(\omega,s),\chi(\omega,s)]\}$. $T(\omega_0,s_0) < \infty$, a.s. and is stochastically increasing in ω.

PROOF: (a) The result is standard, so we merely outline the argument. Let $\Sigma \equiv \mathbb{Z}\times\mathbb{Z}^\infty$ and $u \in \ell_\infty(\Sigma)$, the Banach space of uniformly bounded functions on Σ. Let $\sigma \equiv (\omega,s) \in \Sigma$, and define an operator T, T: $\ell_\infty \longrightarrow \ell_\infty$, pointwise as follows:

$$\text{(A1)}\qquad Tu(\sigma) \equiv \max\left\{\max_x \left[A_\sigma - c_\sigma x + \beta\cdot\sum_{\sigma'\in\Sigma} u_{\sigma'} p(\sigma'\,|\,\sigma,x)\right],\ \Phi\right\},$$

where $p(\sigma'\,|\,\sigma,x) \equiv \sum_{\eta'} q_\omega(\hat{s}'\,|\,s,\eta')\cdot p(\omega'\,|\,\omega,x,\eta')\cdot p_{\eta'}$. A straightforward calculation shows that the conditions of Theorem 5 of Blackwell (1965) are satisfied, so that **T** is a monotone contraction operator on $\ell_\infty(\Sigma)$ with modulus $\beta < 1$. Hence, by the Banach Fixed Point Theorem [D.R. Smart (1974)], there exists a unique function $V: \Sigma \longrightarrow \mathbb{R}$, uniformly bounded, and satisfying $V = TV$ (Bellman's equation, (6)). As V is uniformly bounded, let $\bar{V} \equiv \sup_\sigma V(\sigma)$. Then $V(\sigma) \leq A_\sigma + \beta[\sup_\sigma V(\sigma)] \leq \bar{A} + \beta\bar{V}$, $\forall\ \sigma\epsilon\Sigma$ implies $(1-\beta)\bar{V} \leq \bar{A}$ or $\bar{V} \leq \bar{A}/(1-\beta)$. Clearly $V(\sigma) \geq \phi$, $\forall\sigma$. An optimal policy exists by Theorem 6 of Blackwell (1965). Optimal investment x^*_σ, is uniformly bounded as $\phi \leq A_\sigma - x^*_\sigma + \beta\bar{V}$, $\forall\sigma$, implies that $\bar{x} < \bar{A} + \beta\bar{V} - \phi$. Hence all that remains to be shown is the monotonicity of $V(\omega,s)$ in ω and the uniqueness of $x(\omega,s)$. The latter follows from the strict concavity of the r.h.s. of (3) in x,

$$\text{(A2)} \qquad R(\omega,s;x) + \beta\cdot\left\{\sum_{\omega'}\sum_{\hat{s}'}\sum_{\eta'} V(\omega',\hat{s}'+e_{\omega'})\cdot q_\omega(\hat{s}'\,|\,s,\eta')\cdot p(\omega'\,|\,\omega,x,\eta')\cdot p_{\eta'}\right\},$$

a consequence of Assumptions (A.4) and (A.5), and the monotonicity of $V(\cdot,s)$ in ω.

To show monotonicity in ω, let $\omega_1 \geq \omega_2$. Then, by the contraction property of the linear operator T, $V(\omega,s) = \lim_{n\to\infty} V^n(\omega,s)$ where $V^n(\omega,s) \equiv TV^{n-1}(\omega,s) \equiv$ $\equiv T^n A(\omega,s)$. Thus the proof follows by induction from (A.3). Assuming monotonicity in ω to hold at step n, we see that

$$\begin{aligned}
V^{n+1}(\omega_1,s) - V^{n+1}(\omega_2,s) &= TV^n(\omega_1,s) - TV^n(\omega_2,s) = \\
&= A(\omega_1,s) - A(\omega_2,s) - [c_{\omega_1}x^n(\omega_1,s) - c_{\omega_2}x^n(\omega_2,s)] + \\
&+ \beta\cdot\sum_{\eta'}\left\{\sum_{\omega'}\left[\sum_{\hat{s}_1'} V^n(\omega',\hat{s}_1'+e_{\omega'})\cdot q_{\omega_1}(\hat{s}_1'\,|\,s,\eta')\right]p(\omega'\,|\,\omega_1,x_1,\eta') - \right. \\
&\left. - \sum_{\omega'}\left[\sum_{\hat{s}_2'} V^n(\omega',\hat{s}_2'+e_{\omega'})q_{\omega_2}(\hat{s}_2'\,|\,s,\eta')\right]p(\omega'\,|\,\omega_2,x_2,\eta')]\right\}p_{\eta'}
\end{aligned}$$

(A3)

$$\geq A(\omega_1,s) - A(\omega_2,s) + \beta\cdot\sum_{\eta'}\left\{\sum_{\omega_1'}\left[\sum_{\bar{s}'}\sum_{\omega_2'} V^n(\omega_1',\bar{s}'+e_{\omega_1'}+e_{\omega_2'})q_{\omega_1\omega_2}(\bar{s}'\,|\,s,\eta')p(\omega_2'\,|\,\omega_2,x_1,\eta') - \right.\right.$$

$$\left.\left. - \sum_{\bar{s}'}\sum_{\omega_2'} V^n(\omega_2',\bar{s}'+e_{\omega_1'}+e_{\omega_2'})q_{\omega_1\omega_2}(\bar{s}'\,|\,s,\eta')p(\omega_2'\,|\,\omega_2,x_1,\eta')\right]\cdot p(\omega_1'\,|\,\omega_1,x_1,\eta')]\right\}p_{\eta'} =$$

$$= \sum_{\eta'}\sum_{\omega_1'}\sum_{\omega_2'}\sum_{\bar{s}'}\left[V^n(\omega_1',\bar{s}'+e_{\omega_1'}+e_{\omega_2'}) - V^n(\omega_2',\bar{s}'+e_{\omega_1'}+e_{\omega_2'})\right]\times$$

