# Differentiability properties of Rank Linear Utilities

G. Carlier *

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**Abstract**

We study the differentiability properties of concave functionals defined as integrals of the quantile. These functionals generalize the rank dependent expected utility and are called rank-linear utilities in decision theory. Their superdifferential is described as well as the set of random variables where they are Gâteaux-differentiable. Our results generalize those obtained for the rank dependent expected utility in [1].

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* CEREMADE, UMR CNRS 7534, Université Paris IX Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cedex 16, FRANCE carlier@ceremade.dauphine.fr.

## 1 Introduction

The aim of this paper is to study differentiability properties of some concave quantile-based integral functionals. Such law-invariant utilities are sometimes called rank-linear utilities (henceforth RLU) and are of the form:

$$V(X) := \int_0^1 L(t, F_X^{-1}(t))dt$$

where $F_X^{-1}$ is a version of the quantile of the random variable $X$ and $L$ satisfies some assumptions of concavity and submodularity ensuring that $V$ is concave. Those utilities were studied by Green and Jullien [10] who showed that they are characterized by an axiom of *ordinal independence*, weaker than the von Neumann-Morgenstern independence axiom. We also refer to the papers of Chew and Epstein [5] and Chew and Wakker [6] and the references therein for the decision-theoretic foundations of those utilities.

The issue of differentiability of an RLU naturally arises in a variety of problems : efficient risk sharing rules between RLU agents, demand of an RLU agent for a risky asset, structure of equilibria... Due to the analogy -up to a minus sign- between RLU functionals and law-invariant convex risk measures (although RLU do not fulfill the cash invariance property), we also believe that the results of the present paper may be useful in some risk-measures problems.

When $L(t,x) = f'(1-t)U(x)$, with $f$ a convex distortion satisfying $f(0) = 0$, $f(1) = 1$ and $U$ a concave utility index then $V$ is a rank-dependent utility (RDU), in the linear case $U(x) = x$ then $V$ is a Yaari utility. In the case of a Yaari utility, $V$ is the support function of the core of the distortion of the underlying probability by $f$, hence differentiablity properties of $V$ are tightly linked to the geometry of the core. The differentiability properties of RDU functionals have been studied in [1] using a characterization of the core of convex distortions of a probability. For a more general $L$, the previous approach is not adapted and different arguments have to be developed to compute the superdifferential of $V$ and the set of random variables where $V$ is Gâteaux-differentiable.

Some basic definitions and properties are given in section 2. The superdifferential of an RLU is determined in section 3. Some applications are given in section 4, including the identification of the set where an RLU is Gâteaux-differentiable. A technical lemma, used in the proof of the representation of the superdifferential of an RLU is proved in section 5.

## 2 Rank linear utilities

We recall that a probability space $(\Omega, \mathcal{F}, P)$ is nonatomic if there is no $A \in \mathcal{F}$ such that $P(A) > 0$ and $P(B) \in \{0, P(A)\}$ for every $B \in \mathcal{F}$ such that $B \subset A$. In the sequel, we will always work on a state space $(\Omega, \mathcal{F}, P)$ assumed to be nonatomic. We also recall that if $(\Omega, \mathcal{F}, P)$ is nonatomic, there exists a random variable $U$ on $(\Omega, \mathcal{F}, P)$ such that the probability law of $U$ is the uniform law on $[0, 1]$ (this property is actually a characterization).

Let $X$ be a random variable on $(\Omega, \mathcal{F}, P)$ and let $F_X(t) = P(X \leq t),\ t \in \mathbb{R}$ denote its distribution function. The generalized inverse of $F_X$ is defined by:

$$F_X^{-1}(t) = \inf\{z \in \mathbb{R} : F_X(z) > t\}, \text{ for all } t \in (0, 1)$$

We also define the set or random variables with a uniform probability law:

$$\mathcal{U} := \{U \in L^\infty(\Omega, \mathcal{F}, P) \ : \ F_U^{-1}(t) = t,\ \forall t \in [0, 1]\} \tag{1}$$

