Q Theory Without Adjustment Costs & Cash Flow Effects Without Financing Constraints[1]

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Abstract

Tobin's Q exceeds one, even without any adjustment costs, for a firm that earns rents as a result of monopoly power or of decreasing returns to scale in production. Even when there are no adjustment costs and marginal Q is always equal to one, Tobin's Q is informative about the firm's growth prospects. We show that investment is positively related to Tobin's Q (which is observable average Q). This effect can be quantitatively small, which has been taken as evidence of very high adjustment costs in the empirical literature, but here is consistent with no adjustment costs at all. In addition, cash flow has a positive effect on investment, and this effect is larger for smaller, faster growing and more volatile firms, even though capital markets are perfect. These results provide a new theoretical foundation for Q theory and also cast doubt on evidence of financing constraints based on cash flow effects on investment.

James Tobin (1969) introduced the ratio of the market value of a firm to the replacement cost of its capital stock–a ratio that he called "Q"–to measure the incentive to invest in capital.[1] Tobin's Q, as it has become known, is the empirical implementation of Keynes's (1936) notion that capital investment becomes more attractive as the value of capital increases relative to the cost of acquiring the capital. Neither Keynes nor Tobin provided a formal decision-theoretic analysis underlying the Q theory of investment. Lucas and Prescott (1971) developed a rigorous analysis of the capital investment decision in the presence of convex costs of adjustment, and observed that the market value of capital can be an important element of the capital investment decision, though they did not explicitly make the link to Tobin's Q.

The link between convex costs of adjustment and the Q theory of investment was made explicitly by Mussa (1977) in a deterministic framework and by Abel (1983) in a stochastic framework, though the papers based on convex adjustment costs focused on *marginal* Q–the ratio of the value of an additional unit of capital to its acquisition cost–rather than the concept of *average* Q introduced by Tobin. Hayashi (1982) bridged the gap between the concept of marginal Q dictated by the models based on convex adjustment costs and the concept of average Q, which is readily observable, by providing conditions, in a deterministic framework, under which marginal Q and average Q are equal. Specifically, marginal Q and average Q are equal for a competitive firm with a constant-returns-to-scale production function provided that the adjustment cost function is linearly homogeneous in the rate of investment and the level of the capital stock. Abel and Eberly (1994) extended Hayashi's analysis to the stochastic case and also analyzed the relationship between average Q and marginal Q in some special situations in which these two variables are not equal.

In the current paper, we develop a new theoretical basis for the empirical relationship between investment and Q that differs from the literature based on convex

[1] Brainard and Tobin (1968) introduced the idea that a firm's investment should be positively related to the ratio of its market value to the replacement value of its capital stock, though they did not use the letter Q to denote this ratio.

adjustment costs in two major respects. First, we will dispense with adjustment costs completely, and assume that a firm can instantaneously and completely adjust its capital stock by purchasing or selling capital at an exogenous price, without having to pay any costs of adjustment. Second, average Q and marginal Q will differ from each other. In the literature based on convex adjustment costs, when average Q and marginal Q differ, it is marginal Q that is relevant for the investment decision, which is unfortunate since average Q is more readily observable than marginal Q. In the current paper, it is average Q that is related to the rate of investment; in fact, marginal Q is identically equal to one in this model and hence it cannot be related to fluctuations in investment.

Both average Q and marginal Q would be identically equal to one for a competitive firm with a constant-returns-to-scale production function that can purchase and sell capital at an exogenous price without any cost of adjustment. In order for average Q to exceed one, the firm must earn rents through the ownership or exploitation of a scarce factor. In the traditional Q-theoretic literature, the convex adjustment cost technology is the source of rents for a competitive firm with a constant-returns-to-scale production function. In the current paper, which has no convex adjustment costs, rents are earned as a result of monopoly power or as a result of decreasing returns to scale in the production function. A contribution of this paper is to show that not only do these rents cause average Q to exceed one, but the investment-capital ratio of the firm is positively related to the contemporaneous value of average Q.

An important implication of traditional Q-theoretic models based on convex adjustment costs is that (marginal) Q is a sufficient statistic for the rate of investment. Other variables should not have any marginal explanatory for investment if Q is an explanatory variable. However, many empirical studies of investment and Q have rejected this implication by finding that cash flow has a significant effect on investment, even if Q is included as an explanatory variable. This finding has been interpreted by Fazzari, Hubbard, and Petersen (1988) and others as evidence of financing constraints facing firms. In the model we develop here, there are no financing constraints–capital markets are perfect–yet investment is positively related to cash flow in addition to Q. Furthermore, the effect of cash flow on investment is larger for smaller, faster growing, and more volatile firms, as has been found empirically. Thus the findings reported in the earlier empirical literature cannot be taken as evidence of financing constraints. The interpretation of cash flow effects as evidence of financing constraints is also called into question in a recent paper by Gomes (2001); in his quantitative model, optimal investment is sensitive to both Tobin's Q and cash flow, whether or not a cost of external finance is present. Similarly, Cooper and Ejarque (2001) numerically solve a model with quadratic adjustment costs and a concave revenue function, and also find that investment is sensitive to both Tobin's Q and cash flow in the absence of financing constraints.[2]

The model we develop is designed to be as simple as possible, yet rich enough to deliver interesting time-series variation in the investment-capital ratio, Tobin's Q, and the ratio of cash flow to the capital stock. Section 1 presents the firm's net revenue as an isoelastic function of its capital stock. The revenue function is subject to stochastic shocks that change its growth rate at random points in time. The optimal capital stock is derived in Section 1 and the consequent optimal rate of investment is derived in Section 2. Section 3 derives the value of the firm and Tobin's Q. The relationship among the investment-capital ratio, Tobin's Q, and the cash flow-capital stock ratio is analyzed in Section 4, and the effects of firm size, growth, and volatility on this relationship are analyzed in Section 5. Concluding remarks are presented in Section 6.