$$\times q_{\omega_1\omega_2}(\bar{s}'|s,\eta')\cdot p(\omega_1'|\omega_1,x_1,\eta')\cdot p(\omega_2'|\omega_2,x_1,\eta')\cdot p_{\eta'} \;\geq\; 0,$$

where $x_i \equiv x^n(\omega_i,s)$; $\hat{s}_i$, ω_i' are similarly defined; $\bar{s} \equiv s-e_{\omega_1}-e_{\omega_2}$; $q_{\omega_1\omega_2}$ is the marginal distribution derived from either q_{ω_i}: $q_{\omega_1\omega_2}(\bar{s}'|s,\eta') \equiv \sum_{\omega_j'} q_{\omega_i}(\bar{s}'+e_{\omega_j'}|s,\eta')$, $i \neq j$; and $'$ (prime) indicates next periods (random) realization of the variable. The first inequality in (A3) is due to the use of $x(\omega_1,s)$ at (ω_2,s) and substitution of the appropriate marginal probabilities. The second inequality follows from the monotonicity of $A(\omega,s)$, $c(\omega)$, $V^n(\omega,s)$ in ω, once one observes that the set of s' and their associated probabilities must be identical at both ω, as they arise from the same s and hence their only difference lies in the firms at ω_1 and ω_2 which here invest identically. The first step of the induction follows from an identical argument to that of (A3) with $A(\omega,s)$ in place of $V^n(\omega,s)$. ◻

<u>PROOF</u>: (b) From the first order conditions for the maximization of (A2), given in equation (6.b), we know that $x(\omega,s) > 0$ if and only if $G(\omega,s) \equiv$ $\equiv \beta\cdot\sum_{\eta'}\sum_{\omega'}\hat{V}(\omega'|\omega,s,\eta')\cdot p_x(\omega'|\omega,x(\omega,s),\eta')\cdot p_{\eta'} > c(\omega)$, where $\hat{V}(\omega'|\omega,s,\eta') \equiv$ $\equiv \sum_{\hat{s}'} V(\omega',\hat{s}'+e_{\omega'})q_{\omega}(\hat{s}'|s,\eta')$. Part (a) shows that $V(\cdot) \in [0,\bar{V}]$ and that by monotonicity in ω, $\forall s$, $\lim_{\omega\to-\infty} V(\omega,s) = \phi$ and $\lim_{\omega\to\infty} V(\omega,s) = \bar{V}$. Therefore, as the support of $\sum_{\eta'} p_x(\cdot)\cdot p_{\eta'}$ is finite [k_1+k_2+1 elements] and $\sum_{\omega'}\sum_{\eta'} p_x(\omega'|\cdot)\cdot p_{\eta'} = 0$, $\forall s$, $\lim_{\omega\to\pm\infty} G(\omega,s) = 0$. Indeed, letting $\bar{p} = \max_{\omega'}\{\sum_{\eta'} p_x(\omega'|\cdot)\cdot p_{\eta'}\}$, $G(\omega,s) <$ $< \bar{p}\cdot[\hat{V}(\omega+k_1|\cdot) - \hat{V}(\omega-k_2|\cdot)] = \bar{p}\cdot\epsilon_\omega \downarrow 0$ as $\omega \longrightarrow \pm\infty$. Define $\underline{\omega}(s) := \min\{\,\omega|$ $G(\omega,s) > c(\omega)\}$ and $\bar{\omega}(s) := \max\{\,\omega \geq |\; G(\omega,s) > c(\omega)\}$. Clearly $\underline{\omega}(s)$ and $\bar{\omega}(s)$ are finite, for otherwise V cannot remain bounded, and hence have finite maximum and minimum. Further, $x(\omega,s) = 0$ if $(\omega,s) \in \mathcal{C} \equiv \mathcal{C}_\ell \cup \mathcal{C}_u$, where $\mathcal{C}_\ell \equiv$

$\equiv \{(\omega,s) | \omega < \underline{\omega}(s)\}$ and $C_u \equiv \{(\omega,s) | \omega > \bar{\omega}(s)\}$.

Finally, for a finite termination policy we need to show that, for each s, there exists an $\underline{\underline{\omega}}(s) > -\infty$ such that $V(\omega,s) = \phi$ for all $\omega \leq \underline{\underline{\omega}}(s)$. First notice that when $x(\omega,s) = 0$ the Bellman equation (3) becomes

$$
\text{(A4)} \qquad
\begin{aligned}
V(\omega,s) &= A(\omega,s) + \beta\cdot\left\{\sum_{\eta=-k_2}^{0} p_\eta\cdot\hat{V}(\omega|\omega,s,\eta)\right\} = \\
&= \beta\cdot\left[1-\sum_{-k_2}^{-1} p_\eta\right]\cdot Q_{\omega 0}(s,\cdot)\cdot V(\omega,\cdot) + \beta\cdot\sum_{-k_2}^{-1} p_\eta\cdot Q_{\omega\eta}(s,\cdot)\cdot V(\omega+\eta,\cdot),
\end{aligned}
$$

where $Q_{\omega\eta}(s,\cdot)$ is the s–th row of the finite–dimensional stochastic matrix representing $q_\omega(\hat{s}'|s,\eta)$ and $V(\omega,\cdot)$ is the column vector of firm values at ω for each structure s. Let $\hat{\omega}(s) = \min\{\omega|\ A(\omega,s) \geq (1-\beta)\phi\}$ and $\omega^* = \max_\omega[\{\omega<\hat{\omega}(s)|\ \forall s,\ x(\omega,s) = 0\ \}] > -\infty$, as there are only finitely many $s \in S$ (A.7.b). Then for all $\omega \leq \omega^*$ equation (A4) holds, so we can write in matrix notation

$$
\text{(A5)} \qquad V_\omega = A_\omega + \beta Q_{\omega 0}\cdot V_\omega - \beta\sum_\eta p_\eta\cdot Q_{\omega\eta}\cdot\Delta_\eta V_\omega
$$

