PERFECT COMPETITION IN A BILATERAL MONOPOLY
(In Honor of Martin Shubik)

By

Pradeep Dubey and Dieter Sondermann

September 2005

COWLES FOUNDATION DISCUSSION PAPER NO. 1534

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

PERFECT COMPETITION IN A BILATERAL MONOPOLY
(In honor of Martin Shubik*)

Pradeep Dubey† Dieter Sondermann‡

March 21, 2005

Abstract

We show that if limit orders are required to vary smoothly, then strategic (Nash) equilibria of the double auction mechanism yield competitive (Walras) allocations. It is not necessary to have competitors on any side of any market: smooth trading is a substitute for price wars. In particular, Nash equilibria are Walrasian even in a bilateral monopoly.

Keywords: Limit orders, double auction, Nash equilibria, Walras equilibria, perfect competition, bilateral monopoly, mechanism design

JEL Classification: C72, D41, D42, D44, D61

*It is a pleasure for us to dedicate this paper to Martin Shubik who founded and developed (in collaboration with others, particularly Lloyd Shapley) the field of Strategic Market Games in a general equilibrium framework

†Center for Game Theory, Dept. of Economics, SUNY at Stony Brook and Cowles Foundation, Yale University

‡Department of Economics, University of Bonn, Bonn.

1 Introduction

As is well-known Walrasian analysis is built upon the Hypothesis of Perfect Competition, which can be taken as in Mas-Colell (1980) to state: "...that prices are publicly quoted and are viewed by the economic agents as exogenously given". Attempts to go beyond Walrasian analysis have in particular involved giving "a theoretical explanation of the Hypothesis itself" (Mas-Colell (1980)). Among these the most remarkable are without doubt the 19th century contributions of Bertrand, Cournot and Edgeworth (for an overview, see Stigler (1965)). The Cournot approach was explored intensively, in a general equilibrium framework, in the symposium issue entitled "Non-cooperative Approaches to the Theory of Perfect Competition" (Journal of Economic Theory, Vol. 22 (1980)).
The features common to most of the symposium articles are:

(a) The strategies employed by the agents are of the Cournot type, i.e., consist in quoting quantities.

(b) The (insignificant) size of any agent relative to the market is the key explanatory variable for the tendency of strategic behavior to approximate perfect competition and, in its wake, to lead to Walrasian outcomes (Mas-Colell (1980), p.122).

The extension of pure quantity strategies from Cournot's partial equilibrium model of oligopoly to a general equilibrium framework, however, does raise questions. Underlying the Cournot model is a demand curve for the particular market under consideration which enables the suppliers to relate quantities, via prices, to expected receipts. If such a close relationship is not provided by the market, then it seems more natural to us that an agent will no longer confine himself to quoting quantities, i.e., to pure buy-or-sell market orders. To protect himself against "market uncertainty - or illiquidity, or manipulation by other agents [1]", he will also quote prices limiting the execution of those orders, consenting to sell q units of commodity j only if its price is p or more, or buy $\tilde{q}$ units only if its price is $\tilde{p}$ or less. By sending multiple

[1] to quote from Mertens (2003)

orders of this kind an agent can approximate any monotone demand or supply curve in a market by a step function, as was done in Dubey (1982, 1994). Here we go further and give each agent full manoeuvrability. He places a continuum of infinitesimal limit-price orders, which in effect enables him to send any monotone, continuous demand or supply curve for each commodity. The upshot is a striking result: provided only that all commodity markets are "active" (i.e. there is positive trade in them), and no matter how thin they are, *strategic (Nash) equilibria* (SE) coincide - in outcome space - with *competitive (Walras) equilibria* (CE). Our result thus provides a rationale, based on strategic competition, for Walrasian outcomes even in the case of a bilateral monopoly. This brings it in sharp contrast to Dubey (1982, 1994), where it was necessary to have competition on both sides of each market (in the sense of there being at least two active buyers and two active sellers for each commodity) in order to conclude that SE are CE .

The models in Dubey (1982, 1994) rely on competition that is "cut-throat" in the spirit of Betrand. Any agent can take over a whole chunk of some buy (sell) order from another by quoting an infinitesimally higher (lower) price. Our model is not based on the possibility of such takeovers. Instead it requires that agents' behavior be "smooth", with commodities bought (sold) in infinitesimal increments of continuously non-increasing (non-decreasing) prices. The key point of our paper is that such smooth trading is a substitute for cut-throat price wars, and also gives rise to Walrasian outcomes. A monopolist may be in sole command of his own resource, but nevertheless he will be reduced to behaving as if he had cut-throat rivals, once smooth trading sets in. A related phenomenon[2] was analyzed in Coase (1972) (and following Coase (1972), a long line of literature, see e.g., Bulow (1982), Gaskins (1974), Schmalensee (1979)). There, too, a monopolist was shown to forfeit his power, but this happened in the setting of durable goods which could be sold sequentially over time to infinitely patient customers. In our model the monopolist loses power even with perishable goods which are traded at one instant of time. But we do need, unlike Coase, smooth strategic behavior on both sides of every market as well as convex preferences.

It must be emphasized that our model is based on *decentralized* markets.

[2]We thank John Geanakoplos for this reference.

Each commodity j is traded against fiat money ("unit of account"), and orders sent to the markets $k \neq j$ for *other* commodities k, do not affect how market j functions. Thus we do not allow an agent to link his buy-order for a commodity to whether the sell-order for another commodity goes through.[3] The only connection between different commodity markets is the budget-constraint of agents, requiring them to cover purchases out of their sales receipts. Our model is therefore an order-of-magnitude simpler than that of Mertens (2003), where cross-market limit orders are permitted. In spite of this paucity of our strategy-space compared to Mertens (2003), we exactly implement [4] CE via our mechanism (modulo activity in markets). In contrast, SE form a large superset [5] of CE in Mertens (2003) (though, we hasten to add, the implementation of CE was never the aim there, rather it was to well-define a mechanism that allowed for a rich menu of cross-market limit-orders).

