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Endogenous Prospect Theory

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Abstract. In previous models of (cumulative) prospect theory reference-dependence of preferences is imposed beforehand and the location of the reference point is exogenously determined. This note provides an axiomatization of a new specification of cumulative prospect theory, termed endogenous prospect theory, where reference-dependence is derived from preference conditions and a unique reference point arises endogenously

Keywords: prospect theory, reference point, diminishing sensitivity, loss aversion.

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1 Introduction

Prospect theory is currently the one of the most influential model of decision making under uncertainty and has been applied in various areas like financial markets, consumer choice, and political decision making. Apart from probability weighting, the central innovation of prospect theory is reference-dependence. Reference-dependence means that people do not evaluate final outcomes but instead they base decisions on gains and losses relative to a reference point. Empirically well documented facts supporting reference-dependence are diminishing sensitivity (people are more sensitive to changes near their reference points than to changes remote from it) and loss aversion (a negative deviation from the reference point has a higher impact than a positive deviation of equal size).

While original prospect theory (Kahneman and Tversky, 1979) has been proposed in an ad hoc manner, modern variants of prospect theory like cumulative prospect theory (CPT) and rank- and sign-dependent utility (see e.g. Luce, 1991; Luce and Fishburn, 1991; Tversky and Kahneman 1992; Wakker and Tversky 1993; Chateauneuf and Wakker 1999; Luce, 2000; Zank, 2001; Wakker and Zank 2002; Schmidt and Zank 2008) have been derived from behavioural foundations in terms of preferences. Such behavioural foundations are desirable because they, e.g., clarify the underlying assumptions of the model and set the ground for empirical testing. However, it can be argued that the existing axiomatizations of prospect theory are unsatisfactory. One reason already noted in the literature is the fact that the reference point is assumed to be given exogenously whereas models with endogenous reference points like that of Köszegi and Rabin (2006, 2007) are more successful in explaining behaviour. In our view there exists a second, more fundamental problem: The current axiomatizations of CPT assume the existence of a preference relation defined on gains and losses relative to an exogenously fixed reference point and then impose behavioural conditions on this preference relation. This means that reference-dependence is not derived form preference conditions but remains an ad hoc assumption as in original prospect theory. Additionally, CPT can neither be tested nor applied to concrete choice problems without knowing the location of the reference point.

The goal of the present note is to provide an axiomatization of CPT where reference-dependence is not assumed beforehand but where it is derived from a behavioural foundation in terms of preferences. Additionally, the location of the reference point is determined endogenously in our model, which we call endogenous prospect theory (EPT). This requires a criterion for identifying the location of the reference point since reference-dependence becomes meaningless if nothing would change at the reference point. As mentioned above, according to previous models of prospect theory two criteria can be used to identify the reference point, diminishing sensitivity and loss aversion. In our model we focus on diminishing sensitivity. We do this for two reasons: first, the evidence supporting diminishing sensitivity is extremely strong (see. e.g. Kahneman and Tversky, 1979; Tversky and Kahneman, 1981, 1992; Hershey and Schoemaker, 1985; Camerer, 1989; Currim and Sarin, 1989; Heath, Huddart, and Lang, 1999; Luce, 2000; Abdellaoui, 2000; Abdellaoui, Vossmann, and Weber, 2005; studies in the field of neuroeconomics include Dickhaut et al., 2003; de Martino et al., 2006) whereas evidence on loss aversion depends heavily on assumptions about the location of the non-observable reference point (Harrison and Rutström, 2008) and on which of the various definitions of loss aversion is employed (Abdellaoui, Bleichrdot, and Paraschiv, 2007). Additionally, loss aversion is not always dominant at the individual level (Schmidt and Traub, 2002; Ert and Erev, 2007; Erev, Ert, and Yechiam, 2009) and behavioral foundations of several definitions of loss aversion are often missing or have model-dependent implications (Schmidt and Zank, 2005, 2008). This should not mean that we think that loss aversion is not important and of course our model also allows for loss aversion.