[2] A related, but different, interchange has recently occurred between Kaplan and Zingales (1997, 2000) and Fazzari, Hubbard, and Petersen (2000). Kaplan and Zingales have argued both empirically and theoretically that the sensitivity of investment to cash flow is not a reliable indicator of the degree of financial constraints. This interchange is distinct from the model presented here, since we have assumed no financial constraints at all, yet investment is sensitive to cash flow. Recent empirical work by Gomes, Yaron, and Zhang (2001) also finds no evidence of financing constraints facing firms.

1 The Decision Problem of the Firm

Consider a firm with capital stock K_t at time t, and assume that the firm's revenue (net of labor costs) at time t is

$$R_t = Z_t^{1-\alpha} K_t^{\alpha}, \tag{1}$$

where $0 < \alpha < 1$ and Z_t is a variable that could reflect productivity, the demand for the firm's output, or the price of labor. The assumption that the elasticity of revenue with respect to the contemporaneous capital stock is smaller than one reflects monopoly power or decreasing returns to scale in the production function.[3]

The variable Z_t is exogenous to the firm and has a time-varying growth rate, μ_t, so

$$\frac{dZ_t}{Z_t} = \mu_t dt. \tag{2}$$

[3]Suppose that the production function is $Y_t = A_t \left(K_t^{\gamma} N_t^{1-\gamma}\right)^s$ where Y_t is (nonstorable) output produced at time t, A_t is productivity at time t, N_t is labor employed at time t, $s > 0$ reflects the degree of returns to scale ($s = 1$ for constant returns to scale) and $0 < \gamma < 1$. The inverse demand curve for the firm's output is $P_t = h_t Y_t^{-\frac{1}{\varepsilon}}$ so that the price elasticity of demand is $\varepsilon > 1$. At time t, the firm chooses labor, N_t, to maximize revenue net of labor costs, $R_t = P_t Y_t - w_t N_t$ where w_t is the wage rate at time t. Substituting the production function and the inverse demand curve into the expression for revenue, the firm chooses N_t to maximize $R_t = h_t \left[A_t K_t^{\gamma s} N_t^{(1-\gamma)s}\right]^{1-\frac{1}{\varepsilon}} - w_t N_t$. The first-order condition for this maximization is $(1-\gamma) s \left(1 - \frac{1}{\varepsilon}\right) h_t \left[A_t K_t^{\gamma s} N_t^{(1-\gamma)s}\right]^{1-\frac{1}{\varepsilon}} = w_t N_t$. Therefore, the optimal value of N_t is $N_t = \left[(1-\gamma) s \left(1-\frac{1}{\varepsilon}\right) \frac{h_t}{w_t} \left(A_t K_t^{\gamma s}\right)^{1-\frac{1}{\varepsilon}}\right]^{\frac{1}{1-(1-\gamma)s\left(1-\frac{1}{\varepsilon}\right)}}$. Substitute the optimal value of N_t into the expression for revenue net of labor costs to obtain $R_t = Z_t^{1-\alpha} K_t^{\alpha}$, where $Z_t \equiv \chi^{\frac{1}{1-\alpha}} \left[w_t^{-(1-\gamma)s\left(1-\frac{1}{\varepsilon}\right)} h_t A_t^{1-\frac{1}{\varepsilon}}\right]^{\frac{\varepsilon}{(1-s)\varepsilon+s}}$, $\chi \equiv \left[1 - (1-\gamma) s \left(1-\frac{1}{\varepsilon}\right)\right] \left[(1-\gamma) s \left(1-\frac{1}{\varepsilon}\right)\right]^{\frac{(1-\gamma)s\left(1-\frac{1}{\varepsilon}\right)}{1-(1-\gamma)s\left(1-\frac{1}{\varepsilon}\right)}}$, and $\alpha \equiv \frac{\gamma s\left(1-\frac{1}{\varepsilon}\right)}{1-(1-\gamma)s\left(1-\frac{1}{\varepsilon}\right)}$.

For a competititve firm with constant returns to scale ($\varepsilon = \infty$ and $s = 1$), $\alpha = 1$. However, if the firm has some monopoly power ($\varepsilon < \infty$) or if it faces decreasing returns to scale ($s < 1$), then $\alpha < 1$. Since Z_t is an isoelastic function of h_t, A_t, and w_t (with different, but constant elasticities, with respect to these three variables), the growth rate of Z_t is a weighted average of the growth rates of h_t, A_t, and w_t, with the weights equal to the corresponding elasticities.

If the growth rate μ_t were constant over time, the future growth prospects for the firm would always look the same, and, as we will show, there would be no time-series variation in the present value of the firm's future operating profits relative to current operating profits (the present value in equation (20) would be a constant multiple of contemporaneous Z_t).[4] To introduce some interesting, yet tractable, variation in the firm's growth prospects, we specify a simple form of variation in μ_t. The growth rate μ_t remains constant for a random length of time. A new value of μ_t arrives with constant probability λ; the new value of the growth rate is drawn from an unchanging distribution $F(\mu)$ with finite support $[\mu_L, \mu_H]$. The draws of new values of μ_t are i.i.d.