For better perspective, we consider two somewhat contrasting versions of our model. In the first version agents act under the optimistic illusion that they can exert perfect price discrimination: sell to others, starting at the highest quoted market price (or buy, starting at the lowest). The equilibrium point (EP) that we define does not correspond to a strategic equilibrium (SE) of a standard game, because we allow agents to speculate that they could trade at much better prices, via unilateral deviations, than any proper game form would permit. Nevertheless we think that EP is an interesting concept in its own right.

In the second version we turn to a standard market game, akin to that of Dubey (1982) and Dubey (1994). Here each agent is grimly realistic and realizes that he will be able to buy (sell) only after higher-priced buyers (lower-priced sellers) have been serviced at the market, and that the prices he gets are apropos his own quotations, not the best going.[6] To accommo-

[3] That would be like allowing agents to submit demand functions based on the whole price vector.

[4] Indeed, our result may be interpreted in terms of the mechanism design literature (see Section 3).

[5] For instance, the SE of Shapley's "windows model" (see Sahi and Yao (1989)) are also SE in Mertens' model.

[6] We could make the same assumption also in the first market model. However we would

date economies in which CE consumptions could occur on the boundary[7], it becomes needful here to introduce a “market maker” who has infinitesimal inventories of every good, and stands ready to provide them if sellers renege on their promises of delivery. It turns out that, at our SE, the market maker is never active. But it is important for agents to imagine his presence when they think about what they could get were they to unilaterally deviate.

Though the two versions are built on quite different behaviorial hypotheses, we find their equilibria (the EP and the SE) lead to the same outcomes, namely Walrasian.

Our model shares some of the weaknesses of the Walrasian models. In particular, since it is based on the static concept of a strategic equilibrium, our model does not address the question of what dynamic forces bring the equilibrium about and ensure that individual strategic plans become jointly compatible. But it goes beyond the Walrasian notion in at least three important ways:

(a) It is not assumed that the economic agents face perfectly elastic supply and demand curves.

(b) Prices are not quoted from outside but set by the agents themselves. Each agent, operating in a market, realizes and exerts his ability to influence price.

(c) Strategies of the individuals (i.e. supply and demand curves submitted to the market) need not be based on their true characteristics (preferences and endowments).

---

lose economic insight, as to what happens to the consumers’ and producers’ surplus, when agents behave like monopolists, trying to exert perfect price discrimination .

[7]If we restrict to economies in which CE consumptions are strictly interior, the market maker can be dispensed with.

2 The First Version: Optimistic Conjectures and Equilibrium Points

Let $N = \{1, \ldots, n\}$ be the set of agents who trade in k commodities. Each agent $i \in N$ has an initial endowment $e^i \in \mathbb{R}^k_+ \setminus \{0\}$ and a preference relation $\gtrsim_i$ on $\mathbb{R}^k_+$ that is convex, continuous and monotonic (in the sense that $x \geq y$, $x \neq y$ implies $x \succ_i y$). We assume that $\sum\limits_{i \in N} e^i \gg 0$, i.e. every named commodity is present in the aggregate.

An agent may enter a market either as a buyer or a seller, and submit to each of the k commodity markets a marginal demand or supply curve. Formally, let

$$M^+ = \{f : \mathbb{R}_+ \to \mathbb{R}_{++} |\ f \text{ is continuous and non-decreasing}\}$$
$$M^- = \{f : \mathbb{R}_+ \to \mathbb{R}_{++} |\ f \text{ is continuous and non-increasing}\}.$$

Then a *strategic choice* σ^i of agent i is given by

$$\sigma^i = (d^i_1, s^i_1; \ldots ; d^i_k, s^i_k | d^i_j \in M^-, s^i_j \in M^+, \text{ for } j = 1, \ldots, k).$$

In the interpretation $d^i_j(q^i_j)$ is the price at which agent i is willing to buy an infinitesimal, incremental unit of commodity j, once his level of purchases has reached q^i_j. The supply curve has an analogous meaning. Denote $\sigma \equiv (\sigma_1, \ldots, \sigma_n)$ and let S^σ_j, D^σ_j be the aggregate supply, demand curves.

We suppose that agent i acts under the optimistic conjecture that he can exert perfect price discrimination, i.e., that he can sell (buy) starting at the highest (lowest) prices quoted by the buyers (sellers). This means that agent i calculates his receipts (or expenditures) on the market j as the integral, starting from 0, under the curve D^σ_j (or S^σ_j). The generally non-convex budget-set $B^i(\sigma)$ for $\sigma = (\sigma^1, \ldots, \sigma^n)$, is then obtained by the requiring that (perceived) expenditures do not exceed (perceived) receipts, i.e.,

$$B^i(\sigma) = \{e^i + t \mid t \in \mathbb{R}^k, e^i + t \in \mathbb{R}^k_+, \sum_{j=1}^k E^\sigma_j(t_j) \leq \sum_{j=1}^k R^\sigma_j(t_j)\}$$

where

$$E_j^\sigma(q) = \int_0^q S_j^\sigma \quad \text{if } q > 0,\ 0 \text{ otherwise,}$$

$$R_j^\sigma(q) = \int_0^{|q|} D_j^\sigma \quad \text{if } q < 0,\ 0 \text{ otherwise.}$$

(Note that $t_j^i > 0$ ($t_j^i < 0$) means that i buys (sells) j .)