The next section introduces our framework of decision making under uncertainty and some basic concepts. Section 3 contains our behavioral conditions and the main result: By imposing our central axiom - termed consistent diminshing sensitivity - reference-dependence arises endogenously in our model and the reference point is located at the position at which sensitivity towards changes in outcomes is maximal. In CPT, utility is defined only on deviations from the reference point whereas final wealth has no impact (see Schmidt, 2003, for a detailed analysis). In the terminology of Köszegi and Rabin (2006, 2007) this means that CPT only reflects gain-loss utility but no consumption utility. The utility function in EPT is defined on final wealth and the reference point determines its shape with respect to diminishing sensitivity and possibly loss aversion. Therefore, both gain-loss and consumption utility play a role.

2 Notation and Basic Concepts

We analyze decision problems under uncertainty and consider a finite set S of states of nature.[2] That is, $S = \{s_1, \ldots, s_n\}$ for a natural number $n \geq 3$, and $\mathcal{A} = 2^S$ is the algebra of subsets of S. Elements of $\mathcal{A}$ are called events. An *act* f assigns to each state a real valued *outcome*. The set of acts $\mathcal{F}$ can be identified with the Cartesian product space

[2]Our results can be extended to infinite state spaces by using tools presented in Wakker (1993). Identical results for the case of decision under risk, that is, when (objective) probabilities are given, can be derived by applying the procedure of Köbberling and Wakker (2003, Section 5.3).

$\mathbb{R}^n$, and hence, we write $f = (f_1, \ldots, f_n)$, where f_i is a short notation for $f(s_i)$. An act f is *rank-ordered* if its outcomes are ordered from best to worst: $f_1 \geq \cdots \geq f_n$. For each act f there exists a permutation ρ of $\{1, \ldots, n\}$ such that $f_{\rho(1)} \geq \cdots \geq f_{\rho(n)}$, i.e. such that the outcomes are *rank-ordered with respect to* ρ. For each permutation ρ of $\{1, \ldots, n\}$ the set $\mathbb{R}^n_\rho$ consists of those acts which are rank-ordered with respect to ρ. Acts that can be rank-ordered with respect to the same permutation are called *comonotonic*.

We use the notation $f_E g$ for an act that agrees with the act f on event E and with the act g on the complement E^c. Also, we use $h_i f$ instead of $h_{\{s_i\}} f$ for any state $s_i \in S$. Sometimes we identify constant acts with the corresponding outcome. We may thus write $f_E x$ for an act agreeing with f on E and giving outcome x for states $s \in E^c$.

We consider a preference relation $\succcurlyeq$ on the set of acts. As usually, $f \succcurlyeq g$ means that act f is weakly preferred to act g. The symbols $\succ$ and $\sim$ denote strict preference and indifference, respectively. The preference relation $\succcurlyeq$ is a *weak order* if it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for any acts f, g) and *transitive* ($f \succcurlyeq g$ and $g \succcurlyeq h$ implies $f \succcurlyeq h$). A functional $V : \mathcal{F} \to \mathbb{R}$ represents the preference relation $\succcurlyeq$ if for all $f, g \in \mathcal{F}$ we have $f \succcurlyeq g \Leftrightarrow V(f) \geqslant V(g)$.

An example of a representing functional is Choquet expected utility (CEU) introduced by Schmeidler (1989) and Gilboa (1987). It extends the classical subjective expected utility of Savage (1954) by introducing a non-additive measure for events: a *capacity* v satisfies $v(S) = 1, v(\emptyset) = 0$, and $v(A) \geq v(B)$ if $A \supset B$ and $A, B \in \mathcal{A}$. A capacity v is *strictly monotonic* if $v(A) > v(B)$ for $A \supsetneq B$ and $A, B \in \mathcal{A}$.

Choquet expected utility holds if the preference relation can be represented by the functional

$$CEU(f) = \sum_{i=1}^{n} U(f_i)\pi_i \quad \text{with} \quad \pi_i = v(\{s_1, \ldots, s_i\}) - v(\{s_1, \ldots, s_{i-1}\}).$$

The strictly increasing and continuous utility, U, is cardinal (i.e., it can be replaced by a positive linear transformation of U) and the strictly monotonic capacity, v, is unique. In terms of behavioral conditions, CEU can be derived by restricting Savage (1954)'s sure-thing principle to acts which are pairwise comonotonic, and further by requiring consistency accross states of risk attitudes towards changes in outcomes (see Köbberling and Wakker 2003).