The firm can purchase or sell capital instantaneously and frictionlessly, without any costs of adjustment, at a constant price that we normalize to be one. Because there are no costs of adjustment, we can use Jorgenson's (1963) insight that the optimal path of capital accumulation can be obtained by solving a sequence of static decisions using the concept of the user cost of capital. With the price of capital constant and equal to one, the user cost of capital, υ_t, is

$$\upsilon_t \equiv r + \delta_t \tag{3}$$

where r is the discount rate used by the firm and δ_t is the depreciation rate of capital, which follows a diffusion process that is independent of Z_t. We will discuss the stochastic properties δ_t later in this section. For the narrow goal of studying the relationship between investment and Tobin's Q, we could simply assume that δ_t is constant. Variation in δ_t will be useful when we examine the effect of cash flow on investment.

At time t the firm chooses K_t to maximize operating profit, π_t, which equals revenue less operating costs[5]

[4] If Z_t follows a geometric Brownian motion with constant drift, μ, but continuous shocks, the present value in equation (20) would still be a constant multiple of Z_t. Specifically, for $\tau > 0$, $\frac{1}{\tau} \ln E_t \left(\frac{Z_{t+\tau}}{Z_t} \right) = \mu$ regardless of the variance of the continuous shocks. Time variation in the rate of drift eliminates this feature.

[5] Formally, the firm chooses K_t to maximize $V_t - p_t K_t$, which is equivalent to maximizing

$$\pi_t \equiv R_t - \upsilon_t K_t = Z_t^{1-\alpha} K_t^{\alpha} - \upsilon_t K_t. \tag{4}$$

Differentiating equation (4) with respect to K_t and setting the derivative equal to zero yields the optimal value of the capital stock

$$K_t = Z_t \left(\upsilon_t / \alpha\right)^{\frac{-1}{1-\alpha}}. \tag{5}$$

Substituting the optimal capital stock from equation (5) into equations (4) and (1) yields the optimal level of operating profit

$$\pi_t = (1-\alpha) Z_t \left(\upsilon_t / \alpha\right)^{\frac{-\alpha}{1-\alpha}} \tag{6}$$

and the optimal level of revenue (net of labor cost)

$$R_t = \frac{1}{1-\alpha} \pi_t. \tag{7}$$

Empirical investment equations often use a measure of cash flow, normalized by the capital stock, as an explanator of investment. Since R_t is defined as revenue net of labor costs, it is cash flow before investment expenditure. Let $c_t \equiv R_t / K_t$ be the cash flow before investment normalized by the capital stock, and note, for later use, that

$$c_t = \frac{1}{1-\alpha} \frac{\pi_t}{K_t} = \frac{\upsilon_t}{\alpha}, \tag{8}$$

where the first equality follows from equation (7) and the second equality follows from equations (6) and (5).

Define

$$M_t \equiv \left(\frac{\upsilon_t}{\alpha}\right)^{\frac{-\alpha}{1-\alpha}} = \left(\frac{r+\delta_t}{\alpha}\right)^{\frac{-\alpha}{1-\alpha}}. \tag{9}$$

M_t is a random variable with stochastic properties induced by the stochastic properties of the depreciation rate δ_t. M_t is independent of Z_t because δ_t is independent of Z_t. Instead of specifying the stochastic properties of δ_t, we will specify the stochastic

$E_t \left\{ \int_t^\infty \pi_s e^{-r(s-t)} ds \right\}$ (see equation A.5). The integral $E_t \left\{ \int_t^\infty \pi_s e^{-r(s-t)} ds \right\}$ can be maximized by choosing K_s to maximize π_s at each s.

properties of M_t, which implies stochastic properties of δ_t. Suppose that M_t is the following geometric Brownian motion

$$\frac{dM_t}{M_t} = -\frac{1}{2}\sigma^2 dt + \sigma dz. \tag{10}$$

The variable M_t is a martingale, that is, $E_t\left\{M_{t+\tau}\right\} = M_t$ for $\tau > 0$. We assume that M_t is a martingale to simplify the calculation of present values later in the paper.

Using the definition of M_t, rewrite the expressions for the optimal capital stock and the optimal operating profit in equations (5) and (6) as

$$K_t = Z_t M_t^{\frac{1}{\alpha}} \tag{11}$$

and

$$\pi_t = (1-\alpha)\, Z_t M_t. \tag{12}$$

In Section 2 we examine the firm's investment by analyzing the evolution of the optimal capital stock in equation (11). Then in Section 3 we use the expression for the optimal operating profit in equation (12) to compute the value of the firm.

2 Investment

To calculate net investment normalized by the capital stock, apply Ito's lemma to equation (11) and use equations (2) and (10) to obtain

$$\frac{dK_t}{K_t} = \left[\mu_t + \frac{1}{2}\frac{1-2\alpha}{\alpha^2}\sigma^2\right] dt + \frac{1}{\alpha}\sigma dz. \tag{13a}$$

Adding the depreciation rate of capital, δ_t, to net investment per unit of capital in equation (13a) yields gross investment per unit of capital[6]

$$\frac{dI_t}{K_t} = \left[\mu_t + \delta_t + \frac{1}{2}\frac{1-2\alpha}{\alpha^2}\sigma^2\right] dt + \frac{1}{\alpha}\sigma dz. \tag{14}$$

Investment is a linear function of the growth rate μ_t, the depreciation rate δ_t, and a constant-variance mean-zero disturbance that is independent of μ_t and δ_t. If μ_t and

[6]We write investment as dI_t because the right hand side of equation (14) has infinite variation.