The collection of strategic choices σ will be called an *equilibrium point* (EP) if there exist trade vectors $t^1, \dots, t^n$ in $\mathbb{R}^k$ such that

(i) $$e^i + t^i \text{ is } \gtrsim_i \text{-optimal on } B^i(\sigma) \text{ for } i = 1, \dots, n$$

(ii) $$\sum_{i=1}^{n} t_j^i = 0 \ \text{ for } j = 1, \dots, k$$

(iii) $$\sum_{i:t_j^i>o} t_j^i = sup\{q_j \mid S_j^\sigma(q_j) \leq D_j^\sigma(q_j)\} \text{ for } j = 1, \dots, k$$

Conditions (i) and (ii) require that agents optimize and that markets clear. Condition (iii) says that no trade can be enforced, i.e., it stops when the (marginal) supply price for the first time exceeds the demand price; and, at the same time, in equilibrium all trades compatible with the submitted strategies are actually carried out.

An EP will be called *active* if there is positive trade in each market.

First let us establish that at an active EP all trade $T_j := \sum_{i:t_j^i>0} t_j^i$ in any commodity j takes place at *one* price, p_j.

Lemma 1. *The curves S_j^σ and D_j^σ coincide and are constant on $[0, T_j]$ at any EP .*

Proof. For any j, let $G_j := \{i : t_j^i > 0\}$, $H_j := \{i : t_j^i < 0\}$ Then

$$
\begin{aligned}
(1) \qquad \sum_{i \in H_j} R_j^\sigma(t_j^i) &= \sum_{i \in H_j} \int_0^{|t_j^i|} D_j^\sigma \\
&\geq \int_0^{T_j} D_j^\sigma \\
&\geq D_j^\sigma(T_j) \cdot T_j \\
&\geq S_j^\sigma(T_j) \cdot T_j \\
&\geq \int_0^{T_j} S_j^\sigma \\
&\geq \sum_{i \in G_j} \int_0^{t_j^i} S_j^\sigma \\
&= \sum_{i \in G_j} E_j^\sigma(t_j^i).
\end{aligned}
$$

The third inequality follows from (iii); the other four follow from monotonicity of the supply and demand functions.

Hence

$$(2) \qquad \sum_{i=1}^{n} R_j^\sigma(t_j^i) \geq \sum_{i=1}^{n} E_j^\sigma(t_j^i) \quad \text{for } j = 1, \dots, k.$$

From the monotonicity of preferences, and the fact that each agent has optimized, we have

$$(3) \qquad \sum_{j=1}^{k} R_j^\sigma(t_j^i) = \sum_{j=1}^{k} E_j^\sigma(t_j^i) \quad \text{for } i = 1, \dots, n.$$

(2) and (3) together imply:

$$\text{(4)} \qquad \sum_{i=1}^{n} R_j^\sigma(t_j^i) = \sum_{i=1}^{n} E_j^\sigma(t_j^i) \quad \text{for } j = 1, \ldots, k.$$

From (4) it follows that all the inequalities in (1) must, in fact, be equalities. Therefore

$$\text{(5)} \qquad S_j^\sigma(T_j) = D_j^\sigma(T_j) =: p_j$$

and

$$\text{(6)} \qquad \int_0^{T_j} D_j^\sigma = p_j T_j = \int_0^{T_j} S_j^\sigma.$$

Since by (iii), $D_j^\sigma \geq S_j^\sigma$ on $[0, T_j]$ we get, from (6), and the monotonicity of D and S

$$\text{(7)} \qquad D_j^\sigma = S_j^\sigma \text{ on } [0, T_j].$$

□

In view of the Lemma 1 we can talk not only of the allocation but also the prices produced at an active EP . These are the constant values of S_j^σ, D_j^σ on $[0, T_j]$ for $j = 1, \ldots, k$. Note that these prices are positive by assumption.

Proposition 1. *The prices and allocation at an active equilibrium point are Walrasian.*

Proof. Let σ be an EP with trades $t^1, \ldots, t^n$ and prices p . We need to show that, for each i , $e^i + t^i$ is $\succsim_i$ -optimal on the set

$$B^i(p) := \{e^i + t : t \in \mathbb{R}^k, e^i + t \in \mathbb{R}_+^k,\ p.t = 0\}.$$

W.l.o.g. fix $i = 1$, put

$$
\begin{aligned}
J_1 &:= \{j : t^1_j > 0\} \\
J_2 &:= \{j : t^1_j < 0\} \\
J_3 &:= \{j : t^1_j = 0\} \\
T_j &:= \sum_{i:t^i_j>0} t^i_j \\
\delta_j &:= min[|t^1_j|, T_l : j \in J_1 \cup J_2, l \in J_3] \\
N_j &:= \{\alpha \in \mathbb{R} : |t^1_j - \alpha| < \delta_j\} \\
F_j &:= E_j - R_j
\end{aligned}
$$

(Since the EP is active, $\delta_j > 0$). Now we claim, for $j = 1, \ldots, k$:

(8) F_j is continuously differentiable and strictly increasing on N_j and its derivative at t^1_j is p_j.