Our paper focuses on a variant of cumulative prospect theory (CPT), which also captures the consistency requirement for outcome related risk attitudes, but which generalizes CEU by introducing dependence of these riak attitudes to a reference-point r. In all axiomatic work we are aware of the existence and location of this reference-point is assumed exogenously, in other words it is just assumed ad hoc that preferences are reference-dependent. Formally, previous models considered a preference relation $\succcurlyeq_r$ on acts defined in terms of deviations from r, i.e. act f is considered as $(f_1 - r, ..., f_n - r)$ where f_i is interpreted as gain (loss) if it exceeds (is less than) r.

Cumulative Prospect Theory holds if the representing functional for $\succcurlyeq_r$ has the form

$$CPT(f, r) = \sum_{i=1}^{n} U(f_i - r)\pi_i \text{ with } \pi_i = \begin{cases} v^+(\{s_1, \ldots, s_i\}) - v^+(\{s_1, \ldots, s_{i-1}\}) \text{ if } f_i \geq r \\ v^-(\{s_i, \ldots, s_n\}) - v^-(\{s_{i+1}, \ldots, s_n\}) \text{ if } f_i \leq r \ . \end{cases}$$

The two different capacities v^+ and v^- are uniquely determined and the utility is a ratio scale (i.e., unique up to multiplication by a positive constant) as it is fixed at the reference-point, i.e., $U(r - r) = 0$ holds. Note that in standard presentations of CPT the dependence on r is mostly suppressed. We state it here explicitly in order to clarify that for all previous CPT models reference-dependence of preferences is assumed beforehand, i.e. existence and location of r are not derived from preference conditions.

3 The Model

Let us first recall some standard properties for the preference $\succcurlyeq$, before we then introduce the main preference condition that allows for the identification of the reference-point. The preference relation $\succcurlyeq$ on $\mathcal{F}$ satisfies *monotonicity* if $f \succ g$ whenever $f_i \geq g_i$ for all states s_i with a strict inequality for at least one state. By employing this condition we ensure that the capacities, derived later, are stictly monotone because monotonicity excludes null states, that is, states where the preference is independent of the magnitude of outcomes. Formally, a state s_i is *null* if $x_i f \sim y_i f$ for all acts f and all outcomes x, y.

The continuity condition defined here is continuity with respect to the Euclidean topology on $\mathbb{R}^n$: $\succcurlyeq$ satisfies *continuity* if for any act f the sets $\{g \in \mathcal{F} | g \succcurlyeq f\}$ and $\{g \in \mathcal{F} | g \preccurlyeq f\}$ are closed subsets of $\mathbb{R}^n$.

In what follows we use several indifferences of the form $x_i f \sim y_i g$ with the implicit assumption that all acts involved in such indifferences are rank-ordered with respect to the same permutation ρ. We can now introduce the main condition in the paper: *consistent diminishing sensitivity* holds if for each outcome x one of the following holds:

(I) for any $w, z, y > x$

$$\begin{aligned} \text{if } x_j f &\sim y_j g \text{ and } z_j f \sim w_j g, \\ \text{then } z - x &< w - y \\ \text{and further } x_i f' &\sim y_i g' \text{ implies } z_i f' \sim w_i g'; \text{ or} \end{aligned}$$

(II) for any $w, z, y < x$

$$\begin{aligned} \text{if } x_j f &\sim y_j g \text{ and } z_j f \sim w_j g, \\ \text{then } x - z &< y - w \\ \text{and further } x_i f' &\sim y_i g' \text{ implies } z_i f' \sim w_i g'. \end{aligned}$$

In the presence of weak order, monotonicity and continuity, one can always find acts f and g and distinct outcomes w, z, y, x such that the indifferences $x_j f \sim y_j g$ and $z_j f \sim w_j g$ hold. The first indifference says that the difference in preference between f and g outside state j is off-set by receiving x and y, for the respective acts, if state j occurs. The second indifference says that the difference in preference between f and g outside state j is off-set by receiving z and w, for the respective acts, if state j occurs. One observes that the second indifference is obtained from the first by replacing x and y with z and w, respectively. Consistent diminishing sensitivity puts constraints on the relationship between $z - x$ and $w - y$ as explained next.