δ_t were both observable then we could use OLS to estimate a regression of investment on μ_t and δ_t. However, μ_t is not observable and δ_t may not be well measured. We will show in later sections that movements in μ_t are reflected by movements in Tobin's Q and movements in δ_t are reflected by movments in the firm's cash flow per unit of capital. Thus, Tobin's Q and cash flow per unit of capital can help to explain investment empirically.

3 The Value of the Firm

The value of the firm is the expected present value of its revenues minus expenditures on capital. To ensure that the value of the firm is finite, we impose the following two conditions[7]

$$r + \lambda - \mu_H > 0 \tag{15}$$

and

$$E\left\{\frac{\lambda}{r + \lambda - \mu_t}\right\} < 1. \tag{16}$$

We show in Appendix A that the expected present value of revenues minus expenditures on capital can be written as the value of the replacement cost of the current capital stock, plus the expected present value of operating profits. Therefore, the value of the firm at time t is

$$V_t = K_t + E_t\left\{\int_t^\infty \pi_{t+\tau} e^{-r\tau} d\tau\right\}. \tag{17}$$

As a step toward calculating the present value in equation (17), use equation (12), the independence of M_t and Z_t, and the fact that M_t is a martingale to obtain

$$E_t\left\{\pi_{t+\tau}\right\} = (1-\alpha) M_t E_t\left\{Z_{t+\tau}\right\}. \tag{18}$$

Substituting equation (18) into equation (17) yields

$$V_t = K_t + (1-\alpha) M_t \int_t^\infty E_t\left\{Z_{t+\tau}\right\} e^{-r\tau} d\tau. \tag{19}$$

[7] The condition $r > \mu_H$ is sufficient for the conditions in equations (15) and (16) to hold.

We show in Appendix B that the value of the integral on the right hand side of equation (19) is[8]

$$\int_t^\infty E_t\left\{Z_{t+\tau}\right\}e^{-r\tau}d\tau=\frac{\omega}{r+\lambda-\mu_t}Z_t, \tag{20}$$

where

$$\omega\equiv\left[E\left\{\frac{r-\mu_t}{r+\lambda-\mu_t}\right\}\right]^{-1}>0. \tag{21}$$

Note that when the arrival rate λ is zero, so that the growth rate of Z_t remains μ_t forever, $\omega=1$ and the present value of the stream of $Z_{t+\tau}$ is simply $Z_t/\left(r-\mu_t\right)$. More generally, when the growth rate μ_t varies over time, a high value of μ_t implies a high value of the present value in equation (20).

The value of the firm can now be obtained by substituting equation (20) into equation (19), and recalling from equation (12) that $\pi_t=(1-\alpha)\,Z_tM_t$,to obtain

$$V_t=K_t+\frac{\omega\pi_t}{r+\lambda-\mu_t}. \tag{22}$$

Tobin's Q is ratio of the value of the firm to the replacement cost of the firm's capital stock. Since the price of capital is identically equal to one, the replacement cost of the firm's capital stock is simply K_t. Dividing the value of the firm in equation

[8]To derive the present value of $Z_{t+\tau}$ heuristically, let $P_t=P\left(\mu_t,Z_t\right)\equiv\int_t^\infty E_t\left\{Z_{t+\tau}\right\}e^{-r\tau}d\tau$ be the price of a claim on the infinite stream of $Z_{t+\tau}$. The expected return on this claim over an interval dt of time is $Z_tdt+E_t\{dP_t\}$. Because the path of future growth rates of Z_t is independent of the current value of Z_t, $P\left(\mu_t,Z_t\right)$ can be written as $p\left(\mu_t\right)Z_t$. The expected change in P_t is $E_t\left\{dP_t\right\}=\lambda Z_t\left[p^*-p\left(\mu_t\right)\right]dt+\mu_tZ_tp\left(\mu_t\right)dt$, where p^* is the unconditional expectation of $p\left(\mu_t\right)$ so the first term is the expected change arising from a new drawing of the growth rate μ_t and the second term reflects the fact that the growth rate of Z_t is μ_t. Setting this expected return equal to the required return $rp\left(\mu_t\right)Z_tdt$, and solving yields $\left(r+\lambda-\mu_t\right)p\left(\mu_t\right)=1+\lambda p^*$. Therefore,

$$p\left(\mu_t\right)=\frac{1+\lambda p^*}{r+\lambda-\mu_t}.$$

Taking the unconditional expectation of both sides of this expression yields $p^*=\left(1+\lambda p^*\right)E\left\{\frac{1}{r+\lambda-\mu_t}\right\}$, which can be rearranged to obtain $p^*=\left[E\left\{\frac{r-\mu_t}{r+\lambda-\mu_t}\right\}\right]^{-1}E\left\{\frac{1}{r+\lambda-\mu_t}\right\}$. Therefore $1+\lambda p^*=\left[E\left\{\frac{r-\mu_t}{r+\lambda-\mu_t}\right\}\right]^{-1}$, so $p\left(\mu_t\right)=\frac{1}{r+\lambda-\mu_t}\left[E\left\{\frac{r-\mu_t}{r+\lambda-\mu_t}\right\}\right]^{-1}$.

(22) by K_t yields

$$Q_t \equiv \frac{V_t}{K_t} = 1 + \frac{\omega}{r + \lambda - \mu_t} \frac{\pi_t}{K_t}. \tag{23}$$

Tobin's Q is greater than one because the firm earn's rents π_t. In the absence of rents, Tobin's Q would be identically equal to one because the firm can costlessly and instantaneously purchase and sell capital at a price of one. Sargent (1980?) also develops a model without convex adjustment costs in which Tobin's Q can differ from one. In Sargent's model, Tobin's Q can never exceed one because firms are competitive and do not earn rents, and they can always acquire additional capital instantly at a price of one. However, because investment is irreversible in Sargent's model, Tobin's Q can fall below one.