This follows from the continuity and strict positivity of S_j and D_j, and from Lemma 1 which implies:

(9) $$F_j(q) \text{ coincides with } E_j(q) = p_j q \ \text{ if } j \in J_1,\ 0 \le q \le t^1_j$$

(10) $$F_j(q) \text{ coincides with } -R_j(q) = p_j q \ \text{ if } j \in J_2,\ t^1_j \le q \le 0$$

(11) $$F_j(q) = p_j q \ \text{ if } j \in J_3,\ q \in N_j.$$

W.l.o.g. fix commodity $j = 1$. Since $F_1, \ldots, F_k$ are all strictly increasing and $\sum_{j=1}^{k} F_j(t^1_j) = 0$, and $F'_1(t^1_1) = p_1 > 0$, it follows from the implicit function theorem that there is a neighborhood V of $(t^1_2, \ldots, t^1_k)$ in $N_2 \times \ldots \times N_k$ such that if $(t_2, \ldots, t_k) \in V$ then there is a unique t_1 which satisfies the equation $F_1(t_1) + \ldots + F_k(t_k) = 0$. Thus we have an implicit function $G(t_2, \ldots, t_k) = F_1^{-1}(-F_2(t_2) - \ldots - F_k(t_k))$ defined on V which is clearly continuously differentiable. Finally the point $t^1 = (t^1_1, \ldots, t^1_k)$ belongs by construction to the smooth hypersurface $M = \{(G(t_2, \ldots, t_k), t_2, \ldots, t_k) : (t_2, \ldots, t_k) \in V\} \subset B^1(\sigma)$ and, by (8), the tangent plane H to M at this point has normal p .

Since we are at an EP, $e^1 + t^1$ is $\succsim_1$-optimal on $(e^1 + M) \cap \mathbb{R}^k_+$. Suppose that there is some $x \in H_+ := (e^1 + t^1 + H) \cap \mathbb{R}^k_+$ such that $x \succ_1 e^1 + t^1$. By continuity of $\succsim_1$ we can find a neighborhood Z of x (in $\mathbb{R}^k_+$) with the property: $y \in Z \Rightarrow y \succ_1 e^1 + t^1$. But since M is a smooth surface there exists a point y^* in Z, such that the line segment between y^* and $e^1 + t^1$ pierces $e^1 + M$ at some point $z^* \in (e^1 + M) \cap \mathbb{R}^k_+$ (see Fig.1). By convexity of $\succsim_1$, we have $z^* \succ_1 e^1 + t^1$, contradicting that $e^1 + t^1$ is $\succsim_1$-optimal on $(e^1 + M) \cap \mathbb{R}^k_+$. We conclude that $e^1 + t^1$ is $\succsim_1$-optimal on H_+ . But we have $e^1 \in H_+$ (simply set trades to be zero, i.e., pick $-t^1$ in H). Therefore, in fact, $H_+ = B^1(p)$. Since the choice of $i = 1$ was arbitrary, the proposition follows. □

.... Insert Figure 1 approximately here!................

Proposition 2. *If the trades $t^1, \ldots, t^n$ and prices $p \gg 0$ are Walrasian, then they can be achieved at an EP*

Proof. For any i let

$$\begin{aligned} J^i_1 &= \{j : t^i_j > 0\} \\ J^i_2 &= \{j : t^i_j < 0\} \\ J^i_3 &= \{j : t^i_j = 0\} \\ f^i_j &= \text{any strictly decreasing function with } f^i_j(t^i_j) = p_j \\ g^i_j &= \text{any strictly increasing function with } g^i_j(t^i_j) = p_j \end{aligned}$$

and consider

$$s^i_j(x) = \begin{cases} 0 & \text{if } j \in J^i_1 \cup J^i_3 \\ max\{p_j, g^i_j(x)\} & \text{if } j \in J^i_2\} \end{cases}$$

$$d^i_j(x) = \begin{cases} 0 & \text{if } j \in J^i_2 \cup J^i_3 \\ min\{p_j, f^i_j(x)\} & \text{if } j \in J^i_1\} \end{cases}$$

Then it is readily checked that these strategies constitute a EP and produce the trades $t^1, \ldots, t^n$ at prices p. □

3 Strategic Market Games: Implementing Walras Equilibria with an Infinitesimal Market Maker

The foregoing analysis can be recast in terms of strategic (Nash) equilibria (SE) of a market game. Of course it is well known[8] (see Maskin (1999)) that CE cannot be implemented as SE unless CE consumptions are strictly in the interior of $\mathbb{R}^k_+$. By suitable restrictions on agent characteristics (e.g., $e^i \in \mathbb{R}^k_{++}$ and i's indifference surface through e^i is contained in $\mathbb{R}^k_{++}$, for all $i \in N$), one can consider a smaller domain of economies on which interiority is guaranteed. But we shall place no such restrictions here. Instead we shall imagine a "market maker"who has inventory of $\varepsilon_j > 0$ units of each commodity $j \in K \equiv \{1, ..., k\}$ and who is ready to bring them to market if any seller reneges on his promise to deliver, thereby giving the buyers *something* to look forward to. *No matter how small* $\varepsilon = (\varepsilon_1, ..., \varepsilon_k)$ is, so long as it is positive, CE are implemented as SE. The market maker is not called upon to take any action at the SE of our strategic game. He only lurks in the background. It is enough for every agent i to *believe* that the market maker would make available the infinitesimal inventory ε, were i to unilaterally deviate from SE and thereby trigger a situation in which some sellers of commodity j are unable to deliver on their promises. The belief in the market maker ensures that he is never called upon to prove his existence[9] (somewhat akin to the Federal Reserve's guarantee of private banks, which deters bank runs and eliminates the need for the Federal Reserve to make

[8] We are grateful to Stephen Morris and Andrew Postlewaite for references to the mechanism design literature.

[9] Indeed, we can reinterpret the scenario in terms of "refined" SE of the game without the market maker, relegating the market maker to ε-trembles in the refinement. (See Section 3.7).

good its guarantee). We feel that this role of the market maker is not without economic interest. But the reader who is troubled with the notion can restrict attention to the smaller domain of economies that have only interior CE. Our analysis goes through on this domain *without* the need for a market maker.