We will use in the sequel a decomposition result due to Ryff (see [11]):

**Proposition 1** *Let $X \in L^\infty(\Omega, \mathcal{F}, P)$ there exists $U \in \mathcal{U}$ such that $X = F_X^{-1} \circ U$.*

Let us define

$$\mathcal{U}_X := \{U \in \mathcal{U} \text{ such that } X = F_X^{-1} \circ U\}.$$

Ryff's result implies that $\mathcal{U}_X \neq \emptyset$. Moreover, if $X$ has no atoms (i.e. $F_X$ is continuous) then $\mathcal{U}_X = \{F_X \circ X\}$. A characterization of $\mathcal{U}_X$ can be given using the concept of comonotonicity. Let us first recall the following definition:

**Definition 1** *Two random variables $X$ and $Y$ on $(\Omega, \mathcal{F}, P)$ are said to be comonotone if:*

$$(X(\omega_2) - X(\omega_1))(Y(\omega_2) - Y(\omega_1)) \geq 0,\ P \otimes P \quad a.s. \tag{2}$$

*Similarly $X$ and $Y$ on $(\Omega, \mathcal{F}, P)$ are said to be anticomonotone if $X$ and $-Y$ are comonotone.*

The set $\mathcal{U}_X$ may be characterized as follows:

**Lemma 1** *Let $X \in L^\infty(\Omega, \mathcal{F}, P)$, then:*

$$\mathcal{U}_X = \{U \in \mathcal{U} \text{ such that } U \text{ and } X \text{ are comonotone}\}.$$

*Moreover for every $U \in \mathcal{U}_X$, one has $P$-a.s. $U(\omega) \in [F_X(X(\omega)_-), F_X(X(\omega))]$.*

**Proof.** Since $F_X^{-1}$ is nondecreasing, any element of $\mathcal{U}_X$ is comonotone with $X$. Now assume that $U \in \mathcal{U}$ is comonotone with $X$:

$$(X(\omega_2) - X(\omega_1))(U(\omega_2) - U(\omega_1)) \geq 0,\ P \otimes P \quad \text{a.s.} \tag{3}$$

and let us prove that $X = F_X^{-1} \circ U$. By (3), we obtain that for $P$-a.e. $\omega \in \Omega$, one has:

$$P(\{\omega' \in \Omega \,:\, U(\omega') < U(\omega)\}) = U(\omega) \leq F_X(X(\omega))$$

and

$$P(\{\omega' \in \Omega \,:\, X(\omega') < X(\omega)\}) = F_X(X(\omega)_-) \leq U(\omega).$$

Hence $U(\omega) \in [F_X(X(\omega)_-), F_X(X(\omega))]$. If $F_X$ is continuous at $X(\omega)$, then $U(\omega) = F_X(X(\omega))$ and $X(\omega) = F_X^{-1}(U(\omega))$. Now if $X(\omega) = x$ with $F_X$ discontinuous at $x$, $F_X^{-1}$ is constant equal to $x$ on $[F_X(x_-), F_X(x)]$ hence $F_X^{-1}(U(\omega)) = X(\omega)$ and the proof is complete. □

In the remainder of the paper, we shall study differentiability properties of quantile based-utilities defined by integrals. These utilities generalize the rank dependent expected utility with a convex continuous distortion and are of the form:

$$V(X) := \int_0^1 L(t, F_X^{-1}(t))dt, \quad \text{for all } X \in L^\infty(\Omega, \mathcal{F}, P). \tag{4}$$

where $F_X^{-1}$ is a version of the quantile of the random variable $X$. The previous class of utilities is sometimes refered to as rank-linear utility (henceforth RLU) in decision theory (see [10], [5], [6] and the references therein). We will always assume in the sequel the following properties on $L$:

- $L \in C^0([0,1] \times \mathbb{R}, \mathbb{R})$,
- $L(t,.)$ is concave nondecreasing for every $t \in [0,1]$,
- $L(t,.)$ is differentiable for every $t \in [0,1]$ and $\partial_x L(.,.) \in C^0([0,1]\times, \mathbb{R})$,
- $L$ is submodular i.e for every $(t_1, t_2, x_1, x_2) \in [0,1]^2 \times \mathbb{R}^2$:

$$t_2 \geq t_1,\ x_2 \geq x_1 \Rightarrow L(t_1, x_1) + L(t_2, x_2) \leq L(t_1, x_2) + L(t_2, x_1).$$

A stronger assumption than submodularity is strict submodularity, defined by: for every $(t_1, t_2, x_1, x_2) \in [0,1]^2 \times \mathbb{R}^2$:

$$t_2 > t_1,\ x_2 > x_1 \Rightarrow L(t_1, x_1) + L(t_2, x_2) < L(t_1, x_2) + L(t_2, x_1). \tag{5}$$

When $L$ is of class $C^2$, submodularity of $L$ is equivalent to $\partial_{tx}^2 L \leq 0$ and a sufficient condition for strict submodularity of $L$ is $\partial_{tx}^2 L < 0$. When $L$ is of class $C^1$ a sufficient condition for strict submodularity of $L$ is that $\partial_x L(t,x)$ is decreasing in $t$ for every $x$. Classical examples of submodular $L$'s are given by functions of the form $L(t,x) = f(t)g(x)$ with $f$ nonincreasing and $g$ nondecreasing, $L(t,x) = f(t+x)$ with $f$ concave, $L(x,y) = g(t-x)$

with $g$ convex... When $L(t,x) = g(t)U(x)$ with $g$ nonincreasing and $U$ concave nondecreasing, the corresponding $V$ is an RDU functional with convex distortion.

The assumption of monotonicity clearly ensures monotonicity of $V$ (in the sense that $X \geq Y$ $P$-a.s implies $V(X) \geq V(Y)$)). The assumptions of concavity and submodularity above (which also appear naturally in [10], related to risk aversion) ensure that $V$ is a concave functional (which is not straightforward at first glance), as shown in proposition 2. Actually, more is true: the assumptions of monotonicity, concavity and submodularity above are indeed necessary and sufficient for $V$ to be monotone, concave and u.s.c for the weak $*$ topology of $L^\infty(\Omega, \mathcal{F}, P)$ (see [4] for a proof).

# 3 The superdifferential of an RLU

## 3.1 Preliminary results

In the sequel, we will denote by $\mathbb{E}$, expectation with respect to $P$. Under the previous assumptions of concavity and submodularity, $V$ defined by (4) admits a particular concave representation, as the next result shows:

**Proposition 2** *Under the general assumptions of the paper, for every $X \in L^\infty(\Omega, \mathcal{F}, P)$, one has:*

$$V(X) = \inf_{U \in \mathcal{U}} \mathbb{E}(L(U,X)) \tag{6}$$

*Moreover, defining:*

$$\mathcal{V}_X := \{U \in \mathcal{U} \ : \ V(X) = \mathbb{E}(L(U,X))\}$$

*one has $\mathcal{U}_X \subset \mathcal{V}_X$. In particular, $V$ is concave and the infimum in (6) is a minimum. Finally, under the additional assumption (5), one has $\mathcal{U}_X = \mathcal{V}_X$.*

**Proof.** By definition of $V$ and $\mathcal{U}_X$, we have for every $U \in \mathcal{U}_X$:

$$\mathbb{E}(L(U,X)) = \mathbb{E}(L(U, F_X^{-1} \circ U)) = \int_0^1 L(t, F_X^{-1}(t))dt = V(X).$$

Now for every $U \in \mathcal{U}$, the submodular Hardy-Littlewood inequality (see for instance [2]) implies:

$$V(X) \leq \mathbb{E}(L(U,X)) \tag{7}$$

which implies the representation (6) and $\mathcal{U}_X \subset \mathcal{V}_X$. When (5) is satisfied, inequality (7) is strict unless $U$ and $X$ are comonotone (see for instance [2]) hence $U \in \mathcal{U}_X$ by lemma 1. We then have $\mathcal{U}_X = \mathcal{V}_X$ in this case. $\square$