where $Q_{\omega\eta}$ is the stochastic transition matrix for each exogenous shock η, V_ω is the vector of values at ω, and $\Delta_\eta V_\omega \equiv V_\omega - V_{\omega-\eta}$. Solving (A5) we get

$$
\begin{aligned}
[I-\beta Q_{\omega 0}]\cdot V_\omega &= A_\omega - \beta\cdot\sum_\eta p_\eta\cdot Q_{\omega\eta}\cdot\Delta_\eta V_\omega \\
\text{(A6)} \qquad \vec{\phi} \leq V_\omega &= [I-\beta Q_{\omega 0}]^{-1}A_\omega - \beta[I-\beta Q_{\omega 0}]^{-1}\sum_\eta p_\eta\cdot Q_{\omega\eta}\cdot\Delta_\eta V_\omega \leq \\
&\leq [I-\beta Q_{\omega 0}]^{-1}(1-\beta)\vec{\phi} - \beta[I-\beta Q_{\omega 0}]^{-1}\sum_\eta p_\eta\cdot Q_{\omega\eta}\cdot\Delta_\eta V_\omega \leq \vec{\phi}
\end{aligned}
$$

where $\vec{\phi}$ is a column vector with ϕ for each structure s. Note that $\Delta_\eta V_\omega$ is nonnegative by the monotonicity of V in ω. The first inequality in (A6) follows from the lower bound on V, the second from that on A, and the third from the fact that $[I-\beta Q_{\omega 0}]^{-1}\vec{\phi} = (1-\beta)^{-1}\vec{\phi}$, $\Delta_\eta V_\omega \geq 0$, and $Q_{\omega\eta}$ is a stochastic matrix [the maximal eigenvalue $\rho(Q_\omega) \leq 1$] for all η so $[I-\beta Q_{\omega 0}]$ is invertible and has a positive definite inverse. Hence for all $s \in S$ and $-\infty < \omega \leq \omega^*$, $V(\omega,s) = \phi$. For each s, let $\underline{\underline{\omega}}(s) = \max\{\omega|\ V(\omega,s) = \phi\}$, and let $L = \{(\omega,s)|\ V(\omega,s) = \phi\}$.

Then $\chi(\omega,s) = 0$ on L and $\chi(\omega,s) = 1$ elsewhere. □

PROOF: (c) The proof of this assertion follows from a demonstration that all states $(\omega,s) \notin L$ are transient and hence will never be returned to after some finite (random) time. All states $(\omega,s) \in L$ are recurrent, indeed absorbing, i.e. $\text{Prob}\{\ \exists\ \tau > t \mid (\omega_t,s_t) \in L \text{ and } (\omega_\tau,s_\tau) \notin L\ \} = 0$. Since the probability of a step from ω to $\omega+\eta$, $\eta \in \{-k_2, \ldots, -1\}$ is always strictly positive (A.4), and $\omega^* > -\infty$, it is easy to show that the probability of reaching some state in L is positive and hence the probability of returning to any state NOT in L must be less than 'one'. That is,

$$F_{jj} \leq 1 - F_{jL} = 1 - \sum_{n=1}^{\infty} f_{jL}^{n} < 1,$$

where $j \equiv (\omega,s) \notin L$, F_{jj} is the probability of ever returning to j, $F_{jj} = \sum_{n=1}^{\infty} f_{jj}^{n}$ where $f_{jj}^{n} = \text{Prob}\{\text{first return to } j \text{ occurs at time } n\}$, and F_{jk}, f_{jk} refer similarly to reaching k from j. Transience of all $(\omega,s) \notin L$ implies the existence of the a.s. finite stopping time $T(\omega_0,s_0)$ [Doob (1953), Chapter V.3].

The stochastic monotonicity of $T(\cdot)$ is shown by a coupling argument. To prove that the stopping time is stochastically increasing in ω consider $\omega_2 > \omega_1$ and initial states (ω_2,s_0) and (ω_1,s_0). Denote (for this argument only) the underlying measure space by $\{U,\Sigma,P\}$ with elements u. Let $\omega_t^i(u)$ be the sample path (sequence) arising from initial ω_i at $u \in U$. For each possible $u \in U$ define the stopping time $\tau(u) = \min\{t \geq 0 \mid \omega_t^1(u) = \omega_t^2(u)\}$, and the new sequence

$$\omega_t^*(u) = \begin{cases} \omega_t^2(u) & \text{if } \omega_t^2(u) - \omega_t^1(u) > 0 \text{ for all } t \geq \tau(u) \\ \omega_t^1(u) & \text{otherwise} \end{cases}$$

Note that: (a) the random sequence $\{\omega_t^*,s_t\} \geq \{\omega_t^1,s_t\}$ with probability one; (b) because the random sequence $\{\omega_t^i,s_t\}$ is a Markov process and a stopping time is Markov, the distribution of $\{\omega_t^*,s_t\}$ is the same as that of $\{\omega_t^2,s_t\}$. Property (a), the monotonicity of the value function, and the stopping rule imply that $T(\omega_t^*,s_t) \geq$

$\geq T(\omega_t^1, s_t)$ with probability one. Property (b) and the continuous mapping theorem (Billingsley, 1968) imply that the distribution of $T(\omega_t^2, s_t)$ is the same as that of $T(\omega_t^*, s_t)$. Since the latter stochastically dominates $T(\omega_t^1, s_t)$, the proof is complete.

Q.E.D. □

<u>PROPOSITION 2</u>: Let $s_n(\omega) \equiv s + n \cdot e_\omega$. Under Assumptions (A.0) to (A.6), for all $\omega \in \Omega$, and all $s \in \mathbb{Z}_+^\infty$: $\lim_{n\to\infty} V(\omega, s_n(\omega)) = \phi$, i.e. $\forall \epsilon > 0\ \exists n_\epsilon$ such that $n \geq n_\epsilon$ implies $V(\omega, s_n(\omega)) < \phi + \epsilon$.