The main point of our analysis is not that Maskin's result on the impossibility of (Nash-)implementation of non-interior CE can be overcome with an infinitesimal market maker. Nor is it to add to the list of abstract mechanisms which implement the Walras correspondence. Many such have already been presented (see, e.g., Hurwicz (1979), Hurwicz, Maskin, and Postlewaite (1980), Postlewaite (1985), Schmeidler (1980)) – all of which, incidentally, require at least three agents, in addition to interior CE, and bypass the case of a bilateral monopoly). We are instead inspired by the fact that the "double auction" has a long and rich history, not only in academia, but in real market processes (see Friedman and Rust (1993) for an excellent survey). Our analysis reveals that a "smoothened" version of the double auction will make for efficiency and help to break monopoly power. It thereby implies that, if the "price-jumps" permitted in bidders' strategies are reduced by mandate of the auction-designer, every such reduction will come with efficiency gains. To that extent, we hope that our analysis below will also be of some interest to applied economists who are concerned with the general properties of double auctions.

3.1 The Subeconomy $\mathcal{E}_J$

It will be useful to define subeconomies $\mathcal{E}_J$ of the whole economy $\mathcal{E} = (e^i, \succeq_i)_{i \in N}$ for any subset $J \subset K \equiv \{1, ..., k\}$ of commodities. For a vector $y \in \mathbb{R}^K$, denote $y_J \equiv (y_j)_{j \in J} \in \mathbb{R}^J$. Then the set of agents in $\mathcal{E}_J$ is $\{i \in N : e^i_J \neq 0\}$, with endowments e^i_J and preferences $\succeq_{i,J}$ on $\mathbb{R}^J_+$ given by the rule: $z \succeq_{i,J} y$ iff $(z, e^i_{K \setminus J} \succeq_i (y, e^i_{K \setminus J})$.

3.2 Strategy Sets

There is a market for each commodity, as before. An agent must enter each market either as a buyer or as a seller (and, for simplicity, not both). If i enters as a buyer for commodity j, he must submit a strategic demand function $d^i_j : \mathbb{R}_+ \longrightarrow \mathbb{R}_{++}$ which is weakly decreasing, *and* smooth (i.e., continuously differentiable)[10]. The interpretation is that i is willing to pay $\int_0^x d^i_j(t)dt$ units of "fiat money" in order to purchase x units of commodity j. (There is no endowment of money in our model. But imagine that each agent can borrow money without limit at zero interest rate, from a bank in the background, prior to commodity trade and that the loan is due after trade.)

In the same vein, if i enters market j as a seller he must submit a strategic supply function $s^i_j : \mathbb{R}_+ \longrightarrow \mathbb{R}_{++}$ which is weakly increasing, smooth and (for ease of presentation) satisfies $\lim_{x \to \infty} s^i_j(x) = \infty$. In *addition*, i must put up $\tilde{\theta}^i_j > 0$ (with $\tilde{\theta}^i_j \leq e^i_j$) as "collateral" for his intention to sell j. (If $s^i_j = \phi$, it is understood that $\tilde{\theta}^i_j = 0$.) Finally we stipulate that each agent must enter at least one market as a seller. Thus the strategy set $\sum^i$ of agent i is given by

$$\begin{aligned} \sum\nolimits^i \;=\; & \{(d^i_j, s^i_j, \tilde{\theta}^i_j)_{j \in K} : \text{one, and only one,} \\ & \text{of } d^i_j, s^i_j \text{ is } \phi \text{ for every } j;\; s^i_j \neq \phi \text{ for at} \\ & \text{least one } j;\, 0 < \tilde{\theta}^i_j \leq e^i_j \text{ if } s^i_j \neq \phi; \\ & \tilde{\theta}^i_j = 0 \text{ if } s^i_j = \phi\} \end{aligned}$$

where the functions d^i_j, s^i_j satisfy the conditions mentioned.

3.3 Outcomes

The market maker does a sequence of computations based on the N-tuple $(\sigma_i)_{i \in N} \in \underset{i \in N}{X} \sum^i$ of submitted strategies, in order to impute commodity trades and monetary payments to the agents.

Step 1 Compute the aggregate demand D_j and aggregate supply S_j for each $j \in K$ as before.

Step 2 Compute the set $J \subset K$ of markets in which D_j and S_j intersect[11]

[10]If i does not enter market j as a buyer, we write $d^i_j = \phi$.

[11]At markets $j \in K \backslash J$, the intersection fails to occur either because S_j lies above D_j, or because one of the curves S_j or D_j is missing (which happens if $d^i_j = \phi$ for all i or $s^i_j = \phi$ for all i).

(at, necessarily, a unique price p_j - see Figure 2).

Step 3 In each market $j \in J$, compute sales by agents until the price p_j, rationing proportionately quantities offered for sale at the margin price p_j in the event that there is excess supply at p_j (see Figure 2). Denote these sales $(\theta_j^i)_{i \in N}$. (Some θ_j^i could be zero, provided $s_j^i = \phi$ or $s_j^i(0) > p_j$.)

If $\theta_j^i > \tilde{\theta}_j^i$ for *some* $j \in J$ (i.e., i's collateral fails to cover his imputed sale θ_j^i at some market), then i is declared a "*defaulter*" and forbidden to trade across *all* markets, and his collateral is confiscated at *every* market that he submitted them to.