Let $X \in L^\infty(\Omega, \mathcal{F}, P)$, the superdifferential of $V$ at $X$ is by definition:

$$\partial V(X) := \{\mu \in (L^\infty)' : V(Y) - V(X) \leq \langle \mu, Y - X \rangle, \forall Y \in L^\infty\}.$$

Since $V$ is finite and concave, general convex analysis results (see [7]) imply that $V$ is continuous (for the strong topology of $L^\infty$) hence everywhere superdiffentiable which by definition means $\partial V(X) \neq \emptyset$. Moreover since $V$ is monotone, $\partial V(X)$ is a subset of:

$$(L^\infty)'_+ := \{\mu \in (L^\infty)' : \langle \mu, Y \rangle \geq 0, \forall Y \in L^\infty, Y \geq 0\}.$$

In other words, $\partial V(X)$ consists of finitely additive nonnegative measures on $(\Omega, \mathcal{F}, P)$. More is true: due to the special form (6), $\partial V(X)$ is in fact included in $L^1(\Omega, \mathcal{F}, P)$:

**Lemma 2** *For all $X \in L^\infty(\Omega, \mathcal{F}, P)$, $\partial V(X)$ is a closed convex subset of $L^1(\Omega, \mathcal{F}, P)$.*

**Proof.** Since the convex set $\partial V(X)$ is weak $*$ closed in $(L^\infty)'$, it is enough to show that $\partial V(X) \subset L^1(\Omega, \mathcal{F}, P)$. Let $\mu \in \partial V(X)$, $A \in \mathcal{F}$ and $U_A \in \mathcal{U}_{X - \mathbf{1}_A}$ we then have:

$$\mathbb{E}(L(U_A, X - \mathbf{1}_A)) - \mathbb{E}(L(U_A, X)) \leq V(X - \mathbf{1}_A) - V(X) \leq -\mu(A).$$

By our assumptions on $L$, there exists $c > 0$ such that $L(u, x-1) - L(u, x) \geq -c$ for all $(u, x) \in [0, 1] \times [-\|X\|_{L^\infty}, \|X\|_{L^\infty}]$ which yields:

$$V(X - \mathbf{1}_A) - V(X) = \int_A [L(U_A(\omega), X(\omega) - 1) - L(U_A(\omega), X(\omega))] dP(\omega) \geq -cP(A).$$

We then obtain $0 \leq \mu(A) \leq cP(A)$, which implies that $\mu$ is a $\sigma$-additive nonnegative measure absolutely continuous with respect to $P$, in other words $\mu \in L^1(\Omega, \mathcal{F}, P)$.

$\square$

### 3.2 Main result

In view of formula (6), we see that computing $\partial V(X)$ amounts to computing the superdifferential of a lower envelope. General envelope theorems (see [12]) cover the case where the infimum is taken with respect to a parameter in a compact set. In the present problem (where the parameter space is $\mathcal{U}$), getting some sort of compactness requires to combine carefully a.s. convergence and convergence in law arguments, technical details are defered to section 5. Leaving apart this compactness issue, the following result may be viewed as classical and so are the main lines of its proof.

**Theorem 1** *Let $X \in L^\infty(\Omega, \mathcal{F}, P)$ then:*

$$\partial V(X) := \overline{\text{co}}\{\partial_x L(U, X),\ U \in \mathcal{V}_X\}$$

*where $\overline{\text{co}}$ denotes closed convex hull operation for the $L^1(\Omega, \mathcal{F}, P)$ topology. Under the additional condition (5), we then have:*

$$\partial V(X) := \overline{\text{co}}\{\partial_x L(U, X),\ U \in \mathcal{U}_X\}$$

**Proof.** Define

$$B := \overline{\text{co}}\{\partial_x L(U, X),\ U \in \mathcal{V}_X\}$$

If $U \in \mathcal{V}_X$ and $Y \in L^\infty(\Omega, \mathcal{F}, P)$, by concavity of $L(u, .)$ we have:

$$V(Y) - V(X) \le \mathbb{E}(L(U, Y) - L(U, X)) \le \mathbb{E}(\partial_x L(U, X)(Y - X)$$

hence $\partial_x L(U, X) \in \partial V(X)$. Since $\partial V(X)$ is convex and closed in $L^1$, we then have $B \subset \partial V(X)$.