In the present model, the presence of rents π_t is sufficient to make Tobin's Q greater than one. However, rents alone do not imply that Tobin's Q will vary over time for a firm. If Z_t were to grow at constant rate, so that μ_t were constant, and if the user cost υ_t were constant, so that $\frac{\pi_t}{K_t}$ were constant (see equation 8), then equation (23) shows that Tobin's Q would be constant and greater than one. However, we have modeled the growth rate μ_t as stochastic, and equation (23) shows that Tobin's Q is an increasing function of the contemporaneous growth rate μ_t.

Tobin's Q is often called *average* Q to distinguish it from $\partial V / \partial K$, which is often called *marginal* q. In the q-theoretic literature based on convex adjustment costs for capital, marginal q is the relevant concept for investment. In fact, an optimality condition is that the marginal adjustment is equated with marginal q. In the current model there are no convex adjustment costs. With the price of capital identically equal to one, the marginal adjustment cost equals one. Inspection of equation (22) immediately reveals that marginal q, $\partial V / \partial K$, is also identically equal to one. Though the marginal adjustment cost and marginal q are equal, the equality of these two concepts does not pin down the optimal rate of investment in the absence of nonlinear costs of adjustment.

4 The Effects of Tobin's Q and Cash Flow in Investment

We have shown (equation 14) that the optimal rate of investment depends on the growth rate μ_t and on the depreciation rate δ_t. However, the growth rate μ_t is not observable and the depreciation rate δ_t may not be well measured. In this section, we show that the growth rate μ_t can be written as a function Tobin's Q and cash flow c_t, and that depreciation δ_t is related to cash flow c_t. Thus, to the extent that Q_t and c_t reflect μ_t and δ_t, these variables can help account for movements in investment.

First, we show that the growth rate μ_t can be expressed in terms of the observable variables Q_t and c_t. Use equation (8) to substitute $(1-\alpha)\,c_t$ for $\frac{\pi_t}{K_t}$ in equation (23) to obtain

$$Q_t = 1 + (1-\alpha)\,\omega\frac{c_t}{r+\lambda-\mu_t} \tag{24}$$

Multiply both sides of equation (24) by $r+\lambda-\mu_t$ and rearrange to obtain an expression for the growth rate in terms of the observable values of Tobin's Q and cash flow normalized by the capital stock

$$\mu_t = r+\lambda-(1-\alpha)\,\omega\frac{c_t}{Q_t-1}. \tag{25}$$

Now substitute the expression for μ_t from equation (25) into equation (14) to obtain

$$\frac{dI_t}{K_t} = \left(r+\delta_t+\lambda-(1-\alpha)\,\omega\frac{c_t}{Q_t-1}+\frac{1}{2}\frac{1-2\alpha}{\alpha^2}\sigma^2\right)dt+\frac{1}{\alpha}\sigma dz. \tag{26}$$

Use the definition of the user cost of capital, $\upsilon_t \equiv r+\delta_t$, and the fact that $\upsilon_t = \alpha c_t$ from equation (8) to rewrite equation (26) as

$$\frac{dI_t}{K_t} = \left(\alpha-\frac{(1-\alpha)\,\omega}{Q_t-1}\right)c_t dt+\left(\lambda+\frac{1}{2}\frac{1-2\alpha}{\alpha^2}\sigma^2\right)dt+\frac{1}{\alpha}\sigma dz. \tag{27}$$

Equation (27) shows that investment depends on the observable cash flow, c_t, and the observable value of Tobin's Q, in addition to the stochastic component dz. Let

$$\iota\,(Q_t, c_t) \equiv \left(\alpha-\frac{(1-\alpha)\,\omega}{Q_t-1}\right)c_t dt \tag{28}$$

be the component of investment that depends on the observable variables Q_t and c_t. Use the definition of $\iota(Q_t, c_t)$ in equation (28) to rewrite the investment-capital ratio in equation (27) as

$$\frac{dI_t}{K_t} = \iota(Q_t, c_t) + \left(\lambda + \frac{1}{2}\frac{1-2\alpha}{\alpha^2}\sigma^2\right) dt + \frac{1}{\alpha}\sigma dz. \tag{29}$$

We will analyze the effects of Q_t and c_t on investment by analyzing the effects of these variables on $\iota(Q_t, c_t)$.

First we analyze the effect of Tobin's Q on investment. Let $\beta_Q \equiv \partial\iota(Q_t, c_t)/\partial Q_t$ denote the response of the investment-capital ratio to an increase in Q_t. Partially differentiating $\iota(Q_t, c_t)$ with respect to Q_t yields

$$\beta_Q \equiv \frac{\partial\iota(Q_t, c_t)}{\partial Q_t} = \frac{(1-\alpha)\,\omega c_t}{(Q_t - 1)^2} dt > 0, \tag{30}$$

so that investment is an increasing function of Q_t. The positive relationship between investment and Tobin's Q has some remarkable differences from the relationship in the standard convex adjustment cost framework. The positive relationship between investment and Q_t arises in the standard framework because of the convexity of the adjustment cost function. In addition, the convexity of the adjustment cost function is measured by the coefficient in a regression of the investment-capital ratio on Q_t. The estimated coefficient of the investment-capital ratio on Q_t, which is the analogue of β_Q in equation (30), is typically quite small, which is usually interpreted to mean that adjustment costs are very convex. In the model we present here, investment depends positively on Q_t, that is, $\beta_Q > 0$, *even though there are no convex costs of adjustment*. In addition, it is quite possible for β_Q to be small (if $(1-\alpha)\,\omega c_t$ is small or if Q_t is large). Yet, in this model, without convex adjustment costs, the small value of β_Q cannot indicate strongly convex adjustment costs, as in standard interpretations.