Step 4 At each $j \in J$, define

$$Q_j = \begin{cases} \sum\limits_{i \in N} \theta_j^i, \text{ if there is no seller-default at } j \\ \varepsilon_j + \sum\limits_{i \in N} \min \left\{ \theta_j^i, \tilde{\theta}_j^i \right\} \text{ otherwise} \end{cases}$$

(Recall that ε_j is the market maker's infinitesimal inventory of commodity j). The market maker now allocates Q_j to buyers on D_j, starting at the highest price $D_j(0)$ in D_j and rationing proportionately the demand at the margin price $D_j(Q_j)$ if necessary (i.e., if there is excess demand at this price). Denote these purchases $(\varphi_j^i)_{i \in N.}$ If i is already a defaulter in Step 3, he is ignored; otherwise his net debt is computed:

$$\Delta^i = \sum_{j \in J} \int_0^{\varphi_j^i} d_j^i(t)dt - \sum_{j \in J} \int_0^{\theta_j^i} s_j^i(t)dt$$

(For $d_j = \phi$ or $s_j^i = \phi$, the integral is taken to be zero.) If $\Delta^i > 0$, then again i is declared a "defaulter" and dealt with as before, i.e., his collateral is confiscated at every market in which he put them up and he is forbidden from trading.

.... Insert Figure 2 approximately here!................

3.4 Payoffs

Agents $i \in N$ who are not defaulters (as in Step 3 or in Step 4) buy φ_j^i and sell θ_j^i in markets $j \in J$. They obtain payoff $u^i(x^i)$ where

$$x_j^i = \begin{cases} e_j^i + \varphi_j^i - \theta_j^i \text{ if } j \in J \\ e_j^i \text{ if } j \in j \in K \backslash J \end{cases}$$

Defaulting agents i obtain payoff $u^i(y^i)$ where

$$y_j^i = e_j^i - \tilde{\theta}_j^i \text{ for } j \in K.$$

This well defines a game Γ in strategic form on the player set N. By SE we shall mean a strategic (Nash) equilibrium in pure strategies of the game Γ.

3.5 Active SE are Walrasian

Define a market to be *active* in an SE if there is positive trade at that market.

Proposition 3. *At any SE with active markets J,all trade in $j \in J$ takes place at one price p_j. Moreover these prices and the final allocation constitute a CE of the economy $\mathcal{E}_J$.*

Define an SE to be *active* if all markets are active in it. Then Proposition 3 implies

Proposition 4. *The prices and allocations at at an active SE are Walrasian.*

Proof : We will prove the proposition for the case $J = K$. (The same argument holds for any $J \subset K$ and the corresponding economy $\mathcal{E}_J$.)

First observe that by lowering d_j^i to $\tilde{d}_j^i$ so that $\tilde{d}_j^i(0) < S_j(0)$ and by raising s_j^i to $\tilde{s}_j^i$ so that $\tilde{s}_j^i(0) > D_j(0)$, any agent i can ensure that he does not trade and so end up consuming his initial endowment e^i. But if i defaults, his utility is less than that of e^i, since he loses his collateral in at least one market and purchases nowhere. We conclude that there is no default in an SE.

Next we assert that (at an SE) in each market j all trade must be taking place at the intersection price p_j. The proof of this is similar to that of Lemma 1. Indeed, no more than the money paid out by agent-buyers goes to agent-sellers, implying $\sum_{i \in N} \Delta^i \geq 0$. But no default also implies $\Delta^i \leq 0$ for all $i \in N$. We conclude that $\Delta^i = 0$ for all $i \in N$. Now if any purchase took place above p_j or any sale below p_j in *some* market j, then (since purchases [or, sales] occur at prices $\geq$ [$or, \leq$] the intersection price at *every* market), we would have: total money paid out by agents across all markets $>$ total money received by agents across all markets. This would imply $\Delta^i < 0$ for some i, a contradiction, proving our assertion.

Consider the bundles that an agent i can obtain by unilateral deviation in his own strategy at the SE. First suppose i is a buyer of commodity j at the SE.

Case 1 There exists at least one other active buyer of j at the SE, or else there is excess supply of j at the SE price $p_{j.}$

In this case, i can buy slightly more of j at the price p_j by simply demanding a slightly higher quantity at p_j. (The maneuver works for i even if he is the sole buyer of j and the sellers of j have no collateral left to back further sales. This is on account of the market maker who stands ready to make up for the sellers from his inventory, enabling i to buy a little more).

Case 2 Case 1 fails, i.e., i is the sole buyer of j and there is no excess supply of j at the SE price p_j.

In this case, i can demand a little more at a slightly higher price (i.e., raise the flat part of his demand curve, keeping it flat till it intersects S_j). Since S_j is continuously differentiable, the extra quantity purchased by i will vary smoothly with the rise in the intersection price. (The fact that i can indeed buy a little more is once again assured by the infinitesimal inventory of the market maker.)

By a similar argument, i can sell a little more of any commodity j' that he was selling at the SE, either at the same price or at a price that is slightly lower and varies smoothly with the extra quantity sold.

Clearly i can *reduce* his sale and purchase and get the same price as at the SE.

Thus it is feasible for i to enhance trade a *little* beyond his SE trade in a smooth manner. More precisely, he can get consumption bundles on a smooth ε-extension $M(\varepsilon)$ of the flat part of his achievable set of bundles (where the extension is computed using prices smoothly increasing/decreasing away from p_j in accordance with the D_j/S_j curves). The situation is depicted in Figure 1, with the curved bold line extended only slightly beyond the flat part, and representing $M(\varepsilon)$.

But the argument in the proof of Proposition 1 applies, *no matter how small the smooth extension $M(\varepsilon)$ may be*: if x is not optimal on i's Walrasian budget set, then there exists a point z^* on $M(\varepsilon)$ which yields more utility to i than x, contradicting that i has optimized. This proves Proposition 3.