Given $C$ a closed convex of $L^1(\Omega, \mathcal{F}, P)$, we define the support function of $C$ by:

$$\sigma_C(Y) := \inf_{Z \in C} \mathbb{E}(YZ),\ \forall\, Y \in L^\infty(\Omega, \mathcal{F}, P).$$

Since both $B$ and $\partial V(X)$ are convex and closed in $L^1$, a standard separation argument implies that $\sigma_{\partial V(X)} = \sigma_B$ implies $B = \partial V(X)$. We already know that $\sigma_{\partial V(X)} \le \sigma_B$. To show the converse inequality, we remark that by a standard convex analysis result (see [7]), for all $Y \in L^\infty(\Omega, \mathcal{F}, P)$, one has:

$$\sigma_{\partial V(X)}(Y) = D^+ V(X; Y) := \lim_{t \to 0^+} \frac{1}{t}[V(X + tY) - V(X)] \tag{8}$$

For a given $Y \in L^\infty(\Omega, \mathcal{F}, P)$ and $t > 0$, let $U_t \in \mathcal{U}_{X+tY}$ and let $\theta_t$ be the joint probability law of $(U_t, X + tY, Y)$. There exists a sequence $t_n$ decreasing to 0 such that $\theta_n := \theta_{t_n}$ weakly $*$ converges to some probability measure $\theta$ supported on $[0, 1] \times [-\|X\|_{L^\infty}, \|X\|_{L^\infty}] \times [-\|Y\|_{L^\infty}, \|Y\|_{L^\infty}]$. We claim that there exists $\overline{U} \in \mathcal{V}_X$ such that the joint probability law of $(\overline{U}, X, Y)$ is $\theta$. The proof of this claim is rather long and will be given separately. Let us admit this result and proceed to the end of the proof. By concavity and since $\partial_x L(\overline{U}, X) \in B$, we get:

$$\begin{aligned}
\sigma_{\partial V(X)}(Y) =& \lim_n \frac{1}{t_n}[V(X + t_n Y) - V(X)] \\
\ge& \liminf_n \frac{1}{t_n} \mathbb{E}(L(U_{t_n}, X + t_n Y) - L(U_{t_n}, X)) \\
\ge& \liminf_n \mathbb{E}(\partial_x L(U_{t_n}, X + t_n Y)Y) \\
=& \int_{[0,1] \times \mathbb{R}^2} \partial_x L(u, x, y) y d\theta(u, x, y) \\
=& \mathbb{E}(\partial_x L(\overline{U}, X)Y) \ge \sigma_B(Y).
\end{aligned}$$

this proves that $\partial V(X) = B$. Finally, when (5) is satisfied, it follows from proposition 2 that $\mathcal{V}_X = \mathcal{U}_X$, we thus obtain the desired representation of $\partial V(X)$.

$\square$

In the previous proof, we have used the following technical result that is proved in section 5:

**Lemma 3** *Using the same notations as in the previous proof, there exists $\overline{U} \in \mathcal{V}_X$ such that the joint probability law of $(\overline{U}, X, Y)$ is $\theta$.*

# 4 Applications

## 4.1 Comonotonicity and single-valuedness

Under the assumption (5), we may first deduce from theorem 1 that elements of $\partial V(X)$ are anticomonotone with $X$:

**Proposition 3** *In addition to the general assumptions of the paper, assume that $L$ satisfies (5) and let $X \in L^\infty(\Omega, \mathcal{F}, P)$, then every element of $\partial V(X)$ is anticomonotone with $X$.*