<u>PROOF</u>: Writing s_n for $s_n(\omega_0)$, letting $P^t_{(\omega_0, s_n)}((\omega,s) \mid \{x^*, \chi^*\})$ be the probability of reaching (ω,s) in t steps from (ω_0, s_n) under the optimal investment and shutdown policies $\{x^*, \chi^*\}$, and letting $I_L(\cdot)$ be the indicator function of the shutdown states, we can write

$$
\begin{aligned}
\phi \leq V(\omega_0, s_n) &= \sum_{t=0}^{\infty} \beta^t \sum_{\omega=0}^{K} \sum_{s} \Big[R(\omega,s;x(\omega,s))[1-I_L(\omega,s)] + \phi I_L(\omega,s)\Big] P^t_{(\omega_0,s_n)}((\omega,s) \mid \{x^*,\chi^*\}) \\
&\leq \sum_{t=0}^{\infty} \beta^t \sum_{\omega=0}^{K} \sum_{s} \Big[A(\omega,s)[1-I_L(\omega,s)] + \phi I_L(\omega,s)\Big] P^t_{(\omega_0,s_n)}((\omega,s) \mid \{x^*,\chi^*\}) \leq \\
&\leq \sum_{t=0}^{\infty} \beta^t \sum_{\omega=0}^{K} \sum_{s} [A(\omega,s) \vee (1-\beta)\phi] P^t_{(\omega_0,s_n)}((\omega,s) \mid \{x^*,\chi^*\})
\end{aligned}
$$

where the first inequality is due to ignoring the cost of the optimal investment generating the transition probabilities, and the second from using $(1-\beta)\phi$ in place of $A(\omega,s)$ whenever it is larger. Let $p_\omega(s_n, t, e_t)$ be the probability that a firm starting at (ω_0, s_n) will have $\omega_t \geq \omega$, conditional on a particular t–period sequence, e_t, of realizations of the exogenous process and the decision structure $\{x^*, \chi^*\}$. By (A.3),

$$
A(\omega,s) \leq A_\omega(n) \equiv \sup_{\{s \mid \Sigma_{\omega^* \geq \omega} s_{\omega^*} \geq n\}} A(\omega,s) = (1-\beta)\phi + \theta(n)
$$

for $s \in S_n(\omega)$ where $\theta(n)$ is monotone decreasing to zero in its argument. Hence, for any of the n firms starting at ω_0, we can write

$$\phi \le V(\omega_0, s_n) \le$$

$$\text{(A7)} \quad \le \sum_{t=0}^{\infty} \beta^t \left[\sum_{e_t} \sum_{\omega} p_\omega(s_n, t, e_t) \sum_{k=0}^{n-1} A_\omega(k+1) \binom{n-1}{k} [p_\omega(\cdot)]^k [1-p_\omega(\cdot)]^{n-k-1} P(e_t) \right],$$

$$\le \phi + \sum_{t=0}^{\infty} \beta^t \left[\sum_{e_t} \sum_{\omega} p_\omega(\cdot) \sum_{k=0}^{n-1} \theta(k+1) \binom{n-1}{k} [p_\omega(\cdot)]^k [1-p_\omega(\cdot)]^{n-k-1} P(e_t) \right]$$

where $\binom{n}{k}$ is the number of k–combinations of n objects, and $P(e_t)$ is the probability of the realization, e_t, of the exogenous process. Let $f(n,t)$ be the function in the large square brackets in the second line of (A7). Clearly $f(n,t) \le \bar{A}$ and $\sum_{t=0}^{\infty} \beta^t \cdot \bar{A} = (1-\beta)^{-1} \bar{A} < \infty$. Hence, by the Lebesgue Dominated Convergence Theorem for sums, it suffices to show that for every t, $\lim_{n\to\infty} f(n,t) = 0$, for which it further suffices that for each ω

$$\text{(A8)} \qquad p_\omega(\cdot) \sum_{k=0}^{n-1} \theta(k+1) \binom{n-1}{k} [p_\omega(\cdot)]^k [1-p_\omega(\cdot)]^{n-k-1} \xrightarrow[n\to\infty]{} 0$$

for almost every e_t. Now note that

$$\frac{k+1}{n-1} = \operatorname{argmax}_p \left\{ p^{k+1} (1-p\ )^{n-k-1} \right\}.$$

Thus, for any $N \le n-1$,

$$\sum_{k=0}^{n-1} \theta(k+1) \binom{n-1}{k} [p_\omega(\cdot)]^{k+1} [1-p_\omega(\cdot)]^{n-k-1} \le$$

$$\le \theta \cdot \sum_{k=0}^{N-1} \binom{n-1}{k} \left[\frac{k+1}{n-1}\right]^{k+1} \left[\frac{n-k-2}{n-1}\right]^{n-k-1} + \theta(N+1) =$$

$$\text{(A9)} \quad = \theta \cdot \sum_{k=0}^{N-1} \frac{(n-1)\ \cdots\ (n-k)}{(n-1)^{k+1}} \cdot \frac{(k+1)^{k+1}}{k!} \cdot \left[\frac{n-k-2}{n-1}\right]^{n-k-1} + \theta(N+1) \le$$

$$\le \theta \cdot \left[\sum_{k=0}^{N-1} \frac{(k+1)^{k+1}}{k!} \right] \cdot \frac{1}{n-1} + \theta(N+1).$$

Now fix $\epsilon > 0$ and let n_1 be the minimum n such that $\theta(n+1) \le \frac{\epsilon}{2}$ and n_2 be the smallest $n \ge n_1 - 1$ such that $\theta \cdot \sum_{k=0}^{n_1-1} \frac{(k+1)^{k+1}}{k!} \cdot \frac{1}{n_2 - 1} \le \frac{\epsilon}{2}$. Hence (A8) holds, so that for $n \ge n_2$, $V(\omega_0, s_n) \le \epsilon$ as required. Q.E.D. □