3.6 Walrasian outcomes are achieved at active SE

It is evident that Proposition 4 in fact holds if we allow agents to enter each market both as buyers and as sellers. The mechanism is well-defined, treating buy and sell orders as separate and disregarding the fact that they came from the same individual. Once we enhance the strategy sets in this manner, it is easy to establish along the lines of Proposition 2 :

Proposition 5. *The prices and allocations at any CE can be achieved at an active SE.*

Proof: Consider strategies in which every agents offers to sell his entire endowment at the CE prices (and to sell more at higher prices, as in the proof of proposition 2); and offers to buy his CE consumption bundle at the CE prices (and to buy more at lower prices). It is clear that these strategies constitute an active SE. ■

While Proposition 5 is technically correct it can leave one feeling a little uneasy, because (as its proof makes evident) it is based on “wash sales”, i.e., sale of a commodity by an agent who buys it back at the same price. However, the slightest transaction costs would eliminate such sales. Thus we develop Proposition 7 in the next section as an alternative to Proposition 5.

3.7 Refined Nash Equilibria

It might be useful to couch our results in terms of equilibrium refinement.

Fix the economy $(e^i, \succeq_i)_{i \in N}$ and let Γ_ε denote the strategic market game when the market maker has inventories $\varepsilon = (\varepsilon_1, ...\varepsilon_k) \in \mathbb{R}^K_{++}$ of the various commodities. Thus Γ_0 is the game without the market maker.

We shall say that an SE σ of Γ_0 is *refined* if there exist SE $\sigma(\varepsilon)$ of Γ_ε such that $\sigma(\varepsilon) \longrightarrow \sigma$ as $\varepsilon \longrightarrow 0$.

It is immediate that the market maker can be removed from the foreground and put into the ε-trembles of the refinement process, so that Proposition 4 may be reworded :

Proposition 6. *Active, refined SE of* Γ_0 *coincide in prices and allocations with the CE of the underlying economy* $(e^i, \succeq_i)_{i \in N}$.

In fact the word "active" can be dropped in Proposition 6 by strengthening refinement as follows. Imagine that, in our game Γ_ε, the market maker further endeavors to bolster trade by offering to buy (and, sell) up to $\tilde{\varepsilon}_j > 0$ units of commodity j at some common price $\tilde{p}_j$ and to buy (and, sell) more at smoothly decreasing (and, increasing) prices. Treating the market maker as a strategic dummy, and postulating that he creates the commodity and the money that the mechanism calls upon him to deliver, the game is well-defined even after some subset $J \subset K$ of markets are $\tilde{\varepsilon}_j - \tilde{p}_j$ – perturbed as described. We shall say that an SE $\sigma(\varepsilon)$ of Γ_ε is "*-refined" if there exist $\tilde{\varepsilon}_j - \tilde{p}_j$ – perturbations of the inactive markets in $\sigma(\varepsilon)$ that do not disturb[12] the SE $\sigma(\varepsilon)$. It is then trivial to verify (using the convexity of preferences) that *-refined

[12] A market can be inactive because agents have taken it into their heads to send crazy orders to it (with sellers asking for exorbitant prices and buyers offering absurdly low prices). On the other hand it may be open for business, quoting a single price at which the market maker is ready both to buy and to sell, and nevertheless remain inactive

SE of Γ_ε coincide with the CE of $(u^i, \succeq_i)_{i \in N}$. (i.e., *-refinement eliminates the need to postulate activity in all markets in Proposition 5). Now say that an SE σ of Γ_0 is *strongly refined* if there exist *-refined SE $\sigma(\varepsilon)$ of Γ_ε such that $\lim \sigma(\varepsilon) \longrightarrow \sigma$ as $\varepsilon \longrightarrow 0$. Then we obtain:

Proposition 7. *Strongly refined SE of* Γ_0 *coincide in prices and allocations with the CE of* $(e^i, \succeq_i)_{i \in N}$

3.8 Strong Nash Equilibria

It can be checked that our SE are strong (i.e. no coalition of agents can by coordinatedly changing its strategies - assuming others fixed - Pareto-improve itself). The proof of this is similar to that of the analogous proposition in Dubey (1982), hence omitted.

because agents are choosing voluntarily not to go there. The purpose of *-refinement is to rule out the first kind of inactivity (in which markets are arbitrarily "shut"), but allow for the second kind (in which markets are "open", though no one is coming there).

References

BULOW, J. (1982): "Durable-Goods Monopolists," *The Journal of Political Economy*, 90(2), 314–332.

COASE, R. (1972): "Durability and Monopoly," *The Journal of Law and Economics*, XV(1), 143–149.

DUBEY, P. (1982): "Price-Quantity Strategic Market Games," *Econometrica*, 50(1), 111–126.

——— (1994): "Strategic Market Games: A Survey of Recent Results," *Mertens, J.F. and S.Sorin (eds), Game-Theoretic Methods in General Equilibrium Theory.*

FRIEDMAN, D., AND J. RUST (1993): "The Double Auction Market," *Addison-Wesley.*

GASKINS, D. (1974): "ALCOA Revisted: The Welfare Implications of a Secondhand Market," *Journal of Economic Theory*, 7, 254–271.

HURWICZ, L. (1979): "Outcome Functions yielding Walrasian and Lindahl Allocations at Nash Equilibrium Points," *Review of Economic Studies*, 46, 217–225.

HURWICZ, L., E. MASKIN, AND A. POSTLEWAITE (1980): "Feasible Implementation of Social Choice Correspondences by Nash Equilibria," *Mimeo.*

MAS-COLELL, A. (1980): "Noncooperative Approaches to the Theory of Perfect Competition: Presentation," *Journal of Economic Theory*, 22(2), 121–135.

MASKIN, E. (1999): "Nash Equilibrium and Welfare Optimality," *Review of Economic Studies*, 66(1), 23–38.

MERTENS, J. (2003): "The Limit-Price Mechanism," *Journal of Mathematical Economics*, 39(5-6), 433–528.