Suppose that x is such that the property (I) of consistent diminishing sensitivity holds. Then two features are demanded. First, increasing x in state j of act f to a z is as good as increasing y in state j of act g to a larger outcome w (following monotonicity). The property says that a larger increment than $z - x$ is required to obtain the second indifference, hence, $w - y > z - x$. Further, this "diminishing sensitivity" is required to be independent of the (pair of) acts f and g and the state j, hence the strict inequality is consistent across states. Such a finding is in agreement with risk aversion in the sense of diminishing marginal utility for increments in outcomes.

Suppose, however, that x is such that the property (II) of consistent diminishing sensitivity holds. Then those indifferences say that decreasing x in state j of act f to z is as bad as decreasing outcome y in state j of act g to a smaller w (following monotonicity). The property now requires that a larger decrement than $x - z$ is needed to obtain the second indifference, hence, $y - w > x - z$. Similarly to the previous case, this "diminishing sensitivity" is required to be independent of the acts f and g and the state j. This latter finding is in agreement with risk seeking in the sense of diminishing marginal utility for decrements in outcomes.

Note, however, that consistent diminishing sensitivity does not require a distinction of outcomes into gains and losses. It only says that for each outcome x one of the constraints, (I) or (II) above, must hold. It may, therefore, occur that for all outcomes only the first constraint (I) is satisfied. Or, it may be the case that for all outcomes only the second constraint (I) holds. However, it is worth noting at this stage that, in the presence of the other standard properties, if there exists some x for which constraint (I) is satisfied, then (I) must be satisfied for all $x' > x$; and if there exists some x for which the second constraint (II) is satisfied, then (II) is satisfied for all $x' < x$. It, therefore, follows that if there exists an outcome x^+ for which (I) holds and an outcome x^- for which (II) holds, then there exists a unique outcome r for which both (I) and (II) must hold, and this outcome r acts as a reference point for the preference $\succcurlyeq$.

The following calculus further illustrates the nature of consistent diminishing sensitivity. We distinguish 3 cases: (A) First, suppose that CEU holds and that utility is strictly concave. Then substitution of CEU for the indifferences $x_j f \sim y_j g$ and $z_j f \sim w_j g$ and subtracting the first resulting equality from the second implies

$$U(y) - U(x) = U(w) - U(z).$$

The additional requirement of strict concavity for utility implies that $w - z > y - x$

must hold. Recall that such preferences can be interpreted as CPT preferences with the reference point being at negative infinity (that is, all outcomes are seen as being gains). Further it must hold that $x_i f' \sim y_i g'$ implies $z_i f' \sim w_i g'$ for otherwise the above equality is violated. This implies that for each outcome x constraint (I) of consistent diminishing sensitivity holds.

In the second case (B) we assume that CEU holds with a strictly convex utility. Such preferences can then be interpreted as CPT preferences with the reference point being at infinity (that is, all outcomes are seen as being losses). Similarly to case (A) it follows now that for each outcome x constraint (II) of consistent diminishing sensitivity holds.

For the third case (C) suppose that there exists an outcome r such that preferences are represented by the CPT-like functional[3]

$$EPT(f) = \sum_{i=1}^{n} U(f_i)\pi_i \text{ with } \pi_i = \begin{cases} v^+(\{s_1, \ldots, s_i\}) - v^+(\{s_1, \ldots, s_{i-1}\}) \text{ if } f_i \geq r \\ \\ v^-(\{s_i, \ldots, s_n\}) - v^-(\{s_{i+1}, \ldots, s_n\}) \text{ if } f_i < r, \end{cases}$$

where the cardinal U is strictly concave (convex) for $f(s) \geq (\leq) r$ and the capacities are uniquely determined.[4] Then, substitution of EPT for the indifferences $x_j f \sim y_j g$ and $z_j f \sim w_j g$ and subtracting the first resulting equality from the second implies

$$U(y) - U(x) = U(w) - U(z),$$

whenever $w, z, y > x \geq r$ and the strict concavity of U implies that $w - z > y - x$ must hold. Further, $x_i f' \sim y_i g'$ implies $z_i f' \sim w_i g'$, for otherwise the above equality is violated. It also holds that

$$U(y) - U(x) = U(w) - U(z),$$

[3]The functional is not CPT in the traditional sense because of the interpretation of outcomes as changes in wealth for CPT in contrast to EPT where outcomes have final wealth.