PROOF: [Corollary 1] We use Proposition 2 to show that for any s, $\exists\, M < \infty$ such that $\forall\, m \geq M$, $V^e(s,m) < \epsilon \leq x^e_M$. We do so in two stages; first for $\Omega^e = \{\omega^0 - k_2, \ldots, \omega^0\}$, and then for general Ω^e. In the first case all entry occurs at $\omega^0+\eta$ giving

$$V^e(s,m) = \beta \sum_\eta \left[\sum_{s'} V(\omega^0+\eta, m\cdot e_{\omega^0+\eta} + s') q^0(s'\,|\,s,\eta) \right] p_\eta.$$

For each of a finite number of η's and each s', $V(\cdot) \leq \phi + \epsilon$ for $m \geq M$ by Proposition 2, gives the desired result. In the general case, entry is distributed by π^e on the finite set Ω^e. The proof is immediate by contradiction. Assume that m is unbounded. Then, as π^e is everywhere positive on Ω^e, there will be an unbounded number of firms at each $\omega \in \Omega^e$. But then by Proposition 2 $V(\omega,s) \leq \;\; + \epsilon \;\; \forall\, \omega \in \Omega^e$, showing that there could not be an unbounded number of entrants. □

PROOF: [Corollary 2] Proposition 2 shows that if there are enough firms in any state ω then the value to being in that state is arbitrarily close to ϕ. While this insures that entry in any period must be finite, it is only sufficient to cut off entry if there are sufficiently many firms above the highest entry state. For the industry to remain finite we must show that a sufficiently large number of firms will actually cut off all entry, so that the industry can never become larger. To do so we strengthen Proposition 2 in Lemma 1 to show that there can never be more than a finite number of firms at any $\omega \in \{1,K\}$ without their all desiring to exit the industry immediately. Letting N_ω be that number for each ω, we see that $N = \sum_{\omega=1}^{K} N_\omega$ is sufficiently large that for $\forall\, s \in \hat{S}_n(1)$, $\forall\, n \geq N$, $V^e(s,m) < x^e_1$, $\forall\, m \geq 1$.

LEMMA 1: For each ω, $\exists\, N_\omega$ such that $\forall\, n \geq N_\omega$, $V(\omega, s+n\cdot e_\omega) = \phi$.

PROOF: Proposition 2 gives us an n_ω such that for all $n \geq n_\omega$ $V(\omega, s+n\cdot e_\omega) <$ $< \phi + \epsilon$. We now show that by increasing n sufficiently we can drive the continuation value, $V^c(\cdot)$ [the expression within braces in equation (3)], below ϕ. At each ω there are two cases to consider: (a) $x(\omega,\cdot) = 0$ and (b) $x(\omega,\cdot) > 0$.

Case (a): $x(\omega,\cdot) = 0$ implies $V^c(\omega,s+n\cdot e_\omega) \leq A(\omega,s+n\cdot e_\omega) + \beta\cdot V(\omega,s+n\cdot e_\omega)$. But Assumption (A.3) says that $\exists\ n^*_\omega$ such that $A(\omega,s+n^*_\omega\cdot e_\omega) < (1-\beta)\phi - \epsilon$. Hence, for $n \geq \min\{n_\omega, n^*_\omega\}$, $V^c(\omega,s+n\cdot e_\omega) < (1-\beta)\phi - \epsilon + \beta\phi + \beta\epsilon < \phi$. □

Case (b): $x(\omega,\cdot) > 0$ implies that there is a positive probability of advancing to any $\omega' \in \{\omega-k_2, \dots ,\omega+k_1\}$. Hence there exists an n^* such that with probability $1-\epsilon_1$ there are at least $n_{\omega+k_1}$ firms at $\omega+k_1$. Therefore, letting $\bar{V} = \sup_\Sigma V(\omega,s)$,

$$\begin{aligned} V^c(\omega,s+n^*e_\omega) &\leq A(\omega,s+n^*e_\omega) + \beta(1-\epsilon_1)(\phi+\epsilon) + \beta\epsilon_1\bar{V} \leq \\ &\leq (1-\beta)\phi - \epsilon + \beta(1-\epsilon_1)(\phi+\epsilon) + \beta\epsilon_1\bar{V} < \\ &< \phi - \epsilon - \beta\epsilon_1\phi + \beta\epsilon_1\bar{V} + \beta(1-\epsilon_1)\epsilon = \\ &< \phi + (\bar{V} - \phi)\beta\epsilon_1 - (1-\beta(1-\epsilon_1))\epsilon, \end{aligned}$$

where ϵ comes from (A.3) as in case (a) $[n^* \geq n^*_\omega]$. Hence we need only choose $n > n^*$ so large that $(\bar{V} - \phi)\beta\epsilon_1 < (1-\beta(1-\epsilon_1))\epsilon$. □

Q.E.D. □

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PROPOSITION 3: Under assumptions (A.0) – (A.5), assumption (A.6) holds with $Q(\cdot|\cdot)$ defined using (6.c) and (7), when $q_\omega(\hat{s}'|s,\eta)$ is defined by equation (8).