**Proof.** Let us first remark that the set of $L^1(\Omega, \mathcal{F}, P)$ random variables that are comonotone with $X$ is convex and closed for the $L^1$ topology. Thanks to theorem 1 and lemma 1, it is therefore enough to prove that $\partial_x L(U, X)$ is anticomonotone with $X$ for every $U \in \mathcal{U}_X$. If $U \in \mathcal{U}_X$, $X = F_X^{-1} \circ U$ and $\partial_x L(U, X) = \partial_x L(U, F_X^{-1} \circ U)$. Hence, if we prove that the function $t \mapsto \phi(t) := \partial_x L(t, \gamma(t))$ is nonincreasing on $[0, 1]$ for every nondecreasing $\gamma$, the desired result will follow. Assume first that $L$ is of class $C^2$ and $\gamma$ is differentiable, we then have:

$$\phi'(t) = \partial^2_{tx} L(t, \gamma(t)) + \partial^2_{xx} L(t, \gamma(t))\gamma'(t) \leq 0$$

so that $\phi$ is nondecreasing. In the general case, we can approximate (by convolution for instance) $L$ and $\gamma$ by smooth functions $L_n$ and $\gamma_n$ satisfying $\partial^2_{tx} L_n \leq 0$, $\partial^2_{xx} L_n \leq 0$ and $\gamma_n' \geq 0$. The desired result then follows from letting $n$ go to $+\infty$. $\square$

A second qualitative consequence of theorem 1, is that all the elements of $\partial V(X)$ coincide on the set of $\omega$'s satisfying $P(X = X(\omega)) = 0$:

**Proposition 4** *In addition to the general assumptions of the paper, assume that $L$ satisfies (5) and let $X \in L^\infty(\Omega, \mathcal{F}, P)$. Defining:*

$$\Omega_r := \{\omega \in \Omega \,:\, F_X \text{ is continuous at } X(\omega)\}$$

*then for any* $Z \in \partial V(X)$ *one has:*

$$Z(\omega) = \partial_x L(F_X(X(\omega)), X(\omega)) \text{ for } P\text{-a.e. } \omega \in \Omega_r.$$

**Proof.** The claim immediately follows from theorem 1 and the fact that for every $U \in \mathcal{U}_X$ one has $U = F_X \circ X$ $P$-a.s. on $\Omega_r$ by lemma 1.

□

## 4.2 Gâteaux differentiability

Let us recall that $V$ is Gâteaux-differentiable at $X \in L^\infty(\Omega, \mathcal{F}, P)$, if the map:

$$Y \in L^\infty(\Omega, \mathcal{F}, P) \mapsto D^+V(X;Y) := \lim_{t \to 0^+} \frac{1}{t}[V(X+tY) - V(X)]$$

defines a continuous linear form on $L^\infty(\Omega, \mathcal{F}, P)$, simply denoted $V'(X)$. It follows from (8) that $V$ is Gâteaux-differentiable at $X \in L^\infty(\Omega, \mathcal{F}, P)$, if and only if $\partial V(X)$ consists of a single element. In this case $\partial V(X) = \{V'(X)\}$. Under the following assumption (stronger than (5)):

$$t \mapsto \partial_x L(t,x) \text{ is decreasing on } [0,1], \text{ for all } x \in \mathbb{R}. \tag{9}$$

the set where $V$ is Gâteaux-differentiable is characterized by the following:

**Proposition 5** *In addition to the general assumptions of the paper, assume that* $L$ *satisfies (9) and let* $X \in L^\infty(\Omega, \mathcal{F}, P)$, $V$ *is Gâteaux-differentiable at* $X$ *if and only if* $F_X$ *is continuous, in this case, one has:*