Postlewaite, A. (1985): "Implementation via Nash Equilibria in Economic Environments," *Hurwics, L. and D. Schmeidler and H. Sonnenschein (eds), Social Goals and Social Organization: A Volume in Memory of Elisha Pazner.*

Sahi, S., and S. Yao (1989): "The Non-cooperative Equilibria of a Trading Economy with Complete Markets and Consistent Prices," *Journal of Math. Economics*, 18, 315–346.

Schmalensee, R. (1979): "Market Structure, Durability, and Quality: A Selctive Survey," *Economic Inquiry*, 17, 177–196.

Schmeidler, D. (1980): "Walrasian Analysis via Strategic Outcome Functions," *Econometrica*, 48, 1585–1594.

Stigler, G. (1965): "Perfect Competition, historically contemplated," *Essays in the History of Economics, Quart. 8, University of Chicago Press.*

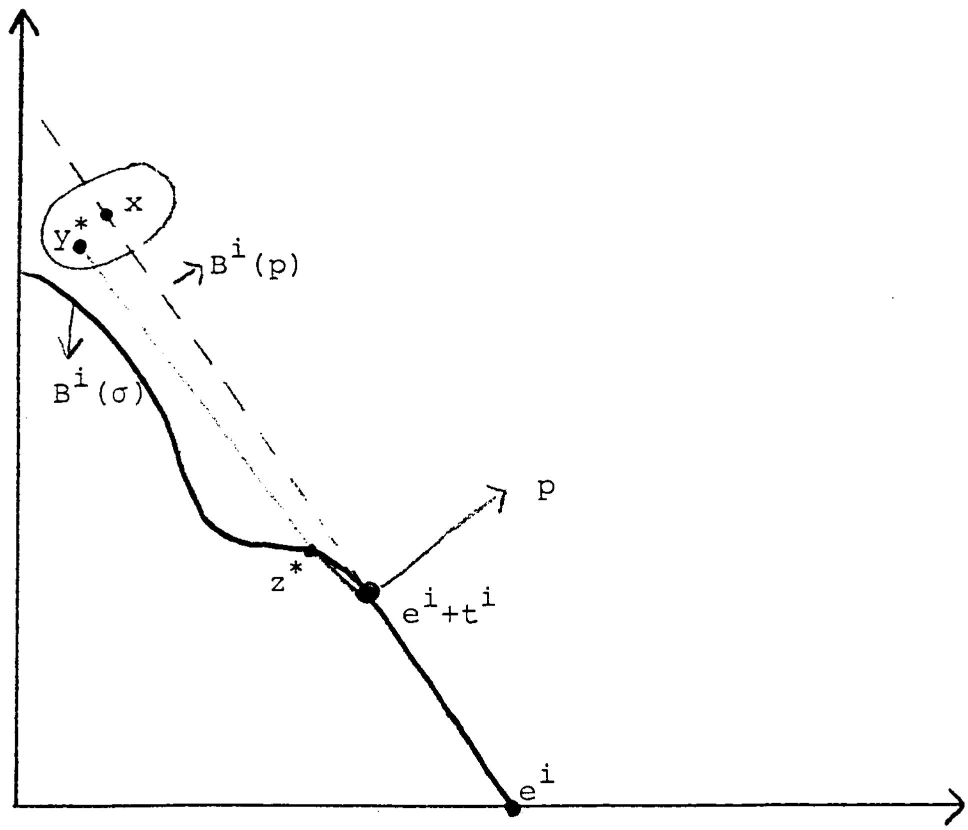
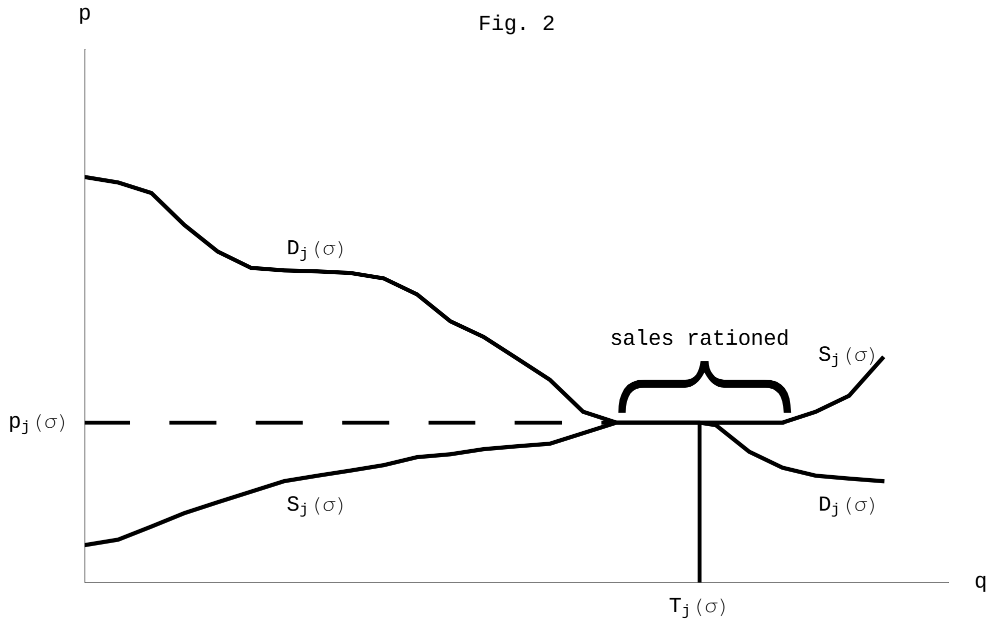


Fig. 1

Fig. 2


p
$D_j(\sigma)$
sales rationed
$S_j(\sigma)$
$p_j(\sigma)$
$S_j(\sigma)$
$D_j(\sigma)$
q
$T_j(\sigma)$