PROOF: We need to show that $Q(\cdot|\cdot)$ is a (probability) density generating a regular conditional probability distribution for a Markov process in S. Let $\mathcal{Q}(s,B) \equiv \equiv \sum_{s'\in B} Q(s'|s)$. Clearly $\mathcal{Q}(s,\cdot)$ is a probability on the σ–field of all the subsets of the finite set S: $\mathcal{Q}(s,\cdot) \geq 0$, $\mathcal{Q}(s,S) = 1$, and $\mathcal{Q}(s,A\cup B) = \mathcal{Q}(s,A) + \mathcal{Q}(s,B)$ for $A \cap B = \emptyset$ and all $s \in S$. Further, $\mathcal{Q}(\cdot,B)$ is evidently measurable in its first argument for each $B \subset S$. Finally, by its definition, the map $s \longrightarrow \mathcal{Q}(s,B)$ is a version of $\text{Prob}\{s' \in B | s\}$. Hence $\mathcal{Q}(\cdot,\cdot)$ is a regular conditional probability distribution [Freedman (1983), Appendix 10]. An easy construction shows the process $\{s_t\}$ to be Markov. Indeed, letting ξ_t be the associated canonical (projection) process, it is clear that $\text{Prob}\{\xi_t = s_t \in S;\ t=0, \dots, n\} = \prod_{t=0}^{n-1} Q(s_{t+1} | s_t)$ and that

$$\text{(A9)} \quad \text{Prob}\{s_{t+1} \in B | s_0, \dots, s_t\} = \sum_{s' \in B} Q(s' | s_t) \equiv \mathcal{Q}(s_t, B) = \text{Prob}\{s_{t+1} \in B | s_t\}.$$

Hence $\mathcal{Q}$ is the transition probability function for a Markov process and $Q(\cdot|\cdot)$ is its kernel. Thus $Q(\cdot|\cdot)$ assumed in (A.6) exists at an equilibrium defined as in (6), and its marginals $q_\omega(\cdot)$, $q_{\omega_1\omega_2}(\cdot)$, are well defined.

Q.E.D. □

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<u>THEOREM 1</u>: Under Assumptions (A.0) – (A.5) there exists an equilibrium (*), satisfying conditions (6.a–e).

<u>PROOF</u>: To prove existence of this rational expectations equilibrium, we need to show the mutual consistency of four fundamental mappings:

(i) $V: \Omega \times S \longrightarrow [\phi, \bar{V}] \subset \mathbb{R}$,

(ii) $x: \Omega \times S \longrightarrow [0, \bar{x}] \subset \mathbb{R}_+$,

(iii) $\mathcal{Q}: S \longrightarrow \Delta^S$,

(iv) $V^e: \hat{M} \times S \longrightarrow [\phi, \bar{V}] \subset \mathbb{R}$,

where $\hat{M} \equiv \{0, 1, \dots, M\}$ is the set of numbers of potential entrants and Δ^S is the set if probability measures with support in the finite set S, i.e. a simplex of dimension $|S|-1$. Finiteness of Ω, S and $\hat{M}$ follow from Propositions 1 and 2 respectively. $\mathcal{Q}$ is the conditional probability distribution generated by the Markov transition kernel, Q [see Proposition 3 proof]. Given a transition kernel Q (6.c) characterizing the behavior of the industry structure, s, individual firm optimization generates an x (6.b) which solves equation (3 or 6.a), yielding both an optimal valuation of ω–states and industry structures, V (6.a), and optimal exit from that structure. V together with Q then generate the value of entering the industry, V^e (4), that determines the number of new entrants, m(s) (5 or 6.d). The optimal investment, exit, and entry decisions of firms in turn define [see (A.4), (A.5)] a transition probability function, $\mathcal{Q}$, for the industry structure through equations (6.c) and (7). An equilibrium will exist iff the resulting $\mathcal{Q}$ is the same as that which determined the optimal valuation and investment functions of firms in the industry. We will use a fixed point argument to show that there exists such a $\mathcal{Q}$, and hence appropriate Q, V, x, and V^e functions also exist, all satisfying the required properties (6.a–d).

First note that each of the mappings, V, x, $\mathcal{P}$, V^e, can be represented by a point in a compact subset of real Euclidean space: $V \in [\phi,\bar{V}]^{\Omega\times S}$, $x \in [0,\bar{x}]^{\Omega\times S}$, $V^e \in [\phi,\bar{V}]^{\hat{M}\times S}$, and $\mathcal{Q} \in \left[\Delta^S\right]^S$, where S and Ω are compact. This is an immediate consequence of Propositions 1 and 2 and the definition of $\mathcal{Q}$ in (6.c). Define a mapping ζ: $\left[\Delta^S\right]^S \longrightarrow [0,\bar{x}]^{\Omega\times S}\times[\phi,\bar{V}]^{\Omega\times S}\times[\phi,\bar{V}]^{\hat{M}\times S}$, which takes a market structure transition function into an optimal investment policy and optimal valuation function for any firm in the industry, and an optimal valuation for any firm considering entry. This mapping is generated by the solution to the Bellman equation (3) for a given transition probability function for industry structures and by equation (4). Define a mapping ψ: $[0,\bar{x}]^{\Omega\times S}\times[\phi,\bar{V}]^{\Omega\times S}\times[\phi,\bar{V}]^{\hat{M}\times S} \longrightarrow \left[\Delta^S\right]^S$, which takes an optimal investment policy and state and entry valuations into a market–structure transition function. This mapping is determined by equations (6.c) and (7). Finally, define the mapping Υ: $\left[\Delta^S\right]^S \longrightarrow \left[\Delta^S\right]^S$ by the composition $\Upsilon = \psi\circ\zeta$. In all cases we work with the appropriate product topology.

LEMMA 2: ζ is a continuous function.

PROOF: As T [see equation (A1)] is a continuous (algebraic) function of the transition kernel Q of $\mathcal{Q}$, V = TV also depends continuously on $\mathcal{Q}$. As in Proposition 1, $x(\omega,s)$ uniquely solves $G(\omega,s) = c(\omega)$, where $G(\omega,s) \equiv \beta\cdot\sum_{\eta'}\sum_{\omega'}\hat{V}(\omega'|\omega,s,\eta')\cdot p_x(\omega'|\omega,x(\omega,s),\eta')\cdot p_{\eta'}$ and $\hat{V}(\omega'|\omega,s,\eta') \equiv \sum_{\hat{s}'}V(\omega',\hat{s}'+e_{\omega'})q_{\omega}(\hat{s}'|s,\eta')$, and satisfies the Kuhn–Tucker condition in equation (6.b). Hence x is clearly a continuous function of V and $\mathcal{Q}$. As V is continuous in the discrete topology on $\Omega\times S$ and all the operations in (4) are continuous, so is V^e. Finally, since all these mappings are indeed functions (single valued), ζ is continuous as the composition and product of continuous functions. □

LEMMA 3: ψ is a continuous function.