Another remarkable difference from standard models based on convex capital adjustment costs is that the investment-capital ratio is related to average Q, $\frac{V_t}{K_t}$, rather than to marginal q, $\frac{\partial V_t}{\partial K_t}$, which equals one in this model.[9] The relationship be-

[9] Caballero and Leahy (1996) and Abel and Eberly (1998) analyze optimal investment in the

tween investment and average Q in our model is noteworthy because average Q is observable, whereas marginal Q is not observable. The link between investment and Tobin's Q arises here because, even in the absence of adjustment costs, investment is a dynamic phenomenon. That is, investment is the growth of the capital stock (plus depreciation) and the growth of the optimal capital stock depends on the growth rate μ_t. Since Q_t also depends on μ_t , it contains information about the growth of the capital stock.

Equation (27) has another remarkable feature. Even after taking account of Q_t on the rate of investment, investment also depends on normalized cash flow c_t. Empirical studies of investment often find that the firm's cash flow is positively related to the rate of investment, even when a measure of Q is included as an explanator of investment. A typical empirical equation, starting from Fazzari, Hubbard, and Petersen (1988), would have the investment-capital ratio as the dependent variable, and Tobin's Q and $c_{t,}$ the ratio of the firm's cash flow to its capital stock, as the dependent variables. The finding of a positive cash flow effect is often interpreted as evidence of a financing constraint facing the firm.

To analyze the effect of cash flow on investment in our model, let $\beta_c \equiv \frac{\partial\iota(Q_t,c_t,)}{\partial c_t}$ denote the response of the investment-capital ratio to an increase in cash flow per unit of capital, c_t. Differentiate equation (28) with respect to c_t to obtain

$$\beta_c \equiv \frac{\partial\iota\left(Q_t, c_t\right)}{\partial c_t} = \alpha - \frac{\left(1-\alpha\right)\omega}{Q_t - 1}. \tag{31}$$

At this level of generality, the sign of the right hand side of equation (31) could be either positive or negative, so the effect of c_t on the investment-capital ratio could be either positive or negative. When the empirical literature finds a significant effect of cash flow on investment, the effect is typically positive, so we present a condition below for the right hand side of equation (31) to be positive.

Use equation equation (24) to substitute $\frac{(1-\alpha)\omega}{r+\lambda-\mu_t}c_t$ for $Q_t - 1$ in equation (31), then use (8) to substitute $\frac{\upsilon_t}{\alpha}$ for c_t, and use the definition of the user cost, $\upsilon_t \equiv r + \delta_t$, to

presence of a fixed cost of investment and find that investment is related to average Q.

obtain

$$\beta_c = \alpha \frac{\mu_t + \delta_t - \lambda}{r + \delta_t}. \tag{32}$$

Condition 1 $\delta_t + \mu_t > \lambda$ *for all t.* [10]

Inspection of equation (32) reveals that Condition 1 is necessary and sufficient for $\beta_c > 0$.

Henceforth we will assume that Condition 1 holds so that $\beta_c > 0$. Although the traditional literature would interpret this positive relationship between cash flow and investment as evidence of a financing constraint, the positive effect arises in this model even though capital markets are perfect and there are no financing constraints. A positive cash flow effect on investment in the absence of financing constraints undermines the logical basis for the common tests of financing constraints in the literature.

The positive time-series relationship between investment and cash flow for a given firm operates through the user cost factor, υ_t. As we discussed in Section 2, an increase in υ_t arising from an increase in the depreciation rate, δ_t, will increase gross investment relative to the capital stock. As is evident from equation (8), an increase in υ_t also increases the ratio of cash flow to the capital stock. Thus, the positive time-series association between cash flow and investment reflects the fact that each of these variables moves in the same direction in response to an increase in the user cost factor.

5 The Effects of Firm Size, Growth and Volatility on the Cash Flow Effect

The empirical literature on investment has found that cash flow has a more significant positive effect on investment for firms that are small, growing quickly, or volatile.

[10] Condition 1 places an upper bound on λ and equation 15 places a lower bound on λ. A nondegenerate range of values of λ will satisfy both of these bounds provided that $\delta_t + \mu_t > r - \mu_H$.

This finding has been interpreted as evidence that these firms face binding financial constraints, while large, slowly growing, stable firms are either less constrained or financially unconstrained. This conclusion is perhaps appealing because it coheres well with the notion that small, rapidly growing, volatile firms do not have as much access to capital markets and external financing as large, slowly growing, stable firms have. In this section we show that in our model, the effect of cash flow on investment, measured by β_c, is larger for firms that are small, rapidly growing, and volatile than for firms that do not display these characteristics, which is consistent with empirical findings, even though there are no financial constraints in our model.[11]

We will measure the size of the firm by R_t, revenue net of labor costs. This is equivalent to measuring firm size by operating profit π_t because R_t is proportional to π_t. To derive an expression for R_t in terms of the exogenous variables Z_t, μ_t, and δ_t, substitute equation (6) into equation (7), and use the definition of the user cost, $\upsilon_t \equiv r + \delta_t$, to obtain

$$R_t = Z_t \left(\frac{r + \delta_t}{\alpha} \right)^{\frac{-\alpha}{1-\alpha}}. \tag{33}$$