$$\partial V(X) = \{V'(X)\} = \{\partial_x L(F_X \circ X, X)\}.$$

**Proof.** If $F_X$ is continuous then $\mathcal{V}_X = \mathcal{U}_X = \{F_X \circ X\}$. The Gâteaux-differentiability result then follows from theorem 1. To prove the "only if part" assume that $F_X$ is discontinuous at $x$ i.e. $P(X = x) > 0$ and let us prove that $\partial V(X)$ contains two different elements. Let $U \in \mathcal{U}_X$, since $U\mathbf{1}_{\{X=x\}}$ has no atom on $\{X = x\}$ and the space $(\{X = x\}, \mathcal{F} \cap \{X = x\}, P/P(\{X = x\}))$ is non atomic, there exists a uniform random variable $W$ on that space and $F$ increasing such that $U\mathbf{1}_{\{X=x\}} = F \circ W$. Define then

$$\widetilde{U} := U\mathbf{1}_{\{X \neq x\}} + F(1-W)\mathbf{1}_{\{X=x\}}.$$

On the one hand, by construction $\widetilde{U} \in \mathcal{U}$ and $V(X) = \mathbb{E}(L(U,X)) = \mathbb{E}(L(\widetilde{U}, X)$ hence $\widetilde{U} \in \mathcal{U}_X$. On the other hand, from theorem 1, both $\partial_x L(U,X)$ and $\partial_x L(\widetilde{U}, X)$ belong to $\partial V(X)$. Finally, by injectivity of $F$ and assumption (9), one has:

$$P(\{\partial_x L(\widetilde{U}, X) \neq \partial_x L(U,X)\}) = P(\{\widetilde{U} \neq U\}) = P(\{X = x\}) > 0$$

which proves that $V$ is not Gâteaux-differentiable at $X$. □

## 5 Proof of lemma 3

It remains to prove the following:

**Lemma 4** *Using the same notations as in the proof of theorem 1, there exists $\overline{U} \in \mathcal{V}_X$ such that the joint probability law of $(\overline{U}, X, Y)$ is $\theta$.*

**Proof.**

**Step1: preliminary remarks**

Let us recall that $\theta$ is a compactly supported probability measure on $\mathbb{R}^3$ whose first marginal (on the variable $u$ say) is the Lebesgue measure on $[0,1]$ and whose marginal on the last two variables $(x,y)$ is $P_{(X,Y)}$, the joint probability law of the pair $(X,Y)$. In the sequel, we shall denote by $\theta^{x,y}$ the conditional probability law of the first component $u$ given $x$ and $y$. To be more precise, these conditional probabilities are characterized by the fact that, for all Borel subsets $A$, $B$, $C$ of $\mathbb{R}$ we have:

$$\theta(A \times B \times C) := \int_{B\times C} \theta^{x,y}(A) dP_{(X,Y)}(x,y).$$

For notational simplicity, we set $(U_n, X_n) := (U_{t_n}, X + t_n Y)$. By definition of $\mathcal{U}_X$, for each $n$, $X_n = F_{X_n}^{-1} \circ U_n$, this can also be written in the form:

$$U_n \in \partial g_n(X_n) = \partial g_n(X + t_n Y) \tag{10}$$

For some convex function $g_n$, that we can assume to be 1-Lipschitz on $\mathbb{R}$ and to satisfy $g_n(0) = 0$. By Ascoli's Theorem, taking if necessary some (not relabeled) subsequence, we may assume that $g_n$ converges uniformly on compact sets to some convex function $g$. Let $S$ be the set where $g$ fails to be differentiable, we may write $S = \{x_i\}_{i\in I}$ with $I$ at most countable. Let us also define:

$$\Omega_i := \{X = x_i\} \ \forall\, i \in I, \text{ and } \Omega_r := \{X \notin S\}.$$

**Step 2: convergence on $\Omega_r$**

Let $\omega \in \Omega_r$, we claim that $U_n(\omega)$ converges to $g'(X(\omega))$. Indeed, $U_n(\omega)$ takes values of $[0,1]$ and if $u \in [0,1]$ is a cluster point of the sequence $U_n(\omega)$, using the fact that $X_n$ converges uniformly to $X$, we easily obtain $u \in \partial g(X(\omega))$ and since $\omega \in \Omega_r$ we deduce that $u = g'(X(\omega))$. This implies that $g'(X(\omega))$ is the unique cluster point of the sequence $U_n(\omega)$, hence $U_n(\omega)$ converges to $g'(X(\omega))$.