PROOF: Any investment policy, $x(\omega,s)$, uniquely determines state transition probabilities through $\pi(\omega'|\omega,x)$ [see assumption (A.4)]. These are the probabilities, $p_{\omega'\omega}(\eta,s)$, given in the transition matrix P [see the Remark following equilibrium definition (6)]. Clearly, by assumption (A.4), they depend continuously on x. Further, the number and distribution of new entrants, i.e. their transition probabilities, are determined by the fixed function $m(\cdot)$ [see equation (5) and Assumption (A.5)] which depends continuously on V^e and hence V. The industry structure transition kernel, Q, is uniquely determined by these probabilities as shown in equations (6.c), (7), and the discussion following (7). The algebraic operations in these equations are all continuous, so $\mathcal{Q}$ depends continuously on x, V, and V^e. Therefore ψ is a continuous function again as the composition of continuous functions. □

LEMMA 4 The function $\mathcal{V}$: $\left[\Delta^S\right]^S \rightarrow \left[\Delta^S\right]^S$ has a fixed point, $\mathcal{Q}^*$, i.e.

$\mathcal{Q}^* \in \left[\Delta^S\right]^S$ such that $\mathcal{Q}^* = \mathcal{V}(\mathcal{Q}^*)$.

PROOF: The function $\mathcal{Y}$ is evidently continuous as the composition of two continuous functions. Further, $\left[\Delta^S\right]^S$ is clearly convex and is compact by Tychonoff's Theorem (as the Cartesian product of compact sets). Thus $\mathcal{Y}$ maps a compact convex set into itself continuously, so that Brouwer's Fixed Point Theorem [Smart (1974)] gives the desired result. □

Thus we have shown that there exists a $\mathcal{Q}$ such that the V and x functions satisfying equations (6.a) and (6.b) generate the transition kernel Q satisfying equations (6.c) and (7). Therefore the first three conditions of the definition of an equilibrium have been shown to be satisfied. The remaining condition (6.d) is an immediate consequence of preceding Propositions, while (6.e) is an arbitrary initial condition. Q.E.D. □

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THEOREM 2: Under Assumptions (A.0) through (A.5) at equilibrium (*) [and (6)]:

a) The stochastic process $s = \{s_t\}_{t=0}^{\infty} \in (S^{\infty}, \otimes \mathscr{S})$ with initial state s^0 is Markov with stationary transitions $Q(s,s')$ and distribution P_{s^0}.

b) The state space, S, contains a unique, positive recurrent communicating class $R \subset S$.

c) There exists a unique, invariant probability measure, μ^*, on S such that

$$\mu^*_s = [mQ(s,s)]^{-1} \text{ for } s \in R, \text{ and } \mu^*_s = 0 \text{ for } s \in S\backslash R,$$

where $mQ(s,s')$ is the P_s–expectation of the time of first reaching state s'.

d) $\forall\, s \in S,\ \mu_n(s) \xrightarrow[n\to\infty]{} \mu^*$.

PROOF: (a) This is an immediate consequence of Proposition 3 when we define the elements of the matrix Q, $Q(s,s')$, to be given by the equilibrium transition kernel:

$Q(s,s') \equiv Q(s'|s)$, incorporating optimal exit and entry, as well as investment, decisions. In the notation of Proposition 3, the Kolmogorov consistency theorem insures that the measure P_s is uniquely given by:

$$\text{(A10)} \qquad P_s\{\xi_t = s_t \text{ for } t=0,1, \dots ,\tau\} = e_s \cdot \prod_{t=0}^{\tau-1} Q(s_t,s_{t+1}). \qquad \square$$

PROOF: (b) The existence of a unique positive recurrent communicating class will be shown through a series of lemmata. First we argue that at least one such class, $R \subset \subset S$, must exist due to the compactness of S. Next we show that it contains a distinguished state, $\bar{s} \in R$, such that all states $s \in S$ communicate with $\bar{s}$, i.e. $\exists\, n \geq 1$ such that $P_s\{\xi_n=\bar{s}\} > 0$. Hence every recurrent state must belong to the same communicating class (i.e. that containing $\bar{s}$) and so must belong to R, proving uniqueness. More formally, we have the following:

LEMMA 5: There exists a positive recurrent communicating class, $R \subset S$.

PROOF: As S is compact (finite), there must exist some state $\hat{s}$ that is essential [Freedman (1983), 1.56].[41] Define $R \equiv \{s \in S \mid s \leftrightarrow \hat{s}\}$ where $i \leftrightarrow j$ iff $i \rightarrow j$ and $j \rightarrow i$, and $i \rightarrow j$ iff $\exists n>0$ such that $Q^n(i,j) > 0$. $R \subset S$ implies that R is finite and therefore compact. Hence any infinite sequence $\{\xi_n\} \subset R$ must contain infinitely many $\bar{s}$, for some $\bar{s} \in R$. Therefore, for any $s \in R$, $\exists\, \bar{s}$ such that $P_s\{\xi_n = \bar{s} \text{ i.o.}\} > 0$. By Theorem 1.51 in Freedman (1983), the probability of ever returning to $\bar{s}$, $fQ(\bar{s},\bar{s}) = 1$, so that $\bar{s}$ is recurrent. But then $\bar{s} \rightarrow s \in R$ implies $fQ(\bar{s},s) = fQ(s,\bar{s}) = fQ(s,s) = 1$, $\forall\, s \in R$. R is therefore a recurrent class, and as it is finite it must be positive recurrent by Theorem 1.78 in Freedman (1983). $\square$

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[41]A state i is essential iff 'i communicates with j' implies that 'j communicates with i'. State i 'communicates' with state j iff $Q^n(i,j) > 0$ for some $n > 0$.

LEMMA 6: There exists an $\bar{s}$ such that, $\forall$ s $\in$ S, $s \rightarrow \bar{s}$.