[4]If one fixes the location parameter of the cardinal utility such that $U(r) = 0$ utility becomes a ratio scale.

whenever $w, z, y < x \leq r$ and the strict convexity of U implies that $z - w > x - y$ must hold. Further, $x_i f' \sim y_i g'$ implies $z_i f' \sim w_i g'$, for otherwise the above equality is violated. We conclude that in this case both (I) and (II) of consistent diminishing sensitivity hold.

The representing functional that agrees with either (A) or (B) or with (C) *endogenous prospect theory* (EPT), and note that consistent diminishing sensitivity is a necessary condition for EPT. The following theorem shows that, in the presence of the other standard preference conditions, consistent diminishing is also sufficient for EPT. This is the main result of the paper:

THEOREM 1 *Suppose that $\succcurlyeq$ is a preference relation on $I\!R^n, n \geq 3$. Then the following two statements are equivalent:*

(i) EPT holds with strictly monotone capacities.

(ii) The preference relation $\succcurlyeq$ is a monotonic continuous weak order satisfying consistent diminishing sensitivity.

Utility is cardinal and the capacities are unique. □

This theorem shows that reference-dependence is implied by our preference conditions and that the reference point is endogenously determined. A further difference to CPT is the fact that the utility function is defined on outcomes and not on deviations from the reference point. The reference point influences, however, the shape of the utility function. Our axiomatization of EPT is entirely based on the revealed preference paradigm and can be tested without assumptions about the location of the reference point.

Appendix: Proof

To prove Theorem **??** we remark that deriving statement (ii) from statement (i) is standard in conjunction with the comments preceding Theorem **??** regarding consistent diminishing sensitivity. Next we assume statement (ii) and derive statement (i). We distinguish three cases:

Case 1: For all outcomes x we have condition (I) of consistent diminishing sensitivity satisfied. In this case the comonotonic tradeoff consistency of Köbberling and Wakker (2003) holds and it follows from their Theorem 8 that CEU holds (with uniqueness results as noted in their Observation 9 (c)). Further, we can always find indifferences $x_j f \sim y_j g$ and $z_j f \sim w_j g$ for acts f, g a state j and outcomes $w, z, y > x$. Substitution of CEU and subtraction of the first resulting equality from the second implies

$$U(y) - U(x) = U(w) - U(z).$$

Constant diminishing sensitivity demands that $w - z > y - x$ must hold in this case. Because this implication must hold for any outcome x (and corresponding $w, z, y > x$), it follows that the utility function must be concave.

Case 2: For all x we have condition (II) of consistent diminishing sensitivity satisfied. Similar to the previous case, the results of Köbberling and Wakker (2003) hold and we obtain CEU. Further, consistent diminishing sensitivity implies that the utility function is convex. Uniqueness results apply as noted in Observation 9 (c) of Köbberling and Wakker (2003).

Case 3: There exist an outcome x^+ for which condition (I) of constant diminishing sensitivity holds and an outcome x^- for which condition (II) of consistent diminishing sensitivity holds. Then there exists a unique outcome r for which both (I) and (II) must hold, which is the reference point for the preference $\succcurlyeq$. In this case consistent diminishing sensitivity implies the sign-comonotonic tradeoff consistency of Köbberling and Wakker (2003), and from their Theorem 12 we obtain that CPT holds. By Proposition 8.2 in

Wakker and Tversky (1993) the gain-loss consistency requirement can be dropped from statement (ii) in Theorem 12 in Köbberling and Wakker's (2003) when the number of states of nature exceeds 2, which is the case here. Similar to cases 1 and 2 above we derive strict concavity of utility for outcomes above r and strict convexity for utility for outcomes below r. Uniqueness results follow from Observation 13 in Köbberling and Wakker (2003).

Together cases 1–3 cover all possibilities and thus statement (i) follows. This completes to proof of the theorem. □

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