Equation (33) implies that small firms have low values of Z_t or high values of the depreciation rate δ_t. Cross-sectional variation in firm size resulting from variation in Z_t alone has no systematic effect on the cash flow coefficient β_c in equation (32). However, cross-sectional variation in firm size resulting from variation in δ_t has a systematic effect on β_c. To analyze this effect, partially differentiate equation (32)

[11] Fazzari, Hubbard, and Petersen (1988), as well as a large subsequent literature, use firm size as a proxy for the severity of the firm's financial constraints. Other commonly used proxies include dividend payouts, debt, interest coverage, and bond ratings, such as in Whited (1992) and Gilchrist and Himmelberg (1995). Later work has also examined the composition of external finance (Kashyap, Stein, and Wilcox (1993)). If these financial variables are unrelated to the real characteristics of the firm that we examine below, then evidence using these variables to identify financial constraints is not subject to the confounding of financial effects and firm characteristics that we develop in this section.

with respect to δ_t to obtain

$$\frac{\partial \beta_c}{\partial \delta_t} = \frac{\alpha}{(r+\delta_t)^2}(r+\lambda-\mu_t) > 0 \tag{34}$$

where we have used equation (15) to determine that the right hand side of equation (34) is positive. Therefore, an increase in the depreciation rate δ_t increases the cash flow coefficient β_c. Thus, firms that are small as the result of high depreciation rates also tend to have large cash flow coefficients, which is consistent with empirical finding that cash flow effects are stronger for smaller firms.

The empirical literature sometimes identifies fast-growing firms as firms likely to face binding financing constraints and finds that these firms have larger cash flow coefficients than slow-growing firms. Inspection of equation (32) reveals that the cash flow coefficient β_c is an increasing function of the growth rate μ_t, which is consistent with the empirical findings. Again, our model without any financing constraints is consistent with empirical findings that have been interpreted as evidence of financing constraints.

6 Concluding Remarks

In this paper we have developed a new explanation for the empirical time-series relationship between investment and Tobin's Q. Traditional explanations of this relationship are based on convex costs of adjusting the capital stock. In this paper, we have assumed that there are no such adjustment costs that drive a wedge between the purchase price of capital and the market value of installed capital. Instead, the wedge between the market value of a firm and the replacement cost of its capital stock is based on rents accruing to market power or to decreasing returns to scale in the production function. The presence of these rents implies that Tobin's Q exceeds one.

Beyond showing that Q exceeds one, we showed that the investment-capital ratio is positively related to Tobin's Q, which is a measure of average Q, rather than to marginal Q, as in the adjustment cost literature. This departure from the adjustment cost literature is particularly important because average Q is readily observable, whereas marginal Q is not directly observable. In the empirical literature, relatively small responses of investment to Q have been taken as evidence of large adjustment costs; here there are no adjustment costs at all, and yet the response of investment to Q is small.

In addition to being consistent with a positive relationship between investment and Tobin's Q, the model in this paper can account for the positive effect of cash flow on investment. The common interpretation of the positive cash flow effect on investment is that it is evidence of financing constraints facing firms. However, the model in this paper has perfect capital markets without financing constraints, and yet cash flow has a positive effect on investment, even after taking account of the effect of Q on investment. Therefore, contrary to the common interpretation, a positive cash flow effect on investment need not be evidence of a financing constraint.

The empirical literature has recognized that the investment regression may be misspecified or mismeasured, leading to spurious cash flow effects. One strategy to address these potentially spurious effects is to split the sample into *a priori* financially constrained and unconstrained firms. Typically, smaller, faster growing, and more volatile firms, which are often classified *a priori* as financially constrained, are found to have larger cash flow effects. The same pattern of cash flow effects emerges in our model, even though there are no financing constraints in the model, which calls into question the interpretation of the empirical findings.

The model in this paper is, by design, very simple and stylized. In order for Tobin's Q to exhibit time-series variation, we assumed that the growth rate of Z_t varies over time, though in a very simple way. In order for the ratio of cash flow to the capital stock to exhibit time-series variation, the user cost factor must vary, and we induced this variation by assuming that the depreciation rate follows a specific stochastic process. We eliminated adjustment costs from the current analysis, not because we believe they are not relevant for an empirical investment model, but rather because they are extraneous to the effects we examine here. The goal of the current paper is to articulate the relationship among investment, Tobin's Q, and cash flow. Empirical findings regarding these relationships have been used to detect the presence of adjustment costs and financing constraints, and to evaluate their importance for investment. Even when these adjustment costs and financing constraints are eliminated, however, we show that investment remains sensitive to both Tobin's Q and cash flow, undermining the power of the empirical argument. An avenue for future research would be to introduce richer and more realistic processes for the various exogenous variables facing the firm. Another direction would be to introduce delivery or gestation lags in the capital investment process. In ongoing research (Abel and Eberly, 2002), we endogenize the growth in technology, summarized here by an exogenous parameter. In that framework, the effects we have examined here also arise.