**Step 3: behavior on $\Omega_i$**

If $\omega \in \Omega_i$, (10) takes the form $U_n(\omega) \in \partial g_n(x_i + t_n Y(\omega))$ which can be rewritten as:

$$U_n(\omega) \in \partial h_{n,i}(Y(\omega)) \text{ with } h_{n,i}(y) := \frac{1}{t_n}(g_n(x_i + t_n y) - g_n(x_i)). \tag{11}$$

Noting that $(h_{n,i})_n$ is a family of 1-Lipschitz convex functions, arguing as in step 1, we may assume (after an extraction depending on $i$) that $h_{n,i}$ converges uniformly on compact subsets to some convex function $h_i$. By a diagonal extraction argument, we may also assume that $h_{n,i}$ converges to some $h_i$ for every $i \in I$. Let $S_i := \{y_{ij}\}_{j\in J_i}$ be the set where $h_i$ fails to be differentiable, and:

$$\Omega_{ij} := \{X = x_i,\ Y = y_{ij}\}\ \forall\, j \in J_i, \text{ and } \Omega_{r,i} := \Omega_i \cap \{Y \notin S_i\}.$$

By the same arguments as in step 1, $U_n$ converges to $h_i'(Y)$ on $\Omega_{r,i}$.

**Step 4: the case of $\Omega_{ij}$**

From the previous steps, the only case where we have no information on the convergence of $U_n(\omega)$ is when $\omega \in \Omega_{ij}$ with $i \in I$, $j \in J_i$ such that $P(\Omega_{ij}) > 0$. In that case, let us remark that $(\Omega_{ij}, \mathcal{F}\cap\Omega_{ij}, P/P(\Omega_{ij}))$ is non atomic hence there exists a random variable $U_{ij}$ on that space whose probability law is $q_{ij}$, where by definition $q_{ij} := \theta^{x_i,y_{ij}}$ denotes the conditional probability of $\theta$ given $x = x_i$ and $y = y_{ij}$.

**Step 5: construction of $\overline{U}$**

Let us define:

$$\overline{U} := \mathbf{1}_{\Omega_r} g'(X) + \sum_{i\in I} \mathbf{1}_{\Omega_{r,i}} h_i'(Y) + \sum_{i\in I} \sum_{j\in J_i\,:P(\Omega_{ij})>0} \mathbf{1}_{\Omega_{ij}} U_{ij} \tag{12}$$

and let us prove that the probability law of $(\overline{U}, X, Y)$ is $\theta$. Let $H \in C^0(\mathbb{R}^3, \mathbb{R})$, we have:

$$\begin{aligned}\int_{\mathbb{R}^3} H d\theta =& \lim_n \mathbb{E}(H(U_n, X_n, Y)) = \int_{\Omega_r} H(g'(X(\omega)), X(\omega), Y(\omega)) dP(\omega)\\ &+ \sum_{i\in I} \int_{\Omega_{r,i}} H(h_i'(Y(\omega)), X(\omega), Y(\omega)) dP(\omega)\\ &+ \sum_{i\in I}\sum_{j\in J_i} P(\Omega_{ij}) \int_0^1 H(u, x_i, y_{ij}) dq_{ij}(u)\\ =& \mathbb{E}(H(\overline{U}, X, Y))\end{aligned}$$

**Step 6: end of the proof**

It remains to prove that $\overline{U} \in \mathcal{V}_X$. The fact that $\overline{U} \in \mathcal{U}$ follows from the fact that the first marginal of $\theta$ is uniform on $[0, 1]$. Finally $V(X_n)$ converges to $V(X)$ so that:

$$V(X) = \lim_n \mathbb{E}(L(U_n, X_n) = \mathbb{E}(L(\overline{U}, X))$$

which proves $\overline{U} \in \mathcal{V}_X$.

□

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