PROOF: Let $\bar{s} \equiv (0, \ldots ,0,N,0, \ldots ,0)$ where $N > 0$ is a finite number of firms at $\omega^0 = \min\Omega^e$. We will show that there exists a finite trajectory, $\{s_0,s_1, \ldots ,s_T\}$, with positive P_s–probability such that $s_0 = s$ and $s_T = \bar{s}$. We do this in two stages.

(i) For all s let s′ be defined as follows: $s'_K = 0$, $s'_\omega = s_{\omega+1}$ for all $\omega \neq \omega^0$, $\omega \geq \underline{\underline{\omega}}(s)$, and $s'_{\omega^0} = s_{\omega^0+1} + m(s)$. Thus competition of all firms outside the industry inexorably advances, while the investments of all active firms fail to yield any success. Then [see Assumptions (A.4) and (A.5)]

$$Q(s,s') = p_{-1}\cdot \prod_{\omega \geq \underline{\underline{\omega}}(s)} [\pi(\omega|\,\omega,x(\omega,s))]^{|s_\omega|}\cdot P(\omega^0)^{|m(s)|} > 0,$$

as must be any finite product of these transition and entry probabilities. Repeat until all active firms have dropped (at some τ_1) to ω^0 or lower:

$$s_{\tau_1} = (n_0,n_1, \ldots ,n_{\omega^0},0, \ldots ,0\}.$$

This occurs in finite time as the initial industry structure is finite (Corollary 2).

(ii) For all $s \in \{s|\ s_\omega{=}0\ \forall\omega{>}\omega^0\}$ let s′ be defined as follows: $s'_\omega = 0$, $\omega > \omega^0$; $s'_{\omega^0} = s_{\omega^0} + m(s)$; $s'_{\omega^0-1} = 0$; $s'_\omega = s_{\omega+1}$, $\omega < \omega^0-1$. Again outside competition advances, while all active inside firms, except those at ω^0, fail to generate any success with their investment. Firms at ω^0 succeed in holding their own. Again such a transition has strictly positive probability [(A.4) and (A.5)]:

$$Q(s,s') = p_{-1}\cdot[\pi(\omega^0{+}1\,|\,\omega^0,x(\omega^0,s))]^{|s_{\omega^0}|}\cdot \prod_{\omega\in W}[\pi(\omega|\,\omega,x(\omega,s))]^{|s_\omega|}\cdot P(\omega^0)^{|m(s)|} > 0,$$

where $W \equiv \{\omega\in\Omega|\ \underline{\underline{\omega}}(s) \leq \omega < \omega^0\}$. Repeat until all firms below ω^0 have exited the industry. Again finiteness of the industry insures that this will occur in finite time τ_2. This yields, at $T = \tau_1 + \tau_2$, $\bar{s} \equiv s_T = (0, \ldots ,0,N,0, \ldots ,0)$, where $N = \text{argmin}_n\{m(0, \ldots ,0,n,0, \ldots ,0) = 0\} = n_{\omega^0} + \sum_{t=\tau_1+1}^{\tau_2} m(s_t)$. □

LEMMA 7: $\bar{s} \in R$.

PROOF: By Lemma 6, $\tilde{s} \longrightarrow \bar{s}$, where $\tilde{s}$ is the recurrent state whose existence was proven in Lemma 5. As $\tilde{s}$ is recurrent, $\bar{s} \longrightarrow \tilde{s}$ must follow [Theorem 1.55 in Freedman (1983)]. Hence $\tilde{s} \longleftrightarrow \bar{s}$ implying $\bar{s} \in R$. □

LEMMA 8: Let $\hat{s}$ be any recurrent state, $\hat{s} \in S$. Then $\hat{s} \in R$, i.e. R is the only recurrent class and $s \notin R$ implies that s is transient.

PROOF: By Lemma 7, $\hat{s} \longrightarrow \bar{s}$. By the definition of recurrence, $\bar{s} \longrightarrow \hat{s}$. Further, $\bar{s} \in$ $\in R$ and so $\hat{s} \in R$ must hold [Theorem 1.55, Freedman (1983)]. Hence R is unique and any $s \notin R$ must be transient. □

PROOF: (c) This is an immediate consequence of the existence of a single positive recurrent class: see Freedman (1983), Theorems 1.81, 1.88. □

PROOF: (d) $\mu_n = \nu Q^n$ (13) and hence converges iff the matrix Q^n does so. By Freedman (1983), Theorem 1.68, $\lim_{n\to\infty} Q^n(s,s') = 0$ if s' is transient $(s' \notin R)$, and by Theorem 1.69(c), if s' is recurrent $(s' \in R)$ then

$$\lim_{n\to\infty} Q^n(s,s') = \frac{\varphi Q(s,s')}{mQ(s',s')}$$

where $\varphi Q(s,s') \equiv P_s\{\xi_n = s' \text{ for some } n \geq 0\}$, P_s is defined in (A10), and $mQ(s',s')$ is defined above. Notice that, for all n, νQ^n is a probability measure. Hence μ_n converges to some probability measure, $\lim \mu_n = \pi$ (say). Now notice that $\pi Q = (\lim \nu Q^n)Q = \nu \cdot \lim Q^n \cdot Q = \nu \cdot \lim Q^n = \lim \nu Q^n = \pi$ so that π is an invariant probability measure for Q. However, by part (c) above, μ^* i the only (unique!) invariant probability measure, and therefore $\pi = \mu^*$. Q.E.D. □

PROOF: (Corollary 3) That $\mu^* Q = \mu^*$ was shown in Theorem 2. That P_{μ^*} (A10) is stationary is an immediate consequence of that fact. Let μ_t be the t–th period distribution starting from μ^*: $\mu_t = \mu_{t-1} Q = \ldots = \mu^* Q^t = \mu^* Q^{t-1} = \ldots = \mu^* Q = = \mu^*$. Q.E.D. □

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