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A Appendix: Calculating the Value Function by Stochastic Integration by Parts

Suppose that u and v are diffusion processes. Ito's lemma implies that $d(uv) = udv + vdu + (du)(dv)$ so that

$$uv \mid_a^b = \int_a^b d(uv) = \int_a^b udv + \int_a^b vdu + \int_a^b (du)(dv). \tag{A.1}$$

Now use this expression to calculate $E_t \left\{ \int_{t_0}^{\infty} e^{-r(t-t_0)} dK_t \right\}$. Let

$$u \equiv K_t$$

and

$$v \equiv e^{-r(t-t_0)}$$

and assume that

$$du \equiv dK_t = \mu_K(t) K_t dt + \sigma_K K_t dz_K$$

and observe that

$$dv = -re^{-r(t-t_0)} dt.$$

Set $a = t_0$ and $b = \infty$, use the facts that $dt^2 = 0 = dtdz_K$, and rearrange equation (A.1) to obtain

$$\int_{t_0}^{\infty} e^{-r(t-t_0)} dK_t = -K_{t_0} + \int_{t_0}^{\infty} rK_t e^{-r(t-t_0)} dt. \tag{A.2}$$

Take the expectation of both sides of equation (A.2) to obtain

$$E_{t_0} \left\{ \int_{t_0}^{\infty} e^{-r(t-t_0)} dK_t \right\} = -K_{t_0} + rE_{t_0} \left\{ \int_{t_0}^{\infty} K_t e^{-r(t-t_0)} dt \right\}. \tag{A.3}$$

The value of the firm is the expected present value of its revenues minus expenditures on capital,

$$V_t = E_t \left\{ \int_t^{\infty} \left[R_s - \delta_s K_s \right] e^{-r(s-t)} ds - \int_t^{\infty} e^{-r(s-t)} dK_s \right\}. \tag{A.4}$$

Substitute equation (A.3) for the second integral on the right-hand side of equation (A.4) and use the definition of operating profits in equation (4) to obtain the following expression for the value of the firm at time t

$$V_t = K_t + E_t \left\{ \int_t^{\infty} \pi_s e^{-r(s-t)} ds \right\}. \tag{A.5}$$

B Appendix: Expected Present Value of a Stream with Variable Drift

Let $P(\mu_t, Z_t) = p(\mu_t) Z_t$, where $p(\mu_t) \equiv E_t \left\{ \int_0^{\infty} \frac{Z_{t+\tau}}{Z_t} e^{-r\tau} d\tau \right\}$. Let $p(\mu_t, T)$ be the value of $p(\mu_t)$ conditional on the assumption that the growth rate of Z_t remains equal to μ_t until time $t+T$, and that a new value of the growth rate is drawn from the unconditional distribution at time $t+T$. Therefore,

$$p(\mu_t, T) = \int_0^T e^{-(r-\mu_t)\tau} d\tau + e^{-(r-\mu_t)T} E_t \left\{ \int_T^{\infty} \frac{Z_{t+\tau}}{Z_{t+T}} e^{-r(\tau-T)} d\tau \right\}. \tag{B.1}$$

Evaluating the first integral on the right hand side of equation (B.1) and rewriting the second integral yields

$$p(\mu_t, T) = \frac{1 - e^{-(r-\mu_t)T}}{r - \mu_t} + e^{-(r-\mu_t)T} E_t \left\{ \int_0^{\infty} \frac{Z_{t+T+\tau}}{Z_{t+T}} e^{-r\tau} d\tau \right\}. \tag{B.2}$$

Let p^* be the expectation of $p(\mu_t)$ when μ_t is drawn from its unconditional distribution, so that equation (B.2) can be written as

$$p(\mu_t, T) = \frac{1 - e^{-(r-\mu_t)T}}{r - \mu_t} + e^{-(r-\mu_t)T} p^*. \tag{B.3}$$

The density of T is

$$f(T) = \lambda e^{-\lambda T} \tag{B.4}$$

and

$$p(\mu_t) = \int_0^{\infty} p(\mu_t, T) f(T) dT. \tag{B.5}$$

Substituting equations (B.3) and (B.4) into equation (B.5) yields

$$p(\mu_t) = \int_0^{\infty} \left[\frac{1 - e^{-(r-\mu_t)T}}{r - \mu_t} + e^{-(r-\mu_t)T} p^* \right] \lambda e^{-\lambda T} dT. \tag{B.6}$$

Equation (B.6) can be rewritten as

$$p(\mu_t) = \frac{1}{r-\mu_t}\left[\int_0^{\infty}\left[1+(rp^* - \mu_t p^* - 1)\,e^{-(r-\mu_t)T}\right]\lambda e^{-\lambda T}dT\right]. \tag{B.7}$$

Evaluating the integral in equation (B.7) yields

$$p(\mu_t) = \frac{1}{r-\mu_t}\left[1+(rp^* - \mu_t p^* - 1)\frac{\lambda}{r+\lambda-\mu_t}\right], \tag{B.8}$$

which can be rearranged to yield

$$p(\mu_t) = \frac{1+\lambda p^*}{r+\lambda-\mu_t}. \tag{B.9}$$

Since $p^* = E\{p(\mu_t)\}$, take the unconditional expectation of both sides of equation (B.9) to obtain

$$p^* = E\left\{\frac{1}{r+\lambda-\mu_t}\right\}(1+p^*\lambda), \tag{B.10}$$

which implies

$$p^* = \left[E\left\{\frac{r-\mu_t}{r+\lambda-\mu_t}\right\}\right]^{-1} E\left\{\frac{1}{r+\lambda-\mu_t}\right\}. \tag{B.11}$$

Substituting equation (B.11) into equation (B.9) yields

$$p(\mu_t) = \frac{\omega}{r+\lambda-\mu_t}, \tag{B.12}$$

where

$$\omega \equiv \left[E\left\{\frac{r-\mu_t}{r+\lambda-\mu_t}\right\}\right]^{-1}. \tag{B.13}$$

Therefore,

$$P(\mu_t, Z_t) = \frac{\omega}{r+\lambda-\mu_t}Z_t. \tag{B.